Analysis of micro-Doppler signatures

V.C. Chen, F. Li, S.-S. Ho and H. Wechsler

Abstract: Mechanical vibration or rotation of a target or structures on the target may induce additional frequency modulations on the returned radar signal which generate sidebands about the target's Doppler frequency, called the micro-Doppler effect. Micro-Doppler signatures enable some properties of the target to be determined. In the paper, the micro-Doppler effect in radar is introduced and the mathematics of micro-Doppler signatures is developed. Computer simulations are conducted and micro-Doppler features in the joint time–frequency domain are exploited.

1 Introduction

Radar transmits a signal to a target, interacts with the target, and returns back to the radar. The change in the properties of the returned signal contains characteristics of interest of the target. When the transmitted signal of a coherent radar system hits moving targets, the carrier frequency of the signal will be shifted, known as the Doppler effect. The Doppler frequency shift reflects the velocity of the moving target. Mechanical vibration or rotation of a target, or structures on the target, may induce additional frequency modulations on the returned radar signal, which generate sidebands about the target's Doppler frequency, called the micro-Doppler effect [1, 2]. Micro-Doppler signatures enable us to determine some properties of the target.

The micro-Doppler effect was originally introduced in coherent laser radar systems. In a coherent system, the phase of a signal returned from a target is sensitive to the variation in range. In many cases, a target or structures on the target may have vibrations or rotations in addition to target translation, such as a rotor on a helicopter or a rotating radar antenna on a ship. Motion dynamics of the rotating rotor or antenna will produce frequency modulation on the back-scattered signals and induce additional Doppler variations to the translation Doppler shift. From the electromagnetic point of view, when a target has vibration, rotation or other nonuniform motions, the radar backscattering is subject to modulations that constitute features in the signature [3, 4]. Micro-Doppler can be regarded as a unique signature of the target and provides additional information that is complementary to existing methods.

To exploit these unique micro-Doppler features, traditional analysis, such as the Fourier transform, or the sliding window or short time Fourier transform, may not possess the necessary resolution for extracting these features. Therefore, high-resolution time–frequency analysis is necessary for extracting the time-varying Doppler signature [2].

2 Mathematics of micro-Doppler effect

Mathematics of the micro-Doppler effect can be derived by introducing vibration or rotation to conventional Doppler analysis. A target can be represented as a set of point scatterers. The point scattering model may simplify the analysis while preserving the micro-Doppler effect.

As shown in Fig. 1, the radar is stationary and located at the origin Q of the radar co-ordinate system (U, V, W). The target is described in the attached local co-ordinate system (x, y, z) and has translation and rotation with respect to the radar co-ordinates. For the purpose of mathematical analysis, a reference co-ordinate system (X, Y, Z) is introduced, which has the same translation as the target local co-ordinates (x, y, z) but has no rotation with respect to the radar co-ordinates (U, V, W). Thus, the reference co-ordinate system shares the same origin O with the target local co-ordinates and is assumed to be at a distance R_0 from the radar.

Assume that the azimuth and elevation angle of the target in the radar co-ordinates (U, V, W) are α and β, respectively, and the unit vector of the radar line of sight (LOS) direction is defined by

$$\boldsymbol{n} = \boldsymbol{R}_0/\|\boldsymbol{R}_0\| = (\cos\alpha\cos\beta, \sin\alpha\cos\beta, \sin\beta)^T \qquad (1)$$

where $\|\cdot\|$ represents the Euclidean norm.

Suppose the target has a translation velocity $\boldsymbol{v}$ with respect to the radar and an angular rotation velocity $\boldsymbol{\omega}$, which can be represented in the reference co-ordinate system as $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)^T$. Thus, a point scatterer P, which is located at $\boldsymbol{r}_0 = (X_0, Y_0, Z_0)^T$, at time $t = 0$ will move to P' at time t. The movement can be considered as, first, a translation from P to P'' with velocity $\boldsymbol{v}$, or $\boldsymbol{OO'} = \boldsymbol{v}t$, and then, a rotation from P'' to P' with an angular velocity $\boldsymbol{\omega}$. The rotation from P'' to P' can be described by a rotation matrix $Rot(\bullet)$ [5, 6]. At time t, the location of P' can be calculated as

$$\boldsymbol{r} = \boldsymbol{O'}P' = Rot(t)\,\boldsymbol{O'}P'' = Rot(t)\cdot\boldsymbol{r}_0 \qquad (2)$$

and the range vector from the radar to the scatterer at P' becomes

IEE Proceedings online no. 20030743

doi: 10.1049/ip-rsn:20030743

Paper first received 20th December 2002 and in revised form 20th June 2003

V.C. Chen is with the Radar Division, Naval Research Laboratory, Code 5311, 4555 Overlook Ave. SW, Washington, DC 20375, USA

F. Li, S.-S. Ho and H. Wechsler are with the Department of Computer Science, George Mason University, Fairfax, VA 22030, USA

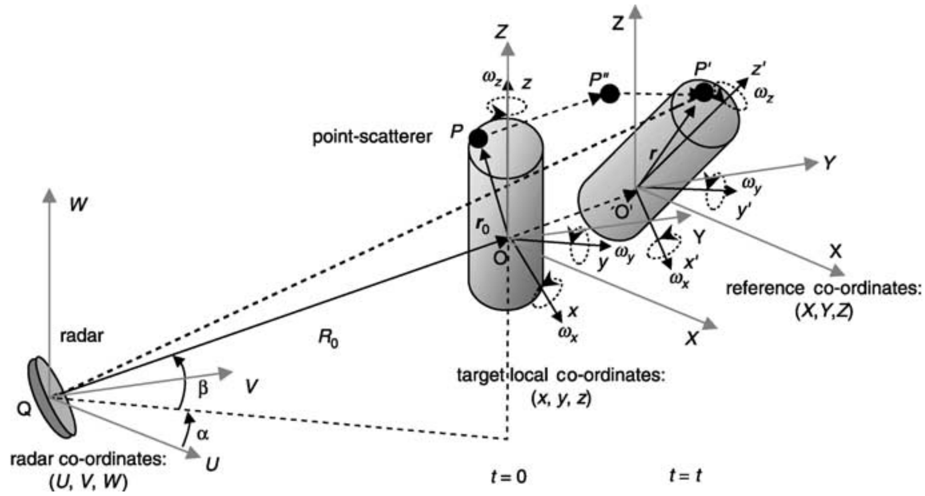

Fig. 1 *Geometry of a radar and a target with translation and rotations*

$$\boldsymbol{QP}' = \boldsymbol{QO} + \boldsymbol{OO}' + \boldsymbol{O}'\boldsymbol{P}' = \boldsymbol{R}_0 + \boldsymbol{v}t + \boldsymbol{r}$$
$$= \boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0 \tag{3}$$

Thus, the scalar range is

$$R_t = R(t) = \|\boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0\| \tag{4}$$

If the radar transmits a sinusoidal waveform with a carrier frequency f, then the baseband of the returned signal from the point scatterer is a function of R_t

$$s(t) = \rho(x,y,z)\exp\left\{j2\pi f\frac{2R_t}{c}\right\} = \rho(x,y,z)\exp\{j\Phi(R_t)\} \tag{5}$$

where $\rho(x,y,z)$ is the reflectivity function of the point scatterer P described in the target local co-ordinates (x, y, z), c is the propagation speed of the electromagnetic wave and the phase of the baseband signal is

$$\Phi(R_t) = 2\pi f\frac{2R_t}{c} \tag{6}$$

By taking the time derivative of the phase, the Doppler frequency shift induced by the target's motion can be obtained

$$\begin{aligned} f_D &= \frac{1}{2\pi}\frac{d\Phi(R_t)}{dt} = \frac{2f}{c}\frac{d}{dt}R_t \\ &= \frac{2f}{c}\frac{1}{2R_t}\frac{d}{dt}[(\boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0)^T \\ &\quad \times (\boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0)] \\ &= \frac{2f}{c}\left[\boldsymbol{v} + \frac{d}{dt}(Rot(t)\cdot\boldsymbol{r}_0)\right]^T \boldsymbol{n}_P \end{aligned} \tag{7}$$

where

$$\boldsymbol{n}_P = \frac{\boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0}{\|\boldsymbol{R}_0 + \boldsymbol{v}t + Rot(t)\cdot\boldsymbol{r}_0\|}$$

is the direction unit vector from the radar to the point scatterer at P'.

The angular rotation velocity vector $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)^T$ defined in the reference co-ordinate system rotates along the unit rotation vector $\boldsymbol{\omega}' = \boldsymbol{\omega}/\|\boldsymbol{\omega}\|$ with a scalar angular velocity $\Omega = \|\boldsymbol{\omega}\|$. Assuming the rotational motion at each time interval can be considered to be infinitesimal, the rotation matrix can be written in terms of the matrix $\hat{\boldsymbol{\omega}}$ as

$$Rot(t) = \exp\{\hat{\boldsymbol{\omega}}t\} \tag{8}$$

where

$$\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_Z & \omega_Y \\ \omega_Z & 0 & -\omega_X \\ -\omega_Y & \omega_X & 0 \end{bmatrix} \tag{9}$$

is called the skew symmetric matrix associated with $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)^T$, which is the linear transformation that computes the cross product of the vector $\boldsymbol{\omega}$ with any other vector, as described in the Appendix.

Thus, the Doppler frequency shift in (7) becomes

$$\begin{aligned} f_D &= \frac{2f}{c}\left[\boldsymbol{v} + \frac{d}{dt}(e^{\hat{\boldsymbol{\omega}}t}\boldsymbol{r}_0)\right]^T \boldsymbol{n}_P = \frac{2f}{c}(\boldsymbol{v} + \hat{\boldsymbol{\omega}}e^{\hat{\boldsymbol{\omega}}t}\boldsymbol{r}_0)^T\boldsymbol{n}_P \\ &= \frac{2f}{c}(\boldsymbol{v} + \hat{\boldsymbol{\omega}}\boldsymbol{r})^T\boldsymbol{n}_P \approx \frac{2f}{c}(\boldsymbol{v} + \hat{\boldsymbol{\omega}} \times \boldsymbol{r})^T\boldsymbol{n} \end{aligned} \tag{10}$$

where, because $\|\boldsymbol{R}_0\| \gg \|\boldsymbol{v}t + Rot(t)\,\boldsymbol{r}\|$, the direction unit vector $\boldsymbol{n}_P$ can be approximated by $\boldsymbol{n} = \boldsymbol{R}_0/\|\boldsymbol{R}_0\| \approx \boldsymbol{n}_P$

Therefore, the Doppler frequency shift is approximately

$$f_D = \frac{2f}{c}[\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r}]_{radial} \tag{11}$$

where the first term is the Doppler shift due to the translation and the second term is the mathematical expression of the micro-Doppler

$$f_{microDoppler} = \frac{2f}{c}[\boldsymbol{\omega} \times \boldsymbol{r}]_{radial} \tag{12}$$

3 Time–frequency analysis of micro-Doppler signatures

A common method to analyse a time domain signal is transforming it from the time domain to the frequency domain by using the Fourier transform. The frequency domain shows the magnitude of different frequencies contained in the signal over the overall time period the signal is analysed. When the radar returned signal from a vibrating or rotating target is viewed in the frequency domain, its micro-Doppler shifts can be seen by their deviation from the centre frequency of the radar returns. Frequency-domain signatures provide information about frequency modulations generated by the vibration or rotation. Although the frequency spectrum may indicate the presence of micro-Doppler shifts and possibly the relative amount of displacement toward each side, because of the lack of time information, it is not easy to tell the vibration or rotation rate from the frequency spectrum alone. Therefore, the time–frequency analysis that provides time-dependent frequency information is more useful and is complementary to the existing time-domain or frequency-domain methods.

To analyse the time-varying frequency characteristics of the micro-Doppler, the radar returned signal should be analysed in the joint time–frequency domain by applying high-resolution time–frequency transforms. From the joint time–frequency domain signature, the frequency and the period of vibration or rotation can be found [2, 7]. The direction of movement of the target at a specific time may also be found by examining the time data and the sign of the micro-Doppler shift caused by the movement of the target.

Time–frequency transforms include linear transforms, such as the short-time Fourier transform (STFT) and bilinear transforms, such as the Wigner–Ville distribution (WVD). With a time-limited window function, the resolution of the STFT is determined by the window size. A larger window has higher frequency resolution but poor time resolution. The bilinear WVD has better characteristics of the time-varying spectrum than any linear transform. However, it suffers the problem of cross-term interference, i.e. the WVD of the sum of two signals is not the sum of their WVDs [8]. To reduce the cross-term interference, the kernel-filtered WVD can be used to preserve the useful properties of the time–frequency transform with a slightly reduced time–frequency resolution and a largely reduced cross-term interference. The WVD with a linear lowpass filter are characterised as the Cohen's class. In our micro-Doppler signature study, the smoothed pseudo Wigner–Ville distribution is used to reduce the cross-term interference and achieve higher resolution [9].

4 Simulation study of micro-Doppler signatures

In this Section, we present examples of vibrations and rotations that can induce micro-Doppler effects. Based on the mathematical analysis, we can calculate theoretical results of micro-Doppler signatures. Simulation study is used to verify the theoretical results.

In the simulation, the point scatterer model [10] is used for modelling targets because it is simple compared to the EM prediction code simulation and it is easy to observe the effect of vibration or rotations and separately study individual movements.

4.1 Micro-Doppler signature of a vibrating point scatterer

The geometry of the radar and a vibrating point-scatterer is illustrated in Fig. 2. The vibration centre O is stationary with azimuth angle α and elevation angle β with respect to the radar. The point-scatterer is vibrating at a vibration frequency f_v with maximum amplitude D_v. The azimuth and elevation angle of the vibration direction described in the reference co-ordinates (X, Y, Z) is α_P and β_P, respectively.

Because of the vibration, the point-scatterer P, which is initially at time $t = 0$ located at $(X_0, Y_0, Z_0)^T$ in (X, Y, Z), will at time t move to

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = r(t)\,\boldsymbol{n}_V + \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \tag{13}$$

where $\boldsymbol{n}_V = [\cos\alpha_P \cos\beta_P, \sin\alpha_P \cos\beta_P, \sin\beta_P]^T$ is the unit vector of the vibration direction.

Therefore, because of the vibration, the velocity of the scatterer P becomes

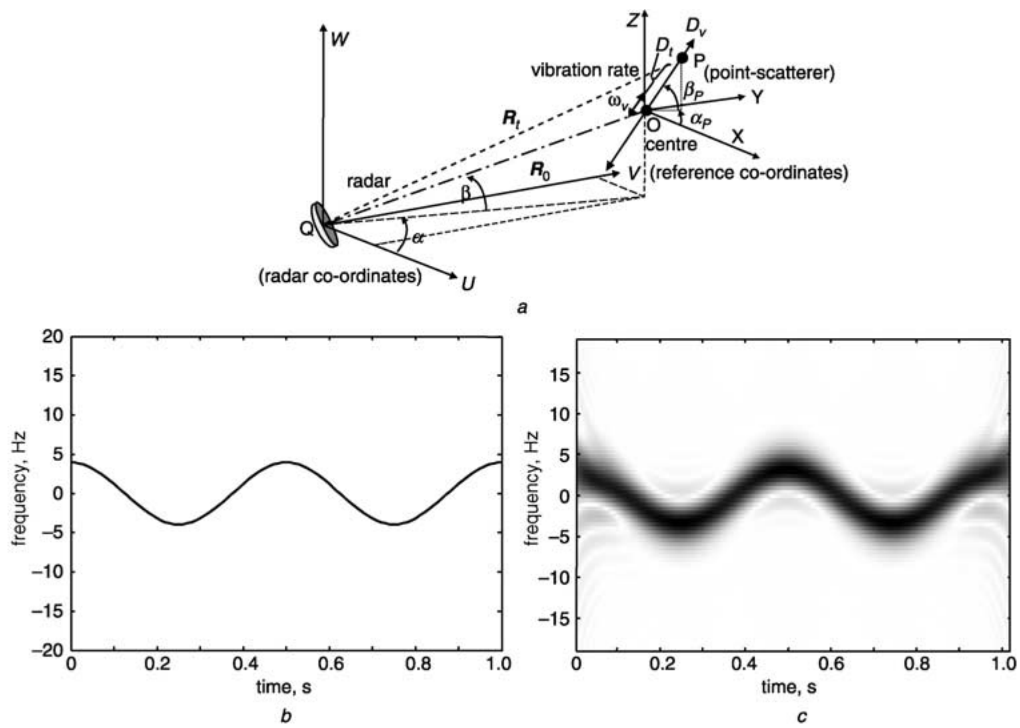

Fig. 2 *Geometry of radar and vibrating point scatterer; and time–frequency micro-Doppler signatures*

a Geometry of a radar and a vibrating point scatterer
b Time–frequency micro-Doppler signature calculated by (15)
c Time–frequency micro-Doppler signature by simulation

$$\frac{d}{dt}\boldsymbol{r}(t) = \frac{d}{dt}r(t)\,\boldsymbol{n}_V$$
$$= 2\pi D_v f_v \cos(2\pi f_v t)$$
$$\times (\cos\alpha_P \cos\beta_P, \sin\alpha_P \cos\beta_P, \sin\beta_P)^T \quad (14)$$

From (7) and using $Rot(t)\cdot\boldsymbol{r}_0 = \boldsymbol{r}$ and $\boldsymbol{n}_P \approx \boldsymbol{n}$, the micro-Doppler shift induced by the vibration is

$$f_{microDoppler} = \frac{2f}{c}\left[\frac{d}{dt}\boldsymbol{r}(t)\right]^T\cdot\boldsymbol{n}$$
$$= \frac{4\pi f f_v D_v}{c}\cos(2\pi f_v t)\boldsymbol{n}_V\cdot\boldsymbol{n} \quad (15)$$

which is a sinusoidal function of time oscillating at the vibration frequency.

Assume the radar operates at $f = 10\,\text{GHz}$ and a point-scatterer is vibrating about a centre point at ($U_0 = 1000\,\text{m}$, $V_0 = 5000\,\text{m}$, $W_0 = 5000\,\text{m}$). Thus, the unit vector from the radar to the vibration centre is

$$\boldsymbol{n} = (U_0, V_0, W_0)^T/(U_0^2 + V_0^2 + W_0^2)^{1/2}$$

If the amplitude and frequency of the vibration is $D_v = 0.01\,\text{m}$ and $f_v = 2\,\text{Hz}$, and the azimuth and elevation angle of the vibration direction are $\alpha_P = 20°$ and $\beta_P = 10°$, respectively, the theoretical result of the micro-Doppler signature calculated from (15) is shown in Fig. 2*b*.

In our simulation study, the pulse radar with a pulse repetition frequency (PRF) of 2000 is assumed and a total of 2048 pulses are used to generate the micro-Doppler signature of the vibrating point-scatterer. The simulation result is shown in Fig. 2*c* and is identical to the theoretical analysis.

4.2 Micro-Doppler signature of a rotating target

The geometry of the radar and a target having three-dimensional rotations is depicted in Fig. 3. The radar co-ordinate system is (U, V, W), the target local co-ordinate system (x, y, z) and the reference co-ordinate system (X, Y, Z) is parallel to the radar co-ordinates (U, V, W) and located at the origin of the target local co-ordinates. The azimuth and elevation angle of the target in the radar co-ordinates (U, V, W) is α and β, respectively.

Because of the target's rotation, any point on the target described in the local co-ordinate system (x, y, z) will move to a new position in the reference co-ordinate system (X, Y, Z). The new position can be calculated from its initial position vector multiplied by an initial rotation matrix Rot_{Init} determined by Euler angles (ϕ, θ, ψ) [6].

In the target local co-ordinate system (x, y, z), when a target rotates about its axes x, y and z with the angular velocity $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^T$, a point-scatterer P at $\boldsymbol{r}_0 = (x_0, y_0, z_0)^T$ in the local co-ordinates will move to a new location in the reference co-ordinates (X, Y, Z) described by $Rot_{Init}\cdot\boldsymbol{r}_0$. The unit vector of the rotation is defined by

$$\boldsymbol{\omega}' = (\omega'_x, \omega'_y, \omega'_z)^T = \frac{Rot_{Init}\cdot\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \quad (16)$$

To compute the 3-D rotation matrix $Rot(t)$ in (8), the Rodrigues' rotation formula [6]

$$R(t) = \exp(\hat{\boldsymbol{\omega}}\,t) = I + \hat{\boldsymbol{\omega}}'\sin\Omega t + \hat{\boldsymbol{\omega}}'2(1-\cos\Omega t) \quad (17)$$

is an efficient method, where $\boldsymbol{I}$ is the identity matrix, the scalar angular velocity $\Omega = \|\boldsymbol{\omega}\|$ and $\hat{\boldsymbol{\omega}}'$ is the skew symmetric matrix associated with $\boldsymbol{\omega}' = (\omega'_x, \omega'_y, \omega'_z)^T$

$$\hat{\boldsymbol{\omega}}' = \begin{bmatrix} 0 & -\omega'_z & \omega'_y \\ \omega'_z & 0 & -\omega'_x \\ -\omega'_y & \omega'_x & 0 \end{bmatrix} \quad (18)$$

Therefore, in the reference co-ordinate system (X, Y, Z), at time t the scatterer P will move from its initial location to a new location $\boldsymbol{r} = Rot_t\cdot Rot_{Init}\cdot r_0$. According to (12), the micro-Doppler frequency shift induced by the rotation is approximately

$$f_{microDoppler} = \frac{2f}{c}[\Omega\boldsymbol{\omega}'\times\boldsymbol{r}]_{radial} = \frac{2f}{c}(\Omega\hat{\boldsymbol{\omega}}'\boldsymbol{r})^T\cdot\boldsymbol{n}$$
$$= \frac{2f}{c}[\Omega\hat{\boldsymbol{\omega}}'\,Rot_t Rot_{Init}\cdot\boldsymbol{r}_0]^T\cdot\boldsymbol{n}$$
$$= \frac{2f\,\Omega}{c}\{[\hat{\boldsymbol{\omega}}'^2\sin\Omega t - \hat{\boldsymbol{\omega}}'^3\cos\Omega t$$
$$+\hat{\boldsymbol{\omega}}'(I+\hat{\boldsymbol{\omega}}'^2)]Rot_{Init}\cdot\boldsymbol{r}_0\}^T\cdot\boldsymbol{n} \quad (19)$$

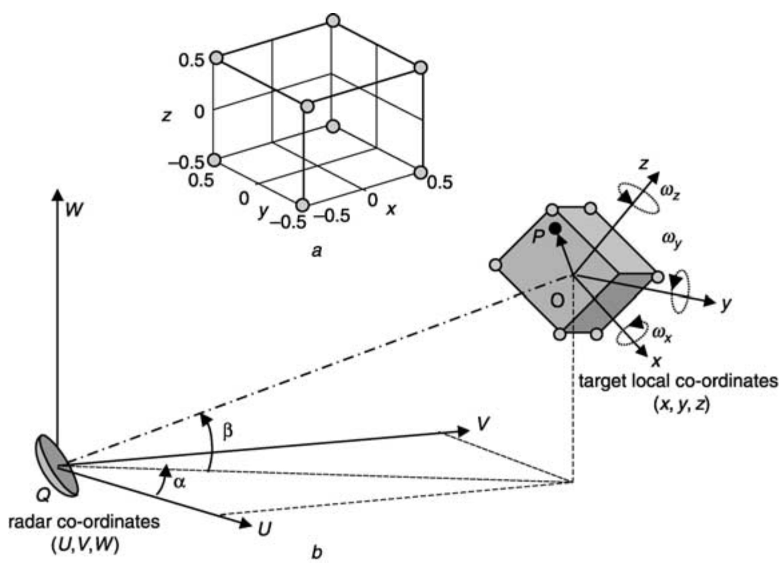

Fig. 3 *Geometry of a radar and a cubic target with eight scatterers*

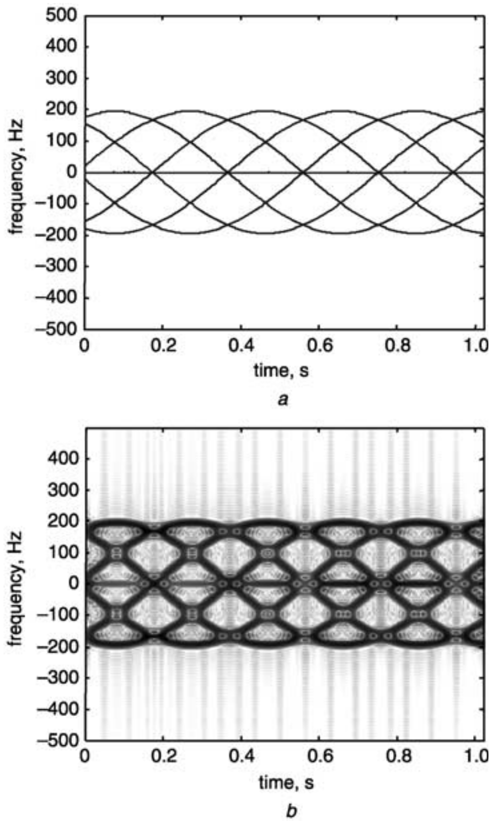

Fig. 4 *Time–frequency micro-Doppler signatures*
a Calculated from (8)
b By simulation

Because the skew symmetric matrix $\hat{\boldsymbol{\omega}}'$ is defined by the unit vector of the rotation $\boldsymbol{\omega}'$, then $\hat{\boldsymbol{\omega}}'^3 = -\hat{\boldsymbol{\omega}}'$ and the rotation-induced micro-Doppler frequency becomes

$$f_{microDoppler} = \frac{2f\,\Omega}{c}\left[\hat{\boldsymbol{\omega}}'(\hat{\boldsymbol{\omega}}'\sin\Omega t + I\cos\Omega t)Rot_{Init}\cdot\boldsymbol{r}_0\right]_{radial} \tag{20}$$

Assume the radar carrier frequency and the initial location of the target centre is the same as described in Section 4.1. The target is assumed to be a cube that consists of eight point-scatterers as illustrated in Fig. 3. The initial Euler angles are ($\phi = 45°$, $\theta = 45°$, $\psi = 45°$). If the target rotates along the x, y and z axes with an angular velocity $\boldsymbol{\omega} = [\pi, \pi, \pi]^T$ rad/s and initial positions of eight scatterers in the target co-ordinate system are

$$\begin{aligned}
P_1 &= (x = 0.5\,\text{m}, y = 0.5\,\text{m}, z = 0.5\,\text{m})\\
P_2 &= (x = -0.5\,\text{m}, y = 0.5\,\text{m}, z = 0.5\,\text{m})\\
P_3 &= (x = -0.5\,\text{m}, y = -0.5\,\text{m}, z = 0.5\,\text{m})\\
P_4 &= (x = 0.5\,\text{m}, y = -0.5\,\text{m}, z = 0.5\,\text{m})\\
P_5 &= (x = 0.5\,\text{m},\ y = 0.5\,\text{m},\ z = -0.5\,\text{m})\\
P_6 &= (x = -0.5\,\text{m}, y = 0.5\,\text{m}, z = -0.5\,\text{m})\\
P_7 &= (x = -0.5\,\text{m}, y = -0.5\,\text{m}, z = -0.5\,\text{m})\\
P_8 &= (x = 0.5\,\text{m}, y = -0.5\,\text{m}, z = -0.5\,\text{m})
\end{aligned}$$

Then, the micro-Doppler frequency shift can be calculated from (20) and is shown in Fig. 4*a*. With a PRF of 2000 and 2048 pulses transmitted within about 1.024 s. of dwell time, the simulated result of the micro-Doppler induced by the rotations is shown in Fig. 4*b*, which is identical to the theoretical result.

From the micro-Doppler signature, the period of the rotation period can be calculated as $T = 2\pi/\|\boldsymbol{\omega}\| = 1.1547$ s. We can see that the micro-Doppler signature in the time–frequency domain is a sinusoid with initial phase and amplitude that depends on the initial positions of the scatterer and the initial Euler angle (ϕ, θ, ψ).

5 Summary

We have shown that the mechanical vibrations or rotations of a target, or structures on the target, can induce additional frequency modulation on radar returned signals and generate the micro-Doppler effect. We derived mathematical formulas for micro-Doppler, and also simulated micro-Doppler signatures of targets undergoing vibrations or rotations. The simulation results confirmed that the mathematical analysis is valid.

6 Acknowledgments

This work was supported in part by the US Office of Naval Research and the Missile Defense Agency.

7 References

1 Zediker, M.S., Rice, R.R., and Hollister, J.H.: 'Method for extending range and sensitivity of a fiber optic micro-Doppler ladar system and apparatus therefor', US Patent no. 6,847,817, 8 Dec. 1998
2 Chen, V.C., and Ling, H.: 'Time-frequency transforms for radar imaging and signal analysis' (Artech House, 2002)
3 Kleinman, R., and Mack, R.B.: 'Scattering by linearly vibrating objects', *IEEE Trans. Antennas Propag.*, 1979, **27**, pp. 344–352
4 Gray, J.E.: 'The Doppler spectrum of accelerating objects'. Proc. IEEE Int. Radar Conf., Arlington, VA, USA, 1990, pp. 385–390
5 Chen, V.C., and Miceli, W.: 'Time-varying spectral analysis for radar imaging of maneuvering targets', *IEE Proc. Radar Sonar Navig.*, 1998, **145**, (5), pp. 262–268
6 Murray, R.M., Li, Z., and Sastry, S.S.: 'A mathematical introduction to robotic manipulation' (CRC Press, Boca Raton, FL, USA, 1994)
7 Chen, V.C.: 'Analysis of radar micro-Doppler signature with time-frequency transform'. Proc. 10th IEEE Workshop on Statistical signal and array processing, Pocono Manor, PA, USA, August 2000, pp. 463–466
8 Cohen, L.: 'Time-frequency analysis' (Prentice–Hall, Englewood Cliffs, NJ, 1995)
9 Auger, F., Flandrin, P., Goncalves, P., and Lemoine, O.: 'Time-frequency toolbox for use with MATLAB', 1996
10 Chen, V.C., and Miceli, W.: 'Simulation of ISAR imaging of moving targets', *IEE Proc., Radar Sonar Navig.*, 2001, **148**, (3), pp. 160–166

8 Appendix

The cross-product of a vector $\boldsymbol{a} = (a_x, a_y, a_z)$ and a vector $\boldsymbol{b} = (b_x, b_y, b_z)$ is

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \hat{\boldsymbol{a}}\,\boldsymbol{b} \tag{21}$$

where

$$\hat{\boldsymbol{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{22}$$

is called the skew symmetric matrix and

$$\hat{\boldsymbol{a}} = -(\hat{\boldsymbol{a}})^T \tag{23}$$

A rotation matrix that belongs to the special orthogonal 3-D rotation matrix group $\mathfrak{R}^{3\times3}$ is denoted by

$$\{Rot \in \mathfrak{R}^{3\times3} | Rot^T Rot = I, \det(Rot) = +1\} \tag{24}$$

By taking a derivative of the constraint $Rot(t)Rot^T(t) = I$ with respect to time t, we have

$$\dot{R}ot(t)Rot^T(t) = -[\dot{R}ot(t)Rot^T(t)]^T \tag{25}$$

This means that the matrix $\dot{R}ot(t)Rot^T(t) \in \mathfrak{R}^{3\times3}$ is a skew symmetric matrix. Therefore, we can find a rotation vector $\boldsymbol{\omega} = (\omega_X, \omega_Y, \omega_Z)$ such that the associated skew symmetric matrix

$$\hat{\boldsymbol{\omega}} = \dot{R}ot(t)Rot^T(t) \tag{26}$$

thus

$$\dot{R}ot(t) = \hat{\omega}\, Rot(t) \tag{27}$$

By solving this linear ordinary differential equation (27), we obtain

$$Rot(t) = \exp\{\hat{\boldsymbol{\omega}} t\} Rot(0)$$

Assuming $Rot(0) = I$ for the initial condition, we have

$$Rot(t) = \exp\{\hat{\boldsymbol{\omega}} t\} \tag{28}$$

The matrix is a 3-D rotation matrix that rotates about the axis $\boldsymbol{\omega}$ by $\|\boldsymbol{\omega}\| t$ rad.