

# Analysis of Nonlinear Traveling Waves by Frequency-Domain Perturbation Method

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## ABSTRACT

We discuss a numerical method for solving the *nonlinear transmission lines* by the frequency-domain perturbation method. To improve the convergence, we introduce two *new* methods of the *compensation* and the *homotopy techniques*, which also help to make the iteration stable. This kind of transmission lines is widely used in the communication circuits such as GaAs integrated circuits, and varactor diode circuits.

## 1. INTRODUCTION

Nonlinear devices such as GaAs nonlinear transmission lines [1], varactor diodes and superconducting transmission lines [2] have the nonlinear capacitors. The velocity of traveling wave is given by

$$v = 1/\sqrt{L(i)C(v)}$$

so that if the capacitor is a decreasing function of the voltages, it will have an interesting property such that the higher part of the waveform is the faster than the lower part of it. The property is used in many purposes such as the picosecond pulse compression [1], picosecond shock-wave generations [3] and so on.

There are some papers discussing the phenomena from the physical points of view [1-5]. In reference [6], the shock-waves of nonlinear transmission line driven by a sinusoidal generator are analyzed by the harmonic balance method, where the nonlinear transmission line is replaced by the finite number of the discrete lumped models. The reference [7] has analyzed the phenomena by the application of the *difference approximation* to the nonlinear partial differential equation. In reference [8], they have analyzed the nonlinear wave propagation phenomena, where the 2-dimensional FFT is applied to solve the circuit equation in the distance and time.

In this paper, we discuss an efficient numerical method for solving the nonlinear transmission lines. The waveform is approximated by the Fourier expansion at each point on the line, and the coefficients are calculated by a perturbation technique. Although, the method is only applied to the weakly nonlinear systems, we have improved it with two techniques of a *compensational* and *homotopy method*.

## 2. FREQUENCY-DOMAIN PERTURBATION METHOD

**2.1. Perturbation Method** Now, consider a nonlinear transmission line terminated by linear subnetworks. The circuit equation is described by the following nonlinear partial differential equations:

$$-\frac{\partial v}{\partial x} = \frac{\partial \phi_L}{\partial t} + v_R, \quad -\frac{\partial i}{\partial x} = \frac{\partial q_C}{\partial t} + i_G \quad (1)$$

where  $\phi_L$  and  $q_C$  are the flux of a nonlinear inductor and the charge of a nonlinear capacitor. We assume the characteristics as follows:

$$i_L = \Gamma \phi_L + \hat{\epsilon} i_L(\phi_L), \quad v_R = R i_R + \epsilon \hat{v}_R(i_R) \quad (2.1)$$

$$v_C = S q_C + \epsilon \hat{v}_C(q_C), \quad i_G = G v_G + \hat{\epsilon} i_G(v_G) \quad (2.2)$$

where  $\epsilon$  means a small constant, and  $\hat{i}_L(\phi_L)$ ,  $\hat{v}_R(i_R)$ ,  $\hat{v}_C(q_C)$  and  $\hat{i}_G(v_G)$  are nonlinear terms. We have the relations of  $i \equiv i_L = i_R$  and  $v \equiv v_C = v_G$ .

To analyze the traveling-waves, consider the responses of nonlinear transmission lines driven by an impulse waveform as follows:

$$e(t) = \begin{cases} E_m \sin\left(\frac{2k\pi}{T}t\right) & : 0 \leq t \leq \frac{T}{2k} \\ 0 & : \frac{T}{2k} \leq t \leq T \end{cases} \quad (3)$$

where  $T$  is a period of the impulse and  $k$  is an integer. Assume that the nonlinear transmission line is terminated by a linear impedance  $Z_L$ . Then, the circuit equations (1) and (2) are mathematically reduced to the nonlinear partial equations constrained by a linear boundary condition. In this paper, we apply the *frequency-domain* approach, and describe the input impulse waveform by the Fourier expansion in the complex domain as follows:

$$e(t) = \sum_{k=-M}^M E_k e^{jk\omega t}, \quad \omega = \frac{2\pi}{T} \quad (4)$$

for a sufficiently large  $M$ . Now, let us assume the solutions at the  $m$ th iteration as follows:

$$v^m(x, t) = \sum_{k=-M}^M V_k^m e^{jk\omega t}, \quad i^m(x, t) = \sum_{k=-M}^M I_k^m e^{jk\omega t} \quad (5.1)$$

$$\phi_L^m(x, t) = \sum_{k=-M}^M \Phi_k^m e^{jk\omega t}, \quad q_C^m(x, t) = \sum_{k=-M}^M Q_k^m e^{jk\omega t} \quad (5.2)$$

Then, we have the following perturbed equations:

$$\frac{dV_k^m}{dx} = -(R + jk\omega L)I_k^m - \epsilon \left( \hat{V}_{R,k}(i^{m-1}) - jk\omega L \hat{I}_{L,k}(\phi_L^{m-1}) \right) \quad (6.1)$$

$$\frac{dI_k^m}{dx} = -(G + jk\omega C)V_k^m - \epsilon \left( \hat{I}_{G,k}(v^{m-1}) - jk\omega C \hat{V}_{C,k}(q_C^{m-1}) \right) \quad (6.2)$$

where  $L = 1/\Gamma$ ,  $C = 1/S$  and

$$\Phi_k^m = L \left( I_k^m - \epsilon \hat{I}_{L,k}(\phi_L^{m-1}) \right) \quad (6.3)$$

$$Q_k^m = C \left( V_k^m - \epsilon \hat{V}_{C,k}(q_C^{m-1}) \right) \quad (6.4)$$

Observe that the relations (6.1) and (6.2) are ordinary differential equation, and the second terms at the right hand side correspond to the forced terms calculated at  $(m-1)$ th iteration. The solutions of (6.1) and (6.2) are written as follows:

$$V_k^m(x) = A_k^m e^{\lambda_k x} + B_k^m e^{-\lambda_k x} + \epsilon \hat{V}_k^m(x) \quad (7.1)$$

$$I_k^m(x) = \frac{1}{Z_{k,0}} \left( -A_k^m e^{\lambda_k x} + B_k^m e^{-\lambda_k x} \right) + \epsilon \hat{I}_k^m(x) \quad (7.2)$$

where the propagation constant  $\lambda_k$  and characteristic impedance  $Z_{k,0}$  are given by

$$\lambda_k = \sqrt{(jk\omega L + R)(jk\omega C + G)}, \quad Z_{k,0} = \sqrt{\frac{jk\omega L + R}{jk\omega C + G}}$$

The first two terms of (7) correspond to the *zero-input responses*, and  $\epsilon \hat{V}_k^m(x)$  and  $\epsilon \hat{I}_k^m(x)$  to the *zero-state responses* calculated by a numerical intergration technique. The constant parameters  $A_k^m$  and  $B_k^m$  in (7) are evaluated by the boundary conditions of the transmission line. Thus, all of the Fourier coefficients  $V_k^m$  and  $I_k^m$  are calculated, and  $\Phi_k^m$  and  $Q_k^m$  are decided by (6.3) and (6.4). The iteration is continued until the variation

$$\delta^m \equiv \left[ \sum_{k=-M}^M (V_k^m(l) - V_k^{m-1}(l))^2 \right]^{1/2}$$

$$+ \sum_{k=-M}^M (I_k^m(l) - I_k^{m-1}(l))^2 \Big]^{1/2} \quad (8)$$

becomes sufficiently small for given a constant  $\delta$ . However, we can not say anything whether the solution waveform is sufficiently accurate or not.

Now, let us define the following *residual error*. The solutions for the  $M$  frequency components be  $v_M(x, t)$ ,  $i_M(x, t)$ . Then, define the *residual error* as follows:

$$\varepsilon_M \equiv \left[ \frac{1}{T} \int_0^T \left\{ (v_M(x, t) - \hat{v}(x, t))^2 + (i_M(x, t) - \hat{i}(x, t))^2 \right\} dt \right]^{1/2} \quad (9)$$

where  $\hat{v}(x, t)$ ,  $\hat{i}(x, t)$  are the exact solutions, and  $v_M(x, t)$ ,  $i_M(x, t)$  are the approximate solutions at  $m$ th iteration with  $M$  frequency components. Since it is impossible to get the exact solutions in (9), we assume the solutions obtained with the larger frequency components  $M'$  as the exact solutions.

The residual error (9) between the solutions with  $M$  and  $M'$  frequency components are given by

$$\varepsilon_M \cong \left[ \sum_{|k|=M}^{M'} (V_k^m(l) - V_k^{m-1}(l))^2 + \sum_{|k|=M}^{M'} (I_k^m(l) - I_k^{m-1}(l))^2 \right]^{1/2}$$

If the residual error  $\varepsilon_M$  is not so small, we need to choose much larger  $M$  for the waveform approximation, and again repeat our perturbation method.

**2.2. Modified Perturbation Method** It is said that the perturbation method is only applied to the weakly nonlinear circuits. In this section, we will modify the method such that it can be applied to the much stronger nonlinear circuits. Assume that the nonlinear characteristics given in (2) are monotone increasing functions, and for simplicity, describe it as follows:

$$u_1 = H u_2 + \epsilon \hat{h}(u_2), \quad H = \text{diag}(\Gamma, R, S, G) \quad (10)$$

where  $u_1 = (i_L, v_R, v_C, i_G)^T$ ,  $u_2 = (\phi_L, i_R, q_C, v_G)^T$  and  $\epsilon \hat{h}(u_2) = \epsilon (\hat{i}_L(\phi_L), \hat{v}_R(i_R), \hat{v}_C(q_C), \hat{i}_G(v_G))^T$ . Since the convergence of our perturbation method largely depends on the magnitude of perturbed terms  $\epsilon \hat{h}(u_2)$ , we need to decrease it compared with the linear term. Thus, we introduce small constant

$$\Delta H \equiv \text{diag}(\Delta\Gamma, \Delta R, \Delta S, \Delta G)$$

as a *compensation parameter* in the following form:

$$u_1 = (H + \Delta H)u_2 + (\epsilon \hat{h}(u_2) - \Delta H u_2) \quad (11)$$

The second term is the nonlinear function corresponding to the perturbed term which can be largely decreased by choosing a suitable  $\Delta H$ . We found from our numerical examples that the convergence ratio is largely improved by introducing of the *compensation parameter*.

*Homotopy method* is sometimes used when the iteration method does not guarantee the convergence. We will apply it to our perturbation method. Let us introduce a parameter  $\rho$  for the nonlinear elements (2).

$$u_1 = (H + \Delta H)u_2 + \rho(\epsilon \hat{h}(u_2) - \Delta H u_2), \quad \rho: 0 \rightarrow 1 \quad (12)$$

It is clear that the relation (12) for  $\rho = 0$  corresponds to linear elements, and the nonlinearity is gradually increased by choosing  $\{\rho: 0 \rightarrow 1\}$ . At  $\rho = 1$ , it reduces to the original function. If we choose a small variation  $\Delta\rho$  per iteration such as

$$\rho^m = \rho^{m-1} + \Delta\rho$$

our perturbation can also get the solution stably.

**2.3. Convergence Condition** Now, let us consider the convergence condition of our algorithm. For simplicity, let us introduce the following symbols for variables in (6) and (7):

(a). For the solutions, we set

$$X_{1,k}^m(x) = \begin{pmatrix} V_k^m(x) \\ I_k^m(x) \end{pmatrix}, \quad X_{2,k}^m(x) = \begin{pmatrix} \Phi_k^m(x) \\ Q_k^m(x) \end{pmatrix}$$

(b). For the perturbed terms, we set

$$\hat{X}_{1,k}(w_1^{m-1}, x) = \begin{pmatrix} \hat{V}_{R,k}(i^{m-1}) \\ \hat{I}_{G,k}(v^{m-1}) \end{pmatrix}$$

$$\hat{X}_{2,k}(w_2^{m-1}, x) = \begin{pmatrix} \hat{I}_{L,k}(\phi_L^{m-1}) \\ \hat{V}_{C,k}(q_C^{m-1}) \end{pmatrix}$$

where  $w_1 = (v, i)^T$ ,  $w_2 = (\phi_L, q_C)^T$ .

Set the coefficient matrices of (6) as follows:

$$A_k = \begin{pmatrix} 0 & R + jk\omega L \\ G + jk\omega C & 0 \end{pmatrix}$$

$$B_k = \begin{pmatrix} jk\omega L & 0 \\ 0 & jk\omega C \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix}$$

Now, describe the *zero-state response* of (7) in the integration form.

$$\begin{aligned} X_{1,k}^m(x) &= C_{1,k}(x)E_k + C_{2,k}(x)X_{1,k}X_{1,k}^m(l) \\ &\quad - \epsilon \int_0^x e^{-A_k(x-s)} \hat{X}_{1,k}(w_1^{m-1}, s) ds \\ &\quad - \epsilon B_k \int_0^x e^{-A_k(x-s)} \hat{X}_{2,k}(w_2^{m-1}, s) ds \end{aligned} \quad (13.1)$$

$$X_{2,k}^m(x) = C_0 X_{1,k}^m(x) - \epsilon C_0 \hat{X}_{2,k}(w_2^{m-1}, x) \quad (13.2)$$

where  $C_{1,k}(x)$ ,  $C_{2,k}(x)$  are decided from the boundary conditions and (7).

Let us adopt the following *norm*

$$\|w\| \equiv \sqrt{\frac{1}{T} \int_0^T w^2(t) dt} \quad (14)$$

Then, we have the following convergence conditions.

**Theorem:** Assume that, for an approximate solution  $(w_1^0, w_2^0)^T$ , there are constants  $D_1, D_2, K_0, L_1$  and  $L_2$  satisfying as follows:

(i) Define the solution domain by:

$$\begin{aligned} \Omega \equiv \{w_1, w_2 \mid &\|w_1 - w_1^0\| \leq D_1, \\ &\|w_2 - w_2^0\| \leq D_2\} \end{aligned} \quad (15.1)$$

(ii) Define norms of the coefficient matrices by:

$$\begin{aligned} \|C_0\| \equiv K_0, \quad K_1 \equiv \max_{-M \leq k \leq M} \frac{\|A_k^{-1}\| L_1 l}{1 - C_{2k}} \\ K_2 \equiv \max_{-M \leq k \leq M} \frac{\|B_k A_k^{-1}\| L_2 l}{1 - C_{2k}} \end{aligned} \quad (15.2)$$

where

$$C_{2k} \equiv \max_{0 \leq x \leq l} \|C_{2,k}(x)\|$$

(iii) Assume the perturbed terms satisfy the following Lipschitz conditions:

$$\begin{aligned} \|\hat{X}_1(w'_1, x) - \hat{X}_1(w''_1, x)\| &\leq L_1 \|w'_1 - w''_1\| \\ \|\hat{X}_2(w'_2, x) - \hat{X}_2(w''_2, x)\| &\leq L_2 \|w'_2 - w''_2\| \end{aligned} \quad (15.3)$$

for all  $0 \leq x \leq l$ , where  $L_1$  and  $L_2$  are Lipschitz constants.

(iv) Set the maximum values of the variables in  $x = [0, l]$  as follows:

$$X_{1,k}^m = \max_{0 \leq x \leq l} \|X_{1,k}^m(x)\|, \quad X_{2,k}^m = \max_{0 \leq x \leq l} \|X_{2,k}^m(x)\| \quad (15.4)$$

Furthermore, if it satisfies the condition,

$$\kappa \equiv \epsilon \|P\| < 1 \quad (15.5)$$

$$\text{for } P = \begin{pmatrix} K_1 & K_2 \\ K_0 K_1 & K_0 L_2 (K_2 + L_2) \end{pmatrix}$$

our perturbation method (13) will converge to the unique solution  $(w_1^M, w_2^M)^T$ .

We found from the *theorem* that our perturbational iteration will be converged to an approximate solution  $X^M(l)$  if the nonlinear term  $\epsilon$  is small.

### 3. ILLUSTRATIVE EXAMPLES

As an example of the nonlinear transmission line, let us consider to analyze the shock-wave to an impulse response. Assume the transmission line is terminated by a resistor  $R_L = 10$  and

$$i_L = 350 \times 10^{11} \phi_L, \quad v_R = 0.015 i_R + 0.1 i_R^3$$

$$v_C = 10^{11} q_C + 0.5 \times 10^{34} q_C^3, \quad i_G = 0.015 v_G + 0.1 v_G^3$$

For the input impulse, we choose  $E_m = 0.25$ ,  $T = 1$  and  $k = 16$  in (3). We found from the example that our perturbation method is largely improved by the combinations of *compensation method* and *homotopy method*.

### 4. CONCLUSIONS AND REMARKS

We have presented a frequency-domain perturbation algorithm for calculating transient responses of a nonlinear transmission lines. We can largely improve the convergence by introducing two methods of the *compensation method* and *homotopy method*.

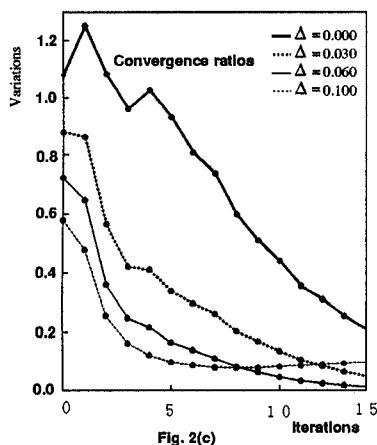


Figure 1: Convergence ratios for various compensational parameter, where variation  $\delta_m = \|X^m(l) - X^{m-1}(l)\|$

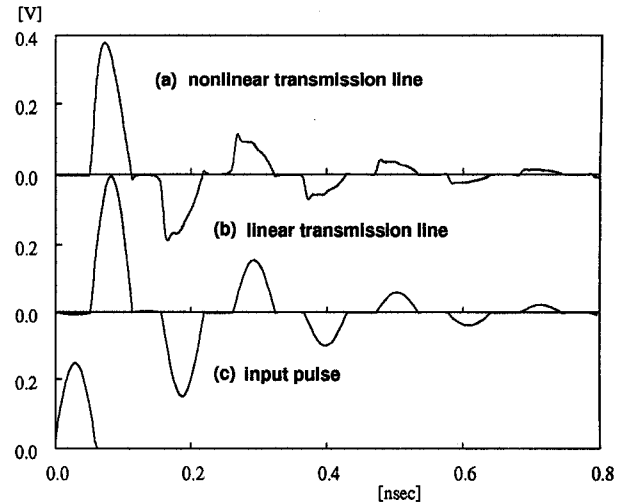


Fig. 2(d)

Figure 2: Impulse responses

(a) The response of nonlinear transmission line (b) The response of linear transmission line where nonlinear terms are set to be zeros (c) Input impulse

### REFERENCES

- [1] M.J.W. Rodwell, "Nonlinear transmission line for picosecond pulse compression and broadband phase modulation," *Electron. Lett.*, vol.23, no.3, pp.109-110, 1987
- [2] R.K. Arora and J.L. Thaker, "Electromagnetic pulse transmission on superconducting interconnects", *Int. J. Electronics*, vol.71, no.1, pp.117-125, 1991
- [3] C.J. Madden, R.A. Marsland, M.J.W. Rodwell, D.M. Bloom and Y.C. Pao, "Hyperabrupt-doped GaAs nonlinear transmission line for picosecond shock-wave generation", *Appl. Phys. Lett.*, vol.54, no.11, pp.1019-1021, 1989
- [4] R. Landauer, "Shock waves in nonlinear transmission lines and their effect on parametric amplification", *IBM Journal*, October, pp.391-401, 1960
- [5] D. Jager and F.-J. Tegude, "Nonlinear wave propagation along periodic-loaded transmission line", *Appl. Phys.*, vol.15, pp.393-397, 1978
- [6] C. Camacho-Penalosa and I. Molina-Fernandez, "Harmonic balance analysis of nonlinear transmission lines", *Electron. Lett.*, vol.24, no.9, pp.1235-1236, 1988
- [7] R.H. Freeman and A.E. Karbowski, "An investigation of nonlinear transmission lines and shock waves", *J. Phys. D: Appl. Phys.*, vol.10, pp.633-643, 1977
- [8] I. Molina-Fernandez, C. Camacho-Penalosa and J. Ramos, "Application of the two-dimensional Fourier transform to nonlinear wave propagation phenomena", *IEEE Trans. on Microwave Theory Tech.*, vol.42, no.6, pp.1079-1085, 1994