IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Date of publication xxxx 00, 0000, date of current version xxxx 00, 0000.

Digital Object Identifier 10.1109/ACCESS.2017.DOI

Analysis of multi-phase flow through porous media for imbibition phenomena by using the LeNN-WOA-NM algorithm

NAVEED AHMAD KHAN[1], M SULAIMAN[*1], ABDULAH JEZA ALJOHANI, (MEMBER, IEEE)[2,3], POOM KUMAM, (MEMBER, IEEE)[*4,5], and Hussam Alrabaiah[6,7]

[1]Department of Mathematics, Abdul Wali Khan University, Mardan 23200, KP, Pakistan (e-mail: msulaiman@awkum.edu.pk, ahmednaveed854477@gmail.com)
[2]Department of Electrical and Computer Engineering, King Abdulaziz University, Jeddah 21589, Saudi Arabia (e-mail: ajaljohani@kau.edu.sa)
[3]Center of Excellence in Intelligent Engineering Systems, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[4]Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
[5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[6]College of Engineering, Al Ain University, Al Ain, United Arab Emirates
[7]Mathematics Department, Tafila Technical University, Tafila, Jordan.

Corresponding author(s): Muhammad Sulaiman (msulaiman@awkum.edu.pk), Poom Kumam (poom.kum@kmutt.ac.th).

This project was funded by the Deanship of Scientific Research, King Abdulaziz University, Jeddah, under grant No. (DF-232-135-1441). The author, therefore, gratefully acknowledge DSR technical and financial supports; and the authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), King Mongkut's University of Technology Thonburi (KMUTT).

ABSTRACT The flow of fluids in multi-phase porous media results due to many interesting natural phenomena. The counter-current water imbibition phenomena, that occur during oil extraction through a cylindrical well is an interesting problem in petroleum engineering. During the secondary oil recovery process, water is injected into a porous media having heterogenous and homogenous characteristics. Due to the difference in viscosities of fluids in oil wells, the counter-current imbibition phenomenon occurs. At that moment, the imbibition equation $V_i = -V_n$ is satisfied by the viscosities of oil and water. In this paper, we have analyzed the governing mathematical model of the imbibition phenomenon occurring during the secondary oil recovery process. A new soft computing algorithm is designed and adapted to analyze the mathematical model of dual-phase flow in detail. Weighted Legendre polynomials based artificial neural networks are hybridized with an efficient global optimizer the Whale Optimization Algorithm (WOA) and a local optimizer the Nelder-Mead algorithm. It is established, that our algorithm LeNN-WOA-NM is efficient and reliable in calculating high-quality solutions in less time. We have compared our experimental outcome with state-of-the-art results. The quality of our solutions is judged based on values of absolute errors, MAD, TIC, and ENSE. It is obvious that LeNN-WOA-NM algorithm can solve real application problems efficiently and accurately.

INDEX TERMS Counter-current imbibition phenomena, homogeneous porous media, heterogeneous porous media, capillary pressure, soft computing algorithm, weighted Legendre neural networks, Nelder-Mead algorithm, whale optimization algorithm.

I. INTRODUCTION

THIS paper models the phenomena of counter-current imbibition in the multi-phase flow of two immiscible fluids through heterogeneous and homogenous porous mediums that occurs during the secondary oil recovery process or water flooding [1]. In applied sciences, various changes occur in porous mediums having multi-phase flows of different fluids. In the past few decades, researchers are focused on studying the flows in underground petroleum reservoirs [2]. The primary recovery process includes the production of oil by simple natural decompression with no effect of external forces at wells [3]. In secondary oil recovery or water flooding process, fluids like water, polymer, steam are injected into a reservoir through injection wells located in rocks that have fluid communication with production wells. This process is called the immiscible displacement process [3].

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

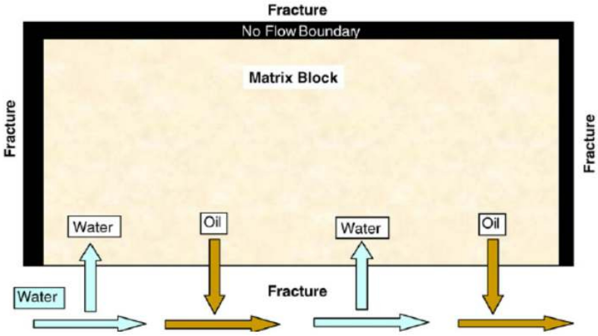


FIGURE 1: Process of counter-current imbibition phenomenon during fracture reservoir [3].

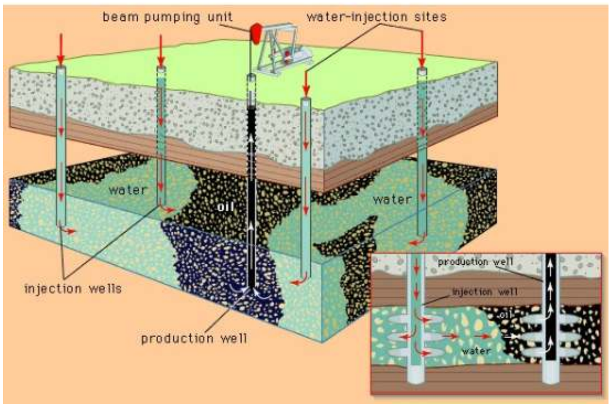


FIGURE 2: Natural mechanism of counter-current imbibition phenomenon [3].

In this paper, we have discussed the imbibition phenomenon occurring during the secondary oil recovery (displacement) process through heterogeneous and homogenous porous mediums with capillary pressure. Due to the difference of two insoluble fluids, each fluid has a distinct wetting ability. Imbibition defined by [4] is the process when a non-wetting fluid displaces the wetting fluid, and vice-versa is called the drainage. In spontaneous imbibition, the non-wetting fluid is removed, and wetting fluid is drawn into rocks by capillary suction with no external pressure. The rate and extent of imbibition depend critically on the viscosity of wetting and non-wetting fluids. Other factors include pore structure, fluid interfacial tension (IFT), relative permeability curves, and the initial water saturation of the rock. The importance of spontaneous imbibition is exemplified as the oil recovery from fractured reservoirs. In spontaneous imbibition-process of sea-water into an oil-filled porous rock, generally, it is considered that flow rates of the oil and sea-water are equal and opposite in the directions [3], as shown in FIGURE 1. The rock properties, like permeability and prosperity, may vary from one place to another in a heterogeneous porous medium.

The imbibition phenomenon arises when fluid like water is injected in the formation of oil to push the oil towards the production wells. If the porous medium (heterogenous/homogenous) with some fluid (oil) is brought in contact with some other fluid from the medium. When the flow of the original fluid (water) and the flow of fluid (oil) injected is in opposite direction then this phenomenon is called counter-current imbibition phenomenon. The secondary oil recovery mechanism in hydrocarbon reservoir is shown in FIGURE 4. The imbibition phenomenon is applicable in oil recovery process, biological sciences, composite materials, surface chemistry, printing process, food industry, textiles and construction.

Investigation of the secondary oil recovery process or water flooding process has been an active area of research for many years. Researchers have discussed the imbibition phenomenon during the water flooding process with a different point of view. Numerical solutions for the oil recovery process was discussed by Kazemi [5] with the help of empirical, exponential transfer functions based on the data provided by Aronofsky [6], Mattax and Kyte [7]. The effect of boundary conditions and sample shape on capillary imbibition was presented by Torsaeter and Silseth [8]. Mehta and Yadav [9] discovered the mathematical model and found the approximate solutions of counter-current imbibition phenomena in a banded porous matrix. Spatially heterogeneous reservoirs are studied by modeling the non-classical flow of spontaneous imbibition [10]. In [11] a heterogeneous porous medium with cracks and small inclination is investigated. A partially covered homogenous hydrocarbon reservoir is studied by modeling the phenomena of co-current spontaneous imbibition during oil recovery [12], [13]. The optimal homotopy analysis method is used to find approximated solutions of imbibition phenomena arising in heterogenous porous media [14]. Another interesting problem of spontaneous imbibition is investigated with a multi-layer porous medium [15]. Capillary pressure is considered in the analysis of counter-current imbibition phenomena in a double phase flow [16]. A study of gravity effects on spontaneous imbibition from cores, by considering two ends open in the periodical frontal flow is accomplished in [17].

Imbibition phenomena for the homogenous porous medium have been discussed formally by many researchers, but the heterogeneous porous medium for imbibition has been discussed by a few researchers. The analytical viewpoint in cracked porous media is discussed by Verma [18]. Counter-current and co-current imbibitions are investigated by Bourblaux and Kalaidjian [19] and conclude the effect of heterogeneous porous medium on the spatial distribution of the fluids. Applications of metaheuristics are gaining momentum in various fields of science and engineering. A plant propagation algorithm (PPA) was designed to solve design engineering problems [20]. A modified version of PPA is presented in [21]. Impacts of different crossover operators are investigated for handling multi-objective problems [22]. An improved version of the genetically adaptive multi-algorithm paradigm is studied in [23]. Plant propagation algorithm is modified and applied to constrained, unconstrained prob-

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

lems, and theoretical analysis are studied in [24]–[26]. A state-of-the-art survey is published in [27], where evolutionary algorithms are investigated in terms of decomposition and indicator functions [28]–[30]. In electrical engineering, several metaheuristics are used to solve complex optimization problems [31]–[34]. Unconstrained single-objective optimization problems are solved by using a hybrid of global and local search procedures [29], [35]. The optimal design of heat fins is proposed in [36]. A study of temperature distribution in heat fins is carried out by using a hybrid of the Cuckoo Search (CS) algorithm and Artificial Neural Network architecture [37], [38]. Neuro-fuzzy modeling is used to predict the summer precipitation in targeted metrological sites [39]. An interesting study of financial market forecasting is accomplished by the ARFIMA-LSTM technique [39]. Fractional order DPSO algorithm is used to solve the corneal model for eye surgery [40]. A novel initialization strategy is introduced in a multi-verse optimization technique, and different design engineering problems are solved in [41]. Nonlinear dusty plasma systems are analyzed with the help of NAR-RBFs neural networks [42]. A neuro-evolutionary algorithm is applied to investigate oscillatory behavior of heart beat [43]. Singular ordinary differential equations are handled by a hybrid of DPSO and artificial neural networks [44]. Fractional differential equations representing the damping materials are analysed by an efficient soft computing algorithm [44].

Continuing the efforts in this field, in the present study, we design a new soft computing algorithm to analyze the mathematical model in detail. Weighted Legendre polynomials based artificial neural networks are hybridized with an efficient global optimizer the Whale Optimization Algorithm (WOA) and a local optimizer the Nelder-Mead algorithm. It is established that our algorithm LeNN-WOA-NM is efficient and reliable in calculating high-quality solutions in less time. We have compared our experimental outcome with state-of-the-art results. The quality of our solutions is judged based on the values of absolute errors, MAD, TIC, and ENSE. It is evident that the LeNN-WOA-NM algorithm can solve real application problems efficiently and accurately. It is worth noting that in this paper the mathematical model is presented to discuss the heterogenous porous media along with homogenous porous media where oil is assumed as the non-wetting fluid and water is considered as wetting fluid.

Our key contributions in this research are summarized as follows:

- A new accurate and efficient soft computing procedure is designed. Weighted Legendre polynomials based neural networks are used to model approximate series solutions for ordinary differential equations. The best sets of unknown weights are determined by constructing a fitness function for each problem. Fitness functions are optimized by minimizing the residual errors. A recently developed global optimizer, the Whale Optimization Algorithm (WOA), and a local optimizer, the Nelder-Mead algorithm are combined to minimize the objective function. Our procedure is named as the LeNN-WOA-NM algorithm.
- To test the efficiency and accuracy of our algorithm, we have analyzed the imbibition phenomenon which occurs during the secondary oil recovery process. It is a real-application problem of multi-phase fluid flow through porous media, see FIGURES 1 and 2.
- Statistical analyses are carried out in terms of mean absolute deviation (MAD), Theil's inequality coefficient(TIC), and Nash Sutcliffe efficiency (NSE). Normal probability graphs are indicating the performance of the proposed technique. Errors in our solutions dictate that LeNN-WOA-NM algorithm calculated the best so far solution for the problem of saturation of injected fluid (water) through heterogeneous and homogenous mediums.
- The results for saturation of injected fluid are shown through different graphs and tables which dictate the dominance and robustness of the proposed (LeNN-WOA-NM) algorithm.

II. STATEMENT OF THE PROBLEM

During the secondary oil recovery process, when fluid like water is injected in oil formation to drive oil towards production well, then some phenomena occur. Let us consider a finite cylindrical piece of a heterogeneous porous matrix of length "L" and saturated with native fluid "N" like oil. Each side of the cylinder is bounded by impermeable surfaces expect at one end of the cylinder, which is labeled as imbibition surface (x = 0). This end is exposed to an adjacent formation of the injected fluid (I) as water [45]. Since water preferentially wets the medium, so such arrangements give rise to the linear counter-current imbibition, which is spontaneous and linear flow of wetting phase of fluid (water) into the medium and counter flow of the resident fluid (oil) from the medium.

The area of our interest in this work is to find the saturation denoted by $S_i(X,T)$ of water (injected fluid) into the oil (Native fluid) in counter-current imbibition phenomenon during the secondary oil recovery process [3]. The graphical overview of the problem is shown in FIGURE 5.

III. MATHEMATICAL MODEL OF THE PROBLEM

A. BASIC EQUATIONS

During the process of injection, let us assume that native fluid "N" and injected fluid "I" are two insoluble (immiscible) fluids that are governed by the Darcy's law, and their seepage velocity is given by [47].

$$V_i = \frac{k_i}{\delta_i} K \frac{\partial p_i}{\partial x}, \tag{1}$$

$$V_n = -\frac{k_n}{\delta_n} K \frac{\partial p_n}{\partial x}, \tag{2}$$

where K is the permeability of both homogenous and heterogeneous porous mediums. Relative permeability of displacing fluids are given by k_i and k_n, which are functions of saturations S_i and S_n. The pressures of native fluid and

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

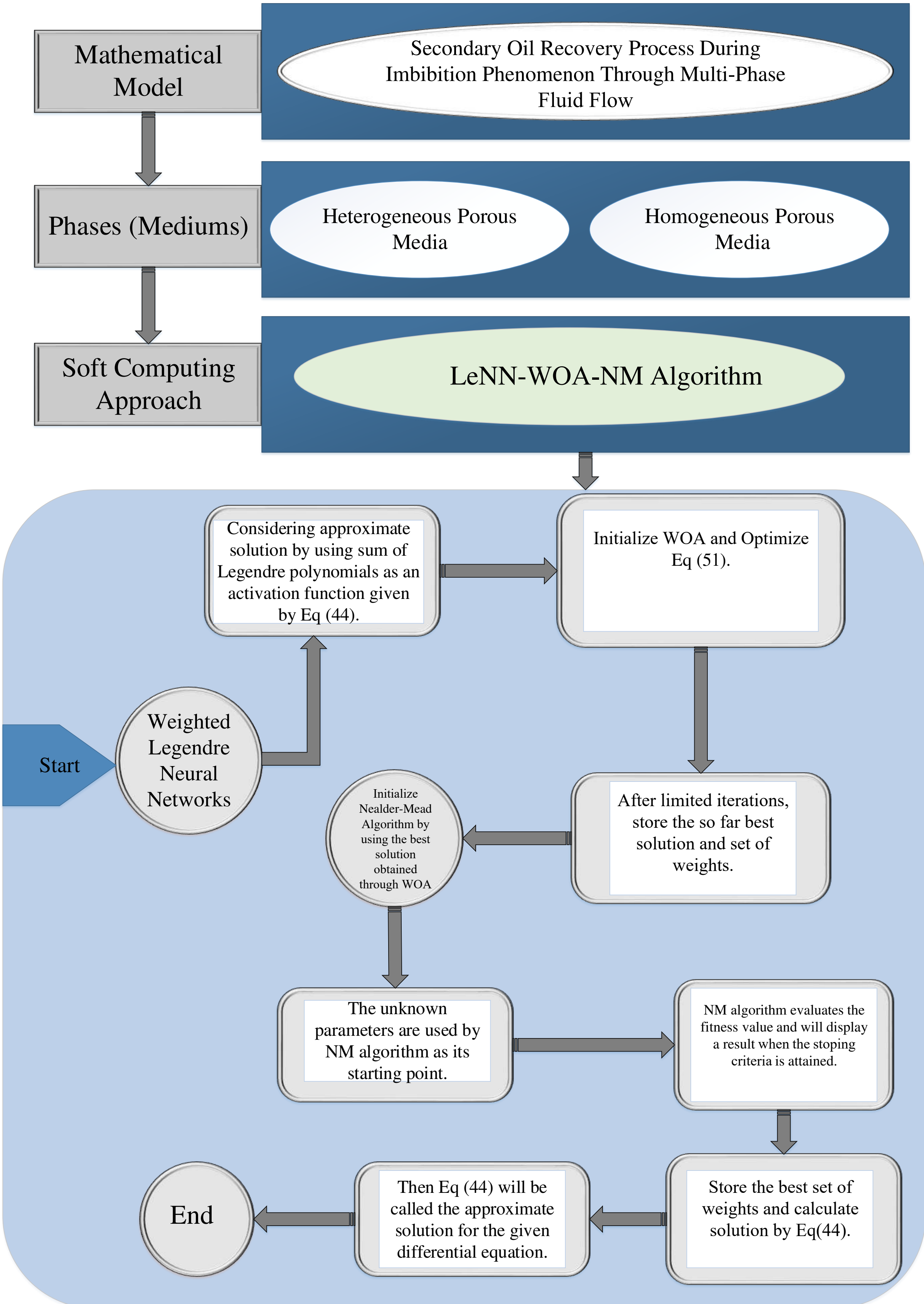


FIGURE 3: Flowchart of LeNN-WOA-NM algorithm for finding the saturation of water (injected fluid) into the oil (native fluid) in the secondary oil recovery process.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

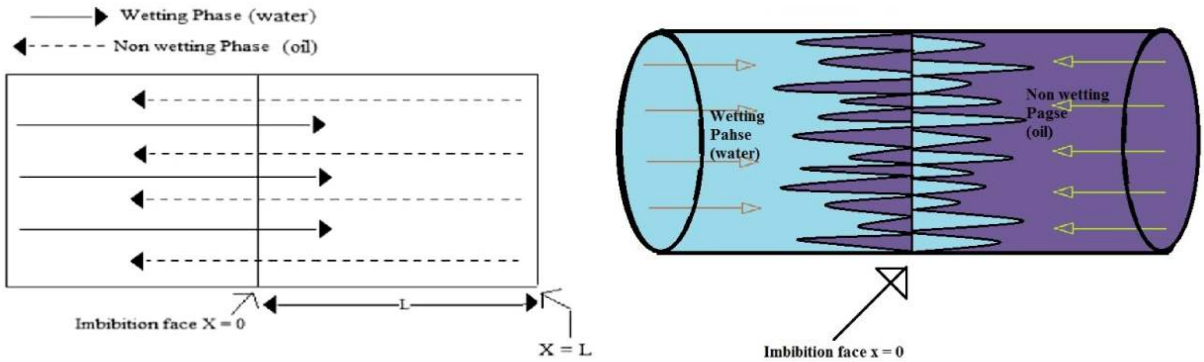


FIGURE 4: Representation of counter-current imbibition in a cylindrical piece during secondary oil recovery or water flooding process [3].

displacing fluid is presented by p_n and p_i. Kinematic viscosities of native fluid (oil) and displacing fluid (water) are represented by δ_n and δ_i, respectively. Equation of continuity for two immiscible fluids can be written as

$$P\frac{\partial S_i}{\partial t}+\frac{\partial V_i}{\partial x}=0, \tag{3}$$

$$P\frac{\partial S_n}{\partial t}+\frac{\partial V_n}{\partial x}=0, \tag{4}$$

where $P=P(x)$ shows the porosity of the two porous mediums. In counter-current imbibition phenomena, the following condition is apparent at a common interface [48],

$$V_i=-V_n,$$

$$V_i+V_n=0. \tag{5}$$

B. ANALYTICAL RELATIONSHIPS

In this section, we present the mathematical descriptions of the motion for saturation of water (injected fluid) through heterogeneous/ homogeneous porous mediums. To describe the problem, we use the following relationships.

1) Capillary pressure

When a fluid (oil) is injected into a displacing fluid (water), the flow of the fluids takes place in interconnected capillaries. Therefore, capillary pressure denoted by p_c at the standard interface is the function of injected fluid saturation.

$$P_c\left(S_i\right)=p_n-p_i. \tag{6}$$

In the imbibition phase, capillary pressure is linear, and in the absence of external force, it is linearly proportional to a common interface. Thus, the directions of water saturation and capillary pressure are the opposite. Thus by Mehta [49] Eq (6) becomes,

$$P_c=-\beta(S_i), \tag{7}$$

where β is a proportionality constant.

2) Phase saturation and relative permeability

Scheidegger and Johnson [50] gave the standard relation between relative permeability and phase saturation which is given as,

$$\begin{aligned} k_i&=S_i,\\ k_n&=1-\alpha S_i, \end{aligned} \tag{8}$$

where α is a constant and its value is 1.11.

3) Characteristics of the medium

Sticking our attention to heterogenous porous media we have applied the laws of variation in the permeability and porosity of the uniform heterogeneous medium, defined as the function of x as follows [3],

$$\begin{aligned} P&=P(x)=\frac{1}{a-bx},\\ K&=K(x)=K_c\left(1+a_1x\right), \end{aligned}$$

where are a ,b ,a_1 and K_c are positive constants. Since we know that permeability and porosity of heterogenous porous medium vary with distance x and time t. Thus, for better results and accuracy we suppose,

$$P=P(x,t)=\frac{-1}{b(t)x-a(t)}, \tag{9}$$

$$K=K(x,t)=K_c\left(a_1(t)x+1\right), \tag{10}$$

where $a(t)$, $b(t)$ and $a_1(t)$ are constants depending on time. Furthermore, we assumes that $-(b(t)x-a(t))\geqslant 1$.

C. SATURATION BY EQUATION OF MOTION

Using values of V_i and V_n in Eq (5), then using Eq (6) and after simplification, we have

$$\frac{\partial p_i}{\partial x}=-\left[\left(\frac{\frac{k_n}{\delta_n}\left(\frac{\partial p_c}{\partial x}\right)}{\frac{k_i}{\delta_i}+\frac{k_n}{\delta_n}}\right)\right], \tag{11}$$

putting value of Eq (11) in Eq (1) we get,

$$V_i=K\left[\frac{\frac{k_i}{\delta_i}\frac{k_n}{\delta_n}}{\frac{k_i}{\delta_i}+\frac{k_n}{\delta_n}}\right]\frac{\partial p_c}{\partial x}. \tag{12}$$

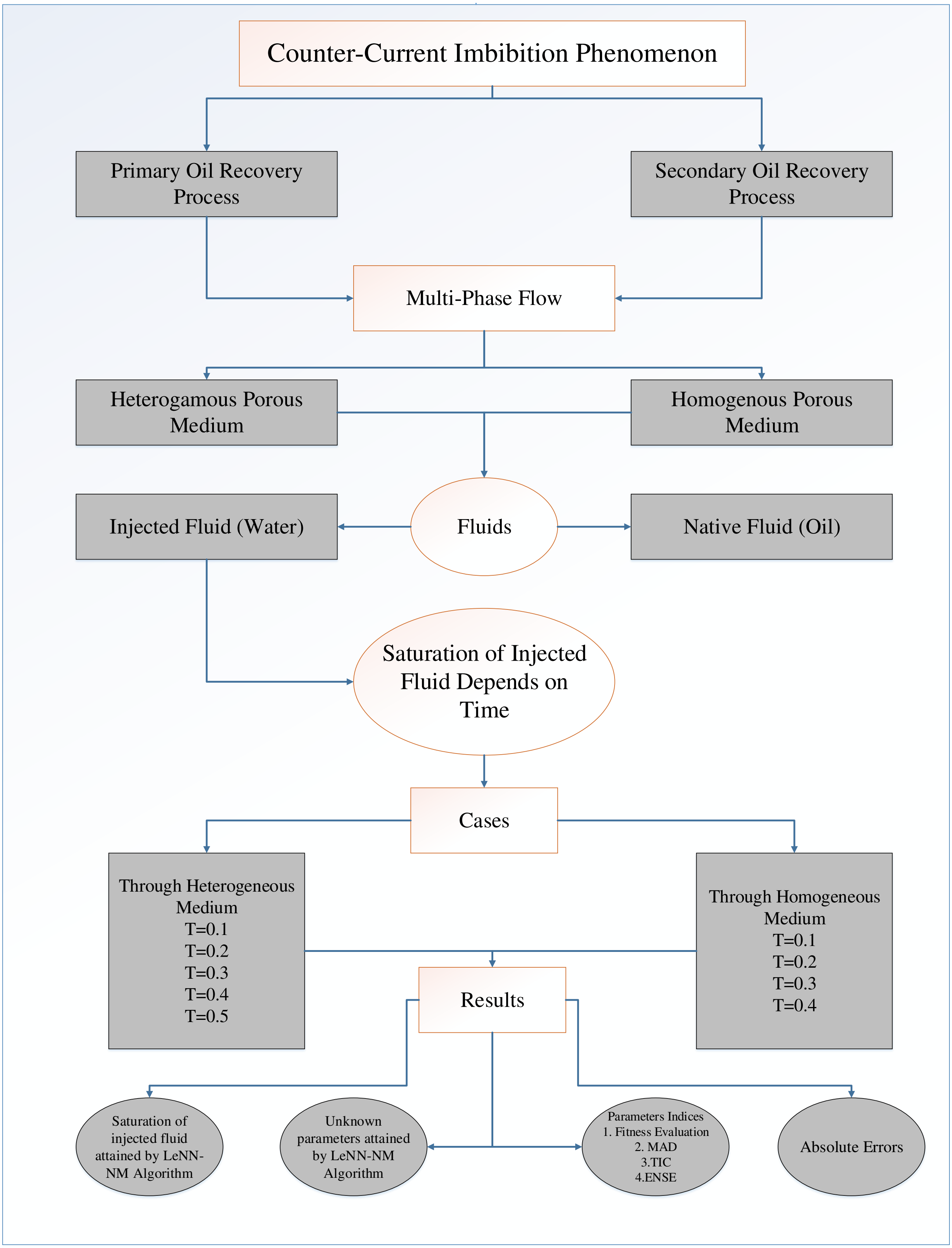


FIGURE 5: Graphical overview of Counter-current Imbibition phenomenon along with the cases discussed in this paper for secondary oil recovery process through multi phases [46].

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

Now using value of V_i in equation of continuity given by Eq (3), we get,

$$P\left(\frac{\partial S_i}{\partial t}\right)+\frac{\partial}{\partial X}\left\{K\frac{(1-\alpha S_i)}{\delta_n}\left(\frac{\partial P_c}{\partial S_i}\right)\left(\frac{\partial S_i}{\partial x}\right)\right\}=0, \tag{13}$$

where, $\frac{\frac{k_n k_i}{\delta_n \delta_i}}{\frac{k_n}{\delta_n}+\frac{k_i}{\delta_i}} \approx \frac{k_n}{\delta_n} = \frac{(1-\alpha S_i)}{\delta_n}$ and $P_c = -\beta S_i$.
Simplifying Eq (13) we get,

$$P\frac{\delta S_i}{\delta t}+\frac{\beta}{\delta_n}\frac{\partial}{\partial x}\left[\left(K\left(\alpha S_i-1\right)\frac{\partial S_i}{\partial x}\right)\right]=0, \tag{14}$$

In order to simplify Eq (14), we consider the relations $P \propto K$ or $K \propto P$, we obtain,

$$K = K_c P,$$

where K_c is a proportionality constant.
Using value of K in Eq (14), we have

$$P\frac{\partial S_i}{\partial t}+\frac{\beta K_c}{\delta_n}\frac{\partial}{\partial x}\left[\left(P\left(\alpha S_i-1\right)\frac{\partial S_i}{\partial x}\right)\right]=0. \tag{15}$$

Eq (15) expresses the counter-current imbibition phenomenon which represents the motion for saturation of water (injected fluid) through heterogenous porous medium. The boundary conditions for saturation of water (injected fluid) into oil (native fluid) at common interface are

$$S_i(0,t) = S_{i0} \text{ at } x = 0 \text{ for } t > 0. \tag{16}$$

At distance x = L the saturation will be

$$S_i(L,t)S_{il} \text{ at } x = L \text{ for } t > 0. \tag{17}$$

Initial condition for saturation of water (injected fluid) into oil is

$$S_i(x,0) = S_{ic} \text{ at } t = 0 \text{ for } x > 0, \quad 0 \leqslant S_{ic} < S_{i0}. \tag{18}$$

IV. SIMILARITY TRANSFORMATION

A. HETEROGENOUS POROUS MEDIUM

Now we use the technique of similarity transformation to reduce the non-linear PDE given by Eq (15) into ODE. Let we choose a dimensionless variable X and T such that

$$X = \frac{x}{L} \text{ and } T = \frac{\alpha\beta K_c}{L^2\delta_n}t.$$

Using values of X and T in the governing PDE given by Eq (15) along with boundary conditions, Eq (16-18), we get the following dimensionless form

$$\frac{\partial S_i}{\partial T}+\frac{\partial}{\partial X}\left[\left(S_i-\frac{1}{\alpha}\right)\frac{\partial S_i}{\partial X}\right]+\frac{1}{P}\frac{\partial P}{\partial X}\left(S_i-\frac{1}{\alpha}\right)\frac{\partial S_i}{\partial X}=0, \tag{19}$$

After simplifying Eq (19),

$$\frac{\partial S_i}{\partial T}+\frac{\partial}{\partial X}\left[\left(S_i+\lambda\right)\frac{\partial S_i}{\partial X}\right]+\frac{1}{P}\frac{\partial P}{\partial X}\left(S_i+\lambda\right)\frac{\partial S_i}{\partial X}=0, \tag{20}$$

where $\lambda = -\frac{1}{\alpha}$ and $\alpha = 1.11$.
In order to simplify Eq (20), the term $\frac{1}{P}\frac{\partial P}{\partial X}$ has been simplified as

$$\frac{1}{P}\frac{\partial P}{\partial X} = \frac{\partial}{\partial X}(\log P) = \frac{\partial}{\partial X}\left[\frac{Lb(T)}{a(T)}X\right],$$

neglecting higher-order terms of X, we get

$$\frac{1}{P}\frac{\partial P}{\partial X} = \frac{\partial}{\partial X}(\log P) = \frac{\partial}{\partial X}\left[\frac{Lb(T)}{a(T)}X\right] = \frac{Lb(T)}{a(T)} = \frac{L}{2\sqrt{T}}, \tag{21}$$

where $\frac{b(T)}{a(T)} = \frac{1}{2\sqrt{T}}$, and $0 < T \leq 1$.
Hence,

$$\frac{\partial S_i}{\partial T}+\frac{\partial}{\partial X}\left[\left(S_i+\lambda\right)\frac{\partial S_i}{\partial X}\right]+\frac{L}{2\sqrt{T}}\left(S_i+\lambda\right)\frac{\partial S_i}{\partial X}=0, \tag{22}$$

$$S_i(0,T) = S_{i0} \text{ at } X = 0 \text{ for } T > 0, \tag{23}$$

$$S_i(X,0) = S_{ic} \text{ at } T = 0 \text{ for } X > 0; \quad 0 \leqslant S_{ic} < S_{i0}. \tag{24}$$

Now choosing the similarity transformation as,

$$S_i(X,T) = f(\eta), \text{ where } \eta = \frac{X}{2\sqrt{T}}.$$

Hence, the partial differential equation, Eq (22), along with boundary conditions Eq (23) and Eq (24), representing the counter-current imbibition phenomena is reduced to an ordinary differential equation given by Eq (25)

$$(f(\eta)+\lambda)f''(\eta)+L(\lambda+f(\eta))f'(\eta) = f'(\eta)[2\eta - f'(\eta)], \tag{25}$$

along with boundary conditions

$$f(0) = S_{i0}, \qquad X = 0, \quad T > 0, \tag{26}$$

$$f(\infty) = S_{ic}, \qquad T = 0, \quad X > 0. \tag{27}$$

Its been noticed that the boundary conditions pose difficulties, because Eq (27) can be satisfied easily. Mehta [49] suggests that family of curves satisfying the initial condition Eq (26) then initially they tends to horizontal asymptote as $\eta \to \infty$. Therefore, Eq (27) is transformed into variational condition.

$$f'(0) = \omega \neq 0, \qquad X = 0 \text{ for any } T > 0. \tag{28}$$

For simplicity, we substitute $F(\eta) = \lambda + f(\eta)$ in Eq (25), thus the boundary conditions given by Eq (26) and Eq (28) becomes

$$F'(\eta)[F'(\eta)-2\eta]+LF(\eta)F'(\eta)+(\eta)F''(\eta)=0, \tag{29}$$

$$F(0) = S_{i0}+\lambda, \tag{30}$$

$$F(0) = \omega. \tag{31}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

B. HOMOGENOUS POROUS MEDIUM

For counter-current imbibition phenomenon through the homogenous medium, we consider Eq (14) representing the motion of saturation of injected fluid (water),

$$P\frac{\partial S_i}{\partial t} = \frac{\beta K}{\delta_n}\frac{\partial}{\partial x}\left[\left((1-\alpha S_i)\frac{\partial S_i}{\partial x}\right)\right], \tag{32}$$

$$S_i(0,t) = S_{i0},\ \text{at } x = 0 \text{ for } t > 0, \tag{33}$$

$$S_i(L,t)S_{i1},\ \text{at } x = L \text{ for } t > 0. \tag{34}$$

Choosing the dimensionless variables as

$$X = \frac{x}{L} \text{ and } T = \frac{\alpha\beta K_c}{L^2\delta_n}t.$$

To covert the governing PDE representing imbibition phenomena into ODE through homogenous medium, the dimensionless form of Eq (32) is given as

$$\alpha\frac{\partial S_i}{\partial T} = \frac{\partial}{\partial X}\left[(1-\alpha S_i)\frac{\partial S_i}{\partial X}\right], \tag{35}$$

Let the similarity transformation be,

$$S_i(X,T) = f(\eta),\ \text{where } \eta = \frac{X}{2\sqrt{T}},$$

then ODE representing imbibition phenomenon through homogenous porous medium is given by Eq (36),

$$[f(\eta)+\lambda]f''(\eta) = f'(\eta)\left[f'(\eta)+2\eta\right], \tag{36}$$

subjecting to the boundary conditions

$$f(0) = S_{i0}, \quad X = 0, \quad T > 0, \tag{37}$$

$$f(\infty) = S_{ic}, \quad T = 0, \quad X > 0, \tag{38}$$

transforming the boundary condition Eq (38) into variational condition, we have

$$f'(0) = \omega \neq 0, \qquad X = 0 \text{ for any } T > 0. \tag{39}$$

Substituting $F(\eta) = \lambda + f(\eta)$ hence, Eq (36) becomes

$$F'(\eta)\left[F'(\eta)+2\eta\right] - F(\eta)F''(\eta) = 0, \tag{40}$$

with boundary conditions

$$F(0) = S_{i0} + \lambda, \tag{41}$$

$$F(0) = \omega. \tag{42}$$

V. APPROXIMATE SOLUTIONS AND WEIGHTED LEGENDRE POLYNOMIALS

Before discussing the solution for counter-current imbibition phenomenon through multi-phases, we first discuss the weighted Legendre polynomials. The Legendre polynomials are denoted by $L_n(t)$, where n represents the order of Legendre polynomials. These polynomials are orthogonal on $[-1,1]$, and so they constitute the set of orthogonal polynomials. The first ten Legendre polynomials are given in TABLE 1.

Higher-order Legendre polynomials are generated by the recursive formula given below

$$L_{n+1}(t) = \frac{1}{n+1}\left[(2n+1)tL_n(t) - nL_{n-1}(t)\right]. \tag{43}$$

We consider an approximate series solution for counter-current imbibitions phenomena through multi-phases as

$$y_{appox}(\eta) = \sum_{n=0}^{N}\zeta_n L_n(\psi_n\eta+\theta_n) - \lambda, \quad \eta = \frac{X}{2\sqrt{T}}, \tag{44}$$

where $0 \le X \le 1$,ζ_n, ψ_n and θ_n are unknown parameters and $\lambda = -0.9$. Since Eq (29) and Eq (36) are twice differentiable. Therefore, first derivative $y'(\eta)$ and second derivative $y''(\eta)$ of Eq (44) are represented by the following equations,

$$y'_{appox}(\eta) = \sum_{n=1}^{N}\zeta_n L'_n(\psi_n\eta+\theta_n), \tag{45}$$

$$y''_{appox}(\eta) = \sum_{n=4}^{N}\zeta_n L''_n(\psi_n\eta+\theta_n), \tag{46}$$

plugging Eq (44-46) in governing ordinary differential equations. Eq (29) and Eq (36) will be converted into an equivalent algebraic system of equations that can be solved for unknown parameters ζ_n, ψ_n and θ_n using LeNN-WOA-NM algorithm. Parameter setting for LeNN algorithm are given in TABLE 2.

VI. NELDER-MEAD (NM) ALGORITHM

To solve unconstrained optimization problems without prior information about their gradients, a single-path following method known as Nelder-Mead (NM) algorithm is used. NM is a local search technique and can find good results if initialized with a better initial solution. A simplex is set up to minimize a function using $n+1$ points, and these points define the vector direction for the next move on n-dimensional search space. Implementation of the NM algorithm is based on four necessary procedures, named as reflection, expansion, contraction, and shrink.

Reflection: In order to calculate the reflection point the following Eq. (47) is used,

$$X_r = \bar{X} + \alpha\left(\bar{X} - X_{n+1}\right), \tag{47}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is the centroid of the best points except X_{n+1}, "α" is reflection coefficient and is greater than zero. The iteration is terminated and we accept the point "X_r" if $f(X_1) \le f(X_r) < f(X_n)$.

Expansion: The expansion point given by Eq. (48),

$$X_e = \bar{X} + \beta\left(X_r - \bar{X}\right), \tag{48}$$

can be evaluated only if $f(X_r) < f(X_1)$, where β is an expansion coefficient and is greater than one. If $f(X_e) \le f(X_r)$, then iteration is terminated and we accept "X_e". If the above criteria is not satisfied then X_r will be accepted

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

TABLE 1: First eleven Legendre polynomials with independent variable t.

n	$L_n(t)$
0	1
1	t
2	$\frac{1}{2}\left(3t^2-1\right)$
3	$\frac{1}{2}\left(5t^3-3t\right)$
4	$\frac{1}{8}\left(35t^4-30t^2+3\right)$
5	$\frac{1}{8}\left(63t^5-70t^3+15t\right)$
6	$\frac{1}{16}\left(231t^6-315t^4+105t^2-5\right)$
7	$\frac{1}{16}\left(429t^7-693t^5+315t^3-35t\right)$
8	$\frac{1}{128}\left(6435t^8-12012t^6+6930t^4-1260t^2+35\right)$
9	$\frac{1}{128}\left(12155t^9-25740t^7+18018t^5-4620t^3+315t\right)$
10	$\frac{1}{256}\left(46189t^{10}-109395t^8+90090t^6-30030t^4+3465t^2-63\right)$

TABLE 2: Parameter setting for weighted Legendre Neural Networks (LeNN) and Nelder-Mead (NM) algorithm.

Algorithm	Parameter	Setting	Parameter	Setting
LeNN	Bounds (lower, upper)	[-50,1]	Maximum iterations	6,000
	Number of search agents	50	Candidate selection	Uniform
NM Algorithm	Initial weights	LeNN-WOA global best	Max. function evaluations	150,000
	Maximum iterations	2,000	X-Tolerance 'TolX'	1.00E-200
	'TolFun'	1.00E-200	Scaling	Objective and constraints

and so the iterations will be stopped.
Contraction: If $f\left(X_r\right) \geq f\left(X_n\right)$, between $\bar{X}$ and the best value in X_{n+1} and X_r, a contraction takes place.

a) If $f\left(X_r\right) < f\left(X_{n+1}\right)$ then the outside contraction is executed by Eq. (49)

$$X_c = \bar{X} + \gamma\left(X_{n+1} - \bar{X}\right), \tag{49}$$

where "γ" indicates the contraction coefficient and belongs to the interval $(0, 1)$. The iteration is terminated and we accept "X_c" if $f\left(X_c\right) \leq f\left(X_r\right)$. Otherwise, it will jump to shrink step.

b) If $f\left(X_r\right) \geq f\left(X_{n+1}\right)$ then inside contraction $X_c = \bar{X} - \gamma\left(\bar{X} - X_{n+1}\right)$ can be performed. The iteration is terminated and we accept X_c if $f\left(X_r\right) \leq f\left(X_{n+1}\right)$. Otherwise, it will jump to shrink step.
Shrink: Eq. (50) is used to perform the shrink operator

$$V_i = X_1 - \delta\left(X_1 - X_i\right), i = 2, \ldots, n+1, \tag{50}$$

where "δ" is in $(0, 1)$ and known as shrink coefficient. The shrink resulted by the simplex is represented as $V = \{X_1, V_1, V_2, \ldots, V_{n+1}\}$ for the succeeding iterations. Parameter setting for Nelder-Mead Algorithm are given in TABLE 2.

VII. LENN-WOA-NM ALGORITHM

The steps for the proposed hybridized algorithm are summarized as:

Step 1: Considering approximate solution by using Legendre polynomials as an activation function, see Eq (44).
Step 2: At random search points known as discrete points approximating the solution and its higher derivatives.
Step 3: Substituting the approximate solution and its derivatives in a given fitness function.
Step 4: Step 3 will result an equivalent algebraic system of equations corresponding to the given differential equation.
Step 5: Solving the system for ζ_n, ψ_n and θ_n.
Step 6: Nelder-Mead algorithm which is capable of good local search will take ζ_n, ψ_n and θ_n as its starting point.
Step 7: Nelder-Mead algorithm will evaluate the fitness function and will display the result when the stopping criteria for the given problem is attained.
Step 8: The best values for ζ_n, ψ_n and θ_n attained by LeNN-WOA-NM will be plugged into Eq (44).
Step 9: Return value obtained in step 8.
Flowchart of the proposed soft computing technique is shown in FIGURE 3.

VIII. FITNESS FUNCTION FORMULATION

An error function is constructed to evaluate each candidate solution and its fitness. It is a summation of mean squared errors at each $t \in [0\ 1]$. LeNN-WOA-NM algorithm is applied to minimize the fitness function for Eq (29), representing counter-current imbibition phenomenon through heterogenous porous media, and is written as

$$\text{Minimize } \epsilon = \epsilon_1 + \epsilon_2, \tag{51}$$

where ϵ_1 and ϵ_2 are the fitness-based errors in solutions for problem in Eq (29) and the boundary conditions Eq (30-31), mathematically it is given as,

$$\epsilon_1 = \frac{1}{N}\sum_{k=1}^{N}\left(y'_{approx}(\eta)\left[y'_{approx}(\eta) - 2\eta\right] + Ly_{approx}(\eta)y'_{approx}(\eta) + y_{approx}(\eta)y''_{approx}(\eta)\right) \tag{52}$$

$$\epsilon_2 = \frac{1}{2}\left((y_{approx}(0) - (S_{i0} + \lambda))^2 + (y_{approx}(0) - \omega))^2\right), \tag{53}$$

where $N = \frac{1}{h}$ and h is a step size.

Similarly, fitness function for the ODE in Eq (40) and boundary conditions in Eqs (41-42) representing the secondary oil recovery process through homogenous porous media, is given as,

$$\epsilon_1 = \frac{1}{N}\sum_{k=1}^{N}\left(y'_{approx}(\eta)\left[y'_{approx}(\eta) - 2\eta\right] - y_{approx}(\eta)y''_{approx}(\eta)\right), \tag{54}$$

$$\epsilon_2 = \frac{1}{2}\left((y_{approx}(0) - (S_{i0} + \lambda))^2 + (y_{approx}(0) - \omega))^2\right), \tag{55}$$

where $N = \frac{1}{h}$ and h is a step size.

IX. PERFORMANCE INDICES

To examine the performance of the proposed algorithm (LeNN-WOA-NM) in calculating the saturation of (water) injected fluid into oil during water flooding process, the performance indicators like Mean Absolute Deviation (MAD), Theil's inequality coefficient (TIC) and Error in Nash Sutcliffe Efficiency (ENSE) are implemented [51]. The formulation of these indices are given as,

$$\text{MAD} = \frac{1}{n}\sum_{m=1}^{n}\left|y(\eta) - y_{approx}(\eta)\right|, \tag{56}$$

$$\text{TIC} = \frac{\sqrt{\frac{1}{n}\sum_{n=1}^{n}\left(y(\eta) - y_{approx}(\eta)\right)^2}}{\left(\sqrt{\frac{1}{n}\sum_{n=1}^{n}(y(\eta))^2} + \sqrt{\frac{1}{n}\sum_{n=1}^{n}(y_{approx}(\eta))^2}\right)}, \tag{57}$$

$$\text{NSE} = \left\{1 - \frac{\sum_{n=1}^{n}((y(\eta) - y_{approx}(t))^2}{\sum_{n=1}^{n}((y(\eta) - y\prime(\eta))^2}, y'(\eta) = \frac{1}{n}\sum_{m=1}^{n}y(\eta)\right. \tag{58}$$

$$ENSE = 1 - NSE, \tag{59}$$

where n denotes the number grid points and $\eta = \frac{X}{2\sqrt{T}}$.

X. NUMERICAL RESULTS

In this paper, we have analyzed the problem of multi-phase flow through porous media with imbibition phenomena. The problem under consideration is divided based on the homogeneous and heterogeneous characteristics of porous media. Imbibition is one of the interesting phenomena which occurs during oil extraction through a cylindrical well and is an important problem arising in petroleum engineering.

We have designed an efficient soft-computing technique, named as LeNN-WOA-NM algorithm. Results obtained by our technique are presented for homogeneous and heterogeneous porous media. The mathematical model for heterogeneous porous media is presented in Eq (29-31), and for homogeneous porous media, the ordinary differential equation together with its conditions is given in Eq (40-42). It is obvious from both models (for homogeneous and heterogenous media) that saturation depends on time T and distance X. We have analysed the problems for T =0.1, 0.2, 0.3, 0.4, and 0.5. The distance variable X is varied from 0 to 1 with a step size of h =0.1.

A. HETEROGENOUS POROUS MEDIUM

The LeNN-WOA-NM algorithm is applied to solve ODE in Eq 29. This problem describes the saturation of injected fluid through heterogeneous media when water is injected during water flooding or secondary oil recovery process in counter-current imbibition phenomena. As shown in Eq 29, the saturation depends on time T and distance X. Therefore, to understand the phenomena deeply, we have discussed five cases by varying T and keeping other values constant, such that $L = 3, \omega = 0.1, S_{i0} = 0.1$.

The approximate solution of the ordinary differential equation (29) describes the saturation of injected fluid through heterogeneous media when water is injected during water flooding or secondary oil recovery process in counter-current imbibition phenomenon. Since the saturation depends on time T and distance X. Therefore, to understand the phenomenon deeply, we have discussed several cases for T and fixed the other parameters as $L = 3$, $\omega = 0.1$, and $S_{i0} = 0.1$.

Numerical results obtained by LeNN-WOA-NM algorithm for each case of T=0.1, 0.2, 0.3, 0.4 and 0.5 are given in TABLES 3 and 4. Absolute errors in solutions obtained by proposed algorithm are dictated in TABLES 5 and 6. Unknown parameters achieved by the LeNN-WOA-NM algorithm for different values of T are given in TABLE 7 and TABLE 8. Fitness values of candidate solutions and results of performance indicators including MAD, TIC and ENSE are presented in TABLES 9 ,10 ,11 and 12 respectively.

Graphical representation of solutions, absolute errors, unknown parameters, convergence analysis in terms of MAD, TIC and ENSE for each case are shown in FIGURES 6-12 respectively. Normal probability plots of for fitness evaluation and performance indicators including MAD, TIC and ENSE for saturation of injected fluid (water) into native fluid (oil) in secondary oil recovery process through heterogenous porous medium are shown in FIGURES 13-16. It is obvious from

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

our tables, graphs and normal probability curves that LeNN-WOA-NM algorithm achieved the best possible solutions for saturation of water in heterogenous porous medium .

B. HOMOGENOUS POROUS MEDIUM

The approximate solution of ordinary differential equation (40) describes the saturation of injected fluid through homogenous media, when water is injected in water flooding or secondary oil recovery process during counter-current imbibition phenomenon. Since, the saturation depends on time T and distance X. Again, we have considered several cases for T and fixed the other parameters as $L = 3, \omega = 0.1$ and $S_{i0} = 0.1$.

Numerical results obtained by the LeNN-WOA-NM algorithm for each case i.e. T=0.1, 0.2, 0.3, and 0.4 are given in TABLES 13 and 14. Absolute errors in soultions calculated by our proposed algorithm are dictated in TABLE 15. Unknown parameters achieved by LeNN-WOA-NM algorithm for different values of T are given in TABLE 16 and TABLE 17. Convergence analysis of fitness evaluation and results of performance measures including MAD, TIC and ENSE are presented in TABLES 18 ,19 ,20 and 21 respectively.

Graphical representation of solutions, absolute errors, unknown parameters, convergence analysis along with MAD, TIC and ENSE for each case are shown in FIGURES 6-12 respectively. Normal probability plots of for fitness evaluation and performance indicators including MAD, TIC and ENSE for saturation of water (injected fluid) into oil (native fluid) in secondary oil recovery (SOR) or water flooding process through homogenous porous medium are shown in FIGURES 24-27. It is evident from our tables, graphs and normal probability curves that the LeNN-WOA-NM algorithm gives best possible solutions for saturation of water into oil in homogenous porous medium as compared to other state-of-the-art method.

TABLE 3: Comparison of results by LeNN-WOA-NM algorithm with power series solution [3], PSO and Cuckoo search algorithm for saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.1				$S_i(X,T)$ T=0.2				$S_i(X,T)$ T=0.3			
Distance (X)	Power	PSO	CS	LeNN-WOA-NM	Power	PSO	CS	LeNN-WOA-NM	Power	PSO	CS	LeNN-WOA-NM
0	0.1	0.105545	0.102365	0.1	0.1	0.100787	0.100656	0.1	0.1	0.100430	0.100693	0.1
0.1	0.1126	0.114056	0.111206	0.1086	0.1095	0.109444	0.109570	0.1086	0.1080	0.107990	0.108817	0.1076
0.2	0.1201	0.120111	0.117706	0.1150	0.1162	0.115809	0.115943	0.1150	0.1140	0.114294	0.114701	0.1139
0.3	0.1247	0.123767	0.122321	0.1196	0.1208	0.120506	0.120389	0.1194	0.1185	0.118857	0.118933	0.1184
0.4	0.1286	0.125749	0.125445	0.1229	0.1241	0.123869	0.123436	0.1225	0.1218	0.122083	0.121970	0.1214
0.5	0.1339	0.126928	0.127422	0.1251	0.1268	0.126104	0.125493	0.1246	0.1243	0.124255	0.124147	0.1233
0.6	0.1428	0.127936	0.128544	0.1267	0.1297	0.127414	0.126848	0.1261	0.1265	0.125620	0.125684	0.1246
0.7	0.1574	0.128956	0.129058	0.1277	0.1336	0.128074	0.127694	0.1271	0.1288	0.126435	0.126713	0.1254
0.8	0.1799	0.129748	0.129170	0.1283	0.1392	0.128422	0.128163	0.1278	0.1316	0.126948	0.127320	0.1259
0.9	0.2124	0.129941	0.129049	0.1287	0.1474	0.128763	0.128375	0.1282	0.1353	0.127321	0.127592	0.1263
1	0.2573	0.129594	0.128840	0.1290	0.1587	0.129148	0.128479	0.1283	0.1404	0.127576	0.127676	0.1266

TABLE 4: Comparison of results by LeNN-WOA-NM algorithm with power series solution [3], PSO and Cuckoo search algorithm for saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.4				$S_i(X,T)$ T=0.5			
Distance (X)	Power	PSO	CS	LeNN-WOA-NM	Power	PSO	CS	LeNN-WOA-NM
0	0.1	0.100015	0.099272	0.1	0.1	0.100293	0.100052	0.1
0.1	0.1071	0.108937	0.107207	0.1051	0.1004	0.107574	0.106649	0.1050
0.2	0.1126	0.114532	0.113115	0.1117	0.1115	0.113165	0.112338	0.1107
0.3	0.1168	0.118526	0.117306	0.1151	0.1156	0.117039	0.116408	0.1150
0.4	0.1201	0.121206	0.120131	0.1181	0.1188	0.119420	0.119218	0.1181
0.5	0.1227	0.122897	0.121942	0.1205	0.1214	0.120701	0.121127	0.1204
0.6	0.1247	0.123921	0.123056	0.1221	0.1235	0.122814	0.122433	0.1220
0.7	0.1267	0.124570	0.123730	0.1231	0.1253	0.123433	0.123348	0.1229
0.8	0.1286	0.125062	0.124151	0.1238	0.1269	0.124003	0.124000	0.1236
0.9	0.1310	0.125510	0.124424	0.1241	0.1287	0.124635	0.124470	0.1241
1	0.1339	0.125876	0.124583	0.1246	0.1308	0.125005	0.124874	0.1246

TABLE 5: Absolute Errors obtained by LeNN-WOA-NM, PSO and Cuckoo search algorithm for saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.1			$S_i(X,T)$ T=0.2			$S_i(X,T)$ T=0.3		
Distance(X)	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM
0	0.000319	6.78E-05	4.57E-06	0.000371	1.55E-05	1.03E-06	1.21E-06	6.67E-06	1.92E-09
0.1	2.06E-05	8.06E-05	1.13E-06	0.000166	4.35E-08	2.78E-07	4.89E-06	6.12E-07	2.95E-08
0.2	1.17E-05	3.64E-05	2.89E-06	0.000170	6.27E-08	3.73E-07	9.68E-06	3.71E-06	3.35E-10
0.3	5.87E-05	3.42E-06	1.25E-06	5.15E-06	7.97E-07	4.04E-08	6.85E-06	6.30E-06	2.01E-08
0.4	7.97E-05	2.41E-06	3.45E-08	4.66E-05	6.67E-06	2.60E-08	1.73E-06	5.18E-07	2.20E-08
0.5	5.30E-05	8.33E-06	3.47E-07	7.72E-05	1.82E-05	7.97E-08	2.21E-07	4.84E-07	7.67E-11
0.6	7.32E-06	3.12E-06	9.35E-07	1.54E-05	3.21E-06	4.32E-08	6.08E-06	9.52E-06	3.99E-08
0.7	1.57E-05	1.40E-06	8.06E-07	1.19E-05	4.12E-06	1.02E-08	1.74E-05	7.27E-05	4.55E-08
0.8	9.54E-05	1.86E-05	2.10E-07	5.58E-05	3.09E-05	1.20E-08	2.11E-05	9.06E-05	1.04E-08
0.9	6.60E-05	2.44E-05	3.86E-08	1.54E-05	7.06E-07	5.28E-08	3.68E-06	2.25E-06	2.61E-07
1	0.00017	2.61E-07	7.54E-07	3.49E-05	4.40E-07	2.76E-07	4.51E-06	1.62E-06	9.34E-08

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

TABLE 6: Absolute Errors obtained by LeNN-WOA-NM, PSO and Cuckoo search algorithm for saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.4			$S_i(X,T)$ T=0.5		
Distance(X)	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM
0	7.31E-08	7.02E-03	1.26E-08	4.33E-07	1.50E-07	1.67E-08
0.1	3.47E-08	4.49E-08	6.50E-08	3.61E-07	1.75E-07	2.17E-08
0.2	1.01E-06	4.40E-06	1.94E-07	2.84E-07	1.48E-07	2.68E-08
0.3	5.94E-08	3.62E-06	1.96E-08	3.35E-09	2.80E-09	4.22E-09
0.4	1.01E-06	4.05E-07	2.94E-08	9.60E-08	5.91E-08	4.38E-08
0.5	8.72E-07	2.08E-07	9.64E-08	1.42E-07	4.43E-07	1.49E-08
0.6	3.02E-07	1.01E-06	5.70E-09	3.28E-08	4.67E-08	5.91E-09
0.7	2.68E-06	8.61E-07	2.25E-09	1.12E-08	1.03E-08	2.57E-08
0.8	6.49E-08	1.66E-08	5.50E-09	7.49E-08	5.17E-08	1.13E-09
0.9	6.96E-06	3.98E-07	3.01E-09	2.90E-08	1.69E-08	1.46E-08
1	1.81E-06	7.78E-07	1.55E-09	4.12E-08	2.78E-09	1.65E-10

TABLE 7: Unknown parameters achieved by LeNN-WOA-NM algorithm for T=0.1, 0.2 and 0.3 during saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium.

	T=0.1			T=0.2			T=0.3		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	0.01763	-0.2107	-2.95688	-0.38579	-0.19418	0.148148	-0.35447	-0.0098	1.352705
2	0.412522	-0.2595	0.031839	-1.02698	0.203766	0.005159	-0.06028	-0.13138	0.944415
3	0.171065	0.337367	0.198844	-0.02777	0.115125	0.435598	-0.26131	-0.00142	1.169471
4	-1.99472	-1.31822	-0.14096	0.179334	0.112779	0.291384	0.001027	0.069496	-0.02386
5	0.003446	0.331003	-0.46838	-0.17869	-0.14256	0.251486	-0.00207	-0.27344	0.361211
6	0.327572	0.37234	-0.28472	-0.12394	-0.28668	0.048143	0.192959	-0.02544	-0.05052
7	0.312701	0.192942	-0.71424	-0.23434	-0.31915	-0.21966	0.76704	0.340761	0.26964
8	0.105513	0.050542	-0.46183	0.298691	-0.03056	0.280348	-0.09465	-0.0028	0.01425
9	0.013762	-0.08958	0.310671	0.443982	0.387168	0.06964	0.057989	-0.0313	0.499758
10	0.140086	0.141691	0.375625	-0.12181	0.03352	0.218292	0.066355	0.368513	0.473414
11	0.276773	0.024553	0.177099	-1.12391	0.143873	0.088844	0.015	0.092094	0.482289

TABLE 8: Unknown parameters achieved by LeNN-WOA-NM algorithm for T=0.4 and 0.5 for saturation of water (Injected fluid) into oil (Native fluid) in Imbibition phenomenon through **heterogenous** medium.

	T=0.4			T=0.5		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	-0.4555	0.453364	-0.24451	-0.87123	0.52798	0.125803
2	0.867359	0.042825	0.261891	0.469971	0.422911	0.137286
3	0.858224	0.442705	-0.1962	-0.34952	-0.48966	-0.11777
4	0.385863	-0.56738	0.67516	-0.1973	-0.44249	-0.18312
5	-0.10786	-0.38937	0.12517	-0.60584	-0.97595	0.256633
6	-0.5107	0.161634	-0.33877	-0.01159	0.112715	-0.1813
7	0.269498	0.00744	0.720273	0.11291	0.029375	0.187627
8	0.302274	0.072251	-0.57267	0.471004	-0.27781	-0.13642
9	0.531677	-0.0923	0.238763	0.413099	-0.18385	0.412885
10	0.167693	-0.0925	-0.20194	-0.63399	0.123904	-0.12001
11	-0.96992	0.04264	-1.20442	0.104785	0.220435	-0.25537

TABLE 9: Minimum value, Standard deviation, Median and Variance of fitness evaluation for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** porous medium.

Fitness Evaluation					
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	1.60E-08	7.67E-05	1.94E-05	6.67E-06	5.30E-10
T=0.2	6.78E-07	2.52E-04	4.91E-05	1.26E-05	2.41E-09
T=0.3	8.04E-07	5.81E-05	3.63E-05	1.20E-05	1.32E-09
T=0.4	7.89E-08	1.46E-04	4.29E-05	1.83E-05	1.84E-09
T=0.5	1.04E-08	2.15E-05	6.54E-06	2.18E-06	4.28E-11

TABLE 10: Minimum value, Standard deviation, Median and Variance of Mean Absolute Deviation for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** porous medium.

Mean Absolute Deviation					
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	6.64E-06	0.000348	9.34E-05	0.000114	9.75E-09
T=0.2	2.95E-16	0.000578	0.000153	0.000175	2.36E-08
T=0.3	2.74E-16	0.000571	0.000171	0.000142	2.93E-08
T=0.4	3.33E-16	0.000444	8.45E-05	0.000148	7.13E-09
T=0.5	2.36E-16	0.000639	0.000172	0.000181	2.96E-08

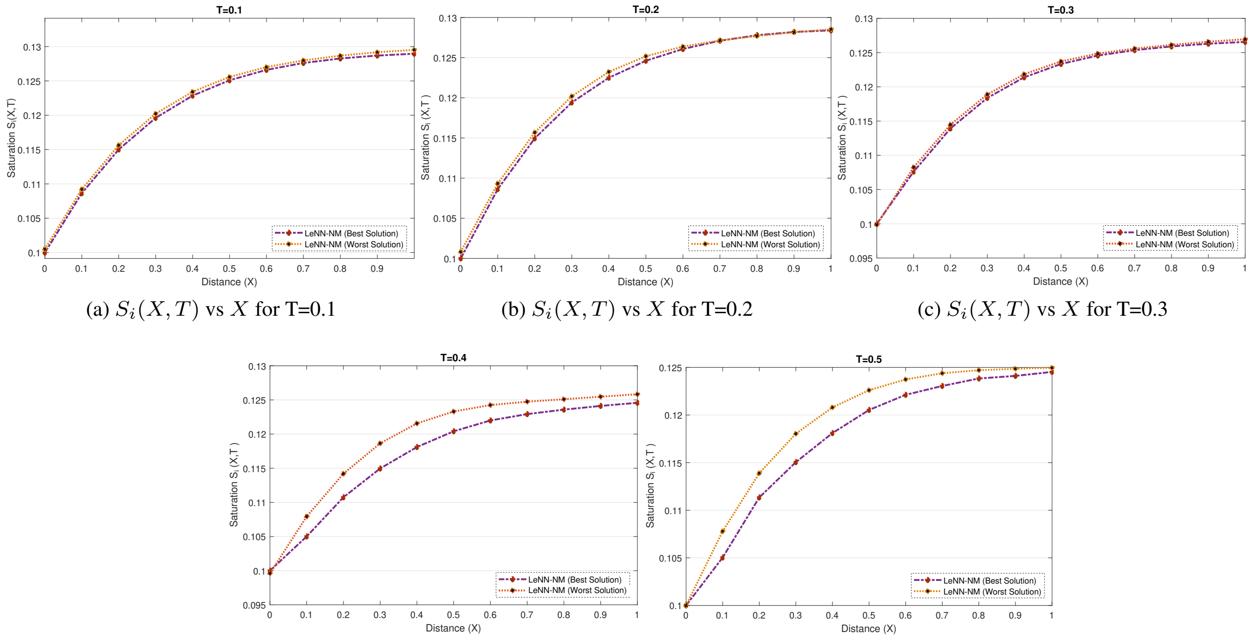
TABLE 11: Minimum value, Standard deviation, Median and Variance of Theil's inequality coefficient for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** porous medium.

Theil's inequality coefficient					
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	1.69E-05	0.000680	0.000195	0.000230	4.21E-08
T=0.2	6.14E-16	0.001276	0.000321	0.000371	1.03E-07
T=0.3	6.18E-16	0.001221	0.000356	0.000297	1.27E-07
T=0.4	7.26E-16	0.000893	0.000184	0.000347	3.39E-08
T=0.5	5.11E-16	0.001246	0.000365	0.000389	1.27E-07

TABLE 12: Minimum value, Standard deviation, Median and Variance of Error in Nash Sutcliffe Efficiency for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** porous medium.

Error in Nash Sutcliffe Efficiency					
Cases	Min.	Mean	Std	Med.	Var.
T=0.1	1.14E-06	0.001865	0.000531	0.000213	3.99E-07
T=0.2	0	0.005905	0.001340	0.000484	1.90E-06
T=0.3	0	0.046144	0.001273	0.002854	1.62E-04
T=0.4	0	0.002742	0.000607	0.000417	3.69E-07
T=0.5	0	0.005154	0.001485	0.000506	2.21E-06

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access



(a) $S_i(X,T)$ vs X for T=0.1

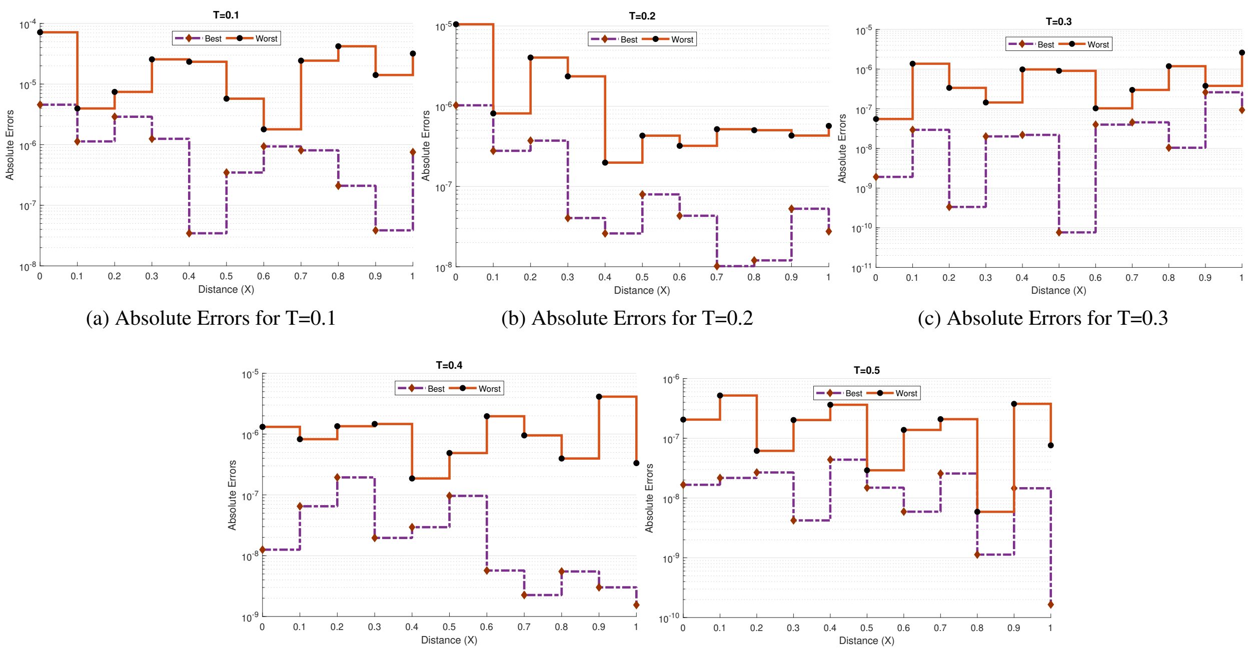
(b) $S_i(X,T)$ vs X for T=0.2

(c) $S_i(X,T)$ vs X for T=0.3

(d) $S_i(X,T)$ vs X for T=0.4

(e) $S_i(X,T)$ vs X for T=0.5

FIGURE 6: Comparison between the best solution and worst solution obtained by LeNN-WOA-NM algorithm for secondary oil recovery (SOR) process in imbibition phenomena through **heterogenous** medium for different cases of T.



(a) Absolute Errors for T=0.1

(b) Absolute Errors for T=0.2

(c) Absolute Errors for T=0.3

(d) Absolute Errors for T=0.4

(e) Absolute Errors for T=0.5

FIGURE 7: Comparison between the maximum and minimum absolute errors obtained by LeNN-WOA-NM algorithm for secondary oil recovery (SOR) process in imbibition phenomena through **heterogenous** medium for different cases of T.

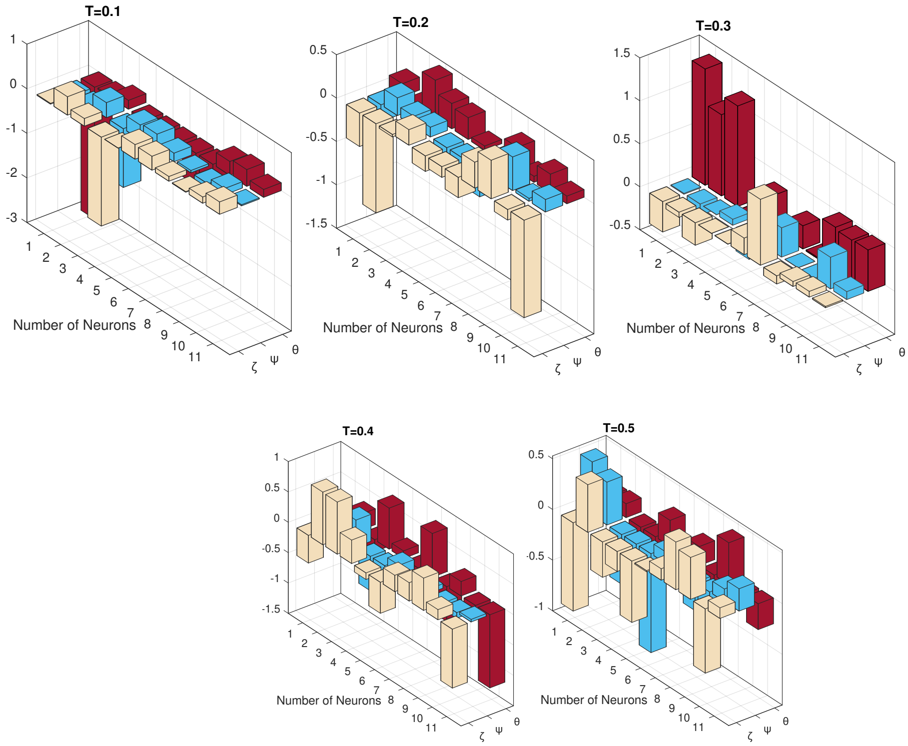


FIGURE 8: Unknown parameters achieved by LeNN-WOA-NM for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** medium for different cases of T.

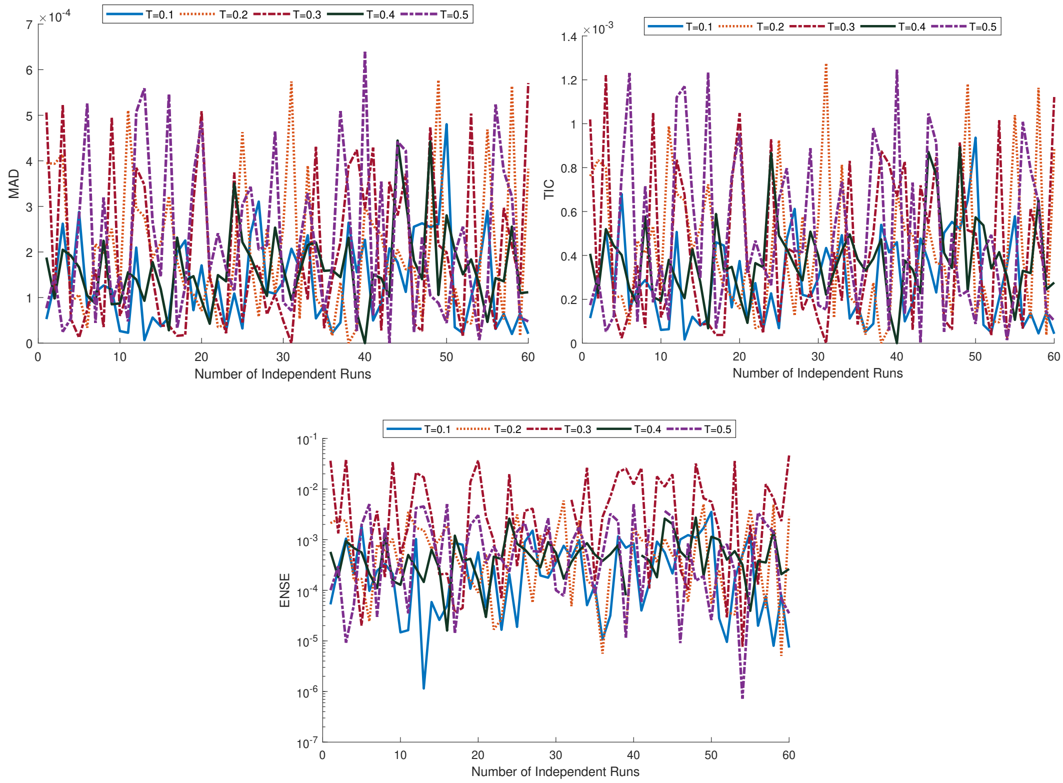


FIGURE 9: MAD, TIC and ENSE with number of independent runs for different cases of T when fluid (water) is injected during Secondary oil recovery process through **heterogenous** medium.

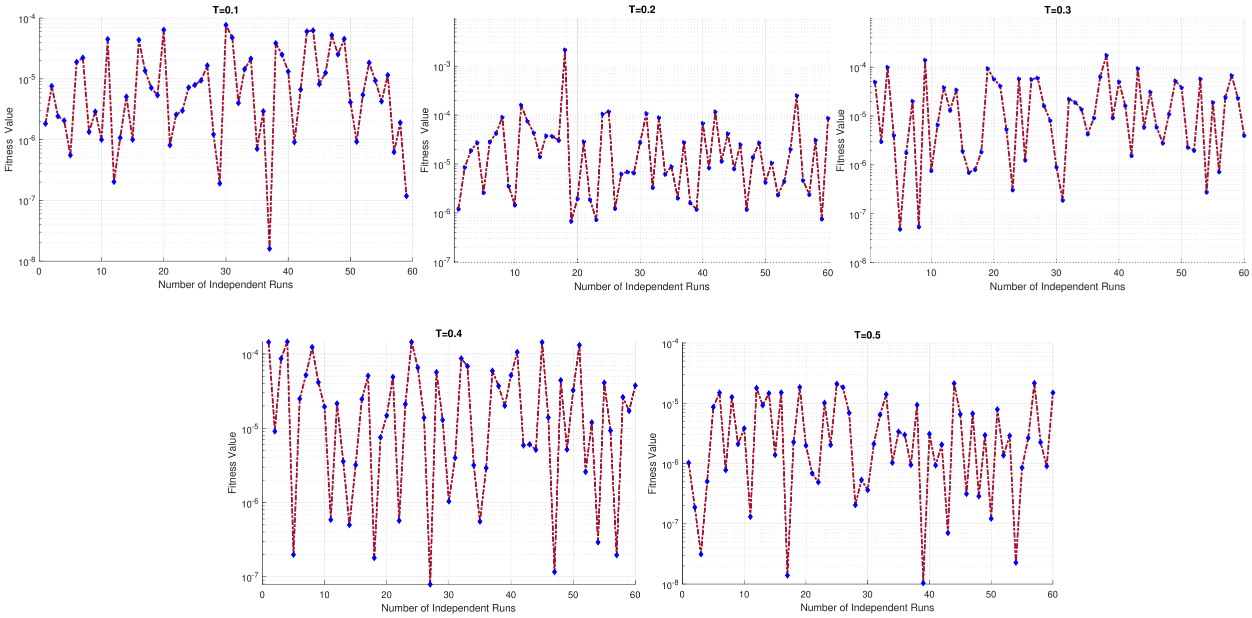


FIGURE 10: Convergence analysis for counter-current imbibition phenomena through **heterogenous** medium for different cases of T.

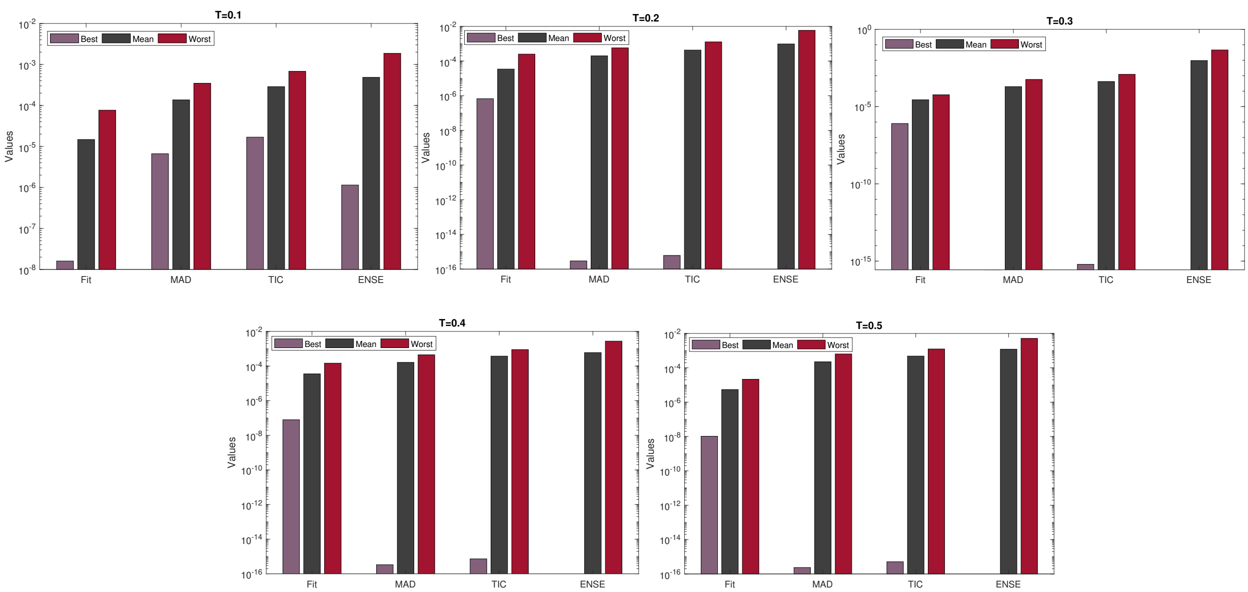


FIGURE 11: Performance measure for saturation of water(injected fluid) obtained by LeNN-WOA-NM algorithm during counter-current imbibition phenomena through **heterogenous** medium for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

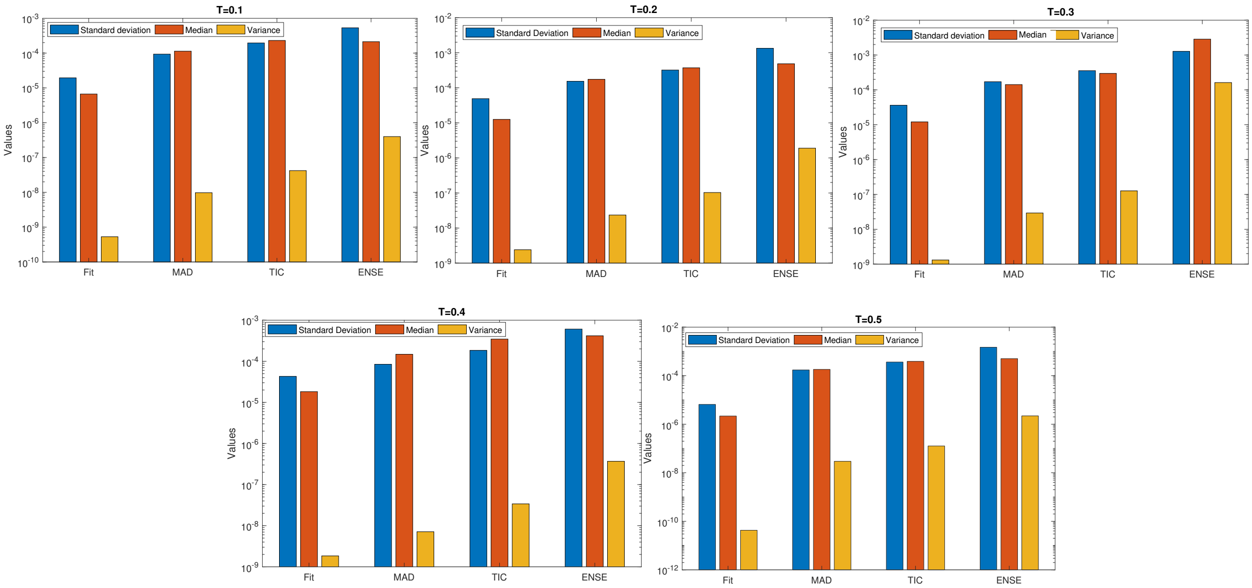


FIGURE 12: Performance measure for saturation of water(injected fluid) obtained by LeNN-WOA-NM algorithm during counter-current imbibition phenomena through **heterogenous** medium for different cases of T.

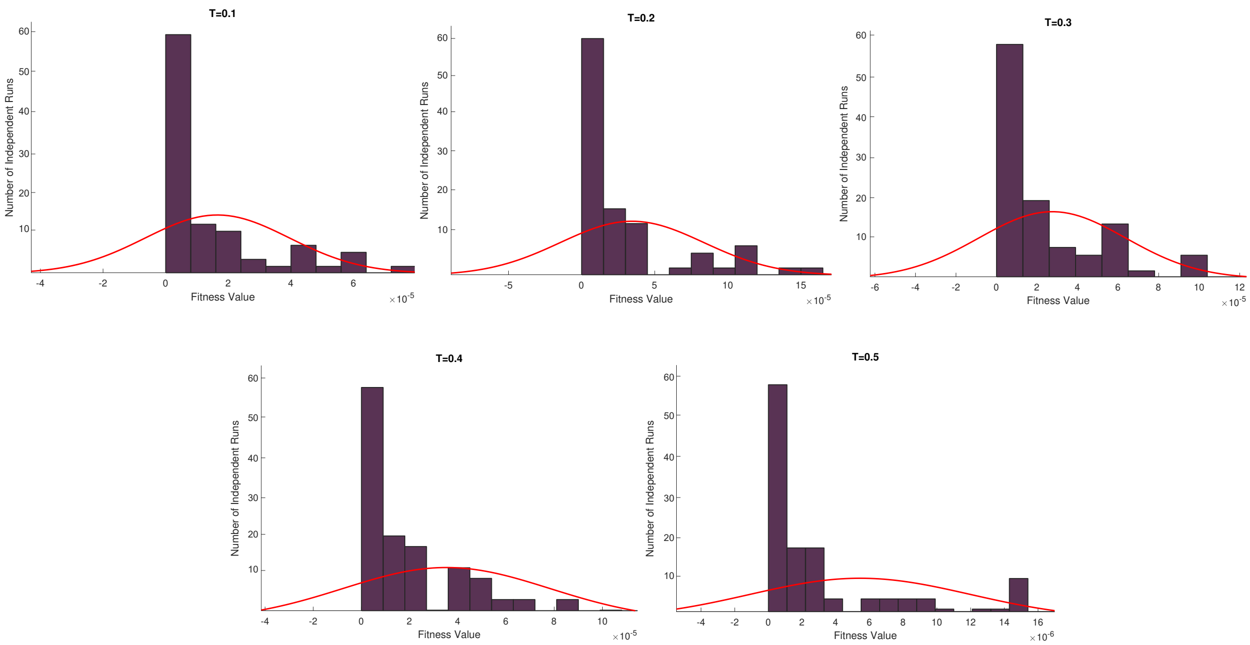


FIGURE 13: Normal probability plots of fitness value attained during 60 runs for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** medium for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

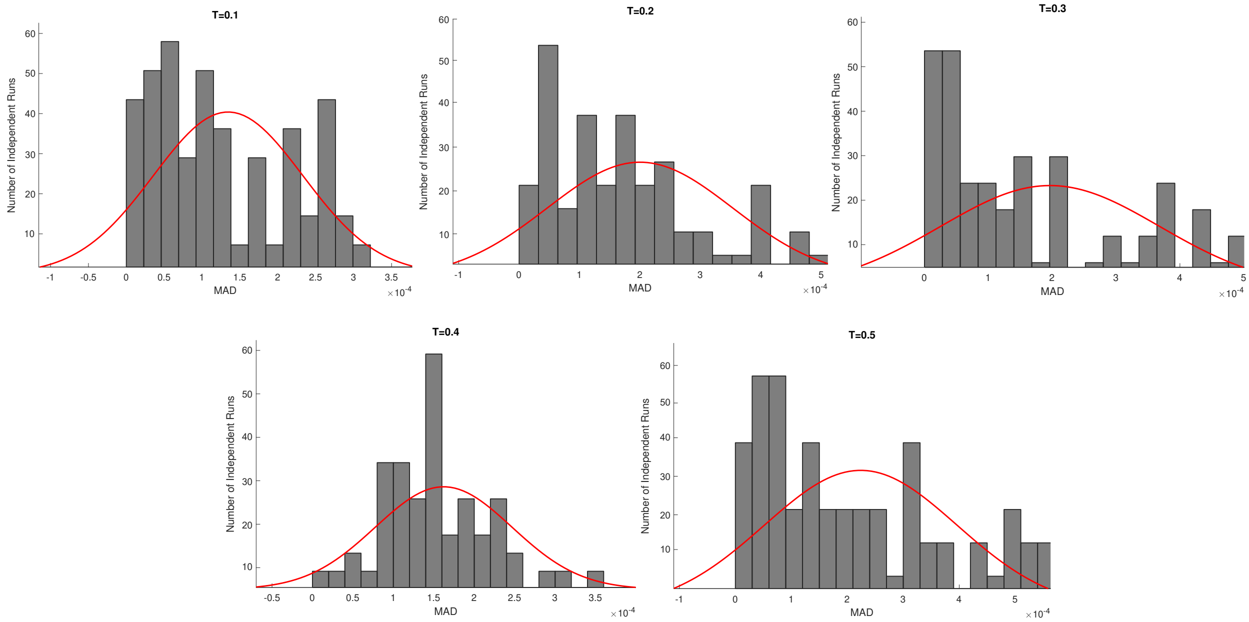


FIGURE 14: Normal probability plots of MAD attained during 60 runs for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** medium for different cases of T.

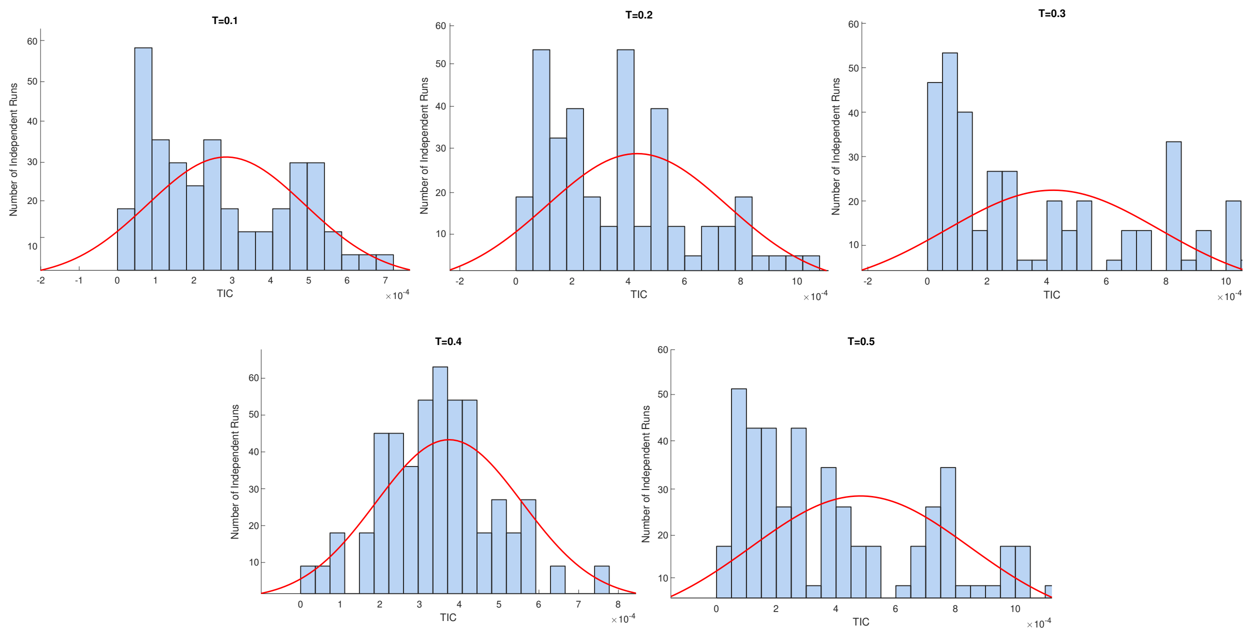


FIGURE 15: Normal probability plots of TIC attained during 60 runs for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** medium for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

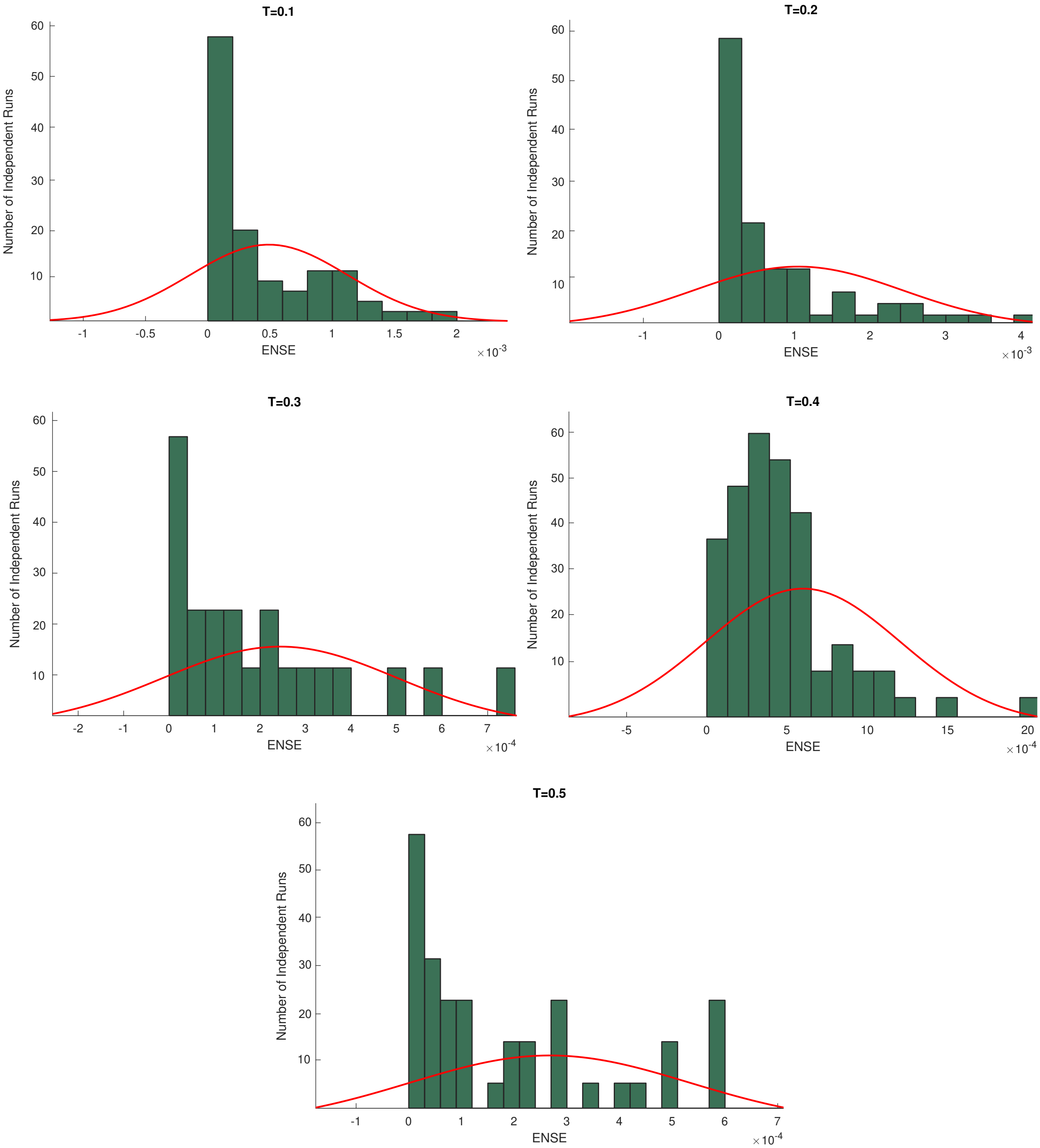


FIGURE 16: Normal probability plots of ENSE attained during 60 runs for saturation of water (injected fluid) into oil during water flooding process through **heterogenous** medium for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

TABLE 13: Comparison of results by LeNN-WOA-NM algorithm with power series solution [3], PSO and Cuckoo Search algorithm for Saturation of water (injected fluid) into oil in Imbibition phenomena through **homogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.1				$S_i(X,T)$ T=0.2			
Distance(X)	Power	PSO	CS	LeNN-WOA-NM	Power	PSO	CS	LeNN-WOA-NM
0	0.1	0.111941	0.100673	0.1	0.1	0.103971	0.100002	0.1
0.1	0.1156	0.121864	0.110589	0.10994	0.1111	0.114076	0.111923	0.10977
0.2	0.1309	0.129772	0.120177	0.11950	0.122	0.123263	0.121768	0.11931
0.3	0.1459	0.137173	0.129260	0.12854	0.1326	0.131917	0.130901	0.12827
0.4	0.1601	0.145060	0.137695	0.13693	0.143	0.140207	0.139103	0.13651
0.5	0.1748	0.152650	0.145374	0.14458	0.1531	0.147555	0.146259	0.14393
0.6	0.1899	0.158670	0.152221	0.15144	0.1629	0.153375	0.152349	0.15048
0.7	0.2031	0.162870	0.158195	0.15746	0.1724	0.157908	0.157429	0.15613
0.8	0.2174	0.166181	0.163292	0.16261	0.1816	0.161983	0.161605	0.16086
0.9	0.2322	0.169305	0.167545	0.16689	0.1905	0.165626	0.164993	0.16470
1	0.2476	0.171564	0.171025	0.17038	0.1992	0.167665	0.167685	0.16770

TABLE 14: Comparison of results by LeNN-WOA-NM algorithm with power series solution [3], PSO and Cuckoo Search algorithm for Saturation of water (injected fluid) into oil in Imbibition phenomena through **homogenous** medium for different cases of T.

	$S_i(X,T)$ T=0.3				$S_i(X,T)$ T=0.4			
Distance(X)	Power	PSO	CS	LeNN-WOA-NM	Power	PSO	CS	LeNN-WOA-NM
0	0.1	0.101324	0.098643	0.1	0.1	0.100123	0.094606	0.1
0.1	0.1091	0.111186	0.108744	0.10912	0.1079	0.109991	0.104676	0.10619
0.2	0.1180	0.120659	0.118485	0.11803	0.1156	0.119292	0.114858	0.11475
0.3	0.1267	0.129405	0.127613	0.12671	0.1232	0.127710	0.122614	0.12256
0.4	0.1352	0.137274	0.135947	0.13462	0.1306	0.135078	0.130129	0.12949
0.5	0.1435	0.144241	0.143380	0.14187	0.1377	0.141321	0.136567	0.13550
0.6	0.1515	0.150351	0.149864	0.14843	0.1446	0.146420	0.141917	0.14054
0.7	0.1591	0.155667	0.155399	0.15423	0.1513	0.150401	0.146206	0.14461
0.8	0.1665	0.160229	0.160019	0.15922	0.1576	0.153337	0.149494	0.14775
0.9	0.1735	0.164041	0.163785	0.16335	0.1635	0.155361	0.151879	0.15002
1	0.1801	0.167079	0.166764	0.16656	0.1691	0.156691	0.153507	0.15151

TABLE 15: Absolute Errors obtained by LeNN-WOA-NM algorithm for Saturation of water (injected fluid) into oil during Imbibition phenomena through **homogenous** medium for different cases of T.

	S_i(X,T) T=0.1			S_i(X,T) T=0.2			S_i(X,T) T=0.3			S_i(X,T) T=0.4		
Distance(X)	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM	PSO	CS	LeNN-WOA-NM
0	2.22E-05	2.56E-06	3.36E-07	7.24E-07	3.06E-07	8.18E-08	1.22E-08	1.46E-06	1.89E-09	4.72E-09	5.00E-09	3.03E-11
0.1	9.79E-06	1.29E-06	2.34E-07	2.61E-07	4.70E-07	2.68E-07	1.71E-08	1.97E-08	1.39E-09	1.04E-09	3.72E-10	1.92E-12
0.2	1.70E-05	1.38E-06	3.06E-07	1.72E-09	5.45E-07	1.50E-07	9.14E-09	4.46E-09	3.37E-09	5.78E-09	4.87E-09	3.32E-09
0.3	3.95E-06	5.60E-07	3.03E-08	3.35E-07	1.49E-07	7.55E-09	1.13E-07	2.44E-10	1.56E-11	1.82E-08	9.99E-09	6.22E-10
0.4	4.18E-07	1.19E-07	6.28E-08	1.16E-06	1.27E-06	2.29E-08	2.08E-07	1.56E-08	4.28E-09	8.24E-09	7.19E-09	9.61E-09
0.5	5.00E-06	6.00E-07	2.20E-07	1.22E-06	9.32E-07	1.02E-07	1.37E-07	3.33E-08	5.27E-09	1.14E-09	9.24E-10	4.49E-10
0.6	4.74E-06	2.56E-07	1.48E-07	1.65E-07	1.01E-07	6.77E-08	2.05E-09	1.38E-10	8.42E-11	2.32E-08	7.93E-08	9.59E-09
0.7	4.66E-07	9.26E-08	2.94E-08	8.64E-07	4.82E-07	6.05E-08	2.08E-07	1.79E-07	1.10E-08	2.27E-08	2.10E-09	2.77E-10
0.8	1.69E-06	1.47E-06	1.40E-07	4.82E-06	1.42E-06	4.31E-07	7.86E-07	3.52E-08	3.72E-09	8.91E-10	1.18E-09	3.31E-09
0.9	4.31E-06	3.82E-07	1.66E-07	2.95E-06	6.64E-08	2.80E-08	4.62E-07	6.71E-08	2.54E-08	5.92E-08	4.07E-08	1.11E-08
1	8.22E-07	2.86E-07	5.26E-08	1.25E-05	2.74E-08	1.43E-08	1.47E-06	8.61E-09	3.57E-09	1.32E-08	2.01E-09	4.04E-10

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

TABLE 16: Unknown parameters achieved by LeNN-WOA-NM algorithm for T=0.1 and 0.2 for Saturation of water (injected fluid) into oil during Imbibition phenomena through **homogenous** medium.

	T=0.1			T=0.2		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	0.2253	0.393245	0.271551	-0.63953	-0.55797	0.785623
2	0.206442	0.062829	0.335276	-0.36515	-0.04758	-0.58515
3	0.229948	0.243532	0.231227	-0.04889	0.292168	0.291422
4	0.158342	0.139987	0.403143	-0.35566	-0.2687	-0.77448
5	0.266312	0.030022	0.292455	-0.00225	-0.2046	-0.50067
6	0.132227	0.186296	0.131959	-0.19405	0.249157	0.281804
7	-2.75181	0.141804	0.546236	-0.54015	0.21766	0.2863
8	0.321214	0.162896	0.12982	0.177789	-0.09419	0.437178
9	0.027507	0.056814	0.232449	-0.11807	-0.165	-0.12936
10	-7.32831	0.210352	0.222878	-0.21697	0.335691	-0.13749
11	0.101217	0.173301	0.19444	-0.22868	0.301901	0.313925

TABLE 17: Unknown parameters achieved by LeNN-WOA-NM algorithm for T=0.3 and 0.4 for Saturation of water (injected fluid) into oil during Imbibition phenomena through **homogenous** medium.

	T=0.3			T=0.4		
index	ζ_n	ψ_n	θ_n	ζ_n	ψ_n	θ_n
1	0.869029	-0.00674	0.056186	-0.85771	-0.15139	-0.2531
2	-3.59196	-0.01049	-0.05866	-0.58133	-0.54686	0.001958
3	0.000335	-0.02673	-2.74564	-0.02437	-0.04096	-0.05071
4	0.06324	0.078882	-1.81773	-0.83669	0.023327	0.040573
5	-0.02443	0.03317	-0.50562	-0.0893	-0.14033	-0.13865
6	0.001536	0.003445	0.030818	-0.12979	-0.04089	-0.56367
7	-1.4718	-0.02774	0.007012	-0.04121	0.042235	-0.17462
8	0.419547	0.276887	0.068675	-0.01223	-0.03705	-0.25882
9	-0.02826	0.050668	0.00293	-0.03856	0.07455	-0.04183
10	0.013296	-0.00795	-0.03449	0.005055	-0.03193	-1.16396
11	-8.60221	0.31899	0.015816	9.18E-05	-0.04724	-0.07238

TABLE 18: Minimum value, Standard deviation, Median and Variance of fitness evaluation for saturation of water (injected fluid) into oil during water flooding process through **homogenous** porous medium.

		Fitness Evaluation			
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	2.01E-09	1.33E-06	2.62E-06	5.48E-07	6.85E-12
T=0.2	5.46E-09	2.33E-06	4.97E-06	7.29E-07	2.47E-11
T=0.3	2.20E-09	3.25E-06	6.35E-06	8.97E-07	4.04E-11
T=0.4	6.35E-09	5.42E-05	0.000218	2.45E-06	4.76E-08

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

TABLE 19: Minimum value, Standard deviation, Median and Variance of Mean Absolute Deviation for saturation of water (injected fluid) into oil during water flooding process through **homogenous** porous medium.

Mean Absolute Deviation					
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	2.49E-16	3.93E-05	0.000101	0.000101	1.54E-09
T=0.2	2.25E-16	6.12E-05	4.66E-05	4.91E-05	2.17E-09
T=0.3	3.04E-16	9.52E-05	0.000105	4.83E-05	1.10E-08
T=0.4	2.81E-07	1.75E-06	3.29E-06	8.48E-07	1.08E-11

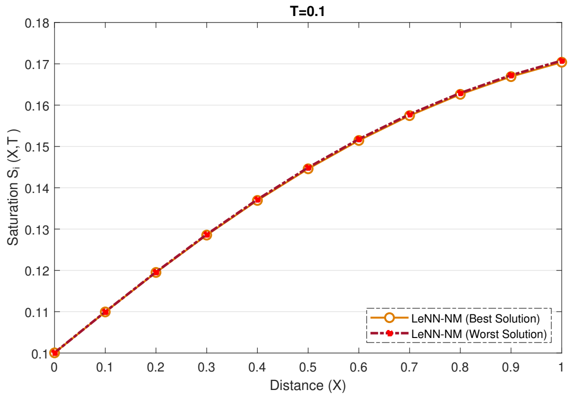
TABLE 20: Minimum value, Standard deviation, Median and Variance of Theil's inequality coefficient for saturation of water (injected fluid) into oil during water flooding process through **homogenous** porous medium.

Theil's inequality coefficient					
Cases	Min.	Mean	Std.	Med.	Var.
T=0.1	4.70E-16	0.000199	7.40E-05	0.000198	5.47E-09
T=0.2	4.41E-16	0.000113	8.61E-05	8.59E-05	7.41E-09
T=0.3	5.31E-16	0.000173	0.000183	9.47E-05	3.37E-08
T=0.4	0.000107	0.000361	0.000146	0.000309	2.14E-08

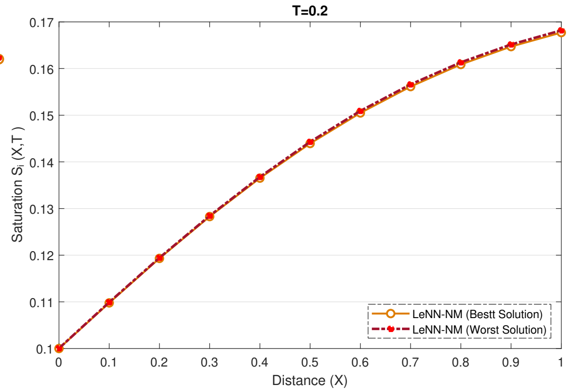
TABLE 21: Minimum value, Standard deviation, Median and Variance of Error in Nash Sutcliffe Efficiency for saturation of water (injected fluid) into oil during water flooding process through **homogenous** porous medium.

Error in Nash Sutcliffe Efficiency					
Cases	Min.	Mean	Std	Med.	Var.
T=0.1	0	0.000269	0.000197	0.000217	3.90E-08
T=0.2	0	0.000134	0.000183	5.52E-05	3.37E-08
T=0.3	0	0.000505	0.0011	5.91E-05	1.27E-06
T=0.4	0	7.38E-05	0.000173	1.39E-05	2.98E-08

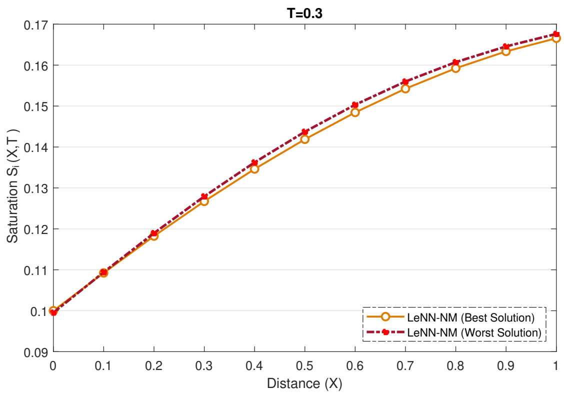
This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access



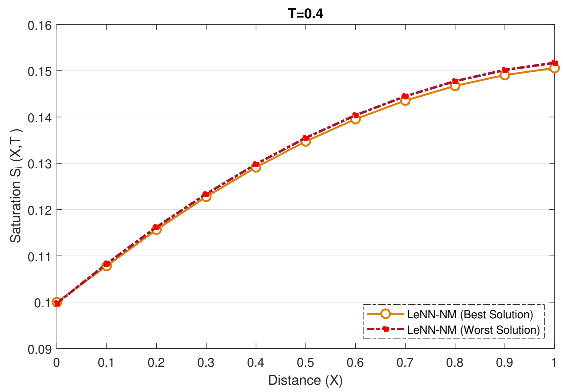
(a) Saturation $S_i(X,T)$ vs Distance (X) for T=0.1



(b) Saturation $S_i(X,T)$ vs Distance (X) for T=0.2

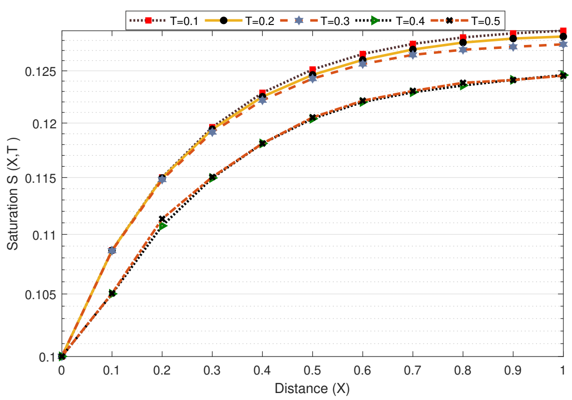


(c) Saturation $S_i(X,T)$ vs Distance (X) for T=0.3

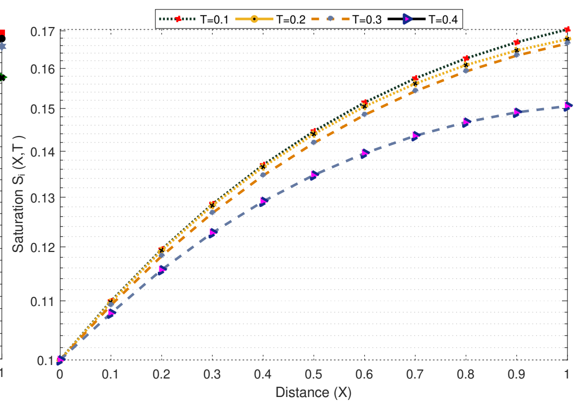


(d) Saturation $S_i(X,T)$ vs Distance (X) for T=0.4

FIGURE 17: Comparison between the best solution and worst solution obtained by LeNN-WOA-NM algorithm for Water flooding process in imbibition phenomena through **homogenous** medium for different cases of T.



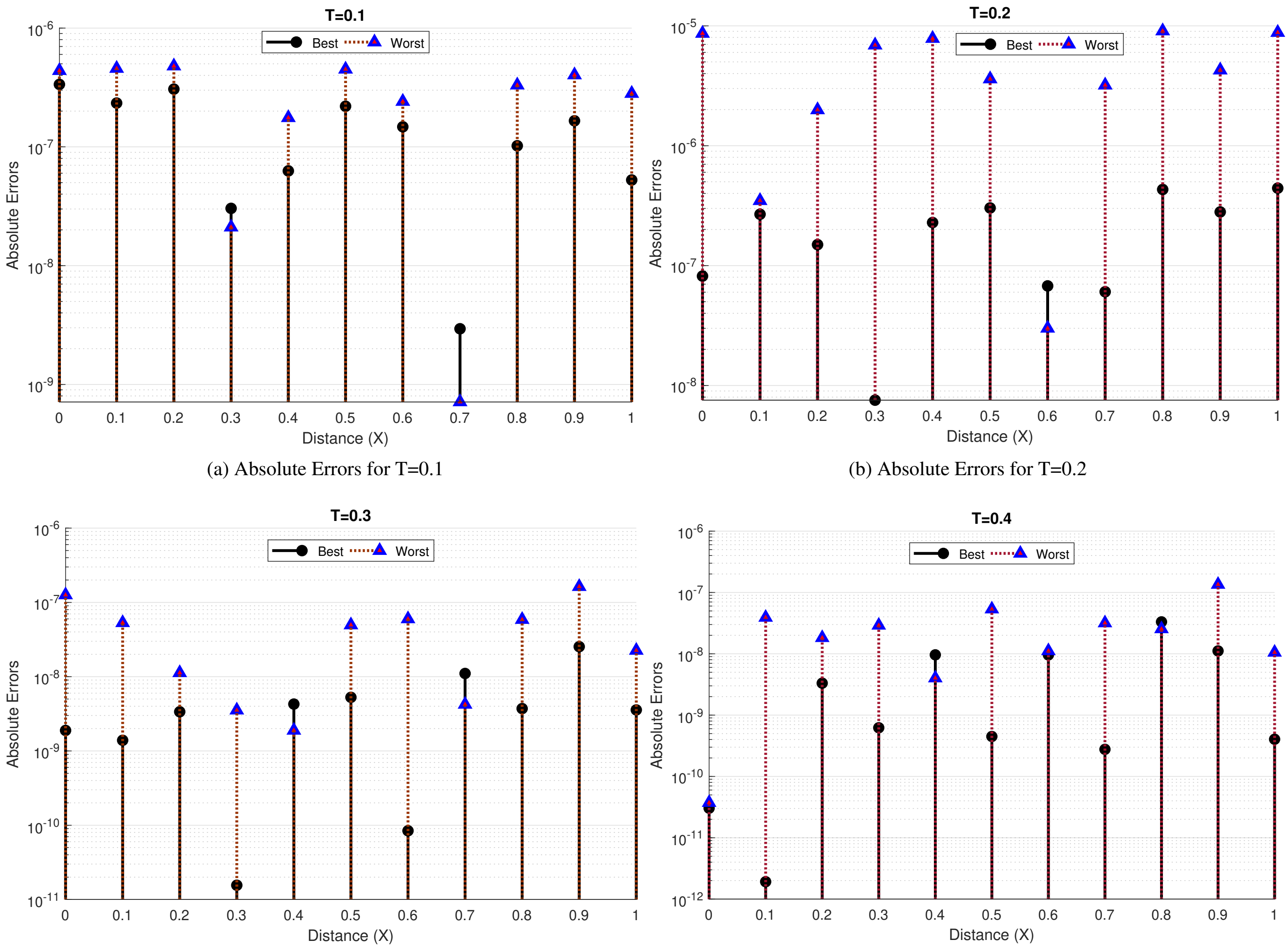
(a) Saturation of injected fluid through heterogenous porous medium



(b) Saturation of injected fluid through homogenous porous medium

FIGURE 18: Comparison between the saturation $S_i(X,T)$ of water (injected fluid) and distance (X) during Secondary Oil Recovery (SOR) process through **heterogenous** and **homogenous** porous mediums for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access



(a) Absolute Errors for T=0.1

(b) Absolute Errors for T=0.2

(c) Absolute Errors for T=0.3

(d) Absolute Errors for T=0.4

FIGURE 19: Comparison between the maximum and minimum absolute errors obtained by LeNN-WOA-NM algorithm for Secondary Oil Recovery (SOR) process in imbibition phenomena through **homogenous** medium for different cases of T.

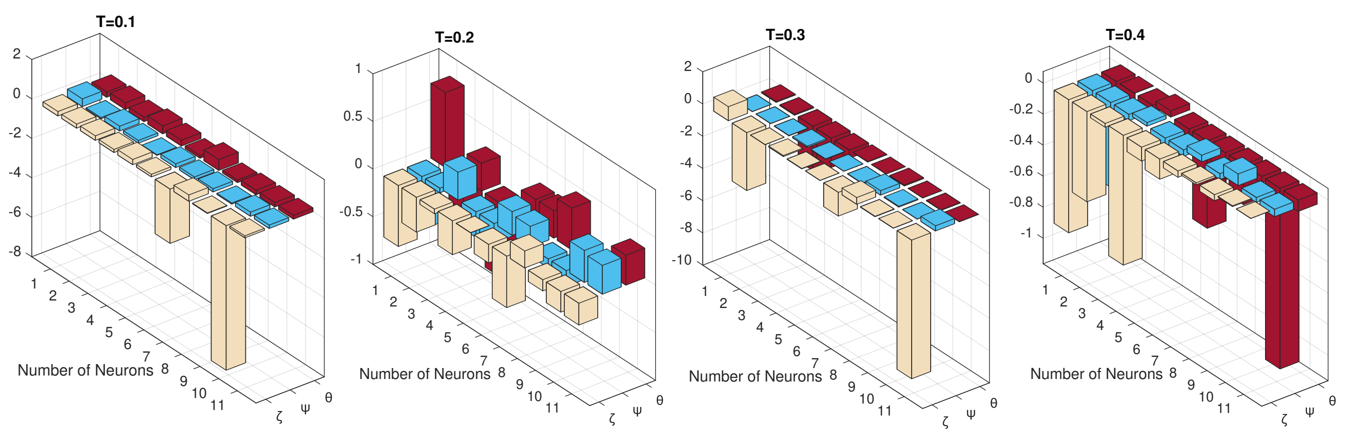


FIGURE 20: Unknown parameters achieved by LeNN-WOA-NM for saturation of water (injected fluid) into oil during water flooding process through **homogenous** medium for different cases of T.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

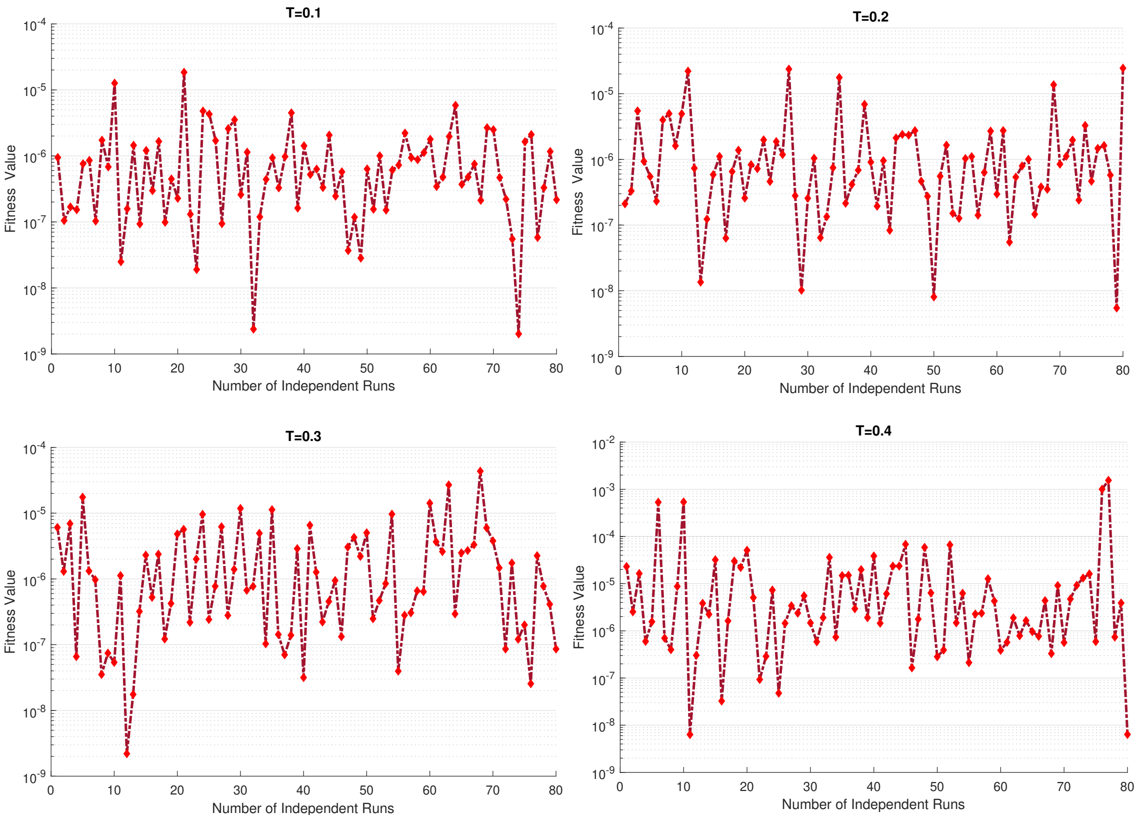


FIGURE 21: Convergence analysis for counter-current imbibition phenomena through **homogenous** medium for different cases of T.

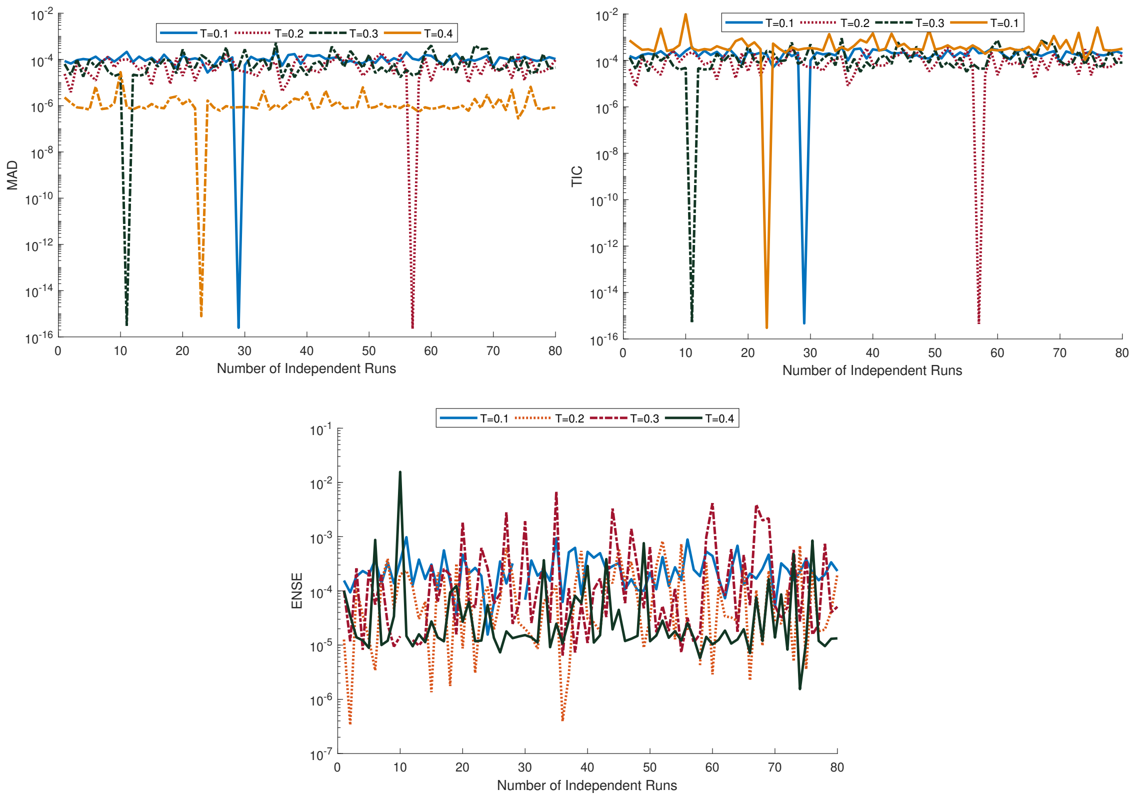


FIGURE 22: MAD, TIC and ENSE with number of independent runs for different cases of T when fluid (water) is injected during Secondary oil recovery process through **homogenous** medium.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

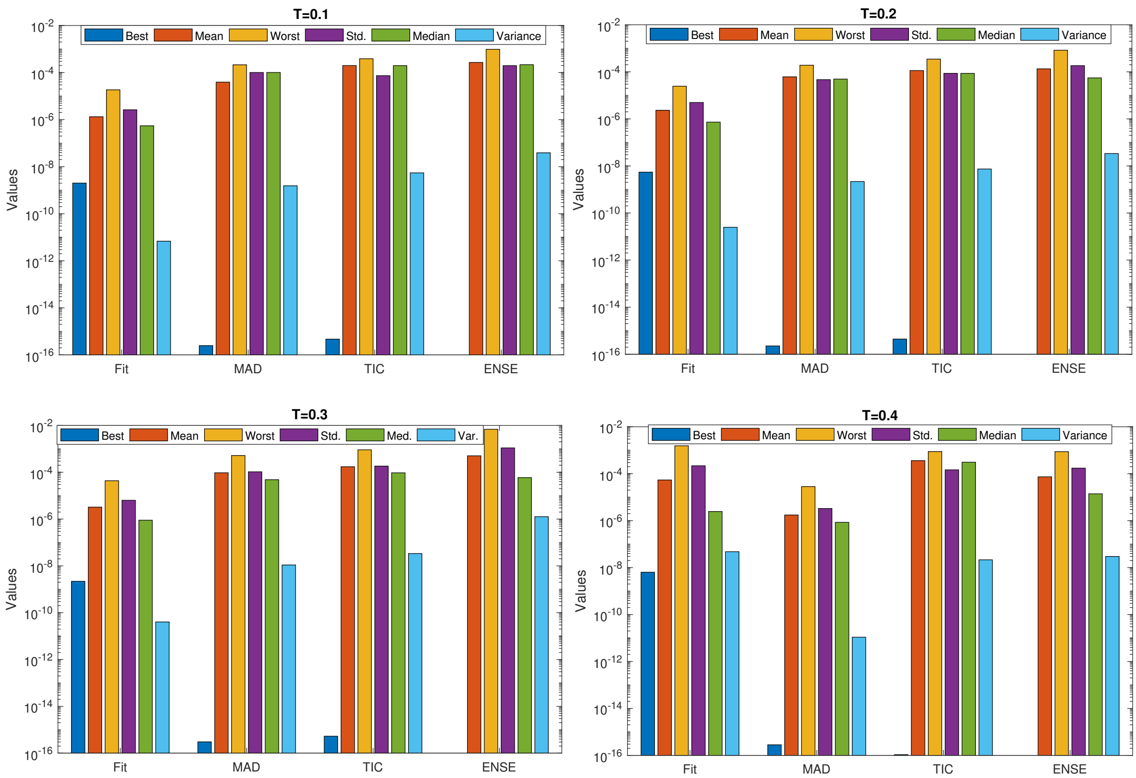


FIGURE 23: Performance measure for saturation of water(injected fluid) obtained by LeNN-WOA-NM algorithm during counter-current imbibition phenomena through **homogenous** medium for different cases of T.

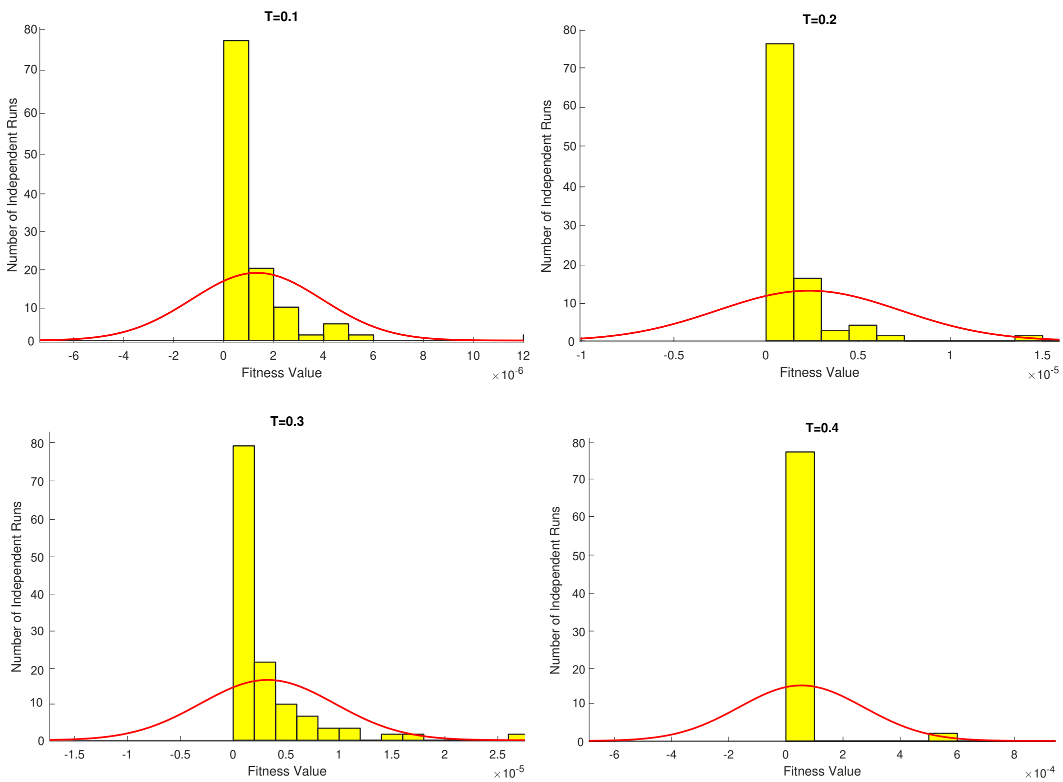


FIGURE 24: Normal probability plots of fitness evaluation attained during 80 runs for saturation of water (injected fluid) into oil during water flooding process through **homogenous** medium for different cases of T.

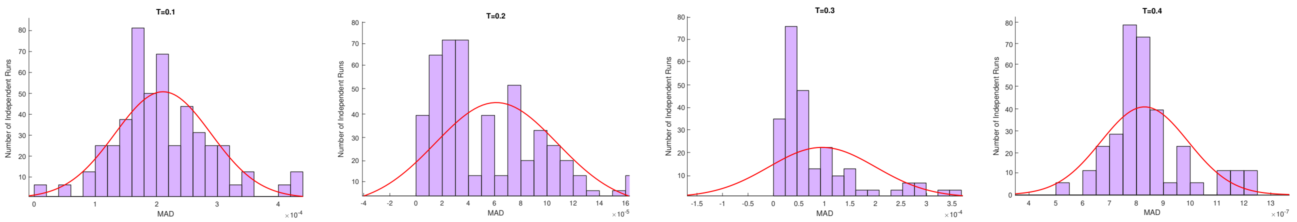


FIGURE 25: Normal probability plots of MAD attained during 80 runs for saturation of water (injected fluid) into oil during water flooding process through **homogenous** medium for different cases of T.

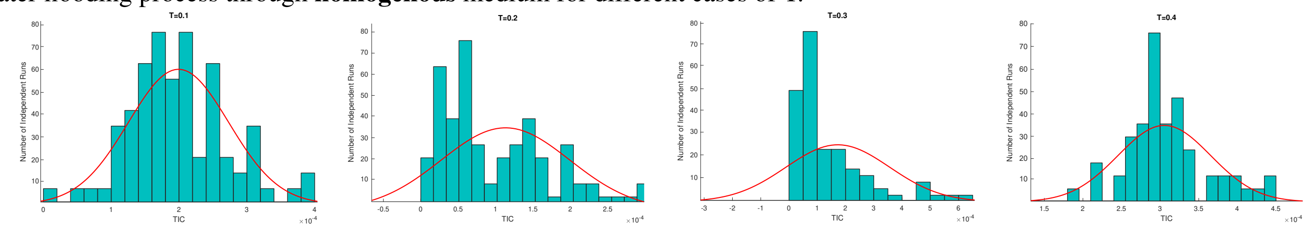


FIGURE 26: Normal probability plots of TIC attained during 80 runs for saturation of water (injected fluid) into oil during water flooding process through **homogenous** medium for different cases of T.

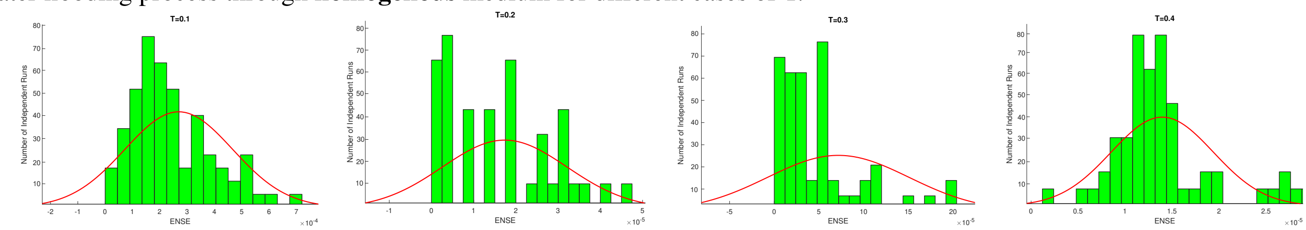


FIGURE 27: Normal probability plots of ENSE attained during 80 runs for saturation of water (injected fluid) into oil during water flooding process through **homogenous** medium for different cases of T.

Nomenclatures

Abbreviation	Descriptions
LeNN	Legendre Neural Networks
NM	Nelder-Mead
MAD	Mean Absolute Deviation
TIC	Theil's inequality coefficient
NSE	Nash Sutcliffe efficiency
L	Length of a cylinder
N	Native Fluid
I	Injected Fluid
K	permeability
k_i ,k_n	Relative permeability
p_n , p_i	Pressures of Native fluid
δ_n and δ_i	Kinematic Viscosities
p_c	Capillary pressure
β	Proportionality constant
a ,b ,a_1 ,K_c	Positive Constants
S_{i0}	Saturation at Common Interface
S_{ic}	Saturation at t=0
X	Distance
T	Time

XI. CONCLUSION

In this work, we have analyzed a mathematical model of counter-current imbibition phenomena in homogenous and heterogeneous porous mediums. The resulting equations are non-linear partial differential equations (PDE) with specific boundary conditions. Similarity transformation is used to model the governing equation of the water-flooding process as ODEs along with boundary conditions.

We have designed a new technique, named as LeNN-WOA-NM algorithm. Weighted Legendre polynomials are used to model approximate series solutions for the saturation $S_i(X,T)$ of (water) injected fluid into a native fluid (oil) during secondary oil recovery (SOR) process, and fitness functions are constructed to evaluate the candidate solutions. The problem has many applications in various fields, including ceramic engineering, water purification, petroleum technology, soil mechanics, powder metallurgy, and the oil recovery process. Series solutions for counter-current imbibition phenomena during the water flooding process represent the saturation of injected fluid (water) $S_i(X,T)$. We summarize our findings as follows:

- Saturation of injected fluid starts with initial value $S_i(0,T) = S_{i0} = 0.1$ and smoothly increases when the distance X increases, for heterogeneous medium see TABLE 3 and FIGURES 6, 27(a). While for homogeneous medium see TABLE 11 and FIGURES 17, 27(b).
- The results obtained by the LeNN-WOA-NM algorithm dictate that saturation of water into oil increases and is higher than the saturation of water in a homogenous porous medium.
- We have compared the saturations $S_i(X,T)$ of fluid in a heterogeneous porous medium with saturations in a homogenous porous medium as shown in FIGURE 18.
- Lower absolute errors in our solutions prove that the LeNN-WOA-NM algorithm is accurate. Moreover, the values of performance indicators MAD, TIC, ENSE are favorable.
- Convergence plots in FIGURES 10, 13, 20, and 23 shows the stability of the LeNN-WOA-NM algorithm.
- It is important to note that the pore size of the heterogeneous medium varies at different time T and distance X. Thus, the diameter of the interconnected capillaries also varies, which may effect the saturation process.

APPENDIX

Approximate solution for the saturation of injected fluid though heterogenous porous medium for T=0.1. where $\eta = \frac{X}{2\sqrt{T}}$

$$
\begin{aligned}
y_{approx} &= -2.95688 + (0.412522\eta - 0.2595)(0.031839) \\
&+ \left(\frac{3(0.171065\eta + 0.337367)^2 - 1}{2}\right)(0.198844) \\
&+ \left(\frac{5(-1.99472\eta - 1.31822)^3 - 3(-1.99472\eta - 1.31822)}{2}\right)(-0.14096) \\
&+ \left(\frac{35(0.003446\eta + 0.331003)^4 - 30(0.003446\eta + 0.331003)^2}{8} + \frac{3}{8}\right)(-0.46838) \\
&+ \left(\frac{63(0.327572\eta + 0.37234)^5 - 70(0.327572\eta + 0.37234)^3}{8}\right. \\
&\left. + \frac{15(0.327572\eta + 0.37234)}{8}\right)(-0.28472) \\
&+ \left(\frac{231(0.31270\eta + 0.19294)^6 - 315(0.31270\eta + 0.19294)^4}{16}\right. \\
&\left. + \frac{105(0.31270\eta + 0.19294)^2 - 5}{16}\right)(-0.7142) \\
&+ \left(\frac{429(0.10551\eta + 0.05054)^7 - 693(0.10551\eta + 0.05054)^5}{16}\right. \\
&\left. + \frac{315(0.10551\eta + 0.05054)^2 - 35(0.10551\eta + 0.05054)}{16}\right)(-0.4618) \\
&+ \left(\frac{6435(0.013762\eta - 0.08958)^8 - 12012(0.013762\eta - 0.08958)^6}{128}\right. \\
&\left. + \frac{6930(0.013762\eta - 0.08958)^4 - 1260(0.013762\eta - 0.08958)^2 + 35}{128}\right)(0.310671) \\
&+ \left(\frac{12155(0.140086\eta + 0.141691)^9 - 25740(0.140086\eta + 0.141691)^7}{128}\right. \\
&+ \frac{18018(0.140086\eta + 0.141691)^5 - 4620(0.140086\eta + 0.141691)^3}{128} \\
&\left. + \frac{315(0.140086\eta + 0.141691)}{128}\right)(0.375625) \\
&+ \left(\frac{46189(0.276773\eta + 0.024553)^{10} - 109395(0.276773\eta + 0.024553)^8}{256}\right. \\
&+ \frac{90090(0.276773\eta + 0.024553)^6 - 30030(0.276773\eta + 0.024553)^4}{256} \\
&\left. + \frac{3465(0.276773\eta + 0.024553)^2 - 63}{256}\right)(0.177099) - \lambda.
\end{aligned} \tag{60}
$$

Approximate solution for the saturation of injected fluid though heterogenous porous medium for T=0.2.

$$
\begin{aligned}
&y_{approx} = 0.148148 + (-1.02698\eta + 0.203766)(0.005159) \\
&+ \Bigg(\frac{3(-0.02777\eta + 0.115125)^2 - 1}{2}\Bigg)(0.435598) \\
&+ \Bigg(\frac{5(0.179334\eta + 0.112779)^3 - 3(0.179334\eta + 0.112779)}{2}\Bigg)(0.291384) \\
&+ \Bigg(\frac{35(-0.17869\eta - 0.14256)^4 - 30(-0.17869\eta - 0.14256^2}{8} + \frac{3}{8}\Bigg)(0.251486) \\
&+ \Bigg(\frac{63(-0.12394\eta - 0.28668)^5 - 70(-0.12394\eta - 0.28668)^3}{8} \\
&+ \frac{15(-0.12394\eta - 0.28668)}{8}\Bigg)(0.048143) \\
&+ \Bigg(\frac{231(-0.2343\eta - 0.3192)^6 - 315(-0.2343\eta - 0.3192)^4}{16} \\
&+ \frac{105(-0.2343\eta - 0.3192)^2 - 5}{16}\Bigg)(-0.2197) \\
&+ \Bigg(\frac{429(0.2987\eta - 0.03056)^7 - 693(0.2987\eta - 0.03056)^5}{16} \\
&+ \frac{315(0.2987\eta - 0.03056)^2 - 35(0.2987\eta - 0.03056)}{16}\Bigg)(0.28035) \\
&+ \Bigg(\frac{6435(0.443982\eta + 0.387168)^8 - 12012(0.443982\eta + 0.387168)^6}{128} \\
&+ \frac{6930(0.443982\eta + 0.387168)^4 - 1260(0.443982\eta + 0.387168)^2 + 35}{128}\Bigg)(0.06964) \\
&+ \Bigg(\frac{12155(-0.12181\eta + 0.03352)^9 - 25740(-0.12181\eta + 0.03352)^7}{128} \\
&+ \frac{18018(-0.12181\eta + 0.03352)^5 - 4620(-0.12181\eta + 0.03352)^3}{128} \\
&+ \frac{315(-0.12181\eta + 0.03352)}{128}\Bigg)(0.218292) \\
&+ \Bigg(\frac{46189(-1.12391\eta + 0.143873)^{10} - 109395(-1.12391\eta + 0.143873)^8}{256} \\
&+ \frac{90090(-1.12391\eta + 0.143873)^6 - 30030(-1.12391\eta + 0.143873)^4}{256} \\
&+ \frac{3465(-1.12391\eta + 0.143873)^2 - 63}{256}\Bigg)(0.088844) - \lambda.
\end{aligned} \tag{61}
$$

Approximate solution for the saturation of injected fluid though heterogenous porous medium for T=0.3.

$$
\begin{aligned}
&y_{approx} = 1.352705 + (-0.06028\eta - 0.13138)(-2.21856) \\
&+ \Bigg(\frac{3(-0.00105\eta + 0.067729)^2 - 1}{2}\Bigg)(0.944415) \\
&+ \Bigg(\frac{5(-0.26131\eta - 0.00142)^3 - 3(-0.26131\eta - 0.00142)}{2}\Bigg)(1.169471) \\
&+ \Bigg(\frac{35(0.001027\eta + 0.069496)^4 - 30(0.001027\eta + 0.069496)^2}{8} + \frac{3}{8}\Bigg)(-0.02386) \\
&+ \Bigg(\frac{63(-0.00207\eta - 0.27344)^5 - 70(-0.00207\eta - 0.27344)^3}{8} \\
&+ \frac{15(-0.00207\eta - 0.27344)}{8}\Bigg)(0.361211) \\
&+ \Bigg(\frac{231(0.7670\eta + 0.34076)^6 - 315(0.7670\eta + 0.34076)^4}{16} \\
&+ \frac{105(0.7670\eta + 0.34076)^2 - 5}{16}\Bigg)(0.2696) \\
&+ \Bigg(\frac{429(-0.0947\eta - 0.0028)^7 - 693(-0.0947\eta - 0.0028)^5}{16} \\
&+ \frac{315(-0.0947\eta - 0.0028)^2 - 35(-0.0947\eta - 0.0028)}{16}\Bigg)(0.0143) \\
&+ \Bigg(\frac{6435(0.057989\eta - 0.0313)^8 - 12012(0.057989\eta - 0.0313)^6}{128} \\
&+ \frac{6930(0.057989\eta - 0.0313)^4 - 1260(0.057989\eta - 0.0313)^2 + 35}{128}\Bigg)(0.499758) \\
&+ \Bigg(\frac{12155(0.066355\eta + 0.368513)^9 - 25740(0.066355\eta + 0.368513)^7}{128} \\
&+ \frac{18018(0.066355\eta + 0.368513)^5 - 4620(0.066355\eta + 0.368513)^3}{128} \\
&+ \frac{315(0.066355\eta + 0.368513)}{128}\Bigg)(0.473414) \\
&+ \Bigg(\frac{46189(0.015\eta + 0.092094)^{10} - 109395(0.015\eta + 0.092094)^8}{256} \\
&+ \frac{90090(0.015\eta + 0.092094)^6 - 30030(0.015\eta + 0.092094)^4}{256} \\
&+ \frac{3465(0.015\eta + 0.092094)^2 - 63}{256}\Bigg)(0.482289) - \lambda.
\end{aligned} \tag{62}
$$

Approximate solution for the saturation of injected fluid though heterogenous porous medium for T=0.4.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

$$
\begin{aligned}
y_{approx} &= -0.24451 + (0.867359\eta + 0.042825)(0.261891)\\
&+ \left(\frac{3(0.858224\eta + 0.442705)^2 - 1}{2}\right)(-0.1962)\\
&+ \left(\frac{5(0.385863\eta - 0.56738)^3 - 3(0.385863\eta - 0.56738)}{2}\right)(0.67516)\\
&+ \left(\frac{35(-0.10786\eta - 0.38937)^4 - 30(-0.10786\eta - 0.38937)^2}{8} + \frac{3}{8}\right)(0.12517)\\
&+ \left(\frac{63(-0.5107\eta + 0.161634)^5 - 70(-0.5107\eta + 0.161634)^3}{8}\right.\\
&\left.+ \frac{15(-0.5107\eta + 0.161634)}{8}\right)(-0.33877)\\
&+ \left(\frac{231(0.2695\eta + 0.00744)^6 - 315(0.2695\eta + 0.00744)^4}{16}\right.\\
&\left.+ \frac{105(0.2695\eta + 0.00744)^2 - 5}{16}\right)(0.7203)\\
&+ \left(\frac{429(0.30227\eta + 0.072251)^7 - 693(0.30227\eta + 0.072251)^5}{16}\right.\\
&\left.+ \frac{315(0.30227\eta + 0.072251)^2 - 35(0.30227\eta + 0.072251)}{16}\right)(-0.5727)\\
&+ \left(\frac{6435(0.531677\eta - 0.0923)^8 - 12012(0.531677\eta - 0.0923)^6}{128}\right.\\
&\left.+ \frac{6930(0.531677\eta - 0.0923)^4 - 1260(0.531677\eta - 0.0923)^2 + 35}{128}\right)(0.238763)\\
&+ \left(\frac{12155(0.167693\eta - 0.0925)^9 - 25740(0.167693\eta - 0.0925)^7}{128}\right.\\
&+ \frac{18018(0.167693\eta - 0.0925)^5 - 4620(0.167693\eta - 0.0925)^3}{128}\\
&\left.+ \frac{315(0.167693\eta - 0.0925)}{128}\right)(-0.20194)\\
&+ \left(\frac{46189(-0.96992\eta + 0.04264)^{10} - 109395(-0.96992\eta + 0.04264)^8}{256}\right.\\
&+ \frac{90090(-0.96992\eta + 0.04264)^6 - 30030(-0.96992\eta + 0.04264)^4}{256}\\
&\left.+ \frac{3465(-0.96992\eta + 0.04264)^2 - 63}{256}\right)(-1.20442) - \lambda.
\end{aligned}
\tag{63}
$$

Approximate solution for the saturation of injected fluid though heterogenous porous medium for T=0.5.

$$
\begin{aligned}
y_{approx} &= 0.125803 + (0.469971\eta + 0.422911)(0.137286)\\
&+ \left(\frac{3(-0.34952\eta - 0.48966)^2 - 1}{2}\right)(-0.11777)\\
&+ \left(\frac{5(-0.1973\eta - 0.44249)^3 - 3(-0.1973\eta - 0.44249)}{2}\right)(-0.18312)\\
&+ \left(\frac{35(-0.60584\eta - 0.97595)^4 - 30(-0.60584\eta - 0.97595)^2}{8} + \frac{3}{8}\right)(0.256633)\\
&+ \left(\frac{63(-0.01159\eta + 0.112715)^5 - 70(-0.01159\eta + 0.112715)^3}{8}\right.\\
&\left.+ \frac{15(-0.01159\eta + 0.112715)}{8}\right)(-0.1813)\\
&+ \left(\frac{231(0.1129\eta + 0.02938)^6 - 315(0.1129\eta + 0.02938)^4}{16}\right.\\
&\left.+ \frac{105(0.1129\eta + 0.02938)^2 - 5}{16}\right)(0.18763)\\
&+ \left(\frac{429(0.4710\eta - 0.2778)^7 - 693(0.4710\eta - 0.2778)^5}{16}\right.\\
&\left.+ \frac{315(0.4710\eta - 0.2778)^2 - 35(0.4710\eta - 0.2778)}{16}\right)(-0.1364)\\
&+ \left(\frac{6435(0.413099\eta - 0.18385)^8 - 12012(0.413099\eta - 0.18385)^6}{128}\right.\\
&\left.+ \frac{6930(0.413099\eta - 0.18385)^4 - 1260(0.413099\eta - 0.18385)^2 + 35}{128}\right)(0.412885)\\
&+ \left(\frac{12155(-0.63399\eta + 0.123904)^9 - 25740(-0.63399\eta + 0.123904)^7}{128}\right.\\
&+ \frac{18018(-0.63399\eta + 0.123904)^5 - 4620(-0.63399\eta + 0.123904)^3}{128}\\
&\left.+ \frac{315(-0.63399\eta + 0.123904)}{128}\right)(-0.12001)\\
&+ \left(\frac{46189(0.104785\eta + 0.220435)^{10} - 109395(0.104785\eta + 0.220435)^8}{256}\right.\\
&+ \frac{90090(0.104785\eta + 0.220435)^6 - 30030(0.104785\eta + 0.220435)^4}{256}\\
&\left.+ \frac{3465(0.104785\eta + 0.220435)^2 - 63}{256}\right)(-0.25537) - \lambda.
\end{aligned}
\tag{64}
$$

Approximate solution for the saturation of injected fluid though homogenous porous medium for T=0.1.

$$
\begin{aligned}
y_{approx} &= 0.271551 + (0.206442\eta + 0.062829)(0.335276)\\
&+ \left(\frac{3(0.229948\eta + 0.243532)^2 - 1}{2}\right)(0.231227)\\
&+ \left(\frac{5(0.158342\eta + 0.139987)^3 - 3(0.158342\eta + 0.139987)}{2}\right)(0.403143)\\
&+ \left(\frac{35(0.266312\eta + 0.030022)^4 - 30(0.266312\eta + 0.030022)^2}{8} + \frac{3}{8}\right)(0.292455)\\
&+ \left(\frac{63(0.132227\eta + 0.186296)^5 - 70(0.132227\eta + 0.186296)^3}{8}\right.\\
&\left.+ \frac{15(0.132227\eta + 0.186296)}{8}\right)(0.131959)\\
&+ \left(\frac{231(-2.7518\eta + 0.14180)^6 - 315(-2.7518\eta + 0.14180)^4}{16}\right.\\
&\left.+ \frac{105(-2.7518\eta + 0.14180)^2 - 5}{16}\right)(0.54624)\\
&+ \left(\frac{429(0.32121\eta + 0.162896)^7 - 693(0.32121\eta + 0.162896)^5}{16}\right.\\
&\left.+ \frac{315(0.32121\eta + 0.162896)^2 - 35(0.32121\eta + 0.162896)}{16}\right)(0.12982)\\
&+ \left(\frac{6435(0.027507\eta + 0.056814)^8 - 12012(0.027507\eta + 0.056814)^6}{128}\right.\\
&\left.+ \frac{6930(0.027507\eta + 0.056814)^4 - 1260(0.027507\eta + 0.056814)^2 + 35}{128}\right)(0.232449)\\
&+ \left(\frac{12155(-7.32831\eta + 0.210352)^9 - 25740(-7.32831\eta + 0.210352)^7}{128}\right.\\
&+ \frac{18018(-7.32831\eta + 0.210352)^5 - 4620(-7.32831\eta + 0.210352)^3}{128}\\
&\left.+ \frac{315(-7.32831\eta + 0.210352)}{128}\right)(0.222878)\\
&+ \left(\frac{46189(0.101217\eta + 0.173301)^{10} - 109395(0.101217\eta + 0.173301)^8}{256}\right.\\
&+ \frac{90090(0.101217\eta + 0.173301)^6 - 30030(0.101217\eta + 0.173301)^4}{256}\\
&\left.+ \frac{3465(0.101217\eta + 0.173301)^2 - 63}{256}\right)(0.19444) - \lambda.
\end{aligned}
\tag{65}
$$

Approximate solution for the saturation of injected fluid though homogenous porous medium for T=0.2.

$$
\begin{aligned}
y_{approx} &= 0.785623 + (-0.36515\eta - 0.04758)(-0.58515)\\
&+ \left(\frac{3(-0.04889\eta + 0.292168)^2 - 1}{2}\right)(0.291422)\\
&+ \left(\frac{5(-0.35566\eta - 0.2687)^3 - 3(-0.35566\eta - 0.2687)}{2}\right)(-0.77448)\\
&+ \left(\frac{35(-0.00225\eta - 0.2046)^4 - 30(-0.00225\eta - 0.2046)^2}{8} + \frac{3}{8}\right)(-0.50067)\\
&+ \left(\frac{63(-0.19405\eta + 0.249157)^5 - 70(-0.19405\eta + 0.249157)^3}{8}\right.\\
&\left.+ \frac{15(-0.19405\eta + 0.249157)}{8}\right)(0.281804)\\
&+ \left(\frac{231(-0.5402\eta + 0.21766)^6 - 315(-0.5402\eta + 0.21766)^4}{16}\right.\\
&\left.+ \frac{105(-0.5402\eta + 0.21766)^2 - 5}{16}\right)(0.2863)\\
&+ \left(\frac{429(0.17779\eta - 0.09419)^7 - 693(0.17779\eta - 0.09419)^5}{16}\right.\\
&\left.+ \frac{315(0.17779\eta - 0.09419)^2 - 35(0.17779\eta - 0.09419)}{16}\right)(0.43718)\\
&+ \left(\frac{6435(-0.11807\eta - 0.1650)^8 - 12012(-0.11807\eta - 0.1650)^6}{128}\right.\\
&\left.+ \frac{6930(-0.11807\eta - 0.1650)^4 - 1260(-0.11807\eta - 0.1650)^2 + 35}{128}\right)(-0.12936)\\
&+ \left(\frac{12155(-0.21697\eta + 0.335691)^9 - 25740(-0.21697\eta + 0.335691)^7}{128}\right.\\
&+ \frac{18018(-0.21697\eta + 0.335691)^5 - 4620(-0.21697\eta + 0.335691)^3}{128}\\
&\left.+ \frac{315(-0.21697\eta + 0.335691)}{128}\right)(-0.13749)\\
&+ \left(\frac{46189(-0.22868\eta + 0.301901)^{10} - 109395(-0.22868\eta + 0.301901)^8}{256}\right.\\
&+ \frac{90090(-0.22868\eta + 0.301901)^6 - 30030(-0.22868\eta + 0.301901)^4}{256}\\
&\left.+ \frac{3465(-0.22868\eta + 0.301901)^2 - 63}{256}\right)(0.313925) - \lambda.
\end{aligned}
\tag{66}
$$

Approximate solution for the saturation of injected fluid though homogenous porous medium for T=0.3.

$$
\begin{aligned}
y_{approx} &= 0.056186 + (-3.59196\eta - 0.01049)(-0.05866) \\
&+ \left(\frac{3(0.000335\eta - 0.02673)^2 - 1}{2}\right)(-2.74564) \\
&+ \left(\frac{5(0.06324\eta + 0.078882)^3 - 3(0.06324\eta + 0.078882)}{2}\right)(-1.81773) \\
&+ \left(\frac{35(-0.02443\eta + 0.03317)^4 - 30(-0.02443\eta + 0.03317)^2}{8} + \frac{3}{8}\right)(-0.50562) \\
&+ \left(\frac{63(0.001536\eta + 0.003445)^5 - 70(0.001536\eta + 0.003445)^3}{8}\right. \\
&\left. + \frac{15(0.001536\eta + 0.003445)}{8}\right)(0.030818) \\
&+ \left(\frac{231(-1.4718\eta - 0.02774)^6 - 315(-1.4718\eta - 0.02774)^4}{16}\right. \\
&\left. + \frac{105(-1.4718\eta - 0.02774)^2 - 5}{16}\right)(0.007012) \\
&+ \left(\frac{429(0.41955\eta + 0.276887)^7 - 693(0.41955\eta + 0.276887)^5}{16}\right. \\
&\left. + \frac{315(0.41955\eta + 0.276887)^2 - 35(0.41955\eta + 0.276887)}{16}\right)(0.068675) \\
&+ \left(\frac{6435(-0.02826\eta + 0.050668)^8 - 12012(-0.02826\eta + 0.050668)^6}{128}\right. \\
&\left. + \frac{6930(-0.02826\eta + 0.050668)^4 - 1260(-0.02826\eta + 0.050668)^2 + 35}{128}\right)(0.00293) \\
&+ \left(\frac{12155(0.013296\eta - 0.00795)^9 - 25740(0.013296\eta - 0.00795)^7}{128}\right. \\
&+ \frac{18018(0.013296\eta - 0.00795)^5 - 4620(0.013296\eta - 0.00795)^3}{128} \\
&\left. + \frac{315(0.013296\eta - 0.00795)}{128}\right)(-0.03449) \\
&+ \left(\frac{46189(-8.60221\eta + 0.31899)^{10} - 109395(-8.60221\eta + 0.31899)^8}{256}\right. \\
&+ \frac{90090(-8.60221\eta + 0.31899)^6 - 30030(-8.60221\eta + 0.31899)^4}{256} \\
&\left. + \frac{3465(-8.60221\eta + 0.31899)^2 - 63}{256}\right)(0.015816) - \lambda.
\end{aligned} \tag{67}
$$

Approximate solution for the saturation of injected fluid though homogenous porous medium for T=0.4.

$$
\begin{aligned}
y_{approx} &= -0.2531 + (-0.58133\eta - 0.54686)(0.001958) \\
&+ \left(\frac{3(-0.02437\eta - 0.04096)^2 - 1}{2}\right)(-0.05071) \\
&+ \left(\frac{5(-0.83669\eta + 0.023327)^3 - 3(-0.83669\eta + 0.023327)}{2}\right)(0.040573) \\
&+ \left(\frac{35(-0.0893\eta - 0.14033)^4 - 30(-0.0893\eta - 0.14033)^2}{8} + \frac{3}{8}\right)(-0.13865) \\
&+ \left(\frac{63(-0.12979\eta - 0.04089)^5 - 70(-0.12979\eta - 0.04089)^3}{8}\right. \\
&\left. + \frac{15(-0.12979\eta - 0.04089)}{8}\right)(-0.56367) \\
&+ \left(\frac{231(-0.0412\eta + 0.042235)^6 - 315(-0.0412\eta + 0.042235)^4}{16}\right. \\
&\left. + \frac{105(-0.0412\eta + 0.042235)^2 - 5}{16}\right)(-0.17462) \\
&+ \left(\frac{429(-0.01223\eta - 0.03705)^7 - 693(-0.01223\eta - 0.03705)^5}{16}\right. \\
&\left. + \frac{315(-0.01223\eta - 0.03705)^2 - 35(-0.01223\eta - 0.03705)}{16}\right)(-0.25882) \\
&+ \left(\frac{6435(-0.03856\eta + 0.07455)^8 - 12012(-0.03856\eta + 0.07455)^6}{128}\right. \\
&\left. + \frac{6930(-0.03856\eta + 0.07455)^4 - 1260(-0.03856\eta + 0.07455)^2 + 35}{128}\right)(-0.04183) \\
&+ \left(\frac{12155(0.005055\eta - 0.03193)^9 - 25740(0.005055\eta - 0.03193)^7}{128}\right. \\
&+ \frac{18018(0.005055\eta - 0.03193)^5 - 4620(0.005055\eta - 0.03193)^3}{128} \\
&\left. + \frac{315(0.005055\eta - 0.03193)}{128}\right)(-1.16396) \\
&+ \left(\frac{46189(9.18E-05\eta - 0.04724)^{10} - 109395(9.18E-05\eta - 0.04724)^8}{256}\right. \\
&+ \frac{90090(9.18E-05\eta - 0.04724)^6 - 30030(9.18E-05\eta - 0.04724)^4}{256} \\
&\left. + \frac{3465(9.18E-05\eta - 0.04724)^2 - 63}{256}\right)(-0.07238) - \lambda.
\end{aligned} \tag{68}
$$

REFERENCES

[1] Kajal K Patel, MN Mehta, and Twinkle R Singh. A homotopy series solution to a nonlinear partial differential equation arising from a mathematical model of the counter-current imbibition phenomenon in a heterogeneous porous medium. European Journal of Mechanics-B/Fluids, 60:119–126, 2016.

[2] Myron B Allen III. Numerical modelling of multiphase flow in porous media. Advances in Water Resources, 8(4):162–187, 1985.

[3] Kinjal R Patel, Manoj N Mehta, and Twinkle R Patel. A mathematical model of imbibition phenomenon in heterogeneous porous media during secondary oil recovery process. Applied Mathematical Modelling, 37(5):2933–2942, 2013.

[4] Lorenzo Adolph Richards. Capillary conduction of liquids through porous mediums. Physics, 1(5):318–333, 1931.

[5] Hossein Kazemi, JR Gilman, AM Elsharkawy, et al. Analytical and numerical solution of oil recovery from fractured reservoirs with empirical transfer functions (includes associated papers 25528 and 25818). SPE Reservoir Engineering, 7(02):219–227, 1992.

[6] JS Aronofsky, L Masse, S Gj Natanson, et al. A model for the mechanism of oil recovery from the porous matrix due to water invasion in fractured reservoirs. Transactions of the AIME, 213(01):17–19, 1958.

[7] Calvin C Mattax, JR Kyte, et al. Imbibition oil recovery from fractured, water-drive reservoir. Society of Petroleum Engineers Journal, 2(02):177–184, 1962.

[8] O Torsaeter and JK Silseth. The effects of sample shape and boundary conditions on capillary imbibition. In North Sea Chalk Symp., Stavanger, 1985.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

[9] S Yadav and MN Mehta. Mathematical model and similarity solution of countercurrent imbibition phenomenon in banded porous matrix. Int. J of Appl. Math and Mech, 5(5):76–86, 2009.
[10] Tufan Ghosh, Debasis Deb, and GP Sekhar. Non-classical flow modeling of spontaneous imbibition in spatially heterogeneous reservoirs. COMPUTATIONAL GEOSCIENCES, 2020.
[11] Juhi Kesarwani and Ramakanta Meher. Modelling of an imbibition phenomenon in a heterogeneous cracked porous medium on small inclination. Special Topics and Reviews in Porous Media: An International Journal, 2020.
[12] Tufan Ghosh, GP Raja Sekhar, and Debasis Deb. Modeling of co-current spontaneous imbibition oil recovery from partially covered homogeneous hydrocarbon reservoir. Transport in Porous Media, 130(3):947–968, 2019.
[13] Tufan Ghosh, GP Raja Sekhar, and Debasis Deb. Mathematical modeling of co-current spontaneous imbibition in heterogeneous porous medium. European Journal of Mechanics-B/Fluids, 76:81–97, 2019.
[14] Shreekant Pathak and Twinkle Singh. Approximate solution of imbibition phenomenon arising in heterogeneous porous media by optimal homotopy analysis method. IJCMS, 8(3):1950014–298, 2019.
[15] Shabina Ashraf and Jyoti Phirani. A generalized model for spontaneous imbibition in a horizontal, multi-layered porous medium. Chemical Engineering Science, 209:115175, 2019.
[16] Hardik S Patel and Ramakanta Meher. Simulation of counter-current imbibition phenomenon in a double phase flow through fracture porous medium with capillary pressure. Ain Shams Engineering Journal, 9(4):2163–2169, 2018.
[17] Qingbang Meng, Huiqing Liu, Jing Wang, and Zhanxi Pang. Effect of gravity on spontaneous imbibition from cores with two ends open in the frontal flow period. Journal of Petroleum Science and Engineering, 141:16–23, 2016.
[18] AP Verma. Imbibition in a cracked porous medium. Canadian journal of Physics, 47(22):2519–2524, 1969.
[19] BJ Bourbiaux and FJ KALAYOJIAN. Experimental study of cocurrent and countercurrent flows in natural porsus media. In Annual technical conference and exhibition, pages 209–222, 1988.
[20] Muhammad Sulaiman, Abdellah Salhi, Birsen Irem Selamoglu, and Omar Bahaaldin Kirikchi. A plant propagation algorithm for constrained engineering optimisation problems. Mathematical problems in engineering, 2014, 2014.
[21] Muhammad Sulaiman and Abdellah Salhi. A seed-based plant propagation algorithm: the feeding station model. The Scientific World Journal, 2015, 2015.
[22] Wali Khan Mashwani, Abdel Salhi, Muhammad Asif Jan, Rasheeda Adeeb Khanum, and M Sulaiman. Impact analysis of crossovers in a multi-objective evolutionary algorithm. Science International, 27(6):4943–4956, 2015.
[23] W Khan, Abdellah Salhi, M Asif, R Adeeb, and Muhammad Sulaiman. Enhanced version of multi-algorithm genetically adaptive for multiobjective optimization. International journal of advanced computer science and applications, 6(12):279–287, 2015.
[24] Muhammad Sulaiman, Abdellah Salhi, Eric S Fraga, Wali Khan Mashwani, and Muhammad M Rashidi. A novel plant propagation algorithm: modifications and implementation. Science International, 28(1):201–209, 2016.
[25] Muhammad Sulaiman, Abdellah Salhi, Asfandyar Khan, Shakoor Muhammad, and Wali Khan. On the theoretical analysis of the plant propagation algorithms. Mathematical Problems in Engineering, 2018, 2018.
[26] Muhammad Sulaiman, Muhammad Sulaman, Abdelwahed Hamdi, and Zubair Hussain Hussain. The plant propagation algorithm for the optimal operation of directional over-current relays in electrical engineering. Mehran University Research Journal of Engineering and Technology, 39(2):223–236, 2020.
[27] Wali Khan Mashwani, Abdellah Salhi, M Jan, R Khanum, and M Suliaman. Evolutionary algorithms based on decomposition and indicator functions: state-of-the-art survey. Advanced Computer Science and Applications (IJACSA), 7(2), 2016.
[28] Wali Khan Mashwani, Alam Zaib, Özgür Yeniay, Habib Shah, Naseer Mansoor Tairan, and Muhammad Sulaiman. Hybrid constrained evolutionary algorithm for numerical optimization problems. Hacettepe Journal of Mathematics and Statistics, 48(3):931–950, 2018.
[29] Hassan Javed, Muhammad Asif Jan, Nasser Tairan, Wali Khan Mashwani, Rashida Adeeb Khanum, Muhammad Sulaiman, Hidayat Ullah Khan, and Habib Shah. On the efficacy of ensemble of constraint handling techniques in self-adaptive differential evolution. Mathematics, 7(7):635, 2019.
[30] Aqib Ali, Salman Qadri, Wali Khan Mashwani, Wiyada Kumam, Poom Kumam, Samreen Naeem, Atila Goktas, Farrukh Jamal, Christophe Chesneau, Sania Anam, et al. Machine learning based automated segmentation and hybrid feature analysis for diabetic retinopathy classification using fundus image. Entropy, 22(5):567, 2020.
[31] Muhammad Sulaiman, Sohail Ahmad, Javed Iqbal, Asfandyar Khan, and Rahim Khan. Optimal operation of the hybrid electricity generation system using multiverse optimization algorithm. Computational Intelligence and Neuroscience, 2019, 2019.
[32] Muhammad Sulaiman, Masihullah Masihullah, Zubair Hussain, Sohail Ahmad, Wali Khan Mashwani, Muhammad Asif Jan, and Rashida Adeeb Khanum. Implementation of improved grasshopper optimization algorithm to solve economic load dispatch problems. Hacettepe Journal of Mathematics and Statistics, 48(5):1570–1589, 2019.
[33] Muhammad Sulaiman, Shakoor Muhammad, Asfandyar Khan, et al. Improved solutions for the optimal coordination of docrs using firefly algorithm. Complexity, 2018, 2018.
[34] Muhammad Sulaiman, Ashfaq Ahmad, Asfandyar Khan, and Shakoor Muhammad. Hybridized symbiotic organism search algorithm for the optimal operation of directional overcurrent relays. Complexity, 2018, 2018.
[35] Rashida Adeeb Khanum, Muhammad Asif Jan, Nasser Tairan, Wali Khan Mashwani, Muhammad Sulaiman, Hidayat Ullah Khan, and Habib Shah. Global evolution commended by localized search for unconstrained single objective optimization. Processes, 7(6):362, 2019.
[36] Muhammad Sulaiman, Ismat Samiullah, Ab Hamdi, and Zubair Hussain. An improved whale optimization algorithm for solving multi-objective design optimization problem of pfhe. Journal of Intelligent & Fuzzy Systems, 37(3):3815–3828, 2019.
[37] Ashfaq Ahmad, Muhammad Sulaiman, Ahmad Alhindi, and Abdulah Jeza Aljohani. Analysis of temperature profiles in longitudinal fin designs by a novel neuroevolutionary approach. IEEE Access, 8:113285–113308, 2020.
[38] Waseem Waseem, Muhammad Sulaiman, Saeed Islam, Poom Kumam, Rashid Nawaz, Muhammad Asif Zahoor Raja, Muhammad Farooq, and Muhammad Shoaib. A study of changes in temperature profile of porous fin model using cuckoo search algorithm. Alexandria Engineering Journal, 59(1):11–24, 2020.
[39] Ayaz Hussain Bukhari, Muhammad Asif Zahoor Raja, Muhammad Sulaiman, Saeed Islam, Muhammad Shoaib, and Poom Kumam. Fractional neuro-sequential arfima-lstm for financial market forecasting. IEEE Access, 8:71326–71338, 2020.
[40] W Waseem, Muhammad Sulaiman, Ahmad Alhindi, and Hosam Alhakami. A soft computing approach based on fractional order dpso algorithm designed to solve the corneal model for eye surgery. IEEE Access, 8:61576–61592, 2020.
[41] Sohail Ahmad, Muhammad Sulaiman, Poom Kumam, Zubair Hussain, Muhammad Asif Jan, Wali Khan Mashwani, and Masih Ullah. A novel population initialization strategy for accelerating levy flights based multiverse optimizer. Journal of Intelligent & Fuzzy Systems, (Preprint):1–17.
[42] Ayaz Hussain Bukhari, Muhammad Sulaiman, Muhammad Asif Zahoor Raja, Saeed Islam, Muhammad Shoaib, and Poom Kumam. Design of a hybrid nar-rbfs neural network for nonlinear dusty plasma system. Alexandria Engineering Journal, 2020.
[43] Adnan Khan, Muhammad Sulaiman, Hosam Alhakami, and Ahmad Alhindi. Analysis of oscillatory behavior of heart by using a novel neuroevolutionary approach. IEEE Access, 8:86674–86695, 2020.
[44] Waseem Waseem, Muhammad Sulaiman, Poom Kumam, Muhammad Shoaib, Muhammad Asif Zahoor Raja, and Saeed Islam. Investigation of singular ordinary differential equations by a neuroevolutionary approach. Plos one, 15(7):e0235829, 2020.
[45] AK Parikh, MN Mehta, and VH Pradhan. Mathematical modeling and analysis of fingero-imbibition phenomenon in vertical downward cylindrical homogeneous porous matrix. In 2013 Nirma University International Conference on Engineering (NUiCONE), pages 1–6. IEEE, 2013.
[46] Shreekant Pathak and Twinkle Singh. Approximate solution of imbibition phenomenon arising in heterogeneous porous media by optimal homotopy analysis method. IJCMS, 8(3):1950014–298, 2019.
[47] J Bear. Dynamics of fluids in porous media. american elsevier publishing company, inc.: New york. 1972.
[48] Adrian E Scheidegger. The physics of flow through porous media. University of Toronto Press, 1957.
[49] MN Mehta. Asymptotic expansion of fluid flow through porous media. South Gujarat University, Surat, India, 1977.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2020.3034053, IEEE Access

[50] Adrian E Scheidegger and Edward F Johnson. The statistical behavior of instabilities in displacement processes in porous media. Canadian Journal of physics, 39(2):326–334, 1961.

[51] Ashfaq Ahmad, Muhammad Sulaiman, Ahmad Alhindi, and Abdulah Jeza Aljohani. Analysis of temperature profiles in longitudinal fin designs by a novel neuroevolutionary approach. IEEE Access, 8:113285–113308, 2020.