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Analysis of Iterative Waterfilling Algorithm for Multiuser Power Control in Digital Subscriber Lines

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We present an equivalent linear complementarity problem (LCP) formulation of the noncooperative Nash game resulting from the DSL power control problem. Based on this LCP reformulation, we establish the linear convergence of the popular distributed iterative waterfilling algorithm (IWFA) for arbitrary symmetric interference environment and for certain asymmetric channel conditions with any number of users. In the case of symmetric interference crosstalk coefficients, we show that the users of IWFA in fact, unknowingly but willingly, cooperate to minimize a common quadratic cost function whose gradient measures the received signal power from all users. This is surprising since the DSL users in the IWFA have no intention to cooperate as each maximizes its own rate to reach a Nash equilibrium. In the case of asymmetric coefficients, the convergence of the IWFA is due to a contraction property of the iterates. In addition, the LCP reformulation enables us to solve the DSL power control problem under no restrictions on the interference coefficients using existing LCP algorithms, for example, Lemke's method. Indeed, we use the latter method to benchmark the empirical performance of IWFA in the presence of strong crosstalk interference.

1. INTRODUCTION

In modern DSL systems, all users share the same frequency band and crosstalk is known to be the dominant source of interference. Since the conventional interference cancellation schemes require access to all users' signals from different vendors in a bundled cable, they are difficult to implement in an unbundled service environment. An alternative strategy for reducing crosstalk interference and increasing system throughput is power control whereby interference is controlled (rather than cancelled) through the judicious choice of power allocations across frequency. This strategy does not require vendor collaboration and can be easily implemented to mitigate the effect of crosstalk interference and maximize total throughput.

A typical measure of system throughput is the sum of all users' rates. Unfortunately the problem of maximizing the sum rate subject to individual power constraints turns out to be nonconvex with many local maxima [1]. To obtain a global optimal power allocation solution, a simulated annealing method was proposed in [2]; however, this method suffers from slow convergence and lacks a rigorous analysis. More recently, a dual decomposition approach [3] was developed to solve the nonconvex rate maximization problem, whose complexity was claimed by the authors to be linear in terms of the number of frequency tones but exponential in the number of users. Notice that all of these approaches require a centralized implementation whereby a spectrum management center collects all the channel and noise information, and calculates rate-maximizing power spectra vectors and send them to individual users for implementation. In a departure from this centralized framework, Yu et al. [4] proposed a distributed game-theoretic approach for the power control problem. The key observation is that each DSL user's data rate is a concave function of its own power spectra vector when the interfering users' power vectors are fixed. Letting each user locally measure the interference plus noise levels and greedily allocate its power to maximize its own rate gives rise to a noncooperative Nash game (called DSL game hereafter) [4, 5]. The resulting distributed power control scheme is known as the *iterative waterfilling algorithm* (IWFA) and has become a popular candidate for the dynamic spectrum management standard for future DSL systems.

Despite its popularity and its apparent convergent behavior in extensive computer simulations, IWFA has only been theoretically shown to converge in limited cases where the crosstalk interferences are weak [6] and/or the number of users is two [4]. The goal of this paper is to present a convergence analysis of IWFA in more realistic channel settings and for arbitrary number of users. Our approach is based on a key new result that establishes a simple reformulation of the noncooperative Nash game (resulting from the distributed power control problem) as a linear complementarity problem (LCP) of the "copositive-plus" type [7]. Based on this equivalent LCP reformulation, we establish the linear convergence of IWFA for arbitrary symmetric interference environment as well as for diagonally dominant asymmetric channel conditions with any number of users. Moreover, in the case of symmetric interference crosstalk coefficients, we show a surprising result that the users of IWFA in fact, unknowingly but willingly, cooperate to minimize a common quadratic cost function whose gradient measures the total received signal power from all users, subject to the constraints that each user must allocate all of its budgeted power across the frequency tones. This "virtual collaborating behavior" is unexpected since the DSL users in IWFA never have any intention nor incentives to cooperate as each simply maximizes its own rate to reach a Nash equilibrium. Another major advantage of this LCP reformulation is that it opens up the possibility to solve the DSL power control problem using the existing well-developed algorithms for LCP, for example, Lemke's method [7, 8]. The latter method requires no restriction on the interference coefficients and therefore can be used to benchmark the performance of IWFA, especially in the presence of strong crosstalk interference which leads to multiple Nash equilibrium solutions. In contrast, there has been no theoretical proof of convergence (to an equilibrium solution) for the IWFA under general interference conditions.

Our current work was partly inspired by the recent work of [9] which presented a nonlinear complementarity problem (NCP) formulation of the DSL game using the Karush-Kuhn-Tucker (KKT) optimality condition for each user's own rate maximization problem. Such an NCP approach can be implemented in a distributed manner despite the need for some small amount of coordination among the DSL users through a spectrum management center. It was shown [9] that the resulting NCP belongs to the P_0 class under certain conditions on the crosstalk interference coefficients among the users relative to the various frequency tones. It was further shown that, under the same conditions, the solution to the NCP is "B-regular" [10]; as a consequence, the NCP can be solved in this case by a host of Newton-type methods as described in the Chapter 9 of the latter monograph. In contrast to [9], our present work shows that the DSL game is basically a linear problem. This simple result has important consequences as we will see.

The rest of this paper is organized as follows. In Section 2, we present the Nash game formulation of the DSL power control problem and develop an equivalent mixed LCP formulation, based on which we obtain a new uniqueness result of the Nash equilibrium solution to the game. In Section 3, we convert the mixed LCP formulation of the DSL game into a standard LCP and show that the well-known Lemke method will successfully compute a Nash equilibrium of the DSL game, under essentially no conditions on the interference and noise coefficients. Section 4 is devoted to the convergence analysis of the IWFA where we apply an existing convergence theory for a symmetric LCP and the contraction principle in the asymmetric case to show the linear convergence of IWFA under two sets of channel conditions. These convergence results significantly enhance those of [4, 6] by allowing arbitrary number of users and more realistic channel conditions. Section 5 reports simulation results of Lemke's algorithm and IWFA. It is observed that the IWFA delivers robust convergent behavior under all simulated channel conditions and achieves superior sum rate performance. Section 6 gives some concluding remarks and suggestions for future work. A brief summary of the LCP and its extension to an affine variational inequality (AVI) is presented in an Appendix.

2. LCP FORMULATION

Let there be m DSL users who wish to communicate with a central office in an uplink multiaccess channel. Let n denote the total number of frequency tones available to the DSL users. Each user i has its own power budget described by the feasible set

$$\mathcal{P}^i = \left\{ p^i \in \mathbb{R}^n \mid 0 \le p^i_k \le \mathrm{CAP}^i_k \quad \forall k = 1,\dots,n,\ \sum_{k=1}^{n} p^i_k \le P^i_{\max} \right\} \tag{1}$$

for some positive constants CAP^i_k and $P^i_{\max}$, where $p^i = (p^i_1, p^i_2, \dots, p^i_n)$ denotes the power spectra vector of user i with p^i_k signifying the power allocated to frequency tone k. In this model, we allow $\mathrm{CAP}^i_k \le \infty$. To avoid triviality, we assume throughout the paper that

$$P^i_{\max} < \sum_{k=1}^{n} \mathrm{CAP}^i_k, \tag{2}$$

which ensures that the budget constraint $\sum_{k=1}^{n} p^i_k \le P^i_{\max}$ is not redundant.

Taking p^j_k for $j \ne i$ as fixed, IWFA lets user i solve the following concave maximization problem in the variables p^i_k for $k = 1,\dots,n$:

$$\begin{aligned} \text{maximize } & f_i(p^1,\dots,p^m) \equiv \sum_{k=1}^{n} \log\left(1 + \frac{p^i_k}{\sigma^i_k + \sum_{j\ne i} \alpha^{ij}_k p^j_k}\right) \\ & \text{subject to } p^i \in \mathcal{P}^i, \end{aligned} \tag{3}$$

where σ^i_k are positive scalars and α^{ij}_k are nonnegative scalars for all $i \ne j$ and all k representing noise power spectra and channel crosstalk coefficients, respectively. A Nash equilibrium of the DSL game is a tuple of strategies $p^* \equiv (p^{*,i})^m_{i=1}$ such that, for every $i = 1,\dots,m$, $p^{*,i} \in \mathcal{P}^i$ and

$$\begin{aligned} & f_i(p^{*,1},\dots,p^{*,i-1},p^{*,i},p^{*,i+1},\dots,p^{*,m}) \\ & \quad \ge f_i(p^{*,1},\dots,p^{*,i-1},p^{i},p^{*,i+1},\dots,p^{*,m}) \\ & \qquad \forall p^i \in \mathcal{P}^i. \end{aligned} \tag{4}$$

The existence of such an equilibrium power vector p^* is well known. Subsequently, we will give some new sufficient conditions for p^* to be unique; see Proposition 2. Our main goal in the paper pertains the computation of p^*. Throughout the paper, we let $\alpha_k^{ii} = 1$ for all i and k.

Letting u_i be the multiplier of the inequality $\sum_{k=1}^n p_k^i \leq P_{\max}^i$, and γ_k^i be the multiplier of the upper bound constraint $p_k^i \leq \mathrm{CAP}_k^i$, we can write down the KKT conditions for user i's problem (3) as follows (where $a \perp b$ means that the two scalars (or vectors) a and b are orthogonal):

$$\begin{aligned} 0 \leq p_k^i \perp -\frac{1}{\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j} + u_i + \gamma_k^i \geq 0 \quad \forall k = 1,\ldots,n, \\ 0 \leq u_i \perp P_{\max}^i - \sum_{k=1}^n p_k^i \geq 0, \\ 0 \leq \gamma_k^i \perp \mathrm{CAP}_k^i - p_k^i \geq 0 \quad \forall k = 1,\ldots,n. \end{aligned} \tag{5}$$

Although the above KKT system is nonlinear, Proposition 1 shows that, under the assumption (2), the system is equivalent to a mixed *linear* complementarity system (see the Appendix for a discussion on the LCP).

Proposition 1. *Suppose that* (2) *holds. The system* (5) *is equivalent to*

$$\begin{aligned} 0 \leq p_k^i \perp \sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j + \nu_i + \varphi_k^i \geq 0 \quad \forall k = 1,\ldots,n, \\ \nu_i \text{ free}, \qquad P_{\max}^i - \sum_{k=1}^n p_k^i = 0, \\ 0 \leq \varphi_k^i \perp \mathrm{CAP}_k^i - p_k^i \geq 0 \quad \forall k = 1,\ldots,n. \end{aligned} \tag{6}$$

Proof. Let (p_k^i, u_i, γ_k^i) satisfy (5). We must have

$$\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j > 0 \quad \forall k = 1,\ldots,n. \tag{7}$$

We claim that $u_i > 0$. Indeed, if $u_i = 0$, then

$$\gamma_k^i \geq \frac{1}{\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j} > 0 \quad \forall k = 1,\ldots,n, \tag{8}$$

which implies $p_k^i = \mathrm{CAP}_k^i$ for all $k = 1,\ldots,n$. Thus

$$P_{\max}^i \geq \sum_{k=1}^n p_k^i = \sum_{k=1}^n \mathrm{CAP}_k^i, \tag{9}$$

which contradicts (2). Hence to get a solution to (6), it suffices to define

$$\nu_i \equiv -\frac{1}{u_i}, \qquad \varphi_k^i \equiv \frac{\gamma_k^i \left(\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j\right)}{u_i}. \tag{10}$$

Conversely, suppose that $(p_k^i, \nu_i, \varphi_k^i)$ satisfies (6). We must have $\nu_i < 0$; otherwise, complementarity yields $p_k^i = 0$ for all $k = 1,\ldots,n$, which contradicts the equality constraint. Consequently, letting

$$u_i \equiv -\frac{1}{\nu_i}, \qquad \gamma_k^i \equiv -\frac{\varphi_k^i}{\nu_i\left(\sigma_k^i + \sum_{j=1}^m \alpha_k^{ij} p_k^j\right)}, \tag{11}$$

we easily see that (5) holds. □

In turn, the mixed LCP (6) is the KKT condition of the AVI defined by the affine mapping $p \equiv (p^i)_{i=1}^m \in \mathbb{R}^{mn} \to \sigma + Mp \in \mathbb{R}^{mn}$ and the polyhedron $X \equiv \prod_{i=1}^m \widehat{\mathcal{P}}^i$, where $\sigma \equiv (\sigma^i)_{i=1}^m$ with σ^i being the n-dimensional noise power vector $(\sigma_k^i)_{k=1}^n$ for user i, M is the block partitioned matrix $(M^{ij})_{i,j=1}^m$ with each $M^{ij} \equiv \mathrm{Diag}(\alpha_k^{ij})_{k=1}^n$ being the $n \times n$ diagonal matrix of power interferences (note: M^{ii} is an identity matrix), and

$$\widehat{\mathcal{P}}^i \equiv \left\{ p^i \in \mathbb{R}^n \mid 0 \leq p_k^i \leq \mathrm{CAP}_k^i \ \ \forall k = 1,\ldots,n, \ \sum_{k=1}^n p_k^i = P_{\max}^i \right\}. \tag{12}$$

(See the Appendix for a discussion on the AVI.) Consequently, the tuple p is a Nash equilibrium to the DSL game if and only if $p \in X$ and

$$(p' - p)^T(\sigma + Mp) \geq 0 \quad \forall p' \in X. \tag{13}$$

We denote this AVI by the triple (X, σ, M). Among its consequences, the above AVI reformulation of the DSL game enables us to obtain some new sufficient conditions for the uniqueness of a Nash equilibrium solution. To present these conditions, we define the $m \times m$ matrix $B = [b_{ij}]$ by

$$b_{ij} \equiv \max_{1 \leq k \leq n} \alpha_k^{ij} \quad \forall i, j = 1,\ldots,m. \tag{14}$$

Note that $b_{ii} = 1$. In what follows, we review some background results in matrix theory, which can be found in [7].

Let B_{dia}, B_{low}, and B^{upp} be the diagonal, strictly lower, and strictly upper triangular parts of B, respectively. Since α_k^{ij} are all nonnegative, B is a nonnegative matrix. Hence $B_{\mathrm{dia}} - B_{\mathrm{low}}$ is a "Z-matrix"; that is, all its off-diagonal entries are nonpositive. Since all principal minors of $B_{\mathrm{dia}} - B_{\mathrm{low}}$ are equal to one, $B_{\mathrm{dia}} - B_{\mathrm{low}}$ is a "P-matrix," and thus a "Minkowski matrix" (also known as an "M-matrix"). It follows that $(B_{\mathrm{dia}} - B_{\mathrm{low}})^{-1}$ exists and is a nonnegative matrix. Therefore, so is the matrix $\Upsilon \equiv (B_{\mathrm{dia}} - B_{\mathrm{low}})^{-1} B^{\mathrm{upp}}$. Let $\rho(\Upsilon)$ denote the spectral radius of Υ, which is equal to its largest eigenvalue, by the well-known Perron-Frobenius theorm for nonnegative matrices. The matrix

$$\bar{B} \equiv B_{\mathrm{dia}} - B_{\mathrm{low}} - B^{\mathrm{upp}} \tag{15}$$

is the "comparison matrix" of B. Notice that $\bar{B}$ is also a Z-matrix. The matrix B is called an *H-matrix* if $\bar{B}$ is also a P-matrix. There are many characterizations for the latter condition to hold; we mention two of these: (a) $\rho(\Upsilon) < 1$ and (b) for every nonzero vector $x \in \mathbb{R}^m$, there exists an index i such that $x_i(\bar{B}x)_i > 0$.

For each $k = 1, \ldots, n$, we call the $m \times m$ matrix M_k, where

$$(M_k)_{ij} \equiv \alpha_k^{ij} \quad \forall i, j = 1, \ldots, m, \tag{16}$$

a *tone matrix*. Notice that the matrix M in the AVI (X, σ, M) is a principal rearrangement of the block diagonal matrix with M_k as its diagonal blocks for $k = 1, \ldots, n$. This rearrangement simply amounts to the alternative grouping of the tuple p by tones, instead of users as done above.

Proposition 2. *Suppose that*

$$\max_{1 \le i \le m} \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k^{ij} p_k^i p_k^j > 0 \quad \forall p \equiv (p^i)_{i=1}^m \ne 0. \tag{17}$$

There exists a unique Nash equilibrium to the DSL game. In particular, this holds if either one of the following two conditions is satisfied:

(a) *for every $k = 1, \ldots, n$, the tone matrix M_k is positive definite;*
(b) $\rho(\Upsilon) < 1$.

Proof. As X is the Cartesian product of the sets $\widehat{\mathcal{P}}^i$, it follows that the AVI (X, σ, M) has a unique solution if M has the "uniform P property" relative to the Cartesian structure of X; see [10]. This property says that for any nonzero tuple $p \equiv (p^i)_{i=1}^m$,

$$\max_{1 \le i \le m} (p^i)^T \sum_{j=1}^{m} M^{ij} p^j > 0. \tag{18}$$

Since $M^{ij} = \text{Diag}(\alpha_k^{ij})_{k=1}^n$, the above condition is precisely (17). Under condition (a), the matrix M is positive definite because it is a principal rearrangement of $\text{Diag}(M_k)_{k=1}^n$. It is easy to verify that

$$p^T M p = \sum_{i=1}^{m} \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k^{ij} p_k^i p_k^j. \tag{19}$$

Hence condition (a) implies (17). To show that condition (b) also implies (17), write

$$\begin{aligned}
\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_k^{ij} p_k^i p_k^j
&= \sum_{k=1}^{n} (p_k^i)^2 + \sum_{j \ne i} \sum_{k=1}^{n} \alpha_k^{ij} p_k^i p_k^j \\
&\ge \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \ne i} \sum_{k=1}^{n} \alpha_k^{ij} |p_k^i| |p_k^j| \\
&\ge \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \ne i} \left(\sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \times \left(\sum_{k=1}^{n} (\alpha_k^{ij} p_k^j)^2 \right)^{1/2} \\
&\ge \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \ne i} \max_{1 \le k \le n} \alpha_k^{ij} \left(\sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \times \left(\sum_{k=1}^{n} (p_k^j)^2 \right)^{1/2} \\
&= \left(\sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \sum_{j=1}^{m} \bar{b}_{ij} \left(\sum_{k=1}^{n} (p_k^j)^2 \right)^{1/2},
\end{aligned} \tag{20}$$

where the first and third inequality are obvious and the second is due to the Cauchy-Schwarz inequality. Hence letting

$$q_i \equiv \left(\sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2}, \tag{21}$$

we have

$$\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_k^{ij} p_k^i p_k^j \ge q_i \sum_{j=1}^{m} \bar{b}_{ij} q_j = q_i (\bar{B} q)_i \quad \forall i = 1, \ldots, m. \tag{22}$$

By what has been mentioned above, condition (b) implies

$$\max_{1 \le i \le m} q_i (\bar{B} q)_i > 0, \tag{23}$$

because q is obviously a nonzero vector; thus (17) holds. □

Proposition 2 significantly extends the current existence and uniqueness result of [4–6] which required $0 \le \alpha_k^{ij} \le 1/n$ for all $i \ne j$ and all k. Under the latter condition, it can be shown that the symmetric part of each tone matrix M_k, $(1/2)(M_k + M_k^T)$, is strictly diagonally dominant; hence each M_k is positive definite. The condition $\rho(\Upsilon) < 1$ is quite broad; for instance, it includes the case where each matrix M_k is "strictly quasi-diagonally dominant," that is, where for each k, there exist positive scalars d_k^j such that

$$d_k^i > \sum_{j=1}^{m} \alpha_k^{ij} d_k^j \quad \forall i = 1, \ldots, m. \tag{24}$$

In Section 4, we will see that the condition $\rho(\Upsilon) < 1$ is responsible for the convergence of the IWFA with asymmetric interference coefficients.

As another application of the AVI formulation of the DSL game, we show that if each tone matrix M_k is positive semidefinite (but not definite), it is still possible to say something about the uniqueness of certain quantities.

Proposition 3. *Suppose that the tone matrices M_k, for $k = 1, \ldots, n$, are all positive semidefinite. Then the set of DSL Nash equilibria is a convex polyhedron; moreover, the quantities*

$$\sum_{j=1}^{m} \left(\alpha_k^{ij} + \alpha_k^{ji} \right) p_k^j, \quad \forall i = 1, \ldots, m; \ k = 1, \ldots, n, \tag{25}$$

are constants among all Nash equilibria.

Proof. Under the given assumption, the matrix M is positive semidefinite. As such, the polyhedrality of the set of Nash equilibria follows from the well-known monotone AVI theory [10]. Furthermore, in this case, the vector $(M+M^T)p$ is a constant among all such equilibria p. By unwrapping the structure of the matrix M, the desired constancy of the displayed quantities follows readily. □

We can interpret $(\alpha_k^{ij}+\alpha_k^{ji})/2$ as the "average interference coefficient" between user i and user j at frequency k. In this way, the invariant quantity $(1/2)\sum_{j=1}^m(\alpha_k^{ij}+\alpha_k^{ji})p_k^j$ represents the average of signal and interference power received and caused by user i across all frequency tones.

3. SOLUTION BY LEMKE'S METHOD

We next discuss the solution of the mixed LCP (6) by the well-known Lemke method [7]. Since this method has a robust theory of convergence, its solution can be used as a benchmark to evaluate the empirical performance of IWFA; see Section 5. For convenience, let us first convert the problem (6) into a standard LCP. Let

$$w_k^i \equiv \sigma_k^i+\sum_{j=1}^m \alpha_k^{ij}p_k^j+\nu_i+\varphi_k^i \quad \forall k=1,\ldots,n, \tag{26}$$

from which we obtain, considering $k=1$ and substituting $p_1^j=P_{\max}^j-\sum_{k=2}^n p_k^j$ for all $j=1,\ldots,m$,

$$\begin{aligned}\nu_i &= -\sigma_1^i+w_1^i-\sum_{j=1}^m\alpha_1^{ij}p_1^j-\varphi_1^i\\ &= -\sigma_1^i+w_1^i-\sum_{j=1}^m\alpha_1^{ij}\left(P_{\max}^j-\sum_{k=2}^n p_k^j\right)+\varphi_1^i\\ &= -\sigma_1^i-\sum_{j=1}^m\alpha_1^{ij}P_{\max}^j+w_1^i+\sum_{j=1}^m\sum_{k=2}^n\alpha_1^{ij}p_k^j-\varphi_1^i.\end{aligned} \tag{27}$$

Substituting this into the expression of w_k^i for $k\ge 2$, we deduce

$$\begin{aligned}w_k^i &\equiv \sigma_k^i-\sigma_1^i-\sum_{j=1}^m\alpha_1^{ij}P_{\max}^j+w_1^i+\sum_{j=1}^m\alpha_k^{ij}p_k^j\\ &\quad+\sum_{j=1}^m\sum_{\ell=2}^n\alpha_1^{ij}p_\ell^j+\varphi_k^i-\varphi_1^i\\ &= \hat{\sigma}_i^i+w_1^i+\sum_{j=1}^m\sum_{\ell=2}^n\left(\alpha_1^{ij}+\alpha_\ell^{ij}\delta_{k\ell}\right)p_\ell^j+\varphi_k^i-\varphi_1^i,\end{aligned} \tag{28}$$

where $\delta_{k\ell}$ is Kronecker delta, that is,

$$\begin{aligned}\delta_{k\ell} &\equiv \begin{cases}1 & \text{if } k=\ell,\\ 0 & \text{otherwise},\end{cases}\\ \hat{\sigma}_k^i &\equiv \sigma_k^i-\sigma_1^i-\sum_{j=1}^m\alpha_1^{ij}P_{\max}^j \quad \forall k=2,\ldots,n.\end{aligned} \tag{29}$$

Consequently, the concatenation of the system (6) for all $i=1,\ldots,m$ is equivalent to the following: for all $i=1,\ldots,m$ and all $k=2,\ldots,n$,

$$\begin{aligned}&0\le p_k^i \perp w_k^i=\hat{\sigma}_k^i+\sum_{j=1}^m\sum_{\ell=2}^n\left(\alpha_1^{ij}+\alpha_\ell^{ij}\delta_{k\ell}\right)\\ &\qquad\qquad\times p_\ell^j+w_1^i+\varphi_k^i-\varphi_1^i\ge 0,\\ &0\le w_1^i\perp p_1^i=P_{\max}^i-\sum_{k=2}^n p_k^i\ge 0,\\ &0\le\varphi_k^i\perp \mathrm{CAP}_k^i-p_k^i\ge 0,\\ &0\le\varphi_1^i\perp \mathrm{CAP}_1^i-P_{\max}^i+\sum_{k=2}^n p_k^i\ge 0.\end{aligned} \tag{30}$$

The above is an LCP of the standard type

$$0\le z\perp \mathbf{q}+\mathbf{M}z\ge 0, \tag{31}$$

where the constant vector $\mathbf{q}$ is given by

$$\mathbf{q}\equiv\begin{pmatrix}\hat{\sigma}_k^i: i=1,\ldots,m;\ k=2,\ldots,n\\ P_{\max}^i: i=1,\ldots,m\\ \mathrm{CAP}_k^i: i=1,\ldots,m;\ k=2,\ldots,n\\ \mathrm{CAP}_1^i-P_{\max}^i: i=1,\ldots,m\end{pmatrix}, \tag{32}$$

z is the vector of variables:

$$z\equiv\begin{pmatrix}p_k^i: i=1,\ldots,m;\ k=2,\ldots,n\\ w_1^i: i=1,\ldots,m\\ \varphi_k^i: i=1,\ldots,m;\ k=2,\ldots,n\\ \varphi_1^i: i=1,\ldots,m\end{pmatrix}, \tag{33}$$

and the matrix $\mathbf{M}$, partitioned in accordance with the vectors $\mathbf{q}$ and z, is of the form

$$\mathbf{M}\equiv\begin{bmatrix}\widehat{M} & N & I & -N\\ -N^T & 0 & 0 & 0\\ -I & 0 & 0 & 0\\ N^T & 0 & 0 & 0\end{bmatrix}, \tag{34}$$

where the leading principal submatrix $\widehat{M}$ is a nonnegative (albeit asymmetric) matrix with positive diagonals and N is a special nonnegative matrix. (The details of the matrices $\widehat{M}$ and N are not important except for the distinctive features mentioned here.) Based on (34), it follows that the matrix $\mathbf{M}$ is copositive-plus (i.e., $z^T\mathbf{M}z\ge 0$ for all $z\ge 0$, and $[z\ge 0,\ z^T\mathbf{M}z=0]$ implies $(\mathbf{M}+\mathbf{M}^T)z=0$). Consequently, Lemke's algorithm can successfully compute a solution to the LCP (31) provided that this LCP is feasible; see [7]. But the latter feasibility condition trivially holds by the nonemptiness of the sets $\widehat{\mathcal{P}}^i$ for $i=1,\ldots,m$, which is a blanket assumption that we have made. Summarizing this discussion, we obtain the following result.

Theorem 1. *Suppose that* (2) *holds and that* $\widehat{\mathcal{P}}^i\neq\emptyset$ *for all* $i=1,\ldots,m$. *For all nonnegative coefficients* α_k^{ij}, $i\neq j$, *and all positive* σ_k^i, *there exists a Nash equilibrium solution which can be obtained by Lemke's algorithm applied to the LCP* (31) *with* $\mathbf{q}$ *and* $\mathbf{M}$ *given by* (32) *and* (34), *respectively.*

This existence result extends that of [4] which required the condition that $\max_k\{\alpha_k^{21}\alpha_k^{12}\} < 1$ and was only for the two user case.

4. CONVERGENCE ANALYSIS OF THE IWFA

The LCP formulation (31) of the DSL game, where each user's variables associated with tone 1 are eliminated, facilitates the computation of a Nash equilibrium by Lemke's method (see Section 5 for numerical results). Nevertheless, for the convergence analysis of the IWFA, it would be convenient to return to the AVI (X, q, M), where all variables are left in the formulation. It is well known [10] that the latter AVI is equivalent to the fixed-point equations: for all $i = 1, \ldots, m$,

$$p^i = \left[p^i - \sigma^i - \sum_{j=1}^{m} M^{ij} p^j\right]_{\widehat{\mathcal{P}}_i} = \left[-\sigma^i - \sum_{j\neq i} M^{ij} p^j\right]_{\widehat{\mathcal{P}}^i}, \tag{35}$$

where $[\cdot]_{\widehat{\mathcal{P}}^i}$ denotes the Euclidean projection operator onto $\widehat{\mathcal{P}}_i$, that is,

$$[x]_{\widehat{\mathcal{P}}^i} = \operatorname{argmin}_{p^i\in\widehat{\mathcal{P}}^i}\|x - p^i\|. \tag{36}$$

As briefly described in Section 2, the IWFA [4–6] is a distributed algorithm for solving the DSL game; it has the attractive feature of not requiring the coordination of the DSL users. In fact, each DSL user i simply maximizes its rate $f_i(p^1, \ldots, p^m)$ on the feasible set $\mathcal{P}^i$ by adjusting its own power vector p^i while assuming other users' powers are fixed but unknown. In so doing, user i measures the aggregated interference powers,

$$\sum_{j\neq i}(M^{ij}p^i)_k = \sum_{j\neq i}\alpha_k^{ij}p_k^j \quad \forall k, \tag{37}$$

locally without the specific knowledge of other users' power allocations p^j or crosstalk coefficients α_k^{ij}, $j \neq i$. Such aggregated interference powers are sufficient for user i to carry out its own rate maximization (3).

Specifically, the iterative waterfilling method can be described as follows: at each iteration, user i measures the aggregated interferences and updates the new iterate by

$$(p^i)^{\text{new}} = \left[-\sigma^i - \Bigg(\underbrace{\sum_{j=1}^{i-1} M^{ij}(p^j)^{\text{new}} + \sum_{j=i+1}^{m} M^{ij}(p^j)^{\text{old}}}_{\text{aggregated interferences}}\Bigg)\right]_{\widehat{\mathcal{P}}^i}. \tag{38}$$

In other words, user i simply projects the negative of the aggregated interferences plus the noise power vector onto the polyhedral set $\widehat{\mathcal{P}}^i$. This simple geometric interpretation of the IWFA is key to its convergence analysis, which we separate into two cases: symmetric and nonsymmetric interferences.

Symmetric interferences

When the DSL users are symmetrically located, the corresponding interference coefficients are symmetric: $\alpha_k^{ij} = \alpha_k^{ji}$ for all i, j, k. In this case, it follows that $M^{ij} = M^{ji}$ for all i, j. Hence the matrix M is symmetric. Consequently, the mixed LCP (6) is precisely the KKT condition for the following quadratic program (QP):

$$\begin{aligned} &\text{minimize } g(p) \equiv \frac{1}{2}p^T M p + \sum_{i=1}^{m}(\sigma^i)^T p^i \\ &\text{subject to } p = (p^i)_{i=1}^m \in \prod_{i=1}^{m}\widehat{\mathcal{P}}^i. \end{aligned} \tag{39}$$

Notice that the gradient of $g(p)$ measures precisely the total received signal power by every user at each frequency. Moreover, the set of Nash equilibrium points for the noncooperative rate maximization game (3) correspond exactly to the set of stationary points of the quadratic minimization problem (39), which is not necessarily convex because the matrix M is not positive semidefinite in general. More importantly, the IWFA (38) can be viewed as a block Gauss-Seidel coordinate descent iteration to solve the QP (39). As such, its convergence behavior can be established by appealing to the following general convergence result for the Gauss-Seidel algorithm [11, Proposition 3.4].

Proposition 4. *Consider the following quadratic minimization problem:*

$$\begin{aligned} &\textit{minimize } \theta(x_1, x_2, \ldots, x_n) \\ &\textit{subject to } x_i \in X_i \quad \forall i = 1, 2, \ldots, n, \end{aligned} \tag{40}$$

with each X_i being a given polyhedral set. Suppose that $X = X_1 \times X_2 \times \cdots \times X_n$ is nonempty and that θ is strongly convex in each variable x_i. Let $\bar{X}$ denote the set of stationary points of (40) and let $x^0, x^1, x^2, \ldots$ be a sequence of iterates generated by the following fixed-point iteration:

$$x_i^{r+1} = \left[x_i^{r+1} - \nabla_{x_i}\theta(x_1^{r+1}, x_2^{r+1}, \ldots, x_i^{r+1}, x_{i+1}^r, \ldots, x_n^r)\right]_{X_i}. \tag{41}$$

Then $\{x^r\}$ converges linearly to an element of $\bar{X}$ and $\{\theta(x^r)\}$ converges linearly and monotonically.

Under the following identifications:

$$x_i \equiv p^i, \qquad X_i \equiv \widehat{\mathcal{P}}^i, \qquad \theta(x) \equiv g(p), \tag{42}$$

iteration (38) is precisely (41). Since M^{ii} is the identity matrix for each i, it follows that the quadratic function $g(p)$ is strongly convex in each variable p^i. Thus, we can invoke Proposition 4 to conclude the following.

Corollary 1. *If the interference coefficients are symmetric, that is, $\alpha_k^{ij} = \alpha_k^{ji}$ for all i, j, k, then the iterates $\{p^\nu \equiv (p^{\nu,i})_{i=1}^m\}$ generated by the IWFA converges linearly to a Nash equilibrium point of the noncooperative DSL game. Moreover, $\{g(p^\nu)\}$ converges linearly and monotonically.*

Notice that in the original IWFA, each user acts greedily to maximize its own rate without coordination. What is surprising is that this seemingly totally distributed algorithm can in fact be viewed equivalently as a coordinate descent algorithm for the minimization of a single quadratic function. In other words, the users actually collaborate, implicitly and willingly, to minimize a common quadratic objective function $g(p)$ whose gradient corresponds to precisely the total received signal power by every user at each frequency. This important insight is the key to the convergence of the IWFA in the symmetric case.

If signal attenuation increases deterministically with the propagation distance, then the symmetric interference assumption used in the above analysis translates directly to the situation that the DSL users are symmetrically located: they are of the same distance to the central office (base station). Such an assumption is obviously idealistic from a practical standpoint. Nonetheless, our analysis of IWFA for this idealized situation may still shed some light on the general behavior of IWFA under arbitrary interferences.

Asymmetric interferences

In general, the DSL users may not be symmetrically located. In this case, the interference matrix M is not symmetric and the aggregated interference power vectors cannot be viewed as the gradient of a scalar function. Thus, Proposition 4 is no longer applicable. More importantly, there is now a lack of an obvious objective function which serves as a monitor for the progress of the IWFA, making the convergence analysis of this algorithm less straightforward. Nevertheless, it is still possible to establish the convergence of the IWFA by imposing the spectral radius condition $\rho(\Upsilon) < 1$ introduced in Proposition 2.

Theorem 2. *Suppose that $\rho(\Upsilon) < 1$. Then the iterates $\{p^\nu \equiv (p^{\nu,i})_{i=1}^m\}$ generated by the IWFA converge linearly to the unique Nash equilibrium of the DSL game.*

Proof. Our proof is by a vector contraction argument; see [7]. Let $p^* \equiv (p^{*,i})_{i=1}^m$ be the unique Nash equilibrium solution, which satisfies

$$\begin{aligned} p^{*,i} &= \left[p^{*,i} - \sigma^i - \sum_{j=1}^m M^{ij} p^{*,j} \right]_{\widehat{\mathcal{P}}^i} \\ &= \left[-\sigma^i - \sum_{j\neq i} M^{ij} p^{*,j} \right]_{\widehat{\mathcal{P}}^i} \qquad \forall i = 1,\ldots,m. \end{aligned} \tag{43}$$

For each ν, we have

$$p^{\nu+1,i} = \left[-\sigma^i - \left(\sum_{j=1}^{i-1} M^{ij} p^{\nu+1,j} + \sum_{j=i+1}^m M^{ij} p^{\nu,j} \right) \right]_{\widehat{\mathcal{P}}^i} \quad \forall i = 1,\ldots,m. \tag{44}$$

Let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^m$. By the nonexpansiveness property of projection operator (i.e., $\|[x]_{\widehat{\mathcal{P}}^i} - [y]_{\widehat{\mathcal{P}}^i}\| \le \|x - y\|$ for all x, y), we have, for all $i = 1,\ldots,m$,

$$\begin{aligned} &\|p^{\nu+1,i} - p^{*,i}\| \\ &= \Bigg\| \left[-\sigma^i - \left(\sum_{j=1}^{i-1} M^{ij} p^{\nu+1,j} + \sum_{j=i+1}^m M^{ij} p^{\nu,j} \right) \right]_{\widehat{\mathcal{P}}^i} \\ &\quad - \left[-\sigma^i - \left(\sum_{j=1}^{i-1} M^{ij} p^{*,j} + \sum_{j=i+1}^m M^{ij} p^{*,j} \right) \right]_{\widehat{\mathcal{P}}^i} \Bigg\| \\ &\le \left\| \sum_{j=1}^{i-1} M^{ij}\left(p^{\nu+1,j} - p^{*,j}\right) + \sum_{j=i+1}^m M^{ij}\left(p^{\nu,j} - p^{*,j}\right) \right\| \\ &\le \sum_{j=1}^{i-1} \|M^{ij}(p^{\nu+1,j} - p^{*,j})\| + \sum_{j=i+1}^m \|M^{ij}(p^{\nu,j} - p^{*,j})\| \\ &\le \sum_{j=1}^{i-1} b_{ij} \|p^{\nu+1,j} - p^{*,j}\| + \sum_{j=i+1}^m b_{ij} \|p^{\nu,j} - p^{*,j}\|. \end{aligned} \tag{45}$$

Hence,

$$\sum_{j=1}^{i} \bar{b}_{ij} \|p^{\nu+1,j} - p^{*,j}\| \le \sum_{j=i+1}^m b_{ij} \|p^{\nu,j} - p^{*,j}\|, \tag{46}$$

where $\bar{B} = [\bar{b}_{ij}]$ is defined by (15). Letting $e^\nu \equiv (e_i^\nu)_{i=1}^m$ with $e_i^\nu \equiv \|p^{\nu,j} - p^{*,j}\|$ and concatenating the above inequalities for all $i = 1,\ldots,m$, we deduce

$$(B_{\text{dia}} - B_{\text{low}}) e^{\nu+1} \le B^{\text{upp}} e^\nu, \tag{47}$$

which implies

$$0 \le e^{\nu+1} \le (B_{\text{dia}} - B_{\text{low}})^{-1} B^{\text{upp}} e^\nu = \Upsilon e^\nu \quad \forall \nu, \tag{48}$$

where we have used the fact that $(B_{\text{dia}} - B_{\text{low}})^{-1}$ is nonnegative entry-wise; see the discussion preceding Proposition 2. Since $\rho(\Upsilon) < 1$, the above inequality implies that the sequence of error vectors $\{e^\nu\}$ contract according to a certain norm. Consequently, the sequence converges to zero, implying that the sequence of waterfilling iterates $\{p^\nu\}$ converges linearly to the unique solution p^* of the DSL game. □

Theorem 2 strengthens the existing convergence results [4, 6]. Specifically, the condition required for convergence is weaker. In particular, it can be seen that the strong diagonal dominance condition ($\alpha_k^{ij} \le 1/(m-1)$) required in [6] and the respective condition for two user case [4] both imply the condition $\rho(\Upsilon) < 1$. Thus, Theorem 2 covers the convergence for a broader class of DSL problems.

5. NUMERICAL SIMULATIONS

In this section, we present some computer simulation results comparing the convergence behavior of IWFA with Lemke's algorithm under various channel conditions and system load (i.e., number of users). In all simulated cases, the channel background noise levels σ_k^i are chosen randomly from the

Table 1: Average sum rate: two user case.

n	$\alpha_k^{12}, \alpha_k^{21} \in (0,1)$		$\alpha_k^{12}, \alpha_k^{21} \in (0,1.5)$	
	Lemke	IWFA	Lemke	IWFA
256	704	698	829.73	826.5787
512	1.402×10^3	1.398×10^3	1.6555×10^3	1.6333×10^3
1024	2.786×10^3	2.811×10^3	3.3028×10^3	3.2968×10^3

Table 2: Average sum rate: $m = 10$ user case.

n	$\alpha_k^{ij} \in (0, 1/(m-1))$	
	Lemke	IWFA
256	2.8216×10^3	2.824×10^3
512	5.6464×10^3	5.6457×10^3
1024	1.1284×10^4	1.1296×10^4

interval $(0, 0.1/(m-1))$ with the uniform distribution, and the total power budgets $P^i_{\max}$ are chosen uniformly from the interval $(n/2, n)$. All sum rates are averaged over 100 independent runs. The IWFA and Lemke's method are both implemented on a Pentium 4 (1.6 GHz) PC using Matlab 6.5 running under Windows XP. For IWFA, we set a maximum of 400 iteration cycles (among all users), while the maximum pivoting steps for Lemke's method is set to be $\min(1000, 25\,mn)$.

Table 1 reports the achieved sum rates (averaged over 100 independent runs) of Lemke's method and IWFA for 2 users and various numbers n of frequency tones. In this case we have chosen crosstalk coefficients $\{\alpha_k^{ij}\}$ from the intervals $(0,1)$ and $(0,1.5)$, respectively, for all k, and all i, j. This represents strong crosstalk interference scenarios. According to the table, the average rates achieved by both algorithms are comparable (within 2%), suggesting that the IWFA is capable of computing approximate Nash solutions with high sum rates. Moreover, the results show that stronger interference actually lead to Nash solutions with higher overall sum rates. This seems to indicate that the well-known Braess paradox [12] exist in DSL games. (In fact, using the QP characterization of Nash game (cf. Section 4), it is possible to construct simple examples whereby more transmission power for individual users do not necessarily lead to Nash solutions with higher sum rate.)

For the case with more ($m = 10$) users, the situation is similar, as shown in Table 2. Indeed, when $\alpha_k^{ij} \in (0, 1/(m-1))$, the condition for the uniqueness of Nash solution is satisfied and the two methods have identical performance. The results in both tables show that IWFA solutions are comparable in quality to the respective solutions generated by the Lemke method. The difference in the solution qualities are due to the finite termination criteria we have used in both algorithms which can cause either algorithm to stop before an equilibrium solution is found.

6. CONCLUSIONS

In this paper we reformulate the DSL Nash game (resulting from the distributed implementation of IWFA) as an equivalent LCP, and apply the rich theory for LCP to analyze the convergence behavior of IWFA. Our analysis not only significantly strengthens the existing convergence results, but also yields surprising insight on IWFA. In particular, in the case of symmetric interference, the users of IWFA in fact collaborate unknowingly to minimize a common quadratic cost, even though their original intention is to maximize their individual rates. Moreover, the LCP reformulation makes it possible to solve the DSL game with existing LCP solvers, such as Lemke's method. With the latter as a benchmark, we show via computer simulations that IWFA tends to converge to good Nash solutions with high sum rates. Our theoretical and simulation work affirms the potential of IWFA as a promising candidate for the dynamic power spectra management in DSL environment.

Several extensions of current work are possible. For example, under either the diagonal dominance condition of $\rho(\Upsilon) < 1$ or the symmetric interference condition, one can establish the linear convergence of a distributed (partially) asynchronous implementation of IWFA. In particular, for the diagonal dominance case, one can use a contraction argument similar to that in [13, page 493], while for the symmetric interference case, use an error bound technique [14] to bound the distance from the iterates to the solution set of the quadratic QP (39). Asynchronous implementation is interesting from a practical standpoint since it does not require the DSL users to coordinate the timing of their power spectra updates.

As a future work, we are interested in further analyzing the behavior of IWFA under no assumptions on the crosstalk coefficients. Perhaps the compactness of the feasible set and the nonnegativity of the crosstalk coefficients already ensure the convergence of IWFA, or at least eliminate the possibility of finite limit cycles. These issues and the design of an efficient optimal power allocation algorithm for the nonconvex sum rate maximization problem are interesting topics for future research.

APPENDIX

BACKGROUND ON LCPs AND AVIs

In this appendix, we briefly summarize some technical background related to the linear complementarity problems and affine variational inequalities. For a comprehensive treatment of these problems, the readers are referred to the two monographs [7, 10].

Unifying linear and quadratic programs and many related problems, the LCP is an inequality system with no objective function to be optimized. Specifically, let M be a given square matrix of order $n \times n$ and q a column vector in $\mathbb{R}^n$. The LCP associated with (q, M) (denoted as $\mathrm{LCP}(q, M)$) is to find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx + q) = 0. \tag{A.1}$$

Let $\mathrm{Sol}(q, M)$ denote the solution set of $\mathrm{LCP}(q, M)$. It is known that $\mathrm{Sol}(q, M)$ is in general equal to a finite union of polyhedral sets. If M is positive semidefinite (not necessarily symmetric), then we say that the corresponding LCP is *monotone*; in this case, the solution set $\mathrm{Sol}(q, M)$ is convex (and polyhedral). If M is symmetric, it can be easily seen that $\mathrm{LCP}(q, M)$ corresponds exactly to the KKT conditions for the following QP:

$$\begin{aligned} &\text{minimize } f(x) \equiv \frac{1}{2}x^T Mx + q^T x \\ &\text{subject to } x \geq 0. \end{aligned} \tag{A.2}$$

Therefore, the stationary points of above QP are precisely the solutions of the $\mathrm{LCP}(q, M)$. Moreover, the gradient vector $\nabla f(x)$ can be shown to be constant on each of the polyhedral piece of $\mathrm{Sol}(q, M)$. (If M is in addition positive semidefinite, then $\mathrm{Sol}(q, M)$ consists of one polyhedral piece, so $\nabla f(x)$ is constant over $\mathrm{Sol}(q, M)$.) When M is not symmetric, the above QP equivalence no longer holds. Instead, we can associate with the $\mathrm{LCP}(q, M)$ the following alternate QP:

$$\begin{aligned} &\text{minimize } x^T(q + Mx) \\ &\text{subject to } q + Mx \geq 0, \quad x \geq 0. \end{aligned} \tag{A.3}$$

In this case, a vector x is a global minimizer of (A.3) with a zero objective value if and only if $x \in \mathrm{Sol}(q, M)$. Unlike the symmetric case, the KKT points of (A.3) are not necessarily the solutions of $\mathrm{LCP}(q, M)$.

The LCP can also be used to model a linear program (LP) via duality. Indeed, the following LP:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax \geq b, \quad x \geq 0 \end{aligned} \tag{A.4}$$

is equivalent to the $\mathrm{LCP}(q, M)$ with

$$q \equiv \begin{pmatrix} c \\ -b \end{pmatrix}, \qquad M \equiv \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}, \tag{A.5}$$

where the matrix M is skew-symmetric, thus positive semidefinite.

There are many algorithms developed for solving an LCP. Among them, Lemke's method is perhaps the most versatile due to its weak requirements for convergence. Algorithmically, Lemke's method is a pivoting algorithm, much like the celebrated simplex method for an LP. As such, it is a finite method but suffers from exponential worst case complexity. Nonetheless, its simplicity and superior average performance have made it a popular choice in practice.

For monotone LCPs, we can also use interior point algorithms which offer polynomial complexity [15]. These algorithms exploit the positive semidefiniteness of M and typically require only a small number of iterations, albeit every iteration requires the solution of a system of linear equations of size $n \times n$. In the absence of monotonicity, interior point algorithms are not guaranteed to converge.

Another popular class of iterative algorithms for solving LCPs consists of the *matrix splitting* algorithms, which are based on the observation that a vector $x \in \mathrm{Sol}(q, M)$ if and only if x satisfies the following fixed point equation:

$$x = [x - \alpha(Mx + q)]_+, \tag{A.6}$$

where $[\cdot]_+$ denotes projection to $\mathbb{R}^n_+$ and $\alpha > 0$ is any constant. This suggests the following simple iterative scheme to compute a solution of $\mathrm{LCP}(q, M)$: for a given *stepsize* $\alpha > 0$ and an initial iterate $x^0 \geq 0$,

$$x^{r+1} = [x^r - \alpha(Mx^r + q)]_+, \quad r = 1, 2, \ldots. \tag{A.7}$$

This iterative scheme is called the *gradient projection* algorithm. If $\{x^r\}$ converges, then the limit must be a solution of $\mathrm{LCP}(q, M)$. More generally, we can split the matrix M as $M = B + C$ for some matrices B and C and generate a sequence according to

$$x^{r+1} = [x^{r+1} - \alpha(Bx^{r+1} + Cx^r + q)]_+, \quad r = 1, 2, \ldots. \tag{A.8}$$

Again, if the sequence $\{x^r\}$ converges, then its limit must be an element of $\mathrm{Sol}(q, M)$. The aforementioned gradient projection is a special matrix splitting algorithm with $B \equiv I/\alpha$ and $C \equiv M - I/\alpha$. If B is taken to be the lower triangular part (including the diagonal) of M while C is taken to be the strict upper triangular part of M, then the resulting matrix splitting algorithm simply corresponds to the well-known Gauss-Seidel method for LCP. In general, to ensure convergence, the matrix splitting $M = B + C$ must satisfy certain conditions. For example, if M is symmetric, B and $B - C$ are both positive definite, then the iterates generated by the resulting matrix splitting algorithm converges linearly to an element of $\mathrm{Sol}(q, M)$.

Much of the theory and algorithms for the LCP can be extended to the AVI of the following form: given the polyhedron,

$$\mathcal{P} \equiv \{x \in \mathbb{R}^n : Ax \geq b\}, \tag{A.9}$$

find $x^* \in \mathcal{P}$ such that

$$(x - x^*)^T(q + Mx^*) \geq 0 \quad \forall x \in \mathcal{P}. \tag{A.10}$$

Within this framework, $\mathrm{LCP}(q, M)$ simply corresponds to the case where $A = I$ and $b = 0$. The solution set of an AVI is also the union of a finite number of polyhedral sets, which becomes a single (convex) polyhedron when M is positive semidefinite (the *monotone* case). In general, a vector x solves the above AVI if and only if x satisfies the following fixed point equation:

$$x = [x - \alpha(Mx + q)]_{\mathcal{P}}, \tag{A.11}$$

where $[\cdot]_{\mathcal{P}}$ denotes the orthogonal projection operator onto $\mathcal{P}$. Similar to the case of LCP, we can devise matrix splitting algorithms for solving the above AVI:

$$x^{r+1} = [x^{r+1} - \alpha(Bx^{r+1} + Cx^r + q)]_{\mathcal{P}}, \quad r = 1, 2, \ldots, \tag{A.12}$$

where $M = B + C$ is a splitting of matrix M. Under conditions similar to those for the LCP, we can also establish linear convergence of the matrix splitting algorithms for solving a symmetric AVI (i.e., $M = M^T$) provided a solution exists; see [11].

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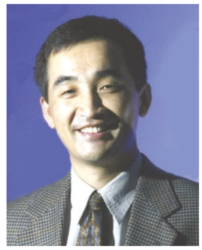

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