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Research Article

Analysis of Nonlinear Coupled Systems of Impulsive Fractional Differential Equations with Hadamard Derivatives

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This work is committed to establishing the assumptions essential for at least one and unique solution of a switched coupled system of impulsive fractional differential equations having derivative of Hadamard type. Using Krasnoselskii's fixed point theorem, the existence, as well as uniqueness results, is obtained. Along with this, different kinds of Hyers–Ulam stability are discussed. For supporting the theory, example is provided.

1. Introduction

Fractional calculus is the field of mathematical analysis that deals with the investigation and applications of integrals and derivatives of arbitrary order. Fractional differential equations (FDEs) have played a significant role in many engineering and scientific disciplines, e.g., as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, capacitor theory, electrical circuits, electron analytical chemistry, biology, control theory, and fitting of experimental data [1–3]. FDEs also serve as an excellent tool for the description of hereditary properties of various materials and processes [4]. The theory of FDEs, involving different kinds of boundary conditions, has been a field of interest in pure and applied sciences. Nonlocal conditions are used to describe certain features of applied mathematics and physics such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, and population dynamics [5–16].

In the classical text [17], it has been mentioned that Hadamard in 1892 [18] suggested a concept of fractional integro-differentiation in terms of the fractional power of the type $(x(d/dx))^p$ in contrast to its Riemann–Liouville counterpart of the form $(d/dx)^p$. The kind of derivative introduced by Hadamard contains the logarithmic function of the arbitrary exponent in the kernel of the integral appearing in its definition. Hadamard construction is invariant in relation to dilation and is well suited to the problems containing half-axes. Coupled systems of FDEs have also been investigated by many authors. Such systems appear naturally in many real-world situations. Some recent results on the topic can be found in a series of papers [19–40].

Another aspect of FDEs which has very recently got attention of the researchers is concerning the Ulam-type stability analysis of the aforesaid equations. The mentioned stability was first pointed out by Ulam [41] in 1940, which was further explained by Hyers [42], over Banach space. Later on, many researchers have done valuable work on the same task and interesting results were formed for linear and nonlinear integral and differential equations; for details see [43, 44]. This stability analysis is very useful in many applications, such as numerical analysis and optimization, where finding the exact solution is quite difficult. For detailed study of Ulam-type stability with different approaches, we recommend papers [44–52].

Existence and uniqueness of Cauchy problems for fractional differential equations involving the Hadamard derivatives have been discussed by Kilbas et al. [53]. Using the contraction principle, existence and uniqueness of the solution of sequential fractional differential equations with Hadamard derivative have been explored by Klimek [54]. Recently, Wang et al. [55] discussed the existence, blowing-up solutions, and Ulam–Hyers stability of fractional differential equations with Hadamard derivative by using some classical methods. Further, Ahmad and Ntouyas [20] and Ma et al. [56] studied two-dimensional fractional differential systems with Hadamard derivative. Wang et al. [57] studied the fractional impulsive Cauchy problem of the form

$$\begin{gathered} {}_H\mathfrak{D}^{\alpha}_{1^+}u(t) - f(t,u(t)) = 0, \\ \alpha \in (0,1),\ t \in (1,e] - \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t_i) = {}_H\mathfrak{J}^{1-\alpha}_{+1}u(t_i^+) - {}_H\mathfrak{J}^{1-\alpha}_{+1}u(t_i^-) = p_i, \\ p_i \in \mathscr{R},\ i = 1,2,\ldots,m, \\ {}_H\mathfrak{J}^{1-\alpha}_{+1}u(1^+) = u_0, \quad u_0 \in \mathscr{R}, \end{gathered} \tag{1}$$

where ${}_H\mathfrak{D}^{\alpha}_{1^+}$ denotes left-sided Hadamard fractional derivative of order α with the lower limit 1 and ${}_H\mathfrak{J}^{1-\alpha}_{+1}$ denotes left-sided Hadamard fractional integral of order $1-\alpha$. In [58], Wang et al. studied the existence and Hyers–Ulam stability of switched coupled problem:

$$\begin{gathered} {}^c\mathfrak{D}^{\sigma}p(t) - f(t)p(t) = \Theta(t, p(t), q(t)), \\ t \in \mathscr{J},\ t \neq t_k, \\ {}^c\mathfrak{D}^{\delta}q(t) - g(t)q(t) = \Psi(t, p(t), q(t)), \\ t \in \mathscr{J},\ t \neq t_k, \\ \Delta p(t)\big|_{t=t_k} = I_k(p_k), \\ \Delta q(t)\big|_{t=t_k} = I_k(q_k), \\ p(t)\big|_{t=0} + \phi(p) = p_0, \\ q(t)\big|_{t=0} + \varphi(q) = q_0, \end{gathered} \tag{2}$$

where ${}^c\mathfrak{D}$ denotes the Caputo derivative of order $\sigma, \delta \in (0,1]$.

Motivated by the work in [58], we consider the following switched coupled impulsive FDEs involving Hadamard derivatives:

$$\begin{gathered} {}_H\mathfrak{D}^{\alpha}p(t) - f(t, p(t), q(t)) = 0, \\ t \in \mathscr{J},\ t \neq t_i,\ i = 1,2,\ldots,m, \\ {}_H\mathfrak{D}^{\beta}q(t) - g(t, p(t), q(t)) = 0, \\ t \in \mathscr{J},\ t \neq t_j,\ j = 1,2,\ldots,n, \\ \Delta p(t_i) = I_i(p(t_i)), \\ \Delta p'(t_i) = \widetilde{I}_i(p(t_i)), \\ i = 1,2,\ldots,m, \\ \Delta q(t_j) = I_j(q(t_j)), \\ \Delta q'(t_j) = \widetilde{I}_j(q(t_j)), \\ j = 1,2,\ldots,n, \\ \mu \ln 2p(2) + \nu \ln 2p'(2) = \phi(p), \\ \mu p(T) + \nu p'(T) = \varphi(p), \\ \mu \ln 2q(2) + \nu \ln 2q'(2) = \phi(q), \\ \mu q(T) + \nu q'(T) = \varphi(q), \end{gathered} \tag{3}$$

where $1 < \alpha, \beta \leq 2$, $f, g : \mathscr{J} \times \mathscr{R}^2 \longrightarrow \mathscr{R}$, and $\phi, \varphi : \mathscr{C}(\mathscr{J}, \mathscr{R}) \longrightarrow \mathscr{R}$ are continuous functions defined as

$$\begin{gathered} \phi(p) = \sum_{i=1}^{\widehat{a}} \hbar_i p(\zeta_i), \\ \varphi(p) = \sum_{i=1}^{\widehat{a}} \wp_i p(\eta_i), \\ \phi(q) = \sum_{j=1}^{\widehat{b}} \hbar_j q(\zeta_j), \\ \varphi(q) = \sum_{j=1}^{\widehat{b}} \wp_j q(\eta_j), \end{gathered} \tag{4}$$

$\zeta_i, \eta_i, \zeta_j, \eta_j \in (0,1)$ for $i = 1,2,\ldots,\widehat{a}$, $j = 1,2,\ldots,\widehat{b}$ and

$$\begin{gathered} \Delta p(t_i) = p(t_i^+) - p(t_i^-), \\ \Delta p'(t_i) = p'(t_i^+) - p'(t_i^-), \\ \Delta q(t_j) = q(t_j^+) - q(t_j^-), \\ \Delta q'(t_j) = q'(t_j^+) - q'(t_j^-). \end{gathered} \tag{5}$$

The notations $p(t_i^+)$, $q(t_j^+)$ are right limits and $p(t_i^-)$, $q(t_j^-)$ are left limits; I_i, $\widetilde{I}_i$, I_j, $\widetilde{I}_j : \mathscr{R} \longrightarrow \mathscr{R}$ are continuous functions; ${}_H\mathfrak{D}^{\alpha}$, ${}_H\mathfrak{D}^{\beta}$ are the Hadamard derivative operators of order α and β, respectively. For some other recent results on Hadamard fractional differential equations, we refer the reader to [59–65].

For system (3), we discuss necessary and sufficient conditions for the existence and uniqueness of a positive solution by using the Krasnoselskii's fixed point and Banach contraction theorems. Further, we investigate various kinds of Hyers–Ulam, generalized Hyers–Ulam, Hyers–Ulam–Rassias, and generalized Hyers–Ulam–Rassias stabilities.

This paper is organized as follows. Section 2 contains basic definitions, auxiliary lemmas, and theorems regarding problem (3). Existence, uniqueness, and at least one solution of the problem (3) are presented in Section 3. Section 4 contains Hyers–Ulam types stability results and Section 5 contains an example, which shows the applicability of our main results.

2. Preliminaries

In this fragment, we are introducing some fundamental descriptions and lemmas, which are used throughout the paper.

Let the norms be $\|p\| = \max\{|p(t)|,\ t \in \mathcal{J}\}$, $\|q\| = \max\{|q(t)|,\ t \in \mathcal{J}\}$ in $\mathcal{PC}(\mathcal{J}, \mathcal{R}_+)$, which is Banach space under these norms, and hence their product is also Banach space with norm $\|(p,q)\| = \|p\| + \|q\|$.

Let $\mathscr{E}_1$ and $\mathscr{E}_2$ denote spaces of the piecewise continuous functions defined as

$$\begin{aligned} \mathscr{E}_1 &= \mathcal{PC}_{2-\alpha,\ln}(\mathcal{J}, \mathcal{R}_+) = \{p : \mathcal{J} \longrightarrow \mathcal{R}_+, p(t_i^+), \\ &\quad p(t_i^-) \text{ and } p'(t_i^+), p'(t_i^-) \text{ exist for } i = 1,2,\ldots,m\}, \\ \mathscr{E}_2 &= \mathcal{PC}_{2-\beta,\ln}(\mathcal{J}, \mathcal{R}_+) = \{q : \mathcal{J} \longrightarrow \mathcal{R}_+, q(t_j^+), \\ &\quad q(t_j^-) \text{ and } q'(t_j^+), q'(t_j^-) \text{ exist for } j = 1,2,\ldots,n\}, \end{aligned} \tag{6}$$

with norms

$$\begin{aligned} \|p\|_{\mathscr{E}_1} &= \sup\{|p(t)(\ln t)^{2-\alpha}|,\ t \in \mathcal{J}\}, \\ \|q\|_{\mathscr{E}_2} &= \sup\{|q(t)(\ln t)^{2-\beta}|,\ t \in \mathcal{J}\}, \end{aligned} \tag{7}$$

respectively. Their product $\mathscr{E} = \mathscr{E}_1 \times \mathscr{E}_2$ is also a Banach space with norm $\|(p,q)\|_{\mathscr{E}} = \|p\|_{\mathscr{E}_1} + \|q\|_{\mathscr{E}_2}$.

We recall the following definitions from [57].

Definition 1. The Hadamard fractional derivative of order $\alpha \in [a-1, a)$, $a \in \mathcal{Z}^+$ of function $p(t)$ is defined by

$$\begin{aligned} ({}_H\mathfrak{D}^\alpha p)(t) &= \frac{1}{\Gamma(a-\alpha)} \left(t\frac{d}{dt}\right)^a \int_1^t \left(\ln \frac{t}{s}\right)^{a-\alpha+1} p(s) \frac{ds}{s}, \\ &\qquad 1 < t \le T, \end{aligned} \tag{8}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. The Hadamard fractional integral of order $\alpha \in \mathcal{R}^+$ of function $p(t)$ is defined by

$$({}_H\mathfrak{J}^\alpha p)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} p(s) \frac{ds}{s}, \qquad 1 < t \le T, \tag{9}$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Lemma 3 (see [66]). *Let $\alpha > 0$ and p be any functions; then the homogenous differential equation along with Hadamard fractional order ${}_H\mathfrak{D}^\alpha p(t) = 0$ has solution*

$$\begin{aligned} p(t) &= b_1 (\ln t)^{\alpha-1} + b_2 (\ln t)^{\alpha-2} + b_3 (\ln t)^{\alpha-3} + \cdots \\ &\quad + b_n (\ln t)^{\alpha-a}, \end{aligned} \tag{10}$$

and the following formula holds:

$$\begin{aligned} {}_H\mathfrak{J}^\alpha\, {}_H\mathfrak{D}^\alpha p(t) &= p(t) + b_1 (\ln t)^{\alpha-1} + b_2 (\ln t)^{\alpha-2} \\ &\quad + b_3 (\ln t)^{\alpha-3} + \cdots + b_n (\ln t)^{\alpha-a}, \end{aligned} \tag{11}$$

where $b_j \in \mathcal{R}$, $j = 1,2,\ldots,a$, and $a-1 < \alpha < a$.

Lemma 4 (see [53]). *Let $0 < \alpha < 1$ and $f : \mathcal{J} \times \mathcal{R} \longrightarrow \mathcal{R}$. A function $p \in \mathscr{C}_{1-\alpha,ln}[a,b]$ is a solution of the fractional differential equation*

$$\begin{aligned} {}_H\mathfrak{D}^\alpha p(t) &= f(t, p(t)), \quad t \in (1, T], \\ {}_H\mathfrak{J}^{1-\alpha} p(1^+) &= p_0, \quad p_0 \in \mathcal{R}, \end{aligned} \tag{12}$$

if and only if p satisfies the following fractional integral equation:

$$\begin{aligned} p(t) &= \frac{p_0}{\Gamma(\alpha)} (\ln t)^{\alpha-1} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, p(s)) \frac{ds}{s}. \end{aligned} \tag{13}$$

Theorem 5 (Altman [67]). *Let $\mathcal{S} \ne \emptyset$ be a convex and closed subset of Banach space $\mathscr{E}$. Consider two operators $\mathbb{F}$, $\mathbb{G}$ such that*

(i) *$\mathbb{F}(p,q) + \mathbb{G}(p,q) \in \mathcal{S}$, where $(p,q) \in \mathcal{S}$.*

(ii) *$\mathbb{F}$ is contractive operator.*

(iii) *$\mathbb{G}$ is completely continuous operator.*

Then the operator system $\mathbb{F}(p,q) + \mathbb{G}(p,q) = (p,q) \in \mathscr{E}$ has a solution $(p,q) \in \mathcal{S}$.

Definition 6 (see [45]). The switched coupled impulsive FDE (3) is said to be Hyers–Ulam stable if there exist $K_{\alpha,\beta} = \max\{K_\alpha, K_\beta\} > 0$ such that, for $\varrho = \max\{\varrho_\alpha, \varrho_\beta\} > 0$ and for every solution $(p,q) \in \mathscr{E}$ of the inequality

$$\begin{aligned} &|{}_H\mathfrak{D}^\alpha p(t) - f(t, p(t), q(t))| \le \varrho_\alpha, \quad t \in \mathcal{J}, \\ &|\Delta p(t_i) - I_i(p(t_i))| \le \varrho_\alpha, \quad i = 1,2,\ldots,m, \\ &|\Delta \widehat{p'}(t_i) - \tilde{I}_i(p(t_i))| \le \varrho_\alpha, \quad i = 1,2,\ldots,m, \\ &|{}_H\mathfrak{D}^\beta q(t) - g(t, p(t), q(t))| \le \varrho_\beta, \quad t \in \mathcal{J}, \\ &|\Delta q(t_j) - I_j(q(t_j))| \le \varrho_\beta, \quad j = 1,2,\ldots,n, \\ &|\Delta \widehat{q'}(t_j) - \tilde{I}_j(q(t_j))| \le \varrho_\beta, \quad j = 1,2,\ldots,n, \end{aligned} \tag{14}$$

there exists a unique solution $(\hat{p}, \hat{q}) \in \mathscr{E}$ with

$$\|(p,q) - (\hat{p}, \hat{q})\|_{\mathscr{E}} \le K_{\alpha,\beta}\varrho, \quad t \in \mathcal{J}. \tag{15}$$

Definition 7 (see [45]). The switched coupled impulsive FDE (3) is said to be generalized Hyers–Ulam stable if there exist $\Phi \in \mathscr{C}(\mathcal{R}^+, \mathcal{R}^+)$ with $\Phi(0) = 0$ such that, for any approximate solution $(p,q) \in \mathscr{E}$ of inequality (14), there exists a unique solution $(\hat{p}, \hat{q}) \in \mathscr{E}$ of (3) satisfying

$$\|(p,q) - (\hat{p}, \hat{q})\|_{\mathscr{E}} \le \Phi(\varrho), \quad t \in \mathcal{J}. \tag{16}$$

Denote $\Psi_{\alpha,\beta} = \max\{\Psi_\alpha, \Psi_\beta\} \in \mathscr{C}(\mathscr{J}, \mathscr{R})$ and $K_{\Psi_\alpha,\Psi_\beta} = \max\{K_{\Psi_\alpha}, K_{\Psi_\beta}\} > 0$.

Definition 8 (see [45]). The switched coupled impulsive FDE (3) is said to be Hyers–Ulam–Rassias stable with respect to $\Psi_{\alpha,\beta}$ if there exists a constant $K_{\Psi_\alpha,\Psi_\beta}$ such that, for some $\varrho > 0$ and for any approximate solution $(p, q) \in \mathscr{E}$ of the inequality

$$\begin{aligned} \left|{}_H\mathfrak{D}^\alpha p(t) - f(t, p(t), q(t))\right| &\le \Psi_\alpha(t)\varrho_\alpha, \quad t \in \mathscr{J}, \\ \left|{}_H\mathfrak{D}^\beta q(t) - g(t, p(t), q(t))\right| &\le \Psi_\beta(t)\varrho_\beta, \quad t \in \mathscr{J}, \end{aligned} \tag{17}$$

there exists a unique solution $(\hat{p}, \hat{q}) \in \mathscr{E}$ with

$$\|(p, q) - (\hat{p}, \hat{q})\|_{\mathscr{E}} \le K_{\Psi_\alpha,\Psi_\beta}\Psi_{\alpha,\beta}\varrho, \quad t \in \mathscr{J}. \tag{18}$$

Definition 9 (see [45]). The switched coupled impulsive FDE (3) is said to be generalized Hyers–Ulam–Rassias stable with respect to $\Psi_{\alpha,\beta}$ if there exists a constant $K_{\Psi_\alpha,\Psi_\beta}$ such that, for any approximate solution $(p, q) \in \mathscr{E}$ of inequality (17), there exists a unique solution $(\hat{p}, \hat{q}) \in \mathscr{E}$ of (3) satisfying

$$\|(p, q) - (\hat{p}, \hat{q})\|_{\mathscr{E}} \le K_{\Psi_\alpha,\Psi_\beta}\Psi_{\alpha,\beta}(t), \quad t \in \mathscr{J}. \tag{19}$$

Remark 10. We say that $(p, q) \in \mathscr{E}$ is a solution of the system of inequalities (14) if there exist functions $\Upsilon_f, \Upsilon_g \in \mathscr{C}(\mathscr{J}, \mathscr{R})$ depending upon p, q, respectively, such that

(I) $|\Upsilon_f(t)| \le \varrho_\alpha$, $|\Upsilon_g(t)| \le \varrho_\beta$, $t \in \mathscr{J}$;

(II)

$$\begin{aligned} {}_H\mathfrak{D}^\alpha p(t) &= f(t, p(t), q(t)) + \Upsilon_f(t), \\ \Delta p(t_i) &= I_i(p(t_i)) + \Upsilon_{f_i}, \\ \Delta p'(t_i) &= \tilde{I}_i(p(t_i)) + \Upsilon_{f_i}, \\ {}_H\mathfrak{D}^\beta q(t) &= g(t, p(t), q(t)) + \Upsilon_g(t). \\ \Delta q(t_j) &= I_j(q(t_j)) + \Upsilon_{g_j}, \\ \Delta q'(t_j) &= \tilde{I}_j(q(t_j)) + \Upsilon_{g_j}. \end{aligned} \tag{20}$$

3. Existence Results

In this fragment, we present our main results.

Theorem 11. *The solution $(p, q) \in \mathscr{E}$ of coupled system*

$$\begin{aligned} {}_H\mathfrak{D}^\alpha p(t) &= f(t), \\ &\qquad t \in \mathscr{J},\ t \ne t_i,\ i = 1, 2, \ldots, m, \\ {}_H\mathfrak{D}^\beta q(t) &= g(t), \\ &\qquad t \in \mathscr{J},\ t \ne t_j,\ j = 1, 2, \ldots, n, \\ \Delta p(t_i) &= I_i(p(t_i)), \\ \Delta p'(t_i) &= \tilde{I}_i(p(t_i)), \\ &\qquad i = 1, 2, \ldots, m, \\ \Delta q(t_j) &= I_j(q(t_j)), \\ \Delta q'(t_j) &= \tilde{I}_j(q(t_j)), \\ &\qquad j = 1, 2, \ldots, n, \\ \mu \ln 2p(2) + \nu \ln 2p'(2) &= \phi(p), \\ \mu p(T) + \nu p'(T) &= \varphi(p), \\ \mu \ln 2q(2) + \nu \ln 2q'(2) &= \phi(q), \\ \mu q(T) + \nu q'(T) &= \varphi(q), \end{aligned} \tag{21}$$

is given by the integral equations

$$\begin{aligned} p(t) &= \frac{(\ln t)^{\alpha-2}\varphi(p)\,{}_\alpha\tilde{\Omega}(t)}{{}_\alpha\Omega} + (\ln t)^{\alpha-2}\phi(p)\,{}_\alpha\mathscr{A}_1(t) \\ &+ \sum_{i=1}^{k}\frac{(\ln t)^{\alpha-2}\,{}_\alpha\mathscr{A}_2^i(t)}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} \\ &+ \sum_{i=1}^{k}(\ln t)^{\alpha-2}\,{}_\alpha\mathscr{A}_2^i(t)\,I_i(p(t_i)) \\ &+ \sum_{i=1}^{k}(\ln t)^{\alpha-2}\,{}_\alpha\mathscr{A}_3^i(t)\,\tilde{I}_i(p(t_i)) \\ &+ \sum_{i=1}^{k}\frac{(\ln t)^{\alpha-2}\,{}_\alpha\mathscr{A}_3^i(t)}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} \\ &- \frac{\mu(\ln t)^{\alpha-2}\,{}_\alpha\tilde{\Omega}(t)}{{}_\alpha\Omega\Gamma(\alpha)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} \\ &- \frac{\nu(\ln t)^{\alpha-2}\,{}_\alpha\tilde{\Omega}(t)}{{}_\alpha\Omega\Gamma(\alpha-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \\ &\qquad k = 1, 2, \ldots, m, \end{aligned}$$

$$\begin{aligned} q(t) &= \frac{(\ln t)^{\beta-2}\varphi(q)\,{}_\beta\tilde{\Omega}(t)}{{}_\beta\Omega} + (\ln t)^{\beta-2}\phi(q)\,{}_\beta\mathscr{A}_1(t) \\ &+ \sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\,{}_\beta\mathscr{A}_2^j(t)}{\Gamma(\beta)}\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-1} g(s)\frac{ds}{s} \\ &+ \sum_{j=1}^{k}(\ln t)^{\beta-2}\,{}_\beta\mathscr{A}_2^j(t)\,I_j(q(t_j)) \\ &+ \sum_{j=1}^{k}(\ln t)^{\beta-2}\,{}_\beta\mathscr{A}_3^j(t)\,\tilde{I}_j(q(t_j)) \end{aligned}$$

$$+\sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_3^{j}(t)}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-2}g(s)\frac{ds}{s}$$
$$-\frac{\mu(\ln t)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}(t)}{{}_{\beta}\Omega\Gamma(\beta)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-1}g(s)\frac{ds}{s}$$
$$-\frac{\nu(\ln t)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}(t)}{{}_{\beta}\Omega\Gamma(\beta-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-2}g(s)\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\beta)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}g(s)\frac{ds}{s},\quad k=1,2,\ldots,n, \tag{22}$$

where

$$\begin{aligned}
{}_{\alpha}\Omega &= (\ln T)^{\alpha-3}\left[\ln T^{\mu}+\frac{\nu(\alpha-2)}{T}-\left(1-{}_{\alpha}\widetilde{\Omega}(T)\right)\left(\ln T^{\mu}+\frac{\nu(\alpha-1)}{T}\right)\right]\neq 0,\\
{}_{\alpha}\widetilde{\Omega}(t) &= 1-\log_2 t\frac{\ln 2^{\mu}+\nu(\alpha-2)/2}{\ln 2^{\mu}+\nu(\alpha-1)/2},\\
{}_{\alpha}\mathscr{A}_1(t) &= \frac{(\ln 2)^{2-\alpha}\left({}_{\alpha}\Omega-{}_{\alpha}\widetilde{\Omega}(t)(\ln T)^{\alpha-2}\left(\mu+\log_2 T^{\nu(\alpha-1)}/\ln 2\right)\right)}{{}_{\alpha}\Omega\left(\ln 2^{\mu}+\nu(\alpha-1)/2\right)},\\
{}_{\alpha}\mathscr{A}_2^{i}(t) &= \frac{(\ln 2)^{2-\alpha}\left({}_{\alpha}\Omega\log_{t_i}\left(t_i^{\alpha-1}/t^{\alpha-2}\right)-(\ln T)^{\alpha-2}\ {}_{\alpha}\widetilde{\Omega}(t)\left(\log_{t_i}\left(t_i^{\alpha-1}/T^{\alpha-2}\right)+\nu(\alpha-1)(\alpha-2)\left(\log_T e-\log_{t_i}e\right)/T\right)\right)}{{}_{\alpha}\Omega},\\
{}_{\alpha}\mathscr{A}_3^{i}(t) &= \frac{t_i(\ln t_i)^{2-\alpha}\left({}_{\alpha}\Omega\ln\left(t/t_i\right)-{}_{\alpha}\widetilde{\Omega}(t)(\ln T)^{\alpha-2}\left(\ln(T/t_i)^{\mu}+\log_T\left(T^{\alpha-1}/t_i^{\alpha-2}\right)^{\nu/T}\right)\right)}{{}_{\alpha}\Omega}.
\end{aligned} \tag{23}$$

Proof. Consider

$${}_{H}\mathfrak{D}^{\alpha}p(t)=f(t),\quad t\in\mathscr{J},\ t\neq t_i. \tag{24}$$

For $t\in[2,t_1]$, applying $\mathfrak{J}^{\alpha}$ on (24), we get

$$p(t)=c_1(\ln t)^{\alpha-1}+c_2(\ln t)^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_2^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}. \tag{25}$$

This implies

$$p'(t)=\frac{c_1(\alpha-1)}{t}(\ln t)^{\alpha-2}+\frac{c_2(\alpha-2)}{t}(\ln t)^{\alpha-3}+\frac{1}{\Gamma(\alpha-1)}\int_2^t\left(\ln\frac{t}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}. \tag{26}$$

Applying the initial condition, we obtain $c_1=(\phi(p)(\ln 2)^{2-\alpha}-c_2(\ln 2^{\mu}+\nu((\alpha-2)/2)))/\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}$. Therefore (25) becomes

$$\begin{aligned}
p(t)&=\frac{\phi(p)(\ln 2)^{2-\alpha}(\ln t)^{\alpha-1}}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}}\\
&\quad+c_2\left((\ln t)^{\alpha-2}-(\ln t)^{\alpha-1}\frac{\ln 2^{\mu}+\nu((\alpha-2)/2)}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}}\right)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_2^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}.
\end{aligned} \tag{27}$$

Now for $t\in(t_1,t_2]$, applying $\mathfrak{J}^{\alpha}$ on (24), we have

$$p(t)=b_1(\ln t)^{\alpha-1}+b_2(\ln t)^{\alpha-2}+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}. \tag{28}$$

This implies

$$p'(t)=\frac{b_1(\alpha-1)}{t}(\ln t)^{\alpha-2}+\frac{b_2(\alpha-2)}{t}(\ln t)^{\alpha-3}+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-2}f(s)\frac{ds}{s}. \tag{29}$$

Using initial impulses

$$\begin{aligned}
p(t_1^-)&=\frac{\phi(p)(\ln 2)^{2-\alpha}(\ln t_1)^{\alpha-1}}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}}+c_2\left((\ln t_1)^{\alpha-2}\right.\\
&\quad\left.-(\ln t_1)^{\alpha-1}\frac{\ln 2^{\mu}+\nu((\alpha-2)/2)}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}}\right)+\frac{1}{\Gamma(\alpha)}\\
&\quad\cdot\int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}f(s)\frac{ds}{s}
\end{aligned} \tag{30}$$

and $p(t_1^+)=b_1(\ln t_1)^{\alpha-1}+b_2(\ln t_1)^{\alpha-2}$.

Using impulsive conditions, we get

$$b_1 = \frac{\phi(p)(\ln 2)^{2-\alpha}}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}} - c_2\frac{\ln 2^{\mu}+\nu((\alpha-2)/2)}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}} - (\alpha-2)(\ln t_1)^{1-\alpha} I_1(p(t_1)) + t_1(\ln t_1)^{2-\alpha}\widetilde{I}_1(p(t_1)) - \frac{(\alpha-2)(\ln t_1)^{1-\alpha}}{\Gamma(\alpha)}\int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} + \frac{t_1(\ln t_1)^{2-\alpha}}{\Gamma(\alpha-1)}\int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s}, \quad (31)$$

$$b_2 = c_2 + (\alpha-1)(\ln t_1)^{2-\alpha} I_1(p(t_1)) - t_1(\ln t_1)^{3-\alpha}\widetilde{I}_1(p(t_1)) + \frac{(\ln t_1)^{2-\alpha}}{\Gamma(\alpha-1)}\int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} - \frac{t_1(\ln t_1)^{3-\alpha}}{\Gamma(\alpha-1)}\int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s}.$$

Substituting the values of b_1, b_2 in (28), we have

$$p(t) = \frac{\phi(p)(\log_2 t)^{\alpha-2}}{\ln 2^{\mu}+\nu((\alpha-1)/2)} + c_2(\ln t)^{\alpha-2} \cdot \left(1 - \log_2 t\frac{\ln 2^{\mu}+\nu((\alpha-2)/2)}{\ln 2^{\mu}+\nu((\alpha-1)/2)}\right) + \left(\log_{t_1} t\right)^{\alpha-2}\log_{t_1}\frac{t_1^{\alpha-1}}{t^{\alpha-2}} I_1(p(t_1)) + t_1\left(\log_{t_1} t\right)^{\alpha-2}\ln\frac{t}{t_1}\widetilde{I}_1(p(t_1)) + \frac{\left(\log_{t_1} t\right)^{\alpha-2}\log_{t_1}\left(t_1^{\alpha-1}/t^{\alpha-2}\right)}{\Gamma(\alpha)} \cdot \int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} + \frac{t_1\left(\log_{t_1} t\right)^{\alpha-2}\ln(t/t_1)}{\Gamma(\alpha-1)} \cdot \int_2^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \cdot \int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s))\frac{ds}{s}. \quad (32)$$

Similarly for $t \in (t_k, T)$, we get

$$p(t) = \frac{\phi(p)(\log_2 t)^{\alpha-2}}{\ln 2^{\mu}+\nu((\alpha-1)/2)} + c_2(\ln t)^{\alpha-2} \cdot \left(1 - \log_2 t\frac{\ln 2^{\mu}+\nu((\alpha-2)/2)}{\ln 2^{\mu}+\nu((\alpha-1)/2)}\right) + \sum_{i=1}^{k}\left(\log_{t_i} t\right)^{\alpha-2}\log_{t_i}\frac{t_i^{\alpha-1}}{t^{\alpha-2}} I_i(p(t_i)) + \sum_{i=1}^{k} t_i\left(\log_{t_i} t\right)^{\alpha-2}\ln\frac{t}{t_i}\widetilde{I}_i(p(t_i)) + \sum_{i=1}^{k}\frac{\left(\log_{t_i} t\right)^{\alpha-2}\log_{t_i}\left(t_i^{\alpha-1}/t^{\alpha-2}\right)}{\Gamma(\alpha)} \cdot \int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} + \sum_{i=1}^{k}\frac{t_i\left(\log_{t_i} t\right)^{\alpha-2}\ln(t/t_i)}{\Gamma(\alpha-1)} \cdot \int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \cdot \int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} f(s))\frac{ds}{s}. \quad (33)$$

This implies

$$p'(t) = \frac{\phi(p)(\alpha-1)(\log_2 t)^{\alpha-1}}{\ln 2^{\ln 2^{\mu}+\nu((\alpha-1)/2)}} + \sum_{i=1}^{k}\frac{(\alpha-1)(\alpha-2)\left(\log_t e - \log_{t_i} e\right)\left(\log_{t_i} t\right)^{\alpha-2}}{t} I_i(p(t_i)) + c_2 \cdot \frac{\left(\alpha - 2 - \log_2 t^{\alpha-1}\left(\left(\ln 2^{\mu}+\nu((\alpha-2)/2)\right)/\left(\ln 2^{\mu}+\nu((\alpha-1)/2)\right)\right)\right)(\ln t)^{\alpha-3}}{t} + \sum_{i=1}^{k}\frac{t_i\left(\log_t\left(t^{\alpha-1}/t_i^{\alpha-2}\right)\right)\left(\log_{t_i} t\right)^{\alpha-2}}{t}\widetilde{I}_i(p(t_i)) + \sum_{i=1}^{k}\frac{(\alpha-1)(\alpha-2)\left(\log_t e - \log_{t_i} e\right)\left(\log_{t_i} t\right)^{\alpha-2}}{t\Gamma(\alpha)} \cdot \int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1} f(s)\frac{ds}{s} + \sum_{i=1}^{k}\frac{t_i\left(\log_t\left(t^{\alpha-1}/t_i^{\alpha-2}\right)\right)\left(\log_{t_i} t\right)^{\alpha-2}}{t\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2} f(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha-1)}$$

$$\cdot \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\alpha-2} f(s)) \frac{ds}{s}. \tag{34}$$

Utilizing $\mu p(T) + \nu p'(T) = \varphi(p)$, we obtain

$$\begin{aligned} c_2 = {} & \frac{\varphi(p)}{{}_{\alpha}\Omega} - \frac{\phi(p)\left(\log_2 T\right)^{\alpha-1}\left(\log_T 2^{\mu} + \log_2 e^{\nu(\alpha-1)}\right)}{{}_{\alpha}\Omega\left(\ln 2^{\mu} + \nu(\alpha-1)/2\right)} - \sum_{i=1}^{k} \widetilde{I}_i(p(t_i)) \frac{t_i\left(\log_{t_i} T\right)^{\alpha-2}\left(\ln(T/t_i)^{\mu} + \log_T\left(T^{\alpha-1}/t_i^{\alpha-2}\right)^{\nu/T}\right)}{{}_{\alpha}\Omega} \\ & - \sum_{i=1}^{k} I_i(p(t_i)) \frac{\left(\log_{t_i} T\right)^{\alpha-2}\left(\log_{t_i}\left(t_i^{\alpha-1}/T^{\alpha-2}\right)^{\mu} + \nu(\alpha-1)(\alpha-2)\left(\log_T e - \log_{t_i} e\right)/T\right)}{{}_{\alpha}\Omega} \\ & - \sum_{i=1}^{k} \frac{\left(\log_{t_i} T\right)^{\alpha-2}\left(\log_{t_i}\left(t_i^{\alpha-1}/T^{\alpha-2}\right)^{\mu} + \nu(\alpha-1)(\alpha-2)\left(\log_T e - \log_{t_i} e\right)/T\right)}{{}_{\alpha}\Omega\Gamma(\alpha)} \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-1} f(s) \frac{ds}{s} \\ & - \sum_{i=1}^{k} \frac{t_i\left(\log_{t_i} T\right)^{\alpha-2}\left(\ln(T/t_i)^{\mu} + \log_T\left(T^{\alpha-1}/t_i^{\alpha-2}\right)^{\nu/T}\right)}{{}_{\alpha}\Omega\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-2} f(s) \frac{ds}{s} - \frac{\mu}{{}_{\alpha}\Omega\Gamma(\alpha)} \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-1} f(s)) \frac{ds}{s} \\ & - \frac{\nu}{{}_{\alpha}\Omega\Gamma(\alpha-1)} \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-2} f(s)) \frac{ds}{s}, \quad k = 1, 2, \ldots, m. \end{aligned} \tag{35}$$

Substituting the value of c_2 in (33), we achieve $p(t)$ of (22). In the same way, we can obtain $q(t)$ of (22). □

Corollary 12. *In view of Theorem 11, our coupled system (3) has the following solution:*

$$\begin{aligned} p(t) = {} & \frac{(\ln t)^{\alpha-2}\, {}_{\alpha}\widetilde{\Omega}(t)\,\varphi(p)}{{}_{\alpha}\Omega} \\ & + (\ln t)^{\alpha-2}\, {}_{\alpha}\mathscr{A}_1(t)\,\phi(p) \\ & + \sum_{i=1}^{k} (\ln t)^{\alpha-2}\, {}_{\alpha}\mathscr{A}_2^{i}(t)\, I_i(p(t_i)) \\ & + \sum_{i=1}^{k} (\ln t)^{\alpha-2}\, {}_{\alpha}\mathscr{A}_3^{i}(t)\, \widetilde{I}_i(p(t_i)) \\ & + \sum_{i=1}^{k} \frac{(\ln t)^{\alpha-2}\, {}_{\alpha}\mathscr{A}_2^{i}(t)}{\Gamma(\alpha)} \\ & \cdot \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-1} f(s, p(s), q(s)) \frac{ds}{s} \\ & + \sum_{i=1}^{k} \frac{(\ln t)^{\alpha-2}\, {}_{\alpha}\mathscr{A}_3^{i}(t)}{\Gamma(\alpha-1)} \\ & \cdot \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-2} f(s, p(s), q(s)) \frac{ds}{s} \\ & - \frac{\nu(\ln t)^{\alpha-2}\, {}_{\alpha}\widetilde{\Omega}(t)}{{}_{\alpha}\Omega\Gamma(\alpha-1)} \\ & \times \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-2} f(s, p(s), q(s)) \frac{ds}{s} \\ & - \frac{\mu(\ln t)^{\alpha-2}\, {}_{\alpha}\widetilde{\Omega}(t)}{{}_{\alpha}\Omega\Gamma(\alpha)} \\ & \cdot \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-1} f(s, p(s), q(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \\ & \cdot \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, p(s), q(s)) \frac{ds}{s}, \\ & \qquad k = 1, 2, \ldots, m, \end{aligned}$$

$$\begin{aligned} q(t) = {} & \frac{(\ln t)^{\beta-2}\, {}_{\beta}\widetilde{\Omega}(t)\,\varphi(q)}{{}_{\beta}\Omega} + (\ln t)^{\beta-2}\, {}_{\beta}\mathscr{A}_1(t)\,\phi(q) \\ & + \sum_{j=1}^{k} (\ln t)^{\beta-2}\, {}_{\beta}\mathscr{A}_2^{j}(t)\, I_j\left(q\left(t_j\right)\right) \\ & + \sum_{j=1}^{k} (\ln t)^{\beta-2}\, {}_{\beta}\mathscr{A}_3^{j}(t)\, \widetilde{I}_j\left(q\left(t_j\right)\right) \\ & + \sum_{j=1}^{k} \frac{(\ln t)^{\beta-2}\, {}_{\beta}\mathscr{A}_2^{j}(t)}{\Gamma(\beta)} \end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{t_{j-1}}^{t_j} \left(\ln\frac{t_j}{s}\right)^{\beta-1} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&+ \sum_{j=1}^{k} \frac{\left(\ln t\right)^{\beta-2}\ {}_{\beta}\mathscr{A}_3^j\left(t\right)}{\Gamma\left(\beta-1\right)} \\
&\cdot \int_{t_{j-1}}^{t_j} \left(\ln\frac{t_j}{s}\right)^{\beta-2} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&- \frac{\nu\left(\ln t\right)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}\left(t\right)}{{}_{\beta}\Omega\Gamma\left(\beta-1\right)} \\
&\times \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\beta-2} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&- \frac{\mu\left(\ln t\right)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}\left(t\right)}{{}_{\beta}\Omega\Gamma\left(\beta\right)} \\
&\cdot \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\beta-1} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} + \frac{1}{\Gamma\left(\beta\right)} \\
&\cdot \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\beta-1} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s}, \\
&\qquad k = 1, 2, \dots, n.
\end{aligned}
\tag{36}
$$

To convert the considered problem into a fixed point problem, we define the operators $\mathbb{F} : \mathscr{E} \longrightarrow \mathscr{E}$ by $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2)$ and $\mathbb{G} : \mathscr{E} \longrightarrow \mathscr{E}$ by $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$ such that

$$
\begin{aligned}
\mathbb{F} = \mathbb{F}_1\left(p\left(t\right)\right) &= \frac{\left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\widetilde{\Omega}\left(t\right)\varphi\left(p\right)}{{}_{\alpha}\Omega} \\
&+ \left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\mathscr{A}_1\left(t\right)\phi\left(p\right) \\
&+ \sum_{i=1}^{k}\left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\mathscr{A}_2^i\left(t\right) I_i\left(p\left(t_i\right)\right) \\
&+ \sum_{i=1}^{k}\left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\mathscr{A}_3^i\left(t\right) \widetilde{I}_i\left(p\left(t_i\right)\right), \\
&\qquad k = 1, 2, \dots, m,
\end{aligned}
\tag{37a}
$$

$$
\begin{aligned}
\mathbb{F} = \mathbb{F}_2\left(q\left(t\right)\right) &= \frac{\left(\ln t\right)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}\left(t\right)\varphi\left(q\right)}{{}_{\beta}\Omega} \\
&+ \left(\ln t\right)^{\beta-2}\ {}_{\beta}\mathscr{A}_1\left(t\right)\phi\left(q\right) \\
&+ \sum_{j=1}^{k}\left(\ln t\right)^{\beta-2}\ {}_{\beta}\mathscr{A}_2^j\left(t\right) I_j\left(q\left(t_j\right)\right) \\
&+ \sum_{j=1}^{k}\left(\ln t\right)^{\beta-2}\ \mathscr{A}_3^j\left(t\right) \widetilde{I}_j\left(q\left(t_j\right)\right), \quad k = 1, 2, \dots, n,
\end{aligned}
\tag{37b}
$$

$$
\begin{aligned}
\mathbb{G} = \mathbb{G}_1\left(p\left(t\right), q\left(t\right)\right) &= \sum_{i=1}^{k} \frac{\left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\mathscr{A}_2^i\left(t\right)}{\Gamma\left(\alpha\right)} \\
&\cdot \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-1} f\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&+ \sum_{i=1}^{k} \frac{\left(\ln t\right)^{\alpha-2}\ {}_{\alpha}\mathscr{A}_3^i\left(t\right)}{\Gamma\left(\alpha-1\right)} \\
&\times \int_{t_{i-1}}^{t_i} \left(\ln\frac{t_i}{s}\right)^{\alpha-2} f\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&- \frac{\mu\ {}_{\alpha}\widetilde{\Omega}\left(t\right)\left(\ln t\right)^{\alpha-2}}{{}_{\alpha}\Omega\Gamma\left(\alpha\right)} \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-1} \\
&\times f\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} - \frac{\nu\ {}_{\alpha}\widetilde{\Omega}\left(t\right)\left(\ln t\right)^{\alpha-2}}{{}_{\alpha}\Omega\Gamma\left(\alpha-1\right)} \\
&\cdot \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\alpha-2} f\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} + \frac{1}{\Gamma\left(\alpha\right)} \\
&\cdot \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\alpha-1} f\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s}, \\
&\qquad k = 1, 2, \dots, m,
\end{aligned}
\tag{38a}
$$

$$
\begin{aligned}
\mathbb{G} = \mathbb{G}_2\left(p\left(t\right), q\left(t\right)\right) &= \sum_{j=1}^{k} \frac{\left(\ln t\right)^{\beta-2}\ {}_{\beta}\mathscr{A}_2^j\left(t\right)}{\Gamma\left(\beta\right)} \\
&\cdot \int_{t_{j-1}}^{t_j} \left(\ln\frac{t_j}{s}\right)^{\beta-1} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&+ \sum_{j=1}^{k} \frac{\left(\ln t\right)^{\beta-2}\ {}_{\beta}\mathscr{A}_3^j\left(t\right)}{\Gamma\left(\beta-1\right)} \\
&\times \int_{t_{j-1}}^{t_j} \left(\ln\frac{t_j}{s}\right)^{\beta-2} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} \\
&- \frac{\mu\ {}_{\beta}\widetilde{\Omega}\left(t\right)\left(\ln t\right)^{\beta-2}}{{}_{\beta}\Omega\Gamma\left(\beta\right)} \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\beta-1} \\
&\times g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} - \frac{\nu\ {}_{\beta}\widetilde{\Omega}\left(t\right)\left(\ln t\right)^{\beta-2}}{{}_{\beta}\Omega\Gamma\left(\beta-1\right)} \\
&\cdot \int_{t_k}^{T} \left(\ln\frac{T}{s}\right)^{\beta-2} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s} + \frac{1}{\Gamma\left(\beta\right)} \\
&\cdot \int_{t_k}^{t} \left(\ln\frac{t}{s}\right)^{\beta-1} g\left(s, p\left(s\right), q\left(s\right)\right) \frac{ds}{s}, \\
&\qquad k = 1, 2, \dots, n,
\end{aligned}
\tag{38b}
$$

respectively.

The following assumptions will be helpful for our results.

($\mathbf{H}_1$) $f, g : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}^+$ are continuous; for all $(p, q), (\tilde{p}, \tilde{q}) \in \mathcal{E}$, and $t \in \mathcal{J}$, there exist $\mathcal{L}_f, \mathcal{L}_g > 0$ such that

$$\begin{aligned} &\left|f(t, p(t), q(t)) - f(t, \tilde{p}(t), \tilde{q}(t))\right| \\ &\quad \leq \mathcal{L}_f \left|(p - \tilde{p}, q - \tilde{q})\right|, \\ &\left|g(t, p(t), q(t)) - g(t, \tilde{p}(t), \tilde{q}(t))\right| \\ &\quad \leq \mathcal{L}_g \left|(p - \tilde{p}, q - \tilde{q})\right|. \end{aligned} \tag{39}$$

($\mathbf{H}_2$) $I_i, \tilde{I}_i, I_j, \tilde{I}_j : \mathcal{R} \longrightarrow \mathcal{R}$ are continuous and there exist $\mathcal{L}_I, \mathcal{L}_{\tilde{I}}, \mathcal{L}'_I, \mathcal{L}'_{\tilde{I}} > 0$ such that for any $(p, q), (\tilde{p}, \tilde{q}) \in \mathcal{E}$

$$\begin{aligned} &\left|I_i(p(t_i)) - I_i(\tilde{p}(t_i))\right| \leq \mathcal{L}_I \left|p - \tilde{p}\right|, \\ &\left|\tilde{I}_i(p(t_i)) - \tilde{I}_i(\tilde{p}(t_i))\right| \leq \mathcal{L}_{\tilde{I}} \left|p - \tilde{p}\right|, \\ &\qquad k = 1, 2, \ldots, m, \\ &\left|I_j(q(t_j)) - I_j(\tilde{q}(t_j))\right| \leq \mathcal{L}'_I \left|q - \tilde{q}\right|, \\ &\left|\tilde{I}_j(q(t_j)) - \tilde{I}_j(\tilde{q}(t_j))\right| \leq \mathcal{L}'_{\tilde{I}} \left|q - \tilde{q}\right|, \\ &\qquad k = 1, 2, \ldots, n. \end{aligned} \tag{40}$$

($\mathbf{H}_3$) $\phi, \varphi : \mathcal{R} \longrightarrow \mathcal{R}$ are continuous, and there exist $\mathcal{L}_\phi, \mathcal{L}_\varphi, \mathcal{L}'_\phi, \mathcal{L}'_\varphi, \mathcal{M}_\phi, \mathcal{M}_\varphi, \mathcal{M}'_\phi, \mathcal{M}'_\varphi > 0$, for any $(p, q), (\tilde{p}, \tilde{q}) \in \mathcal{E}$ such that

$$\begin{aligned} &\left|\phi(p) - \phi(\tilde{p})\right| \leq \mathcal{L}_\phi \left|p - \tilde{p}\right|, \\ &\left|\varphi(p) - \varphi(\tilde{p})\right| \leq \mathcal{L}_\varphi \left|p - \tilde{p}\right|, \\ &\left|\phi(q) - \phi(\tilde{q})\right| \leq \mathcal{L}'_\phi \left|q - \tilde{q}\right|, \\ &\left|\varphi(q) - \varphi(\tilde{q})\right| \leq \mathcal{L}'_\varphi \left|q - \tilde{q}\right|, \\ &\left|\phi(p)\right| \leq \mathcal{M}_\phi, \\ &\left|\varphi(p)\right| \leq \mathcal{M}_\varphi, \\ &\left|\phi(q)\right| \leq \mathcal{M}'_\phi, \\ &\left|\varphi(q)\right| \leq \mathcal{M}'_\varphi. \end{aligned} \tag{41}$$

($\mathbf{H}_4$) $f, g : \mathcal{J} \times \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}^+$ are continuous; for all $(p, q) \in \mathcal{E}$ and $t \in \mathcal{J}$, there exist $\mathcal{M}_f, \mathcal{M}_g > 0$ such that

$$\begin{aligned} &\left|f(t, p(t), q(t))\right| \leq \mathcal{M}_f \left\{|p| + |q|\right\}, \\ &\left|g(t, p(t), q(t))\right| \leq \mathcal{M}_g \left\{|p| + |q|\right\}. \end{aligned} \tag{42}$$

($\mathbf{H}_5$) $I_i, \tilde{I}_i, I_j, \tilde{I}_j : \mathcal{R} \longrightarrow \mathcal{R}$ are continuous and there exist $\mathcal{N}_I, \mathcal{M}_I, \mathcal{N}_{\tilde{I}}, \mathcal{M}_{\tilde{I}}, \mathcal{N}'_I, \mathcal{M}'_I, \mathcal{N}'_{\tilde{I}}, \mathcal{M}'_{\tilde{I}} > 0$ such that for any $(p, q) \in \mathcal{E}$,

$$\begin{aligned} &\left|I_i(p(t_i))\right| \leq \mathcal{N}_I |p| + \mathcal{M}_I, \\ &\left|\tilde{I}_i(p(t_i))\right| \leq \mathcal{N}_{\tilde{I}} |p| + \mathcal{M}_{\tilde{I}}, \\ &\qquad k = 1, 2, \ldots, m, \\ &\left|I_j(q(t_j))\right| \leq \mathcal{N}'_I |q| + \mathcal{M}'_I, \\ &\left|\tilde{I}_j(q(t_j))\right| \leq \mathcal{N}'_{\tilde{I}} |q| + \mathcal{M}'_{\tilde{I}}, \\ &\qquad k = 1, 2, \ldots, n. \end{aligned} \tag{43}$$

Theorem 13. *Under the hypothesis* $(\mathbf{H}_1) - (\mathbf{H}_3)$, *and*

$$\Lambda_f + \Lambda_g < \frac{1}{\mathcal{L}}, \tag{44}$$

(3) has unique solution.

Proof. Define an operator $\Phi = (\Phi_1, \Phi_2) : \mathcal{E} \longrightarrow \mathcal{E}$, i.e., $\Phi(p, q)(t) = (\Phi_1(p, q), \Phi_2(p, q))(t)$, where

$$\begin{aligned} \Phi_1(p, q)(t) &= \frac{\varphi(p)(\ln t)^{\alpha-2} {}_\alpha\tilde{\Omega}(t)}{{}_\alpha\Omega} + \phi(p) \\ &\cdot (\ln t)^{\alpha-2} {}_\alpha\mathcal{A}_1(t) + \sum_{i=1}^{k} (\ln t)^{\alpha-2} {}_\alpha\mathcal{A}_2^i(t) I_i(p(t_i)) \\ &+ \sum_{i=1}^{k} (\ln t)^{\alpha-2} {}_\alpha\mathcal{A}_3^i(t) \tilde{I}_i(p(t_i)) \\ &+ \sum_{i=1}^{k} \frac{(\ln t)^{\alpha-2} {}_\alpha\mathcal{A}_2^i(t)}{\Gamma(\alpha)} \\ &\cdot \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-1} f(s, p(s), q(s)) \frac{ds}{s} \\ &- \frac{\mu {}_\alpha\tilde{\Omega}(t)(\ln t)^{\alpha-2}}{{}_\alpha\Omega\Gamma(\alpha)} \\ &\cdot \int_{t_k}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-1} f(s, p(s), q(s)) \frac{ds}{s} \\ &+ \sum_{i=1}^{k} \frac{(\ln t)^{\alpha-2} {}_\alpha\mathcal{A}_3^i(t)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\alpha-2} \\ &\times f(s, p(s), q(s)) \frac{ds}{s} - \frac{\nu {}_\alpha\tilde{\Omega}(t)(\ln t)^{\alpha-2}}{{}_\alpha\Omega\Gamma(\alpha-1)} \\ &\cdot \int_{t_k}^{T} \left(\ln \frac{T}{s}\right)^{\alpha-2} f(s, p(s), q(s)) \frac{ds}{s} + \frac{1}{\Gamma(\alpha)} \end{aligned}$$

$$\cdot \int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} f\left(s,p(s),q(s)\right)\frac{ds}{s},$$

$$\Phi_2(p,q)(t) = \frac{\varphi(q)(\ln t)^{\beta-2}\ {}_{\beta}\widetilde{\Omega}(t)}{{}_{\beta}\Omega} + \phi(q) \cdot (\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_1(t) + \sum_{j=1}^{k}(\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_2^{j}(t) I_j\left(q\left(t_j\right)\right) + \sum_{j=1}^{k}(\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_3^{j}(t)\widetilde{I}_j\left(q\left(t_j\right)\right) + \sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_2^{j}(t)}{\Gamma(\beta)} \cdot \int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-1} g\left(s,p(s),q(s)\right)\frac{ds}{s} - \frac{\mu\ {}_{\beta}\widetilde{\Omega}(t)(\ln t)^{\beta-2}}{{}_{\beta}\Omega\Gamma(\beta)} \cdot \int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-1} g\left(s,p(s),q(s)\right)\frac{ds}{s} + \sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\ {}_{\beta}\mathscr{A}_3^{j}(t)}{\Gamma(\beta-1)}\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-2} \times g\left(s,p(s),q(s)\right)\frac{ds}{s} - \frac{\nu\ {}_{\beta}\widetilde{\Omega}(t)(\ln t)^{\beta-2}}{{}_{\beta}\Omega\Gamma(\beta-1)} \cdot \int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-2} g\left(s,p(s),q(s)\right)\frac{ds}{s} + \frac{1}{\Gamma(\beta)} \cdot \int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1} g\left(s,p(s),q(s)\right)\frac{ds}{s}. \tag{45}$$

In view of Theorem 13, we have

$$\begin{aligned}\left|\left(\Phi_1(p,q)-\Phi_1(\widetilde{p},\widetilde{q})\right)(\ln t)^{2-\alpha}\right| \le{}& \mathscr{L}\left[\frac{\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|} + \left|{}_{\alpha}\mathscr{A}_1(t)\right| + \sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_2^{i}(t)\right| + \sum_{i=1}^{k}\left|\mathscr{A}_3^{i}(t)\right|\right]\left|p-\widetilde{p}\right| \\ &+ \mathscr{L}\left[\frac{\sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_2^{i}(t)\right|\left(\ln\left(t_i/t_{i-1}\right)\right)^{\alpha}\left|{}_{\alpha}\Omega\right| + (\ln t)^{2-\alpha}\left(\ln\left(t/t_k\right)\right)^{\alpha}\left|{}_{\alpha}\Omega\right| - |\mu|\left(\ln\left(T/t_k\right)\right)^{\alpha}\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)}\right. \\ &\left.+ \frac{\sum_{i=1}^{k}\left|\mathscr{A}_3^{i}(t)\right|\left(\ln\left(t_i/t_{i-1}\right)\right)^{\alpha-1}\left|{}_{\alpha}\Omega\right| - |\nu|\left(\ln\left(T/t_k\right)\right)^{\alpha-1}\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\right]\left|\left(p-\widetilde{p},q-\widetilde{q}\right)\right|.\end{aligned} \tag{46}$$

Taking sup, we achieve

$$\begin{aligned}\left\|\Phi_1(p,q)-\Phi_1(\widetilde{p},\widetilde{q})\right\|_{\mathscr{E}_1} \le{}& \mathscr{L}\left[\frac{\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|} + \left\|{}_{\alpha}\mathscr{A}_1\right\| + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\|\right. \\ &+ \frac{m\left\|{}_{\alpha}\mathscr{A}_2\right\|\left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha}|\Omega| + (\ln t)^{2-\alpha}\left(\ln\left(t/t_m\right)\right)^{\alpha}\left|{}_{\alpha}\Omega\right| - |\mu|\left(\ln\left(T/t_m\right)\right)^{\alpha}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)} \\ &\left.+ \frac{m\left\|{}_{\alpha}\mathscr{A}_3\right\|\left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha-1}\left|{}_{\alpha}\Omega\right| - |\nu|\left(\ln\left(T/t_m\right)\right)^{\alpha-1}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\right]\left\|p-\widetilde{p}\right\| \\ &+ \mathscr{L}\left[\frac{m\left\|{}_{\alpha}\mathscr{A}_2\right\|\left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha}\left|{}_{\alpha}\Omega\right| + (\ln t)^{2-\alpha}\left(\ln\left(t/t_m\right)\right)^{\alpha}\left|{}_{\alpha}\Omega\right| - |\mu|\left(\ln\left(T/t_m\right)\right)^{\alpha}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)}\right. \\ &\left.+ \frac{\left\|{}_{\alpha}\mathscr{A}_3\right\|\left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha-1}\left|{}_{\alpha}\Omega\right| - |\nu|\left(\ln\left(T/t_m\right)\right)^{\alpha-1}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\right]\left\|q-\widetilde{q}\right\| \le \mathscr{L}\Lambda_f\left\|(p,q)-(\widetilde{p},\widetilde{q})\right\|,\end{aligned} \tag{47}$$

where $\Lambda_f = \max\{\Lambda_1, \Lambda_2\}$, such that

$$\begin{aligned}
\Lambda_1 &= \frac{\|{}_\alpha\widetilde{\Omega}\|}{|{}_\alpha\Omega|} + \|{}_\alpha\mathscr{A}_1\| + \frac{m\|{}_\alpha\mathscr{A}_3\|(\ln(t_m/t_{m-1}))^{\alpha-1}|{}_\alpha\Omega| - |\nu|(\ln(T/t_m))^{\alpha-1}\|{}_\alpha\widetilde{\Omega}\|}{|{}_\alpha\Omega|\Gamma(\alpha)} \\
&\quad + \frac{m\|{}_\alpha\mathscr{A}_2\|(\ln(t_m/t_{m-1}))^{\alpha}|\Omega| + (\ln t)^{2-\alpha}(\ln(t/t_m))^{\alpha}|{}_\alpha\Omega| - |\mu|(\ln(T/t_m))^{\alpha}\|{}_\alpha\widetilde{\Omega}\|}{|{}_\alpha\Omega|\Gamma(\alpha+1)} + m\|{}_\alpha\mathscr{A}_2\| + m\|{}_\alpha\mathscr{A}_3\| \\
\Lambda_2 &= \frac{m\|{}_\alpha\mathscr{A}_2\|(\ln(t_m/t_{m-1}))^{\alpha}|{}_\alpha\Omega| + (\ln t)^{2-\alpha}(\ln(t/t_m))^{\alpha}|{}_\alpha\Omega| - |\mu|(\ln(T/t_m))^{\alpha}\|{}_\alpha\widetilde{\Omega}\|}{|{}_\alpha\Omega|\Gamma(\alpha+1)} \\
&\quad + \frac{\|{}_\alpha\mathscr{A}_3\|(\ln(t_m/t_{m-1}))^{\alpha-1}|{}_\alpha\Omega| - |\nu|(\ln(T/t_m))^{\alpha-1}\|{}_\alpha\widetilde{\Omega}\|}{|{}_\alpha\Omega|\Gamma(\alpha)}.
\end{aligned} \tag{48}$$

Similarly

$$\|\Phi_2(p,q) - \Phi_2(\tilde{p},\tilde{q})\|_{\mathscr{E}_2} \le \mathscr{L}\Lambda_g\|(p,q)-(\tilde{p},\tilde{q})\|, \tag{49}$$

where $\Lambda_g = \max\{\Lambda_3, \Lambda_4\}$,

$$\begin{aligned}
\Lambda_3 &= \frac{\|{}_\beta\widetilde{\Omega}\|}{|{}_\beta\Omega|} + \|{}_\beta\mathscr{A}_1\| + \frac{n\|{}_\beta\mathscr{A}_3\|(\ln(t_n/t_{n-1}))^{\beta-1}|{}_\beta\Omega| - |\nu|(\ln(T/t_n))^{\beta-1}\|{}_\beta\widetilde{\Omega}\|}{|{}_\beta\Omega|\Gamma(\beta)} \\
&\quad + \frac{n\|{}_\beta\mathscr{A}_2\|(\ln(t_n/t_{n-1}))^{\beta}|{}_\beta\Omega| + (\ln t)^{2-\beta}(\ln(t/t_n))^{\beta}|{}_\beta\Omega| - |\mu|(\ln(T/t_n))^{\beta}\|{}_\beta\widetilde{\Omega}\|}{|{}_\beta\Omega|\Gamma(\beta+1)} + n\|{}_\beta\mathscr{A}_2\| + n\|{}_\beta\mathscr{A}_3\| \\
\Lambda_4 &= \frac{n\|{}_\beta\mathscr{A}_2\|(\ln(t_n/t_{n-1}))^{\beta}|{}_\beta\Omega| + (\ln t)^{2-\beta}(\ln(t/t_n))^{\beta}|{}_\beta\Omega| - |\mu|(\ln(T/t_n))^{\beta}\|{}_\beta\widetilde{\Omega}\|}{|{}_\beta\Omega|\Gamma(\beta+1)} \\
&\quad + \frac{\|{}_\beta\mathscr{A}_3\|(\ln(t_n/t_{n-1}))^{\beta-1}|{}_\beta\Omega| - |\nu|(\ln(T/t_n))^{\beta-1}\|{}_\beta\widetilde{\Omega}\|}{|{}_\beta\Omega|\Gamma(\beta)}.
\end{aligned} \tag{50}$$

From (47) and (49), we have

$$\begin{aligned}
&\|\Phi(p,q) - \Phi(\tilde{p},\tilde{q})\|_{\mathscr{E}} \\
&\quad \le \mathscr{L}(\Lambda_f + \Lambda_g)\|(p,q)-(\tilde{p},\tilde{q})\|,
\end{aligned} \tag{51}$$

where $\mathscr{L} = \max\{\mathscr{L}_\phi, \mathscr{L}'_\phi, \mathscr{L}_\varphi, \mathscr{L}'_\varphi, \mathscr{L}_I, \mathscr{L}'_I, \mathscr{L}'_{\overline{I}}, \mathscr{L}_{\overline{I}}\}$, which implies that the operator Φ is contraction due to (44). Therefore, (3) has a unique solution. □

Here we use the Krasnoselskii's fixed point theorem to show that the operator $\mathbb{F} + \mathbb{G}$ has at least one fixed point. Therefore, we choose a closed ball

$$\begin{aligned}
\mathscr{E}_r = \Big\{&(p,q) \in \mathscr{E},\ \|(p,q)\| \le r,\ \|p\| \le \frac{r}{2} \text{ and } \|q\| \\
&\le \frac{r}{2}\Big\} \subset \mathscr{E},
\end{aligned} \tag{52}$$

where

$$r \ge \frac{\mathscr{M}\left(\|{}_\alpha\widetilde{\Omega}\|/|{}_\alpha\Omega| + \|{}_\beta\widetilde{\Omega}\|/|{}_\beta\Omega| + \|{}_\alpha\mathscr{A}_1\| + \|{}_\beta\mathscr{A}_1\| + m\|{}_\alpha\mathscr{A}_2\| + n\|{}_\beta\mathscr{A}_2\| + m\|{}_\alpha\mathscr{A}_3\| + n\|{}_\beta\mathscr{A}_3\|\right)}{1 - \mathscr{N}\left(m\|{}_\alpha\mathscr{A}_2\| + n\|{}_\beta\mathscr{A}_2\| + m\|{}_\alpha\mathscr{A}_3\| + n\|{}_\beta\mathscr{A}_3\|\right) - \mathscr{M}(C_\alpha + C_\beta)}. \tag{53}$$

Theorem 14. *If the assumptions* $(\mathbf{H}_1)-(\mathbf{H}_5)$ *are true and* ${}_{\alpha}\mathscr{A}_l=\sup_{1\le i,j\le k}{}_{\alpha}\mathscr{A}_l^{i,j}$, *then (3) has at least one solution.*

Proof. For any $(p,q)\in\mathscr{E}_r$, we have

$$\begin{aligned}\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\|_{\mathscr{E}} &\le \|\mathbb{F}_1(p)\|_{\mathscr{E}_1}+\|\mathbb{F}_2(q)\|_{\mathscr{E}_2}+\|\mathbb{G}_1(p,q)\|_{\mathscr{E}_1}\\ &\quad+\|\mathbb{G}_2(p,q)\|_{\mathscr{E}_2}.\end{aligned} \tag{54}$$

From (37a) and (37b), we get

$$\begin{aligned}\left|\mathbb{F}_1 p(t)(\ln t)^{2-\alpha}\right| &\le \frac{|\varphi(p)|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|}\\ &\quad+|\phi(p)|\left|{}_{\alpha}\mathscr{A}_1(t)\right|\\ &\quad+\sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|\left|I_i(p(t_i))\right|\\ &\quad+\sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|\left|\widetilde{I}_i(p(t_i))\right|, \quad k=1,2,\ldots,m,\\ &\le \frac{\mathscr{M}_{\varphi}\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|}+\mathscr{M}_{\phi}\left|{}_{\alpha}\mathscr{A}_1(t)\right|\\ &\quad+m\left(\mathscr{M}_I+\mathscr{N}_I|p|\right)\left|{}_{\alpha}\mathscr{A}_2^m(t)\right|\\ &\quad+m\left(\mathscr{M}_{\widetilde{I}}+\mathscr{N}_{\widetilde{I}}|p|\right)\left|{}_{\alpha}\mathscr{A}_3^m(t)\right|.\end{aligned} \tag{55}$$

Taking $\sup_{t\in\mathscr{J}}$, we get

$$\begin{aligned}\|\mathbb{F}_1(p)\|_{\mathscr{E}_1} &\le \frac{\mathscr{M}_{\varphi}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|}+\mathscr{M}_{\phi}\left\|{}_{\alpha}\mathscr{A}_1\right\|\\ &\quad+m\left(\mathscr{M}_I+\mathscr{N}_I\|p\|\right)\left\|{}_{\alpha}\mathscr{A}_2\right\|\\ &\quad+m\left(\mathscr{M}_{\widetilde{I}}+\mathscr{N}_{\widetilde{I}}\|p\|\right)\left\|{}_{\alpha}\mathscr{A}_3\right\|.\end{aligned} \tag{56}$$

Similarly, we can obtain

$$\begin{aligned}\|\mathbb{F}_2(q)\|_{\mathscr{E}_2} &\le \frac{\mathscr{M}'_{\varphi}\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|}+\mathscr{M}'_{\phi}\left\|{}_{\beta}\mathscr{A}_1\right\|\\ &\quad+n\left(\mathscr{M}'_I+\mathscr{N}'_I\|q\|\right)\left\|{}_{\beta}\mathscr{A}_2\right\|\\ &\quad+n\left(\mathscr{M}'_{\widetilde{I}}+\mathscr{N}'_{\widetilde{I}}\|q\|\right)\left\|{}_{\beta}\mathscr{A}_3\right\|.\end{aligned} \tag{57}$$

Also, we have

$$\begin{aligned}&\left|\mathbb{G}_1(p,q)(t)(\ln t)^{2-\alpha}\right|\\ &\le \sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}|f(s,p(s),q(s))|\frac{ds}{s}\\ &\quad+\sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\\ &\quad\times|f(s,p(s),q(s))|\frac{ds}{s}\\ &\quad-\frac{|\mu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}|f(s,p(s),q(s))|\frac{ds}{s}\\ &\quad-\frac{|\nu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha-1)}\\ &\quad\times\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2}|f(s,p(s),q(s))|\frac{ds}{s}\\ &\quad+\frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}|f(s,p(s),q(s))|\frac{ds}{s}\\ &\le \sum_{i=1}^{k}\frac{\mathscr{M}_f\left(\ln(t_i/t_{i-1})\right)^{\alpha}\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|}{\Gamma(\alpha+1)}|(p,q)|\\ &\quad+\sum_{i=1}^{k}\frac{\mathscr{M}_f\left(\ln(t_i/t_{i-1})\right)^{\alpha}\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|}{\Gamma(\alpha)}|(p,q)|\\ &\quad-\frac{|\mu|\mathscr{M}_f\left(\ln(T/t_k)\right)^{\alpha}\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)}|(p,q)|\\ &\quad-\frac{|\nu|\mathscr{M}_f\left(\ln(T/t_k)\right)^{\alpha-1}\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}|(p,q)|\\ &\quad+\frac{\mathscr{M}_f(\ln t)^{2-\alpha}\left(\ln(t/t_k)\right)^{\alpha}}{\Gamma(\alpha+1)}|(p,q)|, \quad k=1,2,\ldots,m.\end{aligned} \tag{58}$$

Taking $\sup_{t\in\mathscr{J}}$, we obtain

$$\begin{aligned}\|\mathbb{G}_1(p,q)\|_{\mathscr{E}_1} &\le \mathscr{M}_f\left[\frac{(\ln t)^{2-\alpha}\left(\ln(t/t_m)\right)^{\alpha}\left|{}_{\alpha}\Omega\right|+m\left(\ln(t_m/t_{m-1})\right)^{\alpha}\left\|{}_{\alpha}\mathscr{A}_2\right\|\left|{}_{\alpha}\Omega\right|-|\mu|\left(\ln(T/t_m)\right)^{\alpha}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)}\right.\\ &\quad\left.+\frac{m\left(\ln(t_m/t_{m-1})\right)^{\alpha}\left\|{}_{\alpha}\mathscr{A}_3\right\|\left|{}_{\alpha}\Omega\right|-|\nu|\left(\ln(T/t_m)\right)^{\alpha-1}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\right]\|(p,q)\|\le\mathscr{M}_f C_{\alpha}\|(p,q)\|,\end{aligned} \tag{59}$$

where

$$C_{\alpha} = \frac{(\ln t)^{2-\alpha}(\ln(t/t_m))^{\alpha}\left|{}_{\alpha}\Omega\right| + m(\ln(t_m/t_{m-1}))^{\alpha}\left\|{}_{\alpha}\mathscr{A}_2\right\|\left|{}_{\alpha}\Omega\right| - |\mu|(\ln(T/t_m))^{\alpha}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)} + \frac{m(\ln(t_m/t_{m-1}))^{\alpha}\left\|{}_{\alpha}\mathscr{A}_3\right\|\left|{}_{\alpha}\Omega\right| - |\nu|(\ln(T/t_m))^{\alpha-1}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}. \quad (60)$$

Also

$$\left\|\mathbb{G}_2(p,q)(t)\right\|_{\mathscr{E}_2} \le \mathscr{M}_g C_{\beta}\left\|(p,q)\right\|, \quad (61)$$

where

$$C_{\beta} = \frac{(\ln t)^{2-\beta}(\ln(t/t_n))^{\beta}\left|{}_{\beta}\Omega\right| + n(\ln(t_n/t_{n-1}))^{\beta}\left\|{}_{\beta}\mathscr{A}_2\right\|\left|{}_{\beta}\Omega\right| - |\mu|(\ln(T/t_n))^{\beta}\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|\Gamma(\beta+1)} + \frac{m(\ln(t_n/t_{n-1}))^{\beta}\left\|{}_{\beta}\mathscr{A}_3\right\|\left|{}_{\beta}\Omega\right| - |\nu|(\ln(T/t_n))^{\beta-1}\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|\Gamma(\beta)}. \quad (62)$$

Substituting all inequalities from (59) to (61) in (54), we get

$$\left\|\mathbb{F}(p,q)+\mathbb{G}(p,q)\right\|_{\mathscr{E}} \le \mathscr{M}\left(\frac{\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|} + \frac{\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|} + \left\|{}_{\alpha}\mathscr{A}_1\right\| + \left\|{}_{\beta}\mathscr{A}_1\right\| + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|\right) + \mathscr{N}\left(m\left\|{}_{\alpha}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|\right)r + \mathscr{M}\left(C_{\alpha} + C_{\beta}\right)r \le r, \quad (63)$$

where $\mathscr{M} = \max\{\mathscr{M}_{\phi}, \mathscr{M}'_{\phi}, \mathscr{M}_{\varphi}, \mathscr{M}'_{\varphi}, \mathscr{M}_I, \mathscr{M}'_I, \mathscr{M}_{\bar{I}}, \mathscr{M}'_{\bar{I}}, \mathscr{M}_f, \mathscr{M}_g\}$ and $\mathscr{N} = \max\{\mathscr{N}_I, \mathscr{N}'_I, \mathscr{N}_{\bar{I}}, \mathscr{N}'_{\bar{I}}\}$. Hence, $\mathbb{F}(p,q) + \mathbb{G}(p,q) \in \mathscr{E}_r$.

Next, for any $t \in \mathscr{J}$, $(p,q), (\tilde{p},\tilde{q}) \in \mathscr{E}$

$$\left\|\mathbb{F}(p,q) - \mathbb{F}(\tilde{p},\tilde{q})\right\|_{\mathscr{E}_1} \le \left\|\mathbb{F}_1(p) - \mathbb{F}_1(\tilde{p})\right\|_{\mathscr{E}_1} + \left\|\mathbb{F}_2(q) - \mathbb{F}_2(\tilde{q})\right\|_{\mathscr{E}_1} \le \left(\frac{\mathscr{L}_{\phi}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|} + \mathscr{L}_{\varphi}\left\|{}_{\alpha}\mathscr{A}_1\right\| + m\mathscr{L}_I\left\|{}_{\alpha}\mathscr{A}_2\right\| + m\mathscr{L}_{\bar{I}}\left\|{}_{\alpha}\mathscr{A}_3\right\|\right)\left\|p-\tilde{p}\right\| + \left(\frac{\mathscr{L}'_{\phi}\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|} + \mathscr{L}'_{\varphi}\left\|{}_{\beta}\mathscr{A}_1\right\| + n\mathscr{L}'_I\left\|{}_{\beta}\mathscr{A}_2\right\| + n\mathscr{L}'_{\bar{I}}\left\|{}_{\beta}\mathscr{A}_3\right\|\right)\left\|q-\tilde{q}\right\| \le \mathscr{L}\left(\frac{\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|} + \frac{\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|} + \left\|{}_{\alpha}\mathscr{A}_1\right\| + \left\|{}_{\beta}\mathscr{A}_1\right\| + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|\right)\left\|(p-\tilde{p}, q-\tilde{q})\right\| \le \mathscr{L}\xi^*\left\|(p-\tilde{p}, q-\tilde{q})\right\|, \quad (64)$$

where

$$\xi^* = \frac{\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|} + \frac{\left\|{}_{\beta}\widetilde{\Omega}\right\|}{\left|{}_{\beta}\Omega\right|} + \left\|{}_{\alpha}\mathscr{A}_1\right\| + \left\|{}_{\beta}\mathscr{A}_1\right\| + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|. \quad (65)$$

Therefore, $\mathbb{F}$ is contraction mapping.

Now, we are proving the continuity and compactness of $\mathbb{G}$ and, for this reason, construct a sequence $T_s = (p_s, q_s)$ in $\mathscr{E}_r$ such that $(p_s,q_s) \longrightarrow (p,q)$ for $s \longrightarrow \infty$ in $\mathscr{E}_r$. Thus, we have

$$\left\|\mathbb{G}(p_s,q_s) - \mathbb{G}(p,q)\right\|_{\mathscr{E}} \le \left\|\mathbb{G}_1(p_s,q_s) - \mathbb{G}_1(p,q)\right\|_{\mathscr{E}_1} + \left\|\mathbb{G}_2(p_s,q_s) - \mathbb{G}_2(p,q)\right\|_{\mathscr{E}_2} \le \mathscr{L}_f\left(\frac{|\mu|(\ln(T/t_m))^{\alpha}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha+1)} + \frac{|\nu|(\ln(T/t_m))^{\alpha-1}\left\|{}_{\alpha}\widetilde{\Omega}\right\|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\right.$$

$$
\begin{aligned}
&- \frac{m\left\| {}_{\alpha}\mathscr{A}_2 \right\| \left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha}}{\Gamma(\alpha+1)} \\
&- \frac{m\left\| {}_{\alpha}\mathscr{A}_3 \right\| \left(\ln\left(t_m/t_{m-1}\right)\right)^{\alpha-1}}{\Gamma(\alpha)} \\
&\left. + \frac{(\ln t)^{2-\alpha}\left(\ln\left(t/t_m\right)\right)^{\alpha}}{\Gamma(\alpha+1)}\right) \left\|\left(p_s - p, q_s - q\right)\right\| \\
&+ \mathscr{L}_g \left(\frac{|\mu| \left(\ln\left(T/t_n\right)\right)^{\beta} \left\| {}_{\beta}\widetilde{\Omega} \right\|}{\left| {}_{\beta}\Omega \right| \Gamma(\beta+1)} \right. \\
&+ \frac{|\nu| \left(\ln\left(T/t_n\right)\right)^{\beta-1} \left\| {}_{\beta}\widetilde{\Omega} \right\|}{\left| {}_{\beta}\Omega \right| \Gamma(\beta)} \\
&- \frac{n\left\| {}_{\beta}\mathscr{A}_2 \right\| \left(\ln\left(t_n/t_{n-1}\right)\right)^{\beta}}{\Gamma(\beta+1)} \\
&- \frac{n\left\| {}_{\beta}\mathscr{A}_3 \right\| \left(\ln\left(t_n/t_{n-1}\right)\right)^{\beta-1}}{\Gamma(\beta)} \\
&\left. + \frac{(\ln t)^{2-\beta}\left(\ln\left(t/t_n\right)\right)^{\beta}}{\Gamma(\beta+1)}\right) \left\|\left(p_s - p, q_s - q\right)\right\|.
\end{aligned} \tag{66}
$$

This implies $\|\mathbb{G}(p_n, q_n) - \mathbb{G}(p, q)\|_{\mathscr{E}} \longrightarrow 0$ as $n \longrightarrow \infty$; therefore $\mathbb{G}$ is continuous.

Next, we show that $\mathbb{G}$ is uniformly bounded on $\mathscr{E}_r$. From (59) and (61), we have

$$
\begin{aligned}
\left\|\mathbb{G}(p,q)(t)\right\|_{\mathscr{E}} &\leq \left\|\mathbb{G}_1(p,q)(t)\right\|_{\mathscr{E}_1} + \left\|\mathbb{G}_2(p,q)(t)\right\|_{\mathscr{E}_2} \\
&\leq \left(\mathscr{M}_f C^{*} + \mathscr{M}_g C^{**}\right) r.
\end{aligned} \tag{67}
$$

Thus, $\mathbb{G}$ is uniformly bounded on $\mathscr{E}_r$.

For equi-continuity, take $\tau_1, \tau_2 \in \mathscr{J}$ with $\tau_1 < \tau_2$ and for any $(p, q) \in \mathscr{E}_r \subset \mathscr{E}$, where $\mathscr{E}_r$ is clearly bounded, we have

$$
\begin{aligned}
&\left|\left(\mathbb{G}_1(p,q)(\tau_1) - \mathbb{G}_1(p,q)(\tau_2)\right)(\ln t)^{2-\alpha}\right| \\
&\leq \mathscr{M}_f \left[\left|\widetilde{\Omega}(\tau_1) - \widetilde{\Omega}(\tau_2)\right| \right. \\
&\cdot \left(\frac{|\mu| \left(\ln\left(T/t_m\right)\right)^{\alpha}}{|\Omega| \Gamma(\alpha+1)} + \frac{|\nu| \left(\ln\left(T/t_m\right)\right)^{\alpha-1}}{|\Omega| \Gamma(\alpha)} \right) \\
&- \frac{\sum_{k=1}^{m} \left|\mathscr{A}_2^k(\tau_1) - \mathscr{A}_2^k(\tau_2)\right| \left(\ln\left(t_k/t_{k-1}\right)\right)^{\alpha}}{\Gamma(\alpha+1)} \\
&\left. - \frac{\sum_{k=1}^{m} \left|\mathscr{A}_3^k(\tau_1) - \mathscr{A}_3^k(\tau_2)\right| \left(\ln\left(t_k/t_{k-1}\right)\right)^{\alpha-1}}{\Gamma(\alpha)} \right] \\
&\cdot \left|(p,q)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \left| \int_{t_m}^{\tau_1} \left(\ln\frac{\tau_1}{s}\right)^{\alpha-1} f\left(s, p(s), q(s)\right) \frac{ds}{s} \right. \\
&\left. - \int_{t_m}^{\tau_2} \left(\ln\frac{\tau_2}{s}\right)^{\alpha-1} f\left(s, p(s), q(s)\right) \frac{ds}{s} \right|.
\end{aligned} \tag{68}
$$

This implies that $\|\mathbb{G}_1(p,q)(\tau_1) - \mathbb{G}_1(p,q)(\tau_2)\|_{\mathscr{E}_1} \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. In the same way, we have $\|\mathbb{G}_2(p,q)(\tau_1) - \mathbb{G}_2(p,q)(\tau_2)\|_{\mathscr{E}_2} \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Hence $\|\mathbb{G}(p,q)(\tau_1) - \mathbb{G}(p,q)(\tau_2)\|_{\mathscr{E}} \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Therefore, $\mathbb{G}$ is relatively compact on $\mathscr{E}_r$. By *Arzelä-Ascoli* theorem, $\mathbb{G}$ is compact and hence completely continuous operator, so (3) has at least one solution. □

4. Ulam Stability Analysis

In this portion, we analyze different kinds of stability like Hyers–Ulam, generalized Hyers–Ulam, Hyers–Ulam–Rassias, and generalized Hyers–Ulam–Rassias stability of the proposed system.

Theorem 15. *If assumptions* $(\mathbf{H}_1) - (\mathbf{H}_3)$ *and inequality (44) are satisfied and*

$$
\digamma = 1 - \frac{\mathscr{L}^2 \Lambda_2 \Lambda_4}{\left((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1\right)\left((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3\right)} > 0, \tag{69}
$$

then the unique solution of the coupled system (3) is Hyers–Ulam stable and consequently generalized Hyers–Ulam stable.

Proof. Let $(p, q) \in \mathscr{E}$ be an approximate solution of inequality (14) and let $(\widehat{p}, \widehat{q}) \in \mathscr{E}$ be the unique solution of the coupled system given by

$$
\begin{aligned}
&{}_H\mathfrak{D}^{\alpha}\widehat{p}(t) = f\left(t, \widehat{p}(t), \widehat{q}(t)\right), \\
&\qquad t \in \mathscr{J},\ t \neq t_i,\ i = 1, 2, \ldots, m, \\
&{}_H\mathfrak{D}^{\beta}\widehat{q}(t) = g\left(t, \widehat{p}(t), \widehat{q}(t)\right), \\
&\qquad t \in \mathscr{J},\ t \neq t_j,\ j = 1, 2, \ldots, n, \\
&\Delta\widehat{p}(t_i) = I_i\left(\widehat{p}(t_i)\right), \\
&\Delta\widehat{p}'(t_i) = \widetilde{I}_i\left(\widehat{p}(t_i)\right), \\
&\qquad i = 1, 2, \ldots, m, \\
&\Delta\widehat{q}(t_j) = I_j\left(\widehat{q}(t_j)\right), \\
&\Delta\widehat{q}'(t_j) = \widetilde{I}_j\left(\widehat{q}(t_j)\right), \\
&\qquad j = 1, 2, \ldots, n,
\end{aligned}
$$

$$\begin{aligned}\mu \ln 2\widehat{p}(2)+\nu \ln 2\widehat{p}'(2)&=\phi(\widehat{p}),\\ \mu\widehat{p}(T)+\nu\widehat{p}'(T)&=\varphi(\widehat{p}),\\ \mu \ln 2\widehat{q}(2)+\nu \ln 2\widehat{q}'(2)&=\phi(\widehat{q}),\\ \mu\widehat{q}(T)+\nu\widehat{q}'(T)&=\varphi(\widehat{q}).\end{aligned} \tag{70}$$

By Remark 10 we have

$$\begin{aligned}&{}_H\mathfrak{D}^{\alpha}p(t)=f(t,p(t),q(t))+\Upsilon_f(t),\\ &\qquad t\in\mathcal{J},\ t\neq t_i,\ i=1,2,\ldots,m,\\ &\Delta p(t_i)=I_i(p(t_i))+\Upsilon_{f_i},\\ &\Delta p'(t_i)=\widetilde{I}_i(p(t_i))+\Upsilon_{f_i},\\ &\qquad i=1,2,\ldots,m,\\ &{}_H\mathfrak{D}^{\beta}q(t)=g(t,p(t),q(t))+\Upsilon_g(t),\\ &\qquad t\in\mathcal{J},\ t\neq t_j,\ j=1,2,\ldots,n,\\ &\Delta q(t_j)=I_j(q(t_j))+\Upsilon_{g_j},\\ &\Delta q'(t_j)=\widetilde{I}_j(q(t_j))+\Upsilon_{g_j},\\ &\qquad j=1,2,\ldots,n.\end{aligned} \tag{71}$$

Therefore, the solution of problem (71) is

$$\begin{aligned}p(t)={}&\frac{(\ln t)^{\alpha-2}\ {}_\alpha\widetilde{\Omega}(t)\varphi(p)}{{}_\alpha\Omega}\\ &+\sum_{i=1}^{k}(\ln t)^{\alpha-2}\ {}_\alpha\mathscr{A}_3^i(t)\left(\widetilde{I}_i(p(t_i))+\Upsilon_{f_i}\right)\\ &+\sum_{i=1}^{k}(\ln t)^{\alpha-2}\ {}_\alpha\mathscr{A}_2^i(t)\left(I_i(p(t_i))+\Upsilon_{f_i}\right)\\ &+(\ln t)^{\alpha-2}\ {}_\alpha\mathscr{A}_1(t)\phi(p)+\sum_{i=1}^{k}\frac{(\ln t)^{\alpha-2}\ {}_\alpha\mathscr{A}_2^i(t)}{\Gamma(\alpha)}\\ &\cdot\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\left(f(s,p(s),q(s))+\Upsilon_f(s)\right)\frac{ds}{s}\\ &+\sum_{i=1}^{k}\frac{(\ln t)^{\alpha-2}\ {}_\alpha\mathscr{A}_3^i(t)}{\Gamma(\alpha-1)}\\ &\cdot\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}\left(f(s,p(s),q(s))+\Upsilon_f(s)\right)\frac{ds}{s}\\ &-\frac{\nu(\ln t)^{\alpha-2}\ {}_\alpha\widetilde{\Omega}(t)}{{}_\alpha\Omega\Gamma(\alpha-1)}\\ &\cdot\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2}\left(f(s,p(s),q(s))+\Upsilon_f(s)\right)\frac{ds}{s}\\ &-\frac{\mu(\ln t)^{\alpha-2}\ {}_\alpha\widetilde{\Omega}(t)}{{}_\alpha\Omega\Gamma(\alpha)}\\ &\cdot\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\left(f(s,p(s),q(s))+\Upsilon_f(s)\right)\frac{ds}{s}\\ &+\frac{1}{\Gamma(\alpha)}\\ &\cdot\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\left(f(s,p(s),q(s))+\Upsilon_f(s)\right)\frac{ds}{s},\\ &\qquad k=1,2,\ldots,m,\\ q(t)={}&\frac{(\ln t)^{\beta-2}\ {}_\beta\widetilde{\Omega}(t)\varphi(q)}{{}_\beta\Omega}\\ &+\sum_{j=1}^{k}(\ln t)^{\beta-2}\ {}_\beta\mathscr{A}_3^j(t)\left(\widetilde{I}_j(q(t_j))+\Upsilon_{g_j}\right)\\ &+\sum_{j=1}^{k}(\ln t)^{\beta-2}\ {}_\beta\mathscr{A}_2^j(t)\left(I_j(q(t_j))+\Upsilon_{g_j}\right)\\ &+(\ln t)^{\beta-2}\ {}_\beta\mathscr{A}_1(t)\phi(q)+\sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\ {}_\beta\mathscr{A}_2^j(t)}{\Gamma(\beta)}\\ &\cdot\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-1}\left(g(s,p(s),q(s))+\Upsilon_g(s)\right)\frac{ds}{s}\\ &+\sum_{j=1}^{k}\frac{(\ln t)^{\beta-2}\ {}_\beta\mathscr{A}_3^j(t)}{\Gamma(\beta-1)}\\ &\cdot\int_{t_{j-1}}^{t_j}\left(\ln\frac{t_j}{s}\right)^{\beta-2}\left(g(s,p(s),q(s))+\Upsilon_g(s)\right)\frac{ds}{s}\\ &-\frac{\nu(\ln t)^{\beta-2}\ {}_\beta\widetilde{\Omega}(t)}{{}_\beta\Omega\Gamma(\beta-1)}\\ &\cdot\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-2}\left(g(s,p(s),q(s))+\Upsilon_g(s)\right)\frac{ds}{s}\\ &-\frac{\mu(\ln t)^{\beta-2}\ {}_\beta\widetilde{\Omega}(t)}{{}_\beta\Omega\Gamma(\beta)}\\ &\cdot\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\beta-1}\left(g(s,p(s),q(s))+\Upsilon_g(s)\right)\frac{ds}{s}\\ &+\frac{1}{\Gamma(\beta)}\\ &\cdot\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\beta-1}\left(g(s,p(s),q(s))+\Upsilon_g(s)\right)\frac{ds}{s},\\ &\qquad k=1,2,\ldots,n.\end{aligned} \tag{72}$$

We consider

$$\left|\left(p(t)-\widehat{p}(t)\right)(\ln t)^{\alpha-2}\right| \le \frac{\left|{}_{\alpha}\widetilde{\Omega}(t)\right|\left|\varphi(p)-\varphi(\widehat{p})\right|}{\left|{}_{\alpha}\Omega\right|} + \sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|\left|\widetilde{I}_i(p(t_i))-\widetilde{I}_i(\widehat{p}(t_i))\right| + \sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|\left|I_i(p(t_i))-I_i(\widehat{p}(t_i))\right| + \left|{}_{\alpha}\mathscr{A}_1(t)\right| \cdot\left|\phi(p)-\phi(\widehat{p})\right| + \sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|}{\Gamma(\alpha)} \cdot\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1} \cdot\left|f(s,p(s),q(s))-f(s,\widehat{p}(s),\widehat{q}(s))\right|\frac{ds}{s} + \sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2} \cdot\left|f(s,p(s),q(s))-f(s,\widehat{p}(s),\widehat{q}(s))\right|\frac{ds}{s} - \frac{|\nu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2} \cdot\left|f(s,p(s),q(s))-f(s,\widehat{p}(s),\widehat{q}(s))\right| - \frac{|\mu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1} \cdot\left|f(s,p(s),q(s))-f(s,\widehat{p}(s),\widehat{q}(s))\right| + \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \cdot\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1} \cdot\left|f(s,p(s),q(s))-f(s,\widehat{p}(s),\widehat{q}(s))\right|\frac{ds}{s} + \sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|\left|\Upsilon_{f_i}\right| + \sum_{i=1}^{k}\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|\left|\Upsilon_{f_i}\right| + \sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_2^i(t)\right|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-1}\left|\Upsilon_f(s)\right|\frac{ds}{s} + \sum_{i=1}^{k}\frac{\left|{}_{\alpha}\mathscr{A}_3^i(t)\right|}{\Gamma(\alpha-1)}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\alpha-2}\left|\Upsilon_f(s)\right|\frac{ds}{s} - \frac{|\nu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha-1)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-2}\left|\Upsilon_f(s)\right|\frac{ds}{s} - \frac{|\mu|\left|{}_{\alpha}\widetilde{\Omega}(t)\right|}{\left|{}_{\alpha}\Omega\right|\Gamma(\alpha)}\int_{t_k}^{T}\left(\ln\frac{T}{s}\right)^{\alpha-1}\left|\Upsilon_f(s)\right|\frac{ds}{s} + \frac{(\ln t)^{2-\alpha}}{\Gamma(\alpha)} \cdot\int_{t_k}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\left|\Upsilon_f(s)\right|\frac{ds}{s}. \quad (73)$$

As in Theorem 13, we get

$$\begin{aligned}\|p-\widehat{p}\|_{\mathscr{E}_1} \le{}& \mathscr{L}\Lambda_1(\ln t)^{2-\alpha}\|p-\widehat{p}\|_{\mathscr{E}_1} + \mathscr{L}\Lambda_2(\ln t)^{2-\alpha}\|q-\widehat{q}\|_{\mathscr{E}_1} \\ &+ \left[\Lambda_2 + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\|\right]\varrho_\alpha,\end{aligned} \quad (74)$$

$$\begin{aligned}\|q-\widehat{q}\|_{\mathscr{E}_2} \le{}& \mathscr{L}\Lambda_3(\ln t)^{2-\beta}\|p-\widehat{p}\|_{\mathscr{E}_2} + \mathscr{L}\Lambda_4(\ln t)^{2-\beta}\|q-\widehat{q}\|_{\mathscr{E}_2} \\ &+ \left[\Lambda_4 + n\left\|{}_{\beta}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|\right]\varrho_\beta.\end{aligned} \quad (75)$$

From (74) and (75) we have

$$\begin{aligned}&\|p-\widehat{p}\|_{\mathscr{E}_1} - \frac{\mathscr{L}\Lambda_2}{(\ln t)^{\alpha-2}-\mathscr{L}\Lambda_1}\|q-\widehat{q}\|_{\mathscr{E}_1} \le \frac{\Lambda_2 + m\left\|{}_{\alpha}\mathscr{A}_2\right\| + m\left\|{}_{\alpha}\mathscr{A}_3\right\|}{1-\mathscr{L}\Lambda_1(\ln t)^{2-\alpha}}\varrho_\alpha, \\ &\|q-\widehat{q}\|_{\mathscr{E}_2} - \frac{\mathscr{L}\Lambda_4}{(\ln t)^{\beta-2}-\mathscr{L}\Lambda_3}\|p-\widehat{p}\|_{\mathscr{E}_2} \le \frac{\Lambda_4 + n\left\|{}_{\beta}\mathscr{A}_2\right\| + n\left\|{}_{\beta}\mathscr{A}_3\right\|}{1-\mathscr{L}\Lambda_3(\ln t)^{2-\beta}}\varrho_\beta,\end{aligned} \quad (76)$$

respectively. Let $\mathscr{G}_\alpha = (\Lambda_2 + m\|{}_{\alpha}\mathscr{A}_2\| + m\|{}_{\alpha}\mathscr{A}_3\|)/(1-\mathscr{L}\Lambda_1(\ln t)^{2-\alpha})$ and $\mathscr{G}_\beta = (\Lambda_4 + n\|{}_{\beta}\mathscr{A}_2\| + n\|{}_{\beta}\mathscr{A}_3\|)/(1-\mathscr{L}\Lambda_3(\ln t)^{2-\beta})$. Then the last two inequalities can be written in matrix form as

$$\begin{aligned}&\begin{bmatrix} 1 & -\dfrac{\mathscr{L}\Lambda_2}{(\ln t)^{\alpha-2}-\mathscr{L}\Lambda_1} \\ -\dfrac{\mathscr{L}\Lambda_4}{(\ln t)^{\beta-2}-\mathscr{L}\Lambda_3} & 1\end{bmatrix}\begin{bmatrix}\|p-\widehat{p}\|_{\mathscr{E}_1} \\ \|q-\widehat{q}\|_{\mathscr{E}_2}\end{bmatrix} \le \begin{bmatrix}\mathscr{G}_\alpha\varrho_\alpha \\ \mathscr{G}_\beta\varrho_\beta\end{bmatrix}, \\ &\begin{bmatrix}\|p-\widehat{p}\|_{\mathscr{E}_1} \\ \|q-\widehat{q}\|_{\mathscr{E}_2}\end{bmatrix} \le \begin{bmatrix}\dfrac{1}{\digamma} & \dfrac{\mathscr{L}\Lambda_2}{\left((\ln t)^{\alpha-2}-\mathscr{L}\Lambda_1\right)\digamma} \\ \dfrac{\mathscr{L}\Lambda_4}{\left((\ln t)^{\beta-2}-\mathscr{L}\Lambda_3\right)\digamma} & \dfrac{1}{\digamma}\end{bmatrix}\begin{bmatrix}\mathscr{G}_\alpha\varrho_\alpha \\ \mathscr{G}_\beta\varrho_\beta\end{bmatrix},\end{aligned} \quad (77)$$

where

$$\digamma = 1 - \frac{\mathscr{L}^2\Lambda_2\Lambda_4}{\left((\ln t)^{\alpha-2}-\mathscr{L}\Lambda_1\right)\left((\ln t)^{\beta-2}-\mathscr{L}\Lambda_3\right)} > 0. \quad (78)$$

From system (77) we have

$$\|p-\widehat{p}\|_{\mathscr{E}_1} \leq \frac{\mathscr{G}_\alpha \varrho_\alpha}{\digamma} + \frac{\mathscr{L}\Lambda_2 \mathscr{G}_\beta \varrho_\beta}{\left((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1\right)\digamma},$$
$$\|q-\widehat{q}\|_{\mathscr{E}_2} \leq \frac{\mathscr{G}_\beta \varrho_\beta}{\digamma} + \frac{\mathscr{L}\Lambda_4 \mathscr{G}_\alpha \varrho_\alpha}{\left((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3\right)\digamma}, \quad (79)$$

which implies that

$$\|p-\widehat{p}\|_{\mathscr{E}_1} + \|q-\widehat{q}\|_{\mathscr{E}_2} \leq \frac{\mathscr{G}_\alpha \varrho_\alpha}{\digamma} + \frac{\mathscr{G}_\beta \varrho_\beta}{\digamma} + \frac{\mathscr{L}\Lambda_2 \mathscr{G}_\beta \varrho_\beta}{\left((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1\right)\digamma} + \frac{\mathscr{L}\Lambda_4 \mathscr{G}_\alpha \varrho_\alpha}{\left((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3\right)\digamma}. \quad (80)$$

If $\max\{\varrho_\alpha, \varrho_\beta\} = \varrho$ and $\mathscr{G}_\alpha/\digamma + \mathscr{G}_\beta/\digamma + \mathscr{L}\Lambda_2\mathscr{G}_\beta/((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1)\digamma + \mathscr{L}\Lambda_4\mathscr{G}_\alpha/((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3)\digamma = \mathscr{G}_{\alpha,\beta}$, then

$$\|(p,q)-(\widehat{p},\widehat{q})\|_{\mathscr{E}} \leq \mathscr{G}_{\alpha,\beta}\varrho. \quad (81)$$

This shows that system (3) is Hyers–Ulam stable. Also, if

$$\|(p,q)-(\widehat{p},\widehat{q})\|_{\mathscr{E}} \leq \mathscr{G}_{\alpha,\beta}\Phi(\varrho) \quad (82)$$

with $\Phi(0) = 0$, then the solution of system (3) is generalized Hyers–Ulam stable. □

For the upcoming result, we suppose the following:

($\mathbf{H}_6$) There exist two nondecreasing functions $\overline{w}_\alpha, \overline{w}_\beta \in \mathscr{C}(\mathscr{J}, \mathscr{R}^+)$ such that

$$_H\mathfrak{I}^\alpha \overline{w}_\alpha(t) \leq \mathscr{L}_\alpha \overline{w}_\alpha(t),$$
$$_H\mathfrak{I}^\beta \overline{w}_\beta(t) \leq \mathscr{L}_\beta \overline{w}_\beta(t), \quad (83)$$
$$\text{where } \mathscr{L}_\alpha, \mathscr{L}_\beta > 0.$$

Theorem 16. *If assumptions* ($\mathbf{H}_1$) − ($\mathbf{H}_3$) *and* ($\mathbf{H}_6$) *and inequality (44) are satisfied and*

$$\digamma = 1 - \frac{\mathscr{L}^2\Lambda_2\Lambda_4}{\left((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1\right)\left((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3\right)} > 0, \quad (84)$$

then the unique solution of the coupled system (3) is Hyers–Ulam–Rassias stable and consequently generalized Hyers–Ulam–Rassias stable.

Proof. By using Definitions 9 and 8, we can gain our result to perform the same steps as in Theorem 15. □

5. Example

To testify our results established in the previous section, we provide an adequate problem.

Example 1. Consider

$$_HD^{3/2}p(t) = \frac{t^2 + \sin(|p(t)|) + \cos(|q(t)|)}{50}, \quad t \in \mathscr{J},\ t \neq \frac{5}{2},$$
$$_HD^{3/2}q(t) = \frac{|q(t)| + \cos(|p(t)|)}{70 + t^2}, \quad t \in \mathscr{J},\ t \neq \frac{7}{3},$$
$$\ln 2p(2) - \ln 2p'(2) = \sum_{i=1}^{10} \hbar_i p(\zeta_i),$$
$$\ln 2q(2) - \ln 2q'(2) = \sum_{j=1}^{10} \hbar_j q(\eta_j),$$
$$2 < \eta_i, \zeta_i < 3,\ \hbar_i > 0$$
$$p(e) - p'(e) = \sum_{i=1}^{10} \wp_i p(\zeta_i),$$
$$q(e) - q'(e) = \sum_{j=1}^{10} \wp_j q(\eta_j),$$
$$2 < \eta_j, \zeta_j < 3\ \wp_j > 0,$$
$$\Delta p\left(\frac{5}{2}\right) = I_1\left(p\left(\frac{5}{2}\right)\right) = \frac{|p(5/2)|}{75 + |p(5/2)|},$$
$$\Delta p'\left(\frac{5}{2}\right) = \widetilde{I}_1\left(p\left(\frac{5}{2}\right) = \frac{|p(5/2)|}{25 + |p(5/2)|},\right.$$
$$\Delta q\left(\frac{7}{3}\right) = I_1\left(q\left(\frac{7}{3}\right)\right) = \frac{|q(7/3)|}{75 + |q(7/3)|},$$
$$\Delta q'\left(\frac{7}{3}\right) = \widetilde{I}_1\left(q\left(\frac{7}{3}\right) = \frac{|q(7/3)|}{25 + |q(7/3)|}.\right. \quad (85)$$

In system (85), we see that $\alpha = \beta = 3/2$ and $\phi(p) = \sum_{i=1}^{10} \hbar_i |p(\zeta_i)|$, $\varphi(p) = \sum_{i=1}^{10} \wp_i |p(\zeta_i)|$, $\phi(q) = \sum_{j=1}^{10} \hbar_j |q(\eta_j)|$, and $\varphi(q) = \sum_{j=1}^{10} \wp_j |q(\eta_j)|$, where $\sum_{i=1}^{10} \hbar_i < 1/25$ and $\sum_{i=1}^{10} \wp_i < 1/75$.

For $t \in \mathscr{J} = (2, e]$, and $(p,q), (\widetilde{p}, \widetilde{q}) \in \mathscr{E}$, we gain

$$\left|f(t, p(t), q(t)) - f(t, \widetilde{p}(t), \widetilde{q}(t))\right| \leq \frac{1}{50}\left|(p,q) - (\widetilde{p}, \widetilde{q})\right|,$$
$$\left|g(t, p(t), q(t)) - g(t, \widetilde{p}(t), \widetilde{q}(t))\right| \leq \frac{1}{70}\left|(p,q) - (\widetilde{p}, \widetilde{q})\right|. \quad (86)$$

From this we get $\mathscr{L}_f = 1/50$ and $\mathscr{L}_g = 1/75$. Also,

$$\left|I(p(t)) - I(\widetilde{p}(t))\right| \leq \frac{1}{75}\|(p-\widetilde{p})\|,$$
$$\left|\widetilde{I}(p(t)) - \widetilde{I}(\widetilde{p}(t))\right| \leq \frac{1}{25}\|(p-\widetilde{p})\|,$$

$$|\phi(p(t)) - \phi(\tilde{p}(t))| \le \frac{1}{25}\|(p - \tilde{p})\|,$$
$$|\varphi(p(t)) - \varphi(\tilde{p}(t))| \le \frac{1}{75}\|(p - \tilde{p})\|. \tag{87}$$

From this we get that $\mathscr{L}_I = \mathscr{L}'_I = 1/75$, $\mathscr{L}_{\bar{I}} = \mathscr{L}'_{\bar{I}} = 1/25$, $\mathscr{L}_\phi = 1/25$, $\mathscr{L}_\varphi = 1/75$, $\mathscr{L}'_\phi = 1/25$, $\mathscr{L}'_\varphi = 1/75$, and $m = n = 1$. Finding $\Lambda_1 = 0.51282$, $\Lambda_2 = 0.18059$, $\Lambda_3 = 0.12899$, and $\Lambda_4 = 0.46122$, it is clear that $\Lambda_f = 0.51282$ and $\Lambda_g = 0.46122$. By the help of Theorem 13, the following inequality is true

$$\Lambda_f + \Lambda_g < \frac{1}{\mathscr{L}}, \tag{88}$$

and hence (85) has a unique solution. Also,

$$\digamma = 1 - \frac{\mathscr{L}^2 \Lambda_2 \Lambda_4}{\left((\ln t)^{\alpha-2} - \mathscr{L}\Lambda_1\right)\left((\ln t)^{\beta-2} - \mathscr{L}\Lambda_3\right)} \approx 0.02280 > 0, \tag{89}$$

and hence by Theorem 15 the coupled system (85) is Hyers–Ulam stable and thus generalized Hyers–Ulam stable. Similarly, we can verify the condition of Theorems 16 and 14.

6. Conclusion

In this manuscript, we used the *Arzelä-Ascoli* theorem, Banach contraction principle, and Krasnoselskii's fixed point theorem to achieve the necessary criteria for the existence and uniqueness of the solution of considered switched coupled impulsive fractional differential systems given in (3). Similarly, under particular assumptions and conditions, we have established the Hyers–Ulam stability results of different kinds for the solution of the considered problem in (3). From the obtained results, we conclude that such a method is very powerful, effectual, and suitable for the solution of nonlinear fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," *Journal of Rheology*, vol. 27, pp. 201–210, 1983.

[2] Z. Jiao, Y. Chen, and I. Podlubny, *Distributed-Order Dynamic Systems*, Springer, New York, NY, USA, 2012.

[3] A. Zada and U. Riaz, "Kallman-Rota type inequality for discrete evolution families of bounded linear operators," *Fractional Differential Calculus*, vol. 7, no. 2, pp. 311–324, 2017.

[4] J. Sabatier, O. P. Agrawal, and J. A. Machado, *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[5] G. Adomian and G. E. Adomian, "Cellular systems and aging models," *Computers & Mathematics with Applications*, vol. 11, no. 1–3, pp. 283–291, 1985.

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[7] J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity," *Boundary Value Problems*, vol. 82, pp. 1–15, 2018.

[8] X. Zhang, J. Wu, L. Liu, Y. Wu, and Y. Cui, "Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation," *Mathematical Modelling and Analysis*, vol. 23, no. 4, pp. 611–626, 2018.

[9] J. He, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions," *Boundary Value Problems*, vol. 189, pp. 1–17, 2018.

[10] F. Wang, L. Liu, D. Kong, and Y. Wu, "Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with mixed-type boundary value conditions," *Nonlinear Analysis: Modelling and Control*, vol. 24, no. 1, pp. 73–94, 2019.

[11] L. Guo, L. Liu, and Y. Wu, "Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters," *Nonlinear Analysis: Modelling and Control*, vol. 23, no. 2, pp. 182–203, 2018.

[12] L. Guo and L. Liu, "Maximal and minimal iterative positive solutions for singular infinite-point p-Laplacian fractional differential equations," *Nonlinear Analysis: Modelling and Control*, vol. 23, no. 6, pp. 851–865, 2018.

[13] Y. Cui, "Uniqueness of solution for boundary value problems for fractional differential equations," *Applied Mathematics Letters*, vol. 51, pp. 48–54, 2016.

[14] Y. Cui, W. Ma, Q. Sun, and X. Su, "New uniqueness results for boundary value problem of fractional differential equation," *Nonlinear Analysis: Modelling and Control*, vol. 23, no. 1, pp. 31–39, 2018.

[15] Y. Zou and G. He, "On the uniqueness of solutions for a class of fractional differential equations," *Applied Mathematics Letters*, vol. 74, pp. 68–73, 2017.

[16] Z. Yue and Y. Zou, "New uniqueness results for fractional differential equation with dependence on the first order derivative," *Advances in Difference Equations*, vol. 2019, article 38, pp. 1–9, 2019.

[17] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

[18] J. Hadamard, “Essai sur Ietude des fonctions donnes par leur developpment de taylor,” *Journal de Mathématiques Pures et Appliquées*, vol. 8, pp. 86–101, 1892.

[19] B. Ahmad, J. J. Nieto, and A. Alsaedi, “A coupled system of Caputo-type sequential fractional differential equations with coupled (periodic/anti-periodic type) boundary conditions,” *Mediterranean Journal of Mathematics*, vol. 14, no. 227, pp. 1–15, 2017.

[20] B. Ahmad and S. K. Ntouyas, “A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations,” *Fractional Calculus and Applied Analysis*, vol. 17, no. 2, pp. 348–360, 2014.

[21] W. Cheng, J. Xu, and Y. Cui, “Positive solutions for a system of nonlinear semipositone fractional q-difference equations with q-integral boundary conditions,” *The Journal of Nonlinear Science and Applications*, vol. 10, no. 08, pp. 4430–4440, 2017.

[22] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[23] J. Jiang, L. Liu, and Y. Wu, “Multiple positive solutions of singular fractional differential system involving Stieltjes integral conditions,” *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 43, pp. 1–18, 2012.

[24] J. Jiang, L. Liu, and Y. Wu, “Positive solutions to singular fractional differential system with coupled boundary conditions,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 11, pp. 3061–3074, 2013.

[25] X. Hao, H. Wang, L. Liu, and Y. Cui, “Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator,” *Boundary Value Problems*, vol. 182, Article ID 182, pp. 1–18, 2017.

[26] H. Li and J. Zhang, “Positive solutions for a system of fractional differential equations with two parameters,” *Journal of Function Spaces*, vol. 2018, Article ID 1462505, 9 pages, 2018.

[27] B. Liu, J. Li, L. Liu, and Y. Wang, “Existence and uniqueness of nontrivial solutions to a system of fractional differential equations with Riemann-Stieltjes integral conditions,” *Advances in Difference Equations*, vol. 2018, article 306, pp. 1–15, 2018.

[28] X. Qiu, J. Xu, D. O’Regan, and Y. Cui, “Positive solutions for a system of nonlinear semipositone boundary value problems with Riemann-Liouville fractional derivatives,” *Journal of Function Spaces*, vol. 2018, Article ID 7351653, 10 pages, 2018.

[29] T. Qi, Y. Liu, and Y. Zou, “Existence result for a class of coupled fractional differential systems with integral boundary value conditions,” *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 10, no. 7, pp. 4034–4045, 2017.

[30] T. Qi, Y. Liu, and Y. Cui, “Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions,” *Journal of Function Spaces*, vol. 2017, Article ID 6703860, 9 pages, 2017.

[31] K. Shah and R. A. Khan, “Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions,” *Differential Equations & Applications*, vol. 7, no. 2, pp. 245–262, 2015.

[32] J. Tariboon, S. K. Ntouyas, and W. Sudsutad, “Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions,” *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 9, no. 1, pp. 295–308, 2016.

[33] Y. Wang and J. Jiang, “Existence and nonexistence of positive solutions for the fractional coupled system involving generalized p-Laplacian,” *Advances in Difference Equations*, vol. 2017, article 337, pp. 1–19, 2017.

[34] Y. Wang, L. Liu, X. Zhang, and Y. Wu, “Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection,” *Applied Mathematics and Computation*, vol. 258, pp. 312–324, 2015.

[35] Y. Wang, L. Liu, and Y. Wu, “Positive solutions for a class of higher-order singular semipositone fractional differential systems with coupled integral boundary conditions and parameters,” *Advances in Difference Equations*, vol. 2014, article 268, pp. 1–24, 2014.

[36] X. Zhang, L. Liu, Y. Wu, and Y. Zou, “Existence and uniqueness of solutions for systems of fractional differential equations with Riemann-Stieltjes integral boundary condition,” *Advances in Difference Equations*, vol. 2018, article 204, pp. 1–15, 2018.

[37] X. Zhang, L. Liu, and Y. Zou, “Fixed-point theorems for systems of operator equations and their applications to the fractional differential equations,” *Journal of Function Spaces*, vol. 2018, Article ID 7469868, 9 pages, 2018.

[38] Y. Zhang, “Existence results for a coupled system of nonlinear fractional multi-point boundary value problems at resonance,” *Journal of Inequalities and Applications*, vol. 198, pp. 1–17, 2018.

[39] X. Zhang, L. Liu, and Y. Wu, “The uniqueness of positive solution for a singular fractional differential system involving derivatives,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 6, pp. 1400–1409, 2013.

[40] M. Zuo, X. Hao, L. Liu, and Y. Cui, “Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions,” *Boundary Value Problems*, vol. 161, pp. 1–15, 2017.

[41] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.

[42] D. H. Hyers, “On the stability of the linear functional equation,” *Proceedings of the National Acadamy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[43] Z. Ali, A. Zada, and K. Shah, “On Ulam’s stability for a coupled systems of nonlinear implicit fractional differential equations,” *Bulletin of the Malaysian Mathematical Sciences Society*, pp. 1–19, 2018.

[44] A. Zada, S. Ali, and Y. Li, “Ulam-type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition,” *Advances in Difference Equations*, vol. 2017, article 312, pp. 1–26, 2017.

[45] I. A. Rus, “Ulam stabilities of ordinary differential equations in a Banach space,” *Carpathian Journal of Mathematics*, vol. 26, no. 1, pp. 103–107, 2010.

[46] N. Ahmad, Z. Ali, K. Shah, A. Zada, and G. Rahman, “Analysis of implicit type nonlinear dynamical problem of impulsive fractional differentail equations,” *Complexity*, vol. 2018, Article ID 6423974, 15 pages, 2018.

[47] S. Abbas, M. Benchohra, J.-E. Lazreg, and Y. Zhou, “A survey on Hadamard and HILfer fractional differential equations: analysis and stability,” *Chaos, Solitons & Fractals*, vol. 102, pp. 47–71, 2017.

[48] J. Wang, A. Zada, and W. Ali, “Ulam’s–type stability of first–order impulsive differential equations with variable delay in quasi–Banach spaces,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 19, no. 5, pp. 553–560, 2018.

[49] A. Zada, W. Ali, and S. Farina, "Ulam–Hyers stability of nonlinear differential equations with fractional integrable impulsis," *Mathematical Methods in the Applied Sciences*, vol. 40, pp. 5502–5514, 2017.

[50] A. Zada, F. Khan, U. Riaz, and T. Li, "Hyers–Ulam stability of linear summation equations," *PUJM*, vol. 49, no. 1, pp. 19–24, 2017.

[51] A. Zada, U. Riaz, and F. U. Khan, "Hyers–Ulam stability of impulsive integral equations," *Bollettino dell'Unione Matematica Italiana*, 2018.

[52] C. Urs, "Coupled fixed point theorem and application to periodic boundary value problem," *Miskolc Mathematical Notes*, vol. 14, pp. 323–333, 2013.

[53] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland math. stud., Elsevier Science Publishers, New York, NY, USA, 2006.

[54] M. Klimek, "Sequential fractional differential equations with Hadamard derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 12, pp. 4689–4697, 2011.

[55] J. Wang, Y. Zhou, and M. MedveD, "Existence and stability of fractional differential equations with Hadamard derivative," *Topological Methods in Nonlinear Analysis*, vol. 41, no. 1, pp. 113–133, 2013.

[56] Q. Ma, J. Wang, R. Wang, and X. Ke, "Study on some qualitative properties for solutions of a certain two-dimensional fractional differential system with Hadamard derivative," *Applied Mathematics Letters*, vol. 36, pp. 7–13, 2014.

[57] J. Wang and Y. Zhang, "On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives," *Applied Mathematics Letters*, vol. 39, pp. 85–90, 2014.

[58] J. Wang, K. Shah, and A. Ali, "Existence and Hyers–Ulam stability of fractional nonlinear impulsive switched coupled evolution equations," *Mathematical Methods in the Applied Sciences*, pp. 1–11, 2018.

[59] S. Abbas, M. Benchohra, N. Hamidi, and J. Henderson, "Caputo-Hadamard fractional differential equations in Banach spaces," *Fractional Calculus and Applied Analysis*, vol. 21, no. 4, pp. 1027–1045, 2018.

[60] B. Ahmad and S. K. Ntouyas, "Nonlocal initial value problems for Hadamard-type fractional differential equations and inclusions," *Rocky Mountain Journal of Mathematics*, vol. 48, no. 4, pp. 1043–1068, 2018.

[61] B. Ahmad and S. K. Ntouyas, "On Hadamard fractional integro-differential boundary value problems," *Applied Mathematics and Computation*, vol. 47, no. 1-2, pp. 119–131, 2015.

[62] S. Aljoudi, B. Ahmad, J. J. Nieto, and A. Alsaedi, "A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions," *Chaos, Solitons & Fractals*, vol. 91, pp. 39–46, 2016.

[63] C. Zhai, W. Wang, and H. Li, "A uniqueness method to a new Hadamard fractional differential system with four-point boundary conditions," *Journal of Inequalities and Applications*, vol. 207, pp. 1–16, 2018.

[64] K. Zhang and Z. Fu, "Solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity," *Journal of Function Spaces*, vol. 2019, Article ID 9046472, 7 pages, 2019.

[65] K. Zhang, J. Wang, and W. Ma, "Solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations," *Journal of Function Spaces*, Article ID 2193234, 10 pages, 2018.

[66] P. Thiramanus, S. K. Ntouyas, and J. Tariboon, "Positive solutions for Hadamard fractional differential equations on infinite domain," *Advances in Difference Equations*, vol. 2016, article 83, 2016.

[67] M. Altman, "A fixed point theorem for completely continuous operators in Banach spaces," *Bulletin L'Académie Polonaise des Science*, vol. 3, pp. 409–413, 1955.