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ANALYSIS OF NONLINEAR SYSTEMS WITH TIME VARYING INPUTS AND ITS APPLICATION TO GAIN SCHEDULING

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An analytical framework for analysis of a class of nonlinear systems with time varying inputs is presented. It is shown that the trajectories of the transformed nonlinear systems are uniformly bounded with an ultimate bound under certain conditions shown in this paper. The result obtained is useful for applications, in particular, analysis and design of gain scheduling.

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KEYWORDS: *Gain scheduling; nonlinear system; time varying input*

1. INTRODUCTION

The main idea of gain scheduling is that linear design methods are applied to nonlinear control design provided that the scheduling variable captures the plant's nonlinearity and varies slowly. In spite of its wideapplications [1]–[3], the analytical framework on analysis and design of gain scheduling is still in process [4]–[8] to give the gain scheduling approach a rigorous mathematical justification. Based on the Gronwall-Bellman inequality, a stability theorem has been presented in [4]. In [5]–[8], Liapunov functions have been constructed to investigate the robustness and stability properties. However most of their results are limited to slow variations in the scheduling variables. In this paper, we present a theoretical foundation for gain scheduling without restriction to slow variations in the scheduling variables. We use Liapunov stability to analyze a class of nonlinear systems with time varying inputs and obtain an ultimate bound on the discrepancy between the exact solution and the fixed operating point. Since gain scheduling is a nonlinear feedback whose parameters evolve as functions of exogeneous variables, gain scheduling yields a closed loop system that is nonlinear, at least in the scheduling variables. Thus the ultimate behavior of the resulting nonlinear system with time varying inputs can be analyzed using the Theorem formulated below.

2. MAIN RESULT

Consider the following system.

$$\dot{x} = f(x, u(t)) \tag{1}$$

where $x \in R^n$ and $u(t) \in \Gamma$ which is a bounded, open subset of R^m, $\forall t \geq 0$. Let $\|\cdot\|$ denote the Euclidean norm of a time varying vector, and also the corresponding induced norm on a matrix. Assume that

(i) f is twice continuously differentiable,
(ii) the equation $f(x, u(t)) = 0$, $\forall u(t) \in \Gamma$ has a continuously differentiable solution $x = h(u)$,
(iii) $\|\dot{u}(t)\| < \gamma_1$, $\forall t \geq 0$,
(iv) there exist positive constants σ, k_1, and k_2 such that $\lambda_i(u(t)) + \lambda_i^*(u(t)) \leq -2\sigma, k_1\sigma \leq |\lambda_i(u(t))| \leq k_2\sigma$, $\forall u(t) \in \Gamma$, $i = 1,\ldots, n$ where $\lambda_i(u(t))$ is the eigenvalue of $\frac{\partial f}{\partial x}(h(u), u(t))$ for each fixed $t \geq 0$ and $\lambda_i^*(u(t))$ is the complex conjugate of $\lambda_i(u(t))$.

Since f is continuously differentiable, $f(x, u(t))$ can be rewritten as

$$f(x, u(t)) = f(h(u), u(t)) + \frac{\partial f}{\partial x}(h(u), u(t))(x - h(u))$$

$$+ [\frac{\partial f}{\partial x}(x', u(t)) - \frac{\partial f}{\partial x}(h(u), u(t))](x - h(u)) \tag{2}$$

where x' is a point on the line segment joining x and $h(u)$. Let $y = x - h(u)$, then

$$\dot{y} = f(x, u) - \frac{\partial h(u)}{\partial u}\dot{u}(t) \tag{3}$$

Using the fact that $\frac{\partial f}{\partial t}(h(u), u(t)) = 0$, we have $\frac{\partial f}{\partial x}(h(u), u(t))\frac{\partial h(u)}{\partial u}\dot{u}(t) + \frac{\partial f}{\partial u}(h(u), u(t))\dot{u}(t) = 0$. Since $\det [\frac{\partial f}{\partial x}(h(u), u(t))] \neq 0$, $\forall u(t) \in \Gamma$, we obtain

$$\frac{\partial h(u)}{\partial u}\dot{u}(t) = -[\frac{\partial f}{\partial x}(h(u), u(t))]^{-1}\frac{\partial f}{\partial u}(h(u), u(t))\dot{u}(t) \tag{4}$$

From (1)–(4),

$$\dot{y} = A(t)y + R(t)y + B(t)\dot{u} \tag{5}$$

where $A(t) = \frac{\partial f}{\partial x}(h(u), u(t))$, $R(t) = [\frac{\partial f}{\partial x}(x', u(t)) - \frac{\partial f}{\partial x}(h(u), u(t))]$, *and* $B(t) = [\frac{\partial f}{\partial x}(h(u), u(t))]^{-1}\frac{\partial f}{\partial u}(h(u), u(t))$.

THEOREM: *Consider the system (1) with assumptions (i)–(iv). Then there exists a positive constant* σ_o *such that, for any* $\sigma > \sigma_o$,

$$\|y(t)\| \leq \eta \, \|y(0)\| e^{-\alpha_1(\sigma)t} + \alpha_2(\sigma), \ \forall \|y(0)\| < \alpha_3(\sigma) \tag{6}$$

where η *is a positive constant and* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ *are class-K, class-L, class-K functions (see [8]), respectively.*

The proof of Theorem requires two auxiliary results.

LEMMA 1: *Under assumptions (i)–(iv), let* $Q(t)$ *be a bounded matrix which is defined by*

$$Q(t) = \int_0^{\infty} e^{A^T(t)\tau} e^{A(t)\tau} \, d\tau$$

where $A(t) = \dfrac{\partial f}{\partial x}(h(u), u(t))$ *and let* σ *be a positive constant given in the assumption (iv). Then there exist positive constants* c_1, c_2, c_3 *such that* $\|A(t)\| \leq c_1\sigma$, $\|A^{-1}(t)\| \leq \dfrac{c_2}{\sigma}$, $\|Q(t)\| \leq \dfrac{c_3}{\sigma}$.

Proof: Let $A_o(t) = \dfrac{1}{\sigma} A(t)$, then there exists a nonsingular matrix $P(t)$ such that

$$A(t) = \sigma P^{-1}(t) A_J(t) P(t) \tag{7}$$

where $A_J(t)$ is the Jordan form representation of $A_o(t)$. Since $A(t)$ is bounded, there exists a positive constant k_a such that $\|A(t)\| \leq k_a \, \sigma \|A_J(t)\|$. Note that $\|A_J(t)\| \leq \sqrt{\max_i \sum_{j=1}^{n} |(A_J(t))_{i,j}| \max_j \sum_{i=1}^{n} |(A_J(t))_{i,j}|}$. Let n_s be the order of the largest Jordan block in $A_J(t)$. Then, by simple algebraic manipulations, we have $\|A_J(t)\| \leq k_2$ if $n_s = 1$ and $1 + k_2$ otherwise. Thus $\|A(t)\| \leq k_a \sigma k_2$ if $n_s = 1$ and $k_a\sigma(1 + k_2)$ otherwise.

From (7), $A^{-1}(t) = \dfrac{1}{\sigma} P^{-1}(t) A_J^{-1}(t) P(t)$. Since $A^{-1}(t)$ is also bounded, there exists a positive constant k_b such that $\|A^{-1}(t)\| \leq \dfrac{k_b}{\sigma} \|A_J^{-1}(t)\|$. Based on the same way, we obtain $\|A_J^{-1}(t)\| \leq \sum_{i=1}^{n_s} k_1^{-i}$. Thus $\|A^{-1}(t)\| \leq \dfrac{k_b}{\sigma} \sum_{i=1}^{n_s} k_1^{-i}$.

Again, from (7), $e^{A(t)\tau} = P^{-1}(t) e^{\sigma A_J}(t)\tau \rightarrow e^{\sigma A_J(t)\tau} P(t)$. Since $e^{A(t)\tau}$ is also a bounded matrix, there exists a positive constant k_c such that $\|e^{A(t)\tau}\| \leq k_c \|e^{\sigma A_J}(t)\tau \rightarrow e^{\sigma A_J(t)\tau}$. In the same manner, we obtain $\|e^{\sigma A_J(t)\tau}\| \leq e^{-\sigma\tau} \sum_{i=1}^{n_s} \dfrac{(\sigma\tau)^{i-1}}{(i-1)!}$. Thus we have $\|Q(t)\| \leq k_c^2 \int_0^{\infty} \|e^{\sigma A_J(t)\tau}\|^2 d\tau \leq \dfrac{k_c^2}{2\sigma} \sum_{i=1}^{n_s} \sum_{j=1}^{n_s} \dfrac{(i+j-2)!}{2^{i+j-2}(i-1)!(j-1)!}$.

Q.E.D.

Since $\frac{\partial f}{\partial x}(h(u), u(t))$, $\forall u(t) \in \Gamma$ is continuously differentiable, it follows that there exists a positive constant γ_2 such that $\|\dot{A}(t)\| \leq \gamma_2 \|\dot{u}(t)\|$, $\forall t \geq 0$ where $A(t) = \frac{\partial f}{\partial x}(h(u), u(t))$. Then we present the following lemma (see [6] for details) that will be used in the proof of Theorem.

LEMMA 2: *Given* $\|Q(t)\| \leq \frac{c_3}{\sigma}$, $\|A(t)\| \leq c_1\sigma$, *and* $\|\dot{A}(t)\| \leq \gamma_1\gamma_2$ *where* $Q(t)$ *and* $A(t)$ *are defined in Lemma 1, then the following relationships hold: (i)* $A^T(t)Q(t) + Q(t)A(t) = -I$, (ii) $\frac{1}{2c_1\sigma}\|y\|^2 \leq y^T Q(t)y \leq \frac{c_3}{\sigma}\|y\|^2$, (iii) $\|\dot{Q}(t)\| \leq 2(\frac{c_3}{\sigma})^2\gamma_1\gamma_2$.

Proof of the Theorem: Consider $V(t, y) = y^TQ(t)y$. Then, from (5) and Lemmas 1 & 2,

$$\begin{aligned}\dot{V}(t, y) &= -y^T(A^T(t)Q(t) + Q(t)A(t))y + y^T\dot{Q}(t)y \\ &\quad + 2y^TQ(t)R(t)y + 2y^TQ(t)B(t)\dot{u} \\ &\leq -\|y\|^2 + \|\dot{Q}(t)\|\,\|y\|^2 + 2\|Q(t)\|\,\|R(t)\|\,\|y\|^2 \\ &\quad + 2\|Q(t)\|\,\|B(t)\|\,\|y\|\,\|\dot{u}\|\end{aligned}$$

Since $f(x, u(t))$ is twice continuously differentiable, $\frac{\partial f}{\partial x}(x, u(t))$ is Lipschitz in x on any bounded region $B \in R^n$. Thus there exists a positive constant L such that $\|\frac{\partial f}{\partial x}(x, u(t)) - \frac{\partial f}{\partial x}(h(u), u(t))\| \leq L\|x - h(u)\|$, $\forall x, h(u) \in B$, $\forall u(t) \in \Gamma$. As it follows from (5), since $f(h(u), u(t))$, $\forall u(t) \in \Gamma$ is continuously differentiable, there exists a positive constant γ_3 such that $\|B(t)\| \leq \gamma_3\|A^{-1}(t)\|$, $\forall u(t) \in \Gamma$. Then, from Lemmas 1 & 2, it is easy to show

$$\begin{aligned}\dot{V}(t, y) &< -(\frac{1}{2} - \frac{2c_3^2\gamma_1\gamma_2}{\sigma^2})\|y\|^2 + \frac{2c_2c_3\gamma_1\gamma_3}{\sigma^2}\|y\|, \ \forall \|y\| < \frac{\sigma}{4c_3L} \\ &\leq -(1 - \theta)(\frac{1}{2} - \frac{2c_3^2\gamma_1\gamma_2}{\sigma^2})\|y\|^2, \ \forall y \in D_1, 0 < \theta < 1\end{aligned} \tag{8}$$

where $D_1 = \{y \in R^n : d_2 < \|y\| < d_1\}$ and $d_1 = \frac{\sigma}{4c_3L}$, $d_2 = \frac{4c_2c_3\gamma_1\gamma_3}{\theta(\sigma^2 - 4c_3^2\gamma_1\gamma_2)}$.

Let $N = (1 - \theta)(\frac{1}{2} - \frac{2c_3^2\gamma_1\gamma_2}{\sigma^2})$ and $\sigma_0 = 2\sqrt{c_3^2\gamma_1\gamma_2 + 2c_2c_3\frac{\sqrt{\gamma_1}}{\theta\sqrt{\gamma_2}}\gamma_3L}$. Since $\sigma > \sigma_0$, it is easy to verify that $0 < N < \frac{1}{2}$ and $D_1 \neq \emptyset$. From Lemma 2 and (8), we obtain

$$\dot{V}(t, y) \leq -\frac{\sigma N}{c_3} V(t, y), \forall y \in D_1 \tag{9}$$

Then, from Lemma 2 and (9),

$$\|y(t)\|^2 \leq 2c_1\sigma V(0, y(0))\, e^{-\frac{\sigma Nt}{c_3}} \leq 2c_1c_3\|y(0)\|^2\, e^{-\frac{\sigma Nt}{c_3}}, \forall y \in D_1$$

Hence $\|y(t)\| \leq \sqrt{2c_1c_3}\,\|y(0)\|\, e^{-\frac{\sigma Nt}{2c_3}}, \forall y \in D_1$. Thus, by simple algebraic manipulations, we have

$$\|y(t)\| \leq \sqrt{2c_1c_3}\,\|y(0)\|\, e^{-\frac{\sigma Nt}{2c_3}} + \frac{4c_2c_3\gamma_1\gamma_3}{\theta(\sigma^2 - 4c_3^2\gamma_1\gamma_2)}, \forall y(0) \in D_2 \tag{10}$$

where $D_2 = \{y(0) \in R^n : \|y(0)\| < \frac{d_1 - d_2}{\sqrt{2c_1c_3}}\}$. Q.E.D.

Note that, for the fast varying $u(t)$, the ultimate bound in (10) can be made as small as possible by introducing large σ. If, in addition, $\lim_{t\to\infty} \dot{u}(t) = 0$, then we can show $\lim_{t\to\infty} y(t) = 0$ by repeating the steps of the proof of Theorem.

Example: Consider the following system with the desired output $c(t) = 0$.

$$\dot{x}_1(t) = -30x_1(t) + x_2^2(t) + \sin 2t$$

$$\dot{x}_2(t) = x_1^2(t) + x_2(t) + u_c(t)$$

$$c(t) = x_2(t)$$

where $u_c(t)$ is the control input. Since the operating condition is specified by the exogeneous variable, sin $2t$, it is used as a scheduling variable. For each fixed scheduling variable, the frozen system has a fixed operating point defined by $x = h(u)$. In order to have the desired output as fixed operating points, $h(u) = [\frac{\sin 2t}{30}\ 0]^T$ and the gain scheduled control law is $u_c(t) = lx_2(t) - (\frac{\sin 2t}{30})^2$ where l is the constant to be chosen. Since, as it follows from the above, $f(x, u(t))$ and $h(u)$ are smooth and the eigenvalues of $\frac{\partial f}{\partial x}(h(u), u(t))$ for each fixed $t \geq 0$ have negative real parts for any value of l less than -1, the conditions (i)–(ii) and (iv) are satisfied. It also follows from $u(t) = \sin 2t$ that $u(t)$ satisfies the condition (iii). Thus all the conditions of Theorem are met and as it follows from (10), the ultimate bound on $y(t)$ is less than 0.007 for $l \leq -25$. Fig. 1 shows the trajectories of $x(t)$, $y(t)$ for $l = -25$. It is observed from Fig. 1 that $x(t)$ asymptotically follows $h(u)$ and $y(t)$ is uniformly bounded with the ultimate bound obtained from (10).

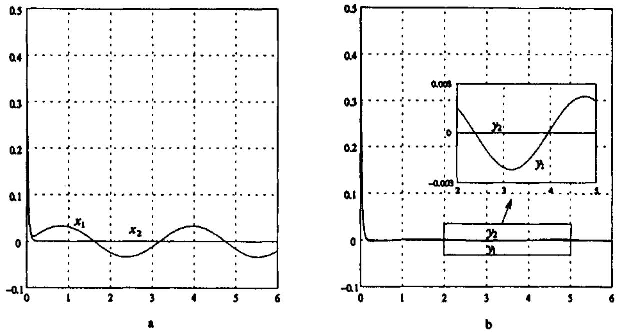

Figure 1 Asymptotic tracking performance
a Trajectory of x(t) for l = −25 with x(0) = $[0.5\ 0.2]^T$
b Trajectory of y(t) for l = −25 with y(0) = $[0.5\ 0.2]^T$

3. CONCLUSION

In this paper, we analyze nonlinear systems with time varying inputs. In particular, we show the effect of input variations on the trajectories of the transformed nonlinear systems in (10). We also show that the result obtained is useful to analysis and design of gain scheduling.

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