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Analysis of Optimal Finite-Element Meshes in R^1

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Dedicated to Professor H. GOERTLER on his seventieth birthday

Abstract. A theory of a posteriori estimates for the finite-element method was developed earlier by the authors. Based on this theory, for a two-point boundary value problem the existence of a unique optimal mesh distribution is proved and its properties analyzed. This mesh is characterized in terms of certain, easily computable local error indicators which in turn allow for a simple adaptive construction of the mesh and also permit the computation of a very effective a posteriori error bound. While the error estimates are asymptotic in nature, numerical experiments show the results to be excellent already for 10% accuracy. The approaches are not restricted to the model problem considered here only for clarity; in fact, they allow for rather straightforward extensions to more general problems in one dimension, as well as to higher-order elements.

1. Introduction. For the numerical solution of boundary value problems by finite-element techniques, the construction of optimal, or near-optimal meshes is of considerable practical importance. The same can be said when finite-difference or collocation methods are used. Many articles in the literature deal with questions that bear a relation to this problem, yet, as observed in [15], even for two-point boundary value problems relatively few address it directly. We shall not attempt to survey this literature.

There are various analyses of the approximation error of a given function by piecewise polynomials with a fixed number of pieces of fixed order (see, e.g., [7], [8], [10], [12], [21] and the references cited there). In principle, such studies may relate to the finite-element method since that method leads to optimal approximations under the energy norm. The mentioned error estimates involve higher derivatives of the given function. With these results as a basis, a number of authors developed methods for the construction of optimal meshes for collocation and finite-difference methods ([11], [13], [14], [17], [22], [23]). For this the needed information about the derivatives of the solution is obtained from the approximate solution, for instance, by means of difference formulas. This procedure can be theoretically justified in the case of regular meshes (see, e.g., [20]). However, when there are abrupt changes in the mesh—as they arise with refinement techniques—then "internal boundary layers"

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appear in the error function (see, e.g., [2], [6]); and hence, the difference formulas caese to approximate well the desired derivatives.

Various results on optimizing finite-element meshes have appeared in the engineering literature. Without entering into any details, we mention, for instance, the articles [9], [18], [19], [25], [26] and [27].

In recent years, for initial value problems for ordinary differential equations, very effective procedures have been designed and analyzed for adapting the stepsizes and the order of the numerical methods (see, e.g., the survey [15]). The principal tool is the availability of an error analysis with a local, a posteriori character. These estimates are asymptotic in nature; yet practical experience has proved their reliability for reasonable tolerances.

In this paper we use a new approach to the construction of optimal finite-element meshes. It is based on a theory of a posteriori estimates for the finite-element method developed in [3], [4] (see also [5]). As in the case of the initial value problems, the estimates are asymptotic in character. More specifically, higher-order terms in the maximal meshsize $\bar{h}$ are neglected; that is, asymptotic expressions of the form $1 + o(1)$ as $\bar{h} \to 0$ are considered to be approximately equal to one. At the same time, all constant factors of these $(1 + o(1))$-terms can be evaluated computationally.

For clarity of presentation, we restrict the discussion to a simple two-point boundary value problem involving a linear, selfadjoint operator of second order. Moreover, for simplicity, we employ only piecewise linear elements. The approaches allow for rather straightforward extensions to a variety of more general problems in one dimension, and there are no essential limitations to the use of higher-order elements. In fact, analogous techniques even permit consideration of elements of different order in different parts of the mesh.

Continuous mesh distribution functions are used to prove the existence of a unique, optimal mesh distribution and to analyze its properties. In particular, it is shown that the value of the optimal error is rather stable under perturbations of the optimal mesh. Hence, it is indeed unnecessary to compute this mesh with excessive accuracy. The optimal mesh is characterized in terms of certain easily computable local error indicators. This allows for a simple adaptive method to construct that mesh (see [4]) and, at the same time, to compute very effective a posteriori error bounds. Although, as mentioned, the error estimates have only asymptotic character, numerical experience shows that, as in the case of initial value problems, the results are excellent already for accuracies of the order of ten percent.

2. Notation. Let $I = I(\alpha, \beta)$, $\alpha < \beta$, be the open interval $\{x \in R^1; \alpha < x < \beta\}$ and $\bar{I} = \bar{I}(\alpha, \beta)$ its closure. As usual, $H^0(I)$ denotes the space of square-integrable functions on $\bar{I}$ and $C^0(I) \subset H^0(I)$ the subspace of continuous functions on $\bar{I}$. The norms on $H^0(I)$ and $C^0(I)$ will be written as ${}_I\|\cdot\|_0$ and ${}_I\|\cdot\|_c$, respectively.

Define $\mathcal{E}(I)$ as the space of real, infinitely differentiable functions on I for which all derivatives have continuous extensions on $\bar{I}$. Moreover, let $\mathcal{D}(I) \subset \mathcal{E}(I)$ be the subspace of all functions with compact support in I. For any integer $k \geq 1$, the spaces

$H^k(I)$ and $C^k(I)$ are the completions of $\mathcal{E}(I)$ under the norms

$$(2.1)\qquad {}_I\|u\|_k^2 = \sum_{i=0}^{k} {}_I\left\|\frac{d^iu}{dx^i}\right\|_0^2$$

and

$$(2.2)\qquad {}_I\|u\|_{c,k} = \sum_{i=0}^{k} {}_I\left\|\frac{d^iu}{dx^i}\right\|_c,$$

respectively. Analogously, the completions of $\mathcal{D}(I)$ under these two norms are the spaces $H_0^k(I)$ and $C_0^k(I)$.

Let $a, b \in C^0(I)$ be given such that $a(x) \geq \underline{a} > 0$, $b(x) \geq 0$, $\forall x \in I$. Then $E(I)$ and $E_0(I)$ shall be the spaces $H^1(I)$ and $H_0^1(I)$, respectively, with their norm replaced by

$$(2.3)\qquad {}_I\|u\|_E^2 = \left[\int_I (au'^2 + bu^2)\,dx\right], \qquad \left(u' = \frac{du}{dx}\right).$$

If $b \equiv 0$ on I, then (2.3) is only a seminorm on $E(I)$. On the other hand, on $E_0(I)$, (2.3) is always a norm which, moreover, is equivalent to ${}_I\|\cdot\|_1$. Obviously, $E_0(I)$ is a Hilbert space and for $b \not\equiv 0$ the same is true for $E(I)$. We denote the inner product in either space by ${}_I(\cdot\,,\cdot)_E$. For $b \equiv 0$ on I, $E(I)$ is a Hilbert space modulo the constant functions.

On $\bar{I} = \bar{I}(\alpha, \beta)$ we consider partitions

$$(2.4)\qquad \Delta(I)\colon \alpha = x_0^\Delta < x_1^\Delta < \cdots < x_{m-1}^\Delta < x_m^\Delta = \beta, \qquad m = m(\Delta) \geq 1,$$

and introduce the notations

$$(2.5)\qquad \left.\begin{aligned} I_j(\Delta) &= I(x_{j-1}^\Delta, x_j^\Delta) \\ h_j(\Delta) &= x_j^\Delta - x_{j-1}^\Delta \end{aligned}\right\}, \qquad j = 1, 2, \ldots, m,$$

$$\bar{h}(\Delta) = \max_{j=1,\ldots,m} h_j(\Delta), \qquad \underline{h}(\Delta) = \min_{j=1,\ldots,m} h_j(\Delta).$$

All partitions Δ which for fixed $\lambda > 0$, $\kappa \geq 1$ satisfy

$$(2.6)\qquad \underline{h}(\Delta) \geq \lambda \bar{h}(\Delta)^\kappa$$

are said to be (λ, κ)-regular.

For given $\Delta = \Delta(I)$, we denote by $S(I, \Delta) \subset H^1(I)$ and $S_0(I, \Delta) \subset H_0^1(I)$ the subspaces of all functions for which the restriction to any $I_j(\Delta)$, $j = 1, \ldots, m$, is linear. Analogously, $P^k(I, \Delta) \subset H^1(I)$ and $P_0^k(I, \Delta) \subset H_0^1(I)$, $k \geq 0$, shall consist of the functions for which the restrictions to $I_j(\Delta)$, $j = 1, \ldots, m$, belong to $C^k(I_j)$.

For later use we note the following well-known lemma (see, e.g., [24]):

LEMMA 2.1. *For given $I = I(\alpha, \beta)$, $\alpha < \beta$, and $\Delta = \Delta(I)$ there exists a positive constant K such that*

$$(2.7)\qquad \inf_{w \in S_0(I,\Delta)} {}_I\|u - w\|_E \leq K\bar{h}(\Delta)\,{}_I\|u\|_2 \quad \forall u \in H^2(I) \cap H_0^1(I).$$

3. A Boundary Value Problem.

3.1. *Basic Formulation.* As mentioned in the introduction, we restrict the discussion to a simple model problem. For ease of notation, the unit interval $I = I(0, 1)$ is used from now on throughout the remainder of the paper. On I we consider the equation

$$L[u] \equiv -\frac{d}{dx}\, a(x)\, \frac{du}{dx} + b(x)u = f(x), \qquad x \in I(0, 1), \tag{3.1}$$

together with the boundary conditions

$$u(0) = u(1) = 0. \tag{3.2}$$

We assume that $a \in C^2(I)$, $b, f \in C^1(I)$, and, as before, that $a(x) \geqslant \underline{a} > 0$, $b(x) \geqslant 0$, $\forall x \in I$.

The weak solution of the problem is the unique $u_0 \in E_0(I)$ with

$${}_I(u_0, v)_E = F_f(v) \quad \forall v \in E_0(I), \tag{3.3}$$

where

$$F_f(v) = \int_I fv\, dx. \tag{3.4}$$

Note that under our differentiability assumptions about a, b, f the solution u_0 of (3.3) belongs to $C^3(I)$ and also satisfies (3.1/2).

With the partition Δ and the space $S_0(I, \Delta) \subset H_0^1(I)$ specified as in Section 2, we consider the finite-element solution $u_\Delta \in S_0(I, \Delta)$ defined by

$${}_I(u_\Delta, v)_E = F_f(v) \quad \forall v \in S_0(I, \Delta). \tag{3.5}$$

Since $u_0 \in H^2(I) \cap H_0^1(I)$, it then follows from Lemma 2.1 that

$${}_I\|u_\Delta - u_0\|_E \leqslant K\bar{h}(\Delta)_I\|u_0\|_2. \tag{3.6}$$

3.2. *A Posteriori Error Analysis.* We consider the residual $r = L(u_\Delta) - f$ on the intervals I_j, that is,

$$r_j(x) = (L(u_\Delta) - f)(x) \quad \forall x \in I_j, j = 1, \ldots, m. \tag{3.7}$$

Let $z_j \in E_0(I_j)$ be the solutions of

$${}_{I_j}(z_j, v)_E = F_{r_j}(v) \quad \forall v \in E_0(I_j),\ j = 1, \ldots, m, \tag{3.8}$$

and set

$$Z(\Delta)^2 = \sum_{j=1}^{m} {}_{I_j}\|z_j\|_E^2. \tag{3.9}$$

The following result was proved in [3]:

THEOREM 3.1. *The error $e = u_\Delta - u_0$ satisfies*

$${}_I\|e\|_E^2 = Z(\Delta)^2(1 + O(\bar{h}(\Delta))) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0, \tag{3.10}$$

where the constant in the bound of the O-term depends on a and b but not on f and Δ.

We analyze the quantity $Z(\Delta)$ of (3.9) in some more detail. Because u_Δ is linear on any I_j, we may write

$$r_j = L(u_\Delta) - f = -a'u'_\Delta + bu_\Delta - f = \rho_j + \tau_j, \tag{3.11}$$

where

$$\rho_j(x) = a(x)u_0''(x), \qquad \tau_j(x) = -a'(x)e'(x) + b(x)e(x) \quad \forall x \in I_j. \tag{3.12}$$

Let $\varphi_j, \psi_j \in E_0(I_j)$ be such that

$$\begin{aligned} &\text{(a)} \quad {}_{I_j}(\varphi_j, v)_E = F_{\rho_j}(v) \quad \forall v \in E_0(I_j), \\ &\text{(b)} \quad {}_{I_j}(\psi_j, v)_E = F_{\tau_j}(v) \quad \forall v \in E_0(I_j) \end{aligned} \tag{3.13}$$

and, therefore, $z_j = \varphi_j + \psi_j$.

The smallest eigenvalue of the differential operator L on I_j with zero boundary conditions is bounded below by the smallest eigenvalue $\underline{a}\pi^2/h_j^2$ of the operator $-\underline{a}d^2/dx^2$ on I_j. Hence, it follows from (3.13(b)) that ${}_{I_j}\|\psi_j\|_0 \leq Ch_j^2\, {}_{I_j}\|\tau_j\|_0$; and therefore,

$$_{I_j}\|\psi_j\|_E^2 = F_{\tau_j}(\psi_j) \leq {}_{I_j}\|\tau_j\|_0\, {}_{I_j}\|\psi_j\|_0 \leq Ch_j^2\, {}_{I_j}\|\tau_j\|_0^2. \tag{3.14}$$

Here as in subsequent estimates C denotes a generic constant which has different values in each instance but is independent of the other essential variables in the same expression. Now note that

$$_{I_j}\|\tau_j\|_0^2 = \int_{I_j} (-a'e' + be)^2\, dx \leq C \int_{I_j} [(a'e')^2 + (be)^2]\, dx \leq C\, {}_{I_j}\|e\|_E^2, \tag{3.15}$$

which together with (3.14) gives

$$_{I_j}\|\psi_j\|_E \leq Ch_j\, {}_{I_j}\|e\|_E. \tag{3.16}$$

We introduce the quantities

$$R(\Delta) = \frac{1}{{}_I\|e\|_E}\left[\sum_{j=1}^m {}_{I_j}\|\psi_j\|_E^2\right]^{1/2}$$

and

$$Q(\Delta)^2 = \sum_{j=1}^m {}_{I_j}\|\varphi_j\|_E^2. \tag{3.17}$$

From Theorem 3.1 we obtain—with some $|\alpha| \leq 1$—

$$\begin{aligned} {}_I\|e\|_E^2 &= Z^2(\Delta)(1 + O(\bar{h})) = \sum_{j=1}^m {}_{I_j}(\varphi_j + \psi_j, \varphi_j + \psi_j)_E(1 + O(\bar{h})) \\ &= [Q(\Delta)^2 + 2\alpha Q(\Delta)\,{}_I\|e\|_E R(\Delta) + {}_I\|e\|_E^2 R(\Delta)^2](1 + O(\bar{h})) \\ &= (Q(\Delta) + \alpha\,{}_I\|e\|_E R(\Delta))^2(1 + O(\bar{h})) + (1 - \alpha^2)\,{}_I\|e\|_E^2 R(\Delta)^2(1 + O(\bar{h})). \end{aligned} \tag{3.18}$$

But by (3.16) we have $R(\Delta) = O(\bar{h})$; and thus

$$_I\|e\|_E^2 = (Q(\Delta) + \alpha\,{}_I\|e\|_E R(\Delta))^2(1 + O(\bar{h})), \tag{3.19}$$

which in turn implies that

$$ {}_I\|e\|_E = Q(\Delta)(1 + O(\bar{h})). \tag{3.20} $$

Therefore, in view of (3.10), we have proved the following result:

THEOREM 3.2. *Let $Z(\Delta)$ and $Q(\Delta)$ be defined by* (3.9) *and* (3.19), *respectively. Then*

$$ Q(\Delta)^2 = Z(\Delta)^2(1 + O(\bar{h})) \quad as\ \bar{h}(\Delta) \longrightarrow 0, \tag{3.21} $$

where the constant in the bound of the O-term depends on a and b but not on f and Δ.

While approximations of $Z(\Delta)$ can be computed and $Q(\Delta)$ is not readily accessible, the quantity Q is better suited than Z for our theoretical studies of optimal partitions Δ.

4. Optimal Partitions—Part I. In this section we restrict ourselves to the case when $u_0'' \neq 0$ on I. This condition will be removed in Section 5.

4.1. *Representations of $Q(\Delta)$.* Recall that under our assumptions about a, b, f we have $u_0 \in C^3(I)$ and, hence, $\rho = au_0'' \in C^1(I)$.

LEMMA 4.1. *Suppose that $u_0''(x) \neq 0$ for all $x \in I$ and set $x_{j-1/2} = (x_j + x_{j-1})/2$, $a_{j-1/2} = a(x_{j-1/2})$, $\bar{\rho}_j = \rho(\xi_j)$, where $|\rho(\xi_j)| = \max\{|\rho(x)|, x \in \bar{I}_j\}$. Then*

$$ Q(\Delta)^2 = \frac{1}{12}\left[\sum_{j=1}^{m} \frac{\bar{\rho}_j^2}{a_{j-1/2}} h_j^3\right](1 + O(\bar{h}(\Delta))) \quad as\ \bar{h}(\Delta) \longrightarrow 0, \tag{4.1} $$

where the constant in the bound of the O-term depends on a, b and ${}_I\|f\|_{C,1}$.

Proof. Set

$$ \sigma_j(x) = \rho(x) - \bar{\rho}_j \quad \forall x \in I_j, j = 1, \ldots, m, \tag{4.2} $$

and define $\varphi_{1,j}, \varphi_{2,j} \in E_0(I_j)$ as the solutions of

$$ \left.\begin{aligned} {}_{I_j}(\varphi_{1,j}, v)_E &= F_{\bar{\rho}_j}(v) \quad \forall v \in E_0(I_j) \\ {}_{I_j}(\varphi_{2,j}, v)_E &= F_{\sigma_j}(v) \quad \forall v \in E_0(I_j) \end{aligned}\right\}, \quad j = 1, \ldots, m, \tag{4.3} $$

respectively. Then we have $\varphi_j = \varphi_{1j} + \varphi_{2j}$.

By assumption there exists a constant $\rho_0 > 0$ such that $|\rho(x)| \geqslant \rho_0$ for all $x \in I$ and, hence, that for all sufficiently small $\bar{h}$

$$ |\sigma_j(x)| \leqslant C\frac{|\bar{\rho}_j|}{\rho_0} h_j \quad \forall x \in I_j. \tag{4.4} $$

Note also that for all $x \in I_j$

$$ \underline{a}_j = \min\{a(x); x \in I_j\} = a_{j-1/2}(1 + O(h_j)) \quad \text{as } h_j \longrightarrow 0. \tag{4.5} $$

Since for any $v \in E_0(I_j)$

$$ {}_{I_j}\|v\|_E^2 = \left[\int_{I_j} a(x)v'(x)^2\,dx\right](1 + O(h_j^2)) \quad \text{as } h_j \longrightarrow 0, $$

it follows from (4.3) and (4.5) that

$$
\begin{aligned}
{}_{I_j}\|\varphi_{1,j}\|_E^2 &= \left[\sup_{v\in E_0(I_j)} \frac{1}{{}_{I_j}\|v\|_E} |F_{\bar\rho_j}(v)|\right]^2 \\
&= \frac{\bar\rho_j^2}{\underline{a}_j}\left[\sup_{v\in E_0(I_j)} \left\{\left|\int_{I_j} v\,dx\right|^2 \Big/ \int_{I_j} (v')^2\,dx\right\}\right](1+O(h_j)) \\
&= \frac{\bar\rho_j^2}{\underline{a}_j}\left[\int_{I_j} (\tilde v')^2\,dx\right](1+O(h_j)),
\end{aligned}
$$

where $\tilde v \in E_0(I_j)$ is the solution of $-\tilde v'' = 1$, $\tilde v(x_{j-1}) = \tilde v(x_j) = 0$. This implies that

(4.6) $$ {}_{I_j}\|\varphi_{1,j}\|_E^2 = \frac{1}{12}\frac{\bar\rho_j^2}{a_{j-1/2}} h_j^3(1+O(h_j)) \quad \text{as } h_j \to 0,\ j = 1, \ldots, m. $$

In order to estimate $\varphi_{2,j}$ we proceed as in the proof of (3.14). The smallest eigenvalue of the operator L on I_j is bounded below by $\underline{a}_j\pi^2/h_j^2$. Hence, by (4.3) it follows that

(4.7) $$ {}_{I_j}\|\varphi_{2,j}\|_E^2 = F_{\sigma_j}(\varphi_{2,j}) \leqslant {}_{I_j}\|\varphi_{2,j}\|_0\ {}_{I_j}\|\sigma_j\|_0 \leqslant \frac{h_j^2}{\underline{a}_j\pi^2}\ {}_{I_j}\|\sigma_j\|_0^2, $$

which together with (4.4) implies that

(4.8) $$ {}_{I_j}\|\varphi_{2,j}\|_E^2 \leqslant \frac{C}{\rho_0^2\pi^2}\frac{\bar\rho_j^2}{a_{j-1/2}} h_j^5(1+O(h_j)) \quad \text{as } h_j \to 0,\ j = 1, \ldots, m. $$

Combining this with (4.6), we obtain—with some $|\alpha| \leqslant 1$—

(4.9) $$
\begin{aligned}
{}_{I_j}\|\varphi_j\|_E^2 &= {}_{I_j}\|\varphi_{1,j}\|_E^2 + 2\alpha\,{}_{I_j}\|\varphi_{1,j}\|_E\ {}_{I_j}\|\varphi_{2,j}\|_E + {}_{I_j}\|\varphi_{2,j}\|_E^2 \\
&= \frac{1}{12}\frac{\bar\rho_j^2}{a_{j-1/2}} h_j^3(1+O(h_j)) \quad \text{as } h_j \to 0,\ j = 1, \ldots, m.
\end{aligned}
$$

By definition (3.17) of $Q(\Delta)$ this proves the lemma.

A partition Δ shall be a (ξ, m)-partition if Δ is a refinement of a fixed partition Δ^* and

(4.10) $$ \xi(x_j) = j/m, \qquad j = 0, 1, \ldots, m, $$

for some function ξ, independent of m, with

(4.11) $$ \xi \in P^2(I, \Delta^*), \xi'(x) \geqslant \delta > 0\ \forall x \in I_j^*, j = 1, \ldots, m, \xi(0) = 0, \xi(1) = 1. $$

For example, for given Δ^*, a class of (ξ, m)-partitions is defined by the piecewise linear function $\xi \in S(I, \Delta^*)$

(4.12) $$ \xi(x) = \frac{j-1}{m^*} + \frac{1}{m^* h_j^*}(x - x_{j-1}) \quad \forall\, x \in \bar I_j^*, j = 1, \ldots, m^*. $$

For a (ξ, m)-partition we have

$$\frac{1}{m} = \int_{I_j} \xi'(t)\, dt = h_j \xi'(x_{j-1/2})(1 + O(h_j)) \quad \text{as } h_j \to 0, \tag{4.13}$$

where the constant in the bound of the O-term depends on ξ but not on m.

In terms of (ξ, m)-partitions our Lemma 4.1 can be rewritten as follows:

THEOREM 4.2. *For the* (ξ, m)*-partition* Δ *we have*

$$Q(\Delta)^2 = \frac{1}{12m^2}\left[\int_0^1 \left(\frac{\rho(x)}{\xi'(x)}\right)^2 \frac{1}{a(x)}\, dx\right](1 + O(\bar{h}(\Delta))) \quad \text{as } \bar{h}(\Delta) \to 0, \tag{4.14}$$

where the constant in the bound of the O-term depends on a, b, f and ξ *but not on m.*

Because $\rho \in C^1(I)$ and $1/\xi' \in P^1(I, \Delta)$, the proof follows directly from the fact that the expression for $Q(\Delta)^2$ in Lemma 4.1 is a Riemann sum of (4.14).

By combining Theorems 3.1, 3.2, and 4.2 we obtain the following result:

THEOREM 4.3. *For the* (ξ, m)*-partition* Δ *the error satisfies*

$$_I\|e\|_E^2 = \frac{1}{12m^2}\left[\int_0^1 \left(\frac{\rho(x)}{\xi'(x)}\right)^2 \frac{1}{a(x)}\, dx\right](1 + O(\bar{h}(\Delta))) \quad \text{as } \bar{h}(\Delta) \to 0, \tag{4.15}$$

where the constant in the bound of the O-term depends on a, b, f and ξ *but not on m.*

4.2. *Optimal Partitions.* The error formula of Theorem 4.3 suggests that we consider minimizing the variational integral

$$J(\xi) = \int_0^1 \left(\frac{\rho(x)}{\xi'(x)}\right)^2 \frac{1}{a(x)}\, dx \tag{4.16a}$$

subject to the boundary conditions

$$\xi(0) = 0, \quad \xi(1) = 1. \tag{1.16b}$$

The Euler equation is directly solvable and the functions

$$\xi(x, \gamma) = \gamma \int_0^x \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt, \qquad x \in I, \tag{4.17}$$

form a field of extremals in $0 < x, \gamma < 1$. A standard application of the E-function test (see, e.g., [1]) then proves the following result:

THEOREM 4.4. *For all functions* (4.10) *we have*

$$J(\xi) \geq J(\xi_0) = 1/\gamma_0^3, \tag{4.18}$$

where $\xi_0 = \xi(\cdot\,, \gamma_0)$ *with*

$$\gamma_0 = \left[\int_0^1 \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt\right]^{-1}. \tag{4.19}$$

Moreover, equality holds in (4.18) *exactly when* $\xi = \xi_0$.

Note that the function ξ_0 belongs to the class of functions (4.11); in fact, we have $\xi_0 \in C^2(I)$ and $\xi_0'(x) \geq \delta > 0$ for $x \in I_j$ because $|\rho(x)| \geq \rho_0$ in that interval. For the partition Δ_0 given by ξ_0 we obtain from Theorems 4.3 and 4.4 the following error formula:

THEOREM 4.5. *The* (ξ_0, m)*-partition* Δ_0 *is asymptotically optimal with*

$$(4.20)\qquad {}_I\|e\|_E^2 = \frac{1}{12m^2\gamma_0^3}(1 + O(\bar{h}(\Delta_0))) \quad \text{as } \bar{h}(\Delta) \to 0,$$

where the constant in the bound of the O-term depends on ξ_0 *but not on m.*

By "asymptotic optimality" we mean here that for any other (ξ, m)-partition with sufficiently small $\bar{h}$ the error is larger than (4.20).

For any (ξ, m)-partition Δ set

$$(4.21)\qquad \vartheta_j(\xi, m)^2 = \frac{1}{12m^2}\int_{I_j}\left[\frac{\rho(t)}{\xi'(t)}\right]^2 \frac{1}{a(t)}\,dt, \qquad j = 1, \ldots, m.$$

These ϑ_j are related to the functions φ_j of (3.13) by

$$(4.22)\qquad {}_{I_j}\|\varphi_j\|_E^2 = \vartheta_j(\xi, m)^2(1 + O(h_j)) \quad \text{as } h_j \to 0,\ j = 1, \ldots, m.$$

This follows directly from the fact that the expression (4.9) for the norms of φ_j is—up to a factor $(1 + O(h_j))$—a Riemann sum of (4.22).

For the optimal partition Δ_0 we have

$$(4.23)\qquad \begin{aligned} \vartheta_j(\xi_0, m)^2 &= \frac{1}{12m^2\gamma_0^2}\int_{I_j}\left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt \\ &= \frac{1}{12m^2\gamma_0^3}\int_{I_j}\xi_0'(t)\,dt = \frac{1}{12m^3\gamma_0^3}, \qquad j = 1, \ldots, m. \end{aligned}$$

In other words, for ξ_0 all ϑ_j are exactly equal and—by (4.22)—all ${}_{I_j}\|\varphi_j\|_E$ are asymptotically equal.

Since the ϑ_j are not readily computable, we turn now to the quantities

$$(4.24)\qquad \mu_j(\xi, m) = {}_{I_j}\|z_j\|_E, \qquad j = 1, \ldots, m,$$

which can be calculated. For the optimal partition Δ_0 we obtain from (3.16), (4.13) and Theorem 4.5 that

$$(4.25)\qquad {}_{I_j}\|\psi_j\|_E^2 = \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h}(\Delta_0)))(O(\bar{h}(\Delta_0))) \quad \text{as } \bar{h}(\Delta_0) \to 0,$$

where the constant in the O-term depends on a, b, f and ξ_0 but not on m. Thus, it follows from (4.22), (4.23) and (4.25) that

$$\begin{aligned} \mu_j(\xi_0, m)^2 = {}_{I_j}\|\varphi_j + \psi_j\|_E^2 &= \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h})) \\ &+ \frac{1}{\sqrt{12}m^{3/2}\gamma_0^{3/2}}\,\frac{1}{\sqrt{12}m^{3/2}\gamma_0^{3/2}}(1 + O(\bar{h}))O(\bar{h}^{1/2}) \\ &+ \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h}))O(\bar{h}) = \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h}^{1/2})). \end{aligned}$$

We summarize this in the following form:

THEOREM 4.6. *For the optimal partition Δ_0 we have*

$$\mu_j(\xi_0, m)^2 = \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h}(\Delta_0)^{1/2})) \quad \text{as } \bar{h}(\Delta_0) \longrightarrow 0, \tag{4.26}$$

where the constant in the bound of the O-term depends on ξ_0 but not on m.

Thus, we see that also the quantities $\mu_j(\xi_0, m)$ are asymptotically equal.

From (4.13) we find for the steps h_j of the optimal partition Δ_0 that

$$h_j = \frac{1}{\gamma_0 m}\left[\frac{a_{j-1/2}}{\bar{\rho}_j^2}\right]^{1/3}\left(1 + O\left(\frac{1}{m}\right)\right) \quad \text{as } m \longrightarrow \infty \tag{4.27}$$

and, hence, that $\underline{h}(\Delta_0)/\bar{h}(\Delta_0) \geqslant \lambda_0 > 0$ with a constant λ_0 that does not depend on m but only on the problem and ρ_0. In other words, the optimal partition is $(\lambda_0, 1)$-regular.

Conversely, it turns out that asymptotically the optimal partition Δ_0 is characterized by the asymptotic equality of the μ_j. This is the content of the following theorem:

THEOREM 4.7. *For the partition Δ suppose that*

$$_{I_j}\|z_j(\Delta)\|_E = \mu(1 + O(\bar{h}^{1/2})) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m, \tag{4.28}$$

where μ does not depend on j. Then

$$_I\|e(\Delta)\|_E = {}_I\|e(\Delta_0)\|_E(1 + O(\bar{h}^{1/2})) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m, \tag{4.29}$$

and

$$|x_j^\Delta - x_j^{\Delta_0}| = O(\bar{h}^{1/2}) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m. \tag{4.30}$$

Proof. We show first that Δ is $(\lambda, 1)$-regular. For ease of notation let

$$\eta_j = \left[\frac{1}{12}\frac{\bar{\rho}_j^2}{a_{j-1/2}}\right]^{1/2}, \quad j = 1, \ldots, m.$$

For any $j = 1, \ldots, m$, we have by (4.9), with some $|\alpha| \leqslant 1$,

$$\begin{aligned} {}_{I_j}\|z_j\|_E^2 &= {}_{I_j}\|\varphi_j + \psi_j\|_E^2 \\ &= \eta_j^2 h_j^3(1 + O(\bar{h})) + 2\alpha\eta_j h_j^{3/2}\,{}_{I_j}\|\psi_j\|_E(1 + O(\bar{h})) + {}_{I_j}\|\psi_j\|_E^2. \end{aligned} \tag{4.31}$$

Let now $h_{j_0} = \bar{h}$. Then (3.16) and (3.6) show that

$$_{I_{j_0}}\|\psi_{j_0}\|_E \leqslant C\bar{h}\ {}_{I_{j_0}}\|e\|_E \leqslant C\bar{h}^2$$

or

$$_{I_{j_0}}\|\psi_{j_0}\|_E = \bar{h}^{3/2}O(\bar{h}^{1/2}) \quad \text{as } \bar{h} \longrightarrow 0. \tag{4.32}$$

Hence, we obtain from (4.31) and (4.28) that

$$\mu^2(1 + O(\bar{h}^{1/2})) = \eta_{j_0}^2\bar{h}^3(1 + O(\bar{h})) + \bar{h}^3 O(\bar{h}^{1/2}) + \bar{h}^3 O(\bar{h})$$

or

$$\mu^2 = \eta_{j_0}^2\bar{h}^3(1 + O(\bar{h}^{1/2})) \quad \text{as } \bar{h} \longrightarrow 0. \tag{4.33}$$

Now let $h_{j_1} = \underline{h}$. Then we have instead of (4.32)

$$ {}_{I_{j_1}}\|\psi_{j_1}\|_E = \underline{h}\,\overline{h}^{1/2}O(\overline{h}^{1/2}); $$

and hence, by (4.31), (4.28), and (4.33)

$$ \begin{aligned} \eta_{j_0}^2\overline{h}^3(1+O(\overline{h}^{1/2})) = \eta_{j_1}^2\underline{h}^3(1+O(\overline{h})) + 2\alpha\eta_{j_1}\underline{h}^{5/2}\overline{h}^{1/2}O(\overline{h}^{1/2}) \\ + \underline{h}^2\overline{h}O(\overline{h}). \end{aligned} \tag{4.34} $$

In other words, $z^2 = \overline{h}/\underline{h}$ satisfies the polynomial equation

$$ \eta_{j_0}^2(1 + O(\overline{h}^{1/2}))z^6 + O(\overline{h})z^4 + O(\overline{h}^{1/2})z - \eta_{j_1}^2(1 + O(\overline{h})) = 0. \tag{4.35} $$

By comparing (4.35) with the equation

$$ \eta_{j_0}^2(1 + O(\overline{h}^{1/2}))z^6 - \eta_{j_1}^2(1 + O(\overline{h})) = 0, \tag{4.36} $$

Rouche's theorem shows that for sufficiently small $\overline{h}$ we have

$$ |z| \leqslant 2(\eta_{j_1}/\eta_{j_0})^{1/3}. \tag{4.37} $$

This shows that Δ is indeed $(\lambda, 1)$-regular.

Our assumption about a and ρ ensure that for any fixed j, $1 \leqslant j \leqslant m$,

$$ \sum_{i=1}^m \eta_i^2 h_i \leqslant C\eta_j^2. $$

We represent Δ by the piecewise linear function ξ of (4.12). Then from Theorems 3.1, 3.2, and Lemma 4.1 it follows that

$$ {}_I\|e\|_E^2 = \left[\sum_{i=1}^m \eta_i^2 h_i^3\right](1 + O(\overline{h})) \leqslant C\eta_j^2\overline{h}^2, \tag{4.38} $$

and hence, by (3.16) that

$$ {}_{I_j}\|\psi_j\|_E^2 \leqslant Ch_j^2\,{}_{I_j}\|e\|_E^2 \leqslant C\eta_j^2h_j^2\overline{h}^2 \leqslant C\eta_j^2h_j^3\overline{h}, \tag{4.39} $$

where in the last inequality the regularity of Δ was used. Therefore, we have ${}_{I_j}\|\psi_j\|_E = \eta_jh_j^{3/2}O(\overline{h}^{1/2})$, whence by (4.31)

$$ {}_{I_j}\|z_j\|_E^2 = \eta_j^2h_j^3[(1 + O(\overline{h})) + O(\overline{h}^{1/2}) + O(\overline{h})] = \eta_j^2h_j^3(1 + O(\overline{h}^{1/2})); $$

that is, by (4.28)

$$ \mu^2 = \eta_j^2h_j^3(1 + O(\overline{h}^{1/2})). \tag{4.40} $$

Since

$$ [12\eta_j^2]^{1/3}h_j = \left[\frac{\overline{\rho}_j^2}{a_{j-1/2}}\right]^{1/3}h_j = \left[\int_{I_j}\left(\frac{\rho(t)^2}{a(t)}\right)^{1/3}dt\right](1 + O(\overline{h})) \quad \text{as } \overline{h} \longrightarrow 0, $$

(4.40) implies that

$$ \int_{I_j}\left[\frac{\rho(t)^2}{a(t)}\right]^{1/3}dt = (12\mu^2)^{1/3}(1 + O(\overline{h}^{1/2})), \tag{4.41} $$

which by (4.19) shows that

$$\frac{1}{\gamma_0} = \int_0^1 \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt = m(12\mu^2)^{1/3}(1 + O(\bar{h}^{1/2})). \tag{4.42}$$

By (4.38) and (4.40) we obtain now ${}_I\|e\|_E^2 = m\mu^2(1 + O(\bar{h}^{1/2}))$, which by (4.42) and Theorem 4.5 gives (4.29). Finally, from (4.41) it follows by summation over the first j intervals that

$$\int_0^{x_j^\Delta} \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt = \frac{j}{m\gamma_0}(1 + O(\bar{h}^{1/2})).$$

On the other hand, we obtain from (4.23)

$$\int_0^{x_j^{\Delta_0}} \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt = \frac{j}{m\gamma_0}(1 + O(\bar{h}^{1/2})).$$

Thus, we have

$$\left|\int_{x_j^{\Delta_0}}^{x_j^\Delta} \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt\right| = O(\bar{h}^{1/2}),$$

which implies (4.30).

4.3. *Computational Aspects.* Suppose that the optimal partition function ξ_0 is changed to some partition function $\xi = \xi_0 + \epsilon$ satisfying (4.11). By (4.13) the derivatives control the stepsizes; and hence, we assume that ${}_I\|\epsilon'\|_c$ is small. For any given m, let ${}_I\|e\|_E$ and ${}_I\|e_0\|_E$ be the errors associated with the (ξ, m)-partition and (ξ_0, m)-partition, respectively. Then by Theorem 4.3 and (4.16a) we have

$$|{}_I\|e\|_E^2 - {}_I\|e_0\|_E^2| = O(1/m^2)|J(\xi) - J(\xi_0)| \quad \text{as } m \longrightarrow \infty.$$

Since the variational integral J is stationary at ξ_0, we have $J'(\xi_0) = 0$; and hence, it follows from the mean-value theorem that $|J(\xi) - J(\xi_0)| = O({}_I\|\epsilon'\|_c^2)$. Therefore, because (4.15) implies that

$${}_I\|e\|_E + {}_I\|e_0\|_E = O(1/m) \quad \text{as } m \longrightarrow \infty,$$

we obtain

$$|{}_I\|e\|_E - {}_I\|e_0\|_E| = O(1/m)O({}_I\|\epsilon'\|_c^2) \quad \text{as } {}_I\|\epsilon'\|_c \longrightarrow 0,\ m \longrightarrow \infty.$$

In other words, a change of ξ_0' by some small ${}_I\|\epsilon'\|_c$ leads to a change of the error proportional to ${}_I\|\epsilon'\|_c^2$. This shows that the value of the optimal error is rather stable under perturbations of the optimal partition. On the other hand, the optimal partition itself is not too stable and, hence, needs to be computed only with relatively low accuracy.

By Theorem 4.7 the optimal partition is characterized by the asymptotic equality of the quantities $\mu_j = \mu_j(\xi_0, m)$ of (4.24). Let r_j again denote the residual $r = L(u_\Delta) - f$ on the subinterval I_j and set

$$\nu_j^2 = \int_{I_j} r_j(x)^2\, dx, \qquad j = 1, \ldots, m. \tag{4.43}$$

We call the quantities

$$\epsilon_j^2 = \frac{1}{12}\frac{\nu_j^2 h_j^2}{a_{j-1/2}}, \quad j = 1, 2, \ldots, m, \tag{4.44}$$

the error indicators for the intervals $I_1, \ldots, I_m$, and set

$$\epsilon(\Delta) = \Big(\sum_{j=1}^{m} \epsilon_j(\Delta)^2\Big)^{1/2}. \tag{4.45}$$

All these quantities are directly computable once the finite-element solution is known. The next theorem shows that $\epsilon(\Delta)$ is asymptotically equal to the error ${}_I\|e\|_E$.

THEOREM 4.8. (a) *For any partition* Δ *we have*

$$ {}_I\|e\|_E^2 = \epsilon(\Delta)^2(1 + O(\bar h)) \quad as\ \bar h(\Delta) \longrightarrow 0. \tag{4.46a}$$

(b) *If* Δ *is* (λ, κ)*-regular*, $1 \leqslant \kappa < 2$, *then*

$$\nu_j^2 = \bar\rho_j^2 h_j(1 + O(\bar h^{\epsilon})) \quad as\ \bar h(\Delta) \longrightarrow 0,\ j = 1, \ldots, m, \tag{4.46b}$$

where $\epsilon = 1 - \pi/2$.

Proof. (a) The definition of $\bar\rho_j$ given in Lemma 4.1 provides that

$$|\bar\rho_j| = \max_{x\in I_j} |\rho_j(x)| \geqslant \rho_0 > 0,$$

and by (4.4), and (3.15) we have

$$|\sigma_j(x)| \leqslant C\frac{1}{\rho_0}\bar\rho_j h_j, \qquad {}_{I_j}\|\tau_j\|_0 \leqslant C\ {}_{I_j}\|e\|_E.$$

Hence, by (3.11) and (4.2) we obtain with some $|\alpha_i| \leqslant 1, i = 1, 2, 3$, that

$$\begin{aligned}\nu_j^2 &= \int_{I_j} (\bar\rho_j + \sigma_j(x) + \tau_j(x))^2\,dx \\ &= \bar\rho_j^2 h_j + \frac{1}{\rho_0^2}\bar\rho_j^2 h_j^2 O(h_j) + {}_{I_j}\|\tau_j\|_0^2 \\ &\quad + 2\alpha_1\frac{1}{\rho_0}\bar\rho_j^2 h_j O(h_j) + 2\alpha_2\bar\rho_j h_j^{1/2}\ {}_{I_j}\|\tau_j\|_0 + 2\alpha_3\frac{1}{\rho_0}\bar\rho_j h_j^{3/2}\ {}_{I_j}\|\tau_j\|_0.\end{aligned} \tag{4.47}$$

With

$$S(\Delta)^2 = \frac{1}{12}\sum_{j=1}^{m}\frac{\bar\rho_j^2 h_j^3}{a_{j-1/2}}, \qquad T(\Delta)^2 = \sum_{j=1}^{m} {}_{I_j}\|\tau_j\|_0^2$$

it then follows that

$$\begin{aligned}\epsilon(\Delta)^2 &= S(\Delta)^2 + \frac{1}{\rho_0^2}S(\Delta)^2 O(\bar h^2) + T(\Delta)^2 O(\bar h^2) \\ &\quad + 2\frac{\alpha_1}{\rho_0}S(\Delta)^2 O(\bar h) + 2\alpha_2 S(\Delta)\,T(\Delta)O(\bar h) + 2\alpha_3 S(\Delta)T(\Delta)O(\bar h^2).\end{aligned}$$

This proves (4.46a) since by Lemma 4.1 and Theorems 3.1 and 3.2

$$S(\Delta)^2 = {}_I\|e\|_E^2(1 + O(\bar h)), \qquad T(\Delta)^2 \leqslant C\sum_{j=1}^{m} {}_{I_j}\|e\|_E^2 = C\ {}_I\|e\|_E^2.$$

(b) For (λ, κ)-regular Δ with $1 \leqslant \kappa < 2$ we have by (3.6)

$$_{I_j}\|\tau_j\|_0 \leqslant C\, _{I_j}\|e\|_E \leqslant C\bar{h} \leqslant C h_j^{1/2}\bar{h}^{\epsilon}.$$

Hence, (4.47) implies that

$$\nu_j^2 = \bar{\rho}_j^2 h_j\left[1 + \frac{1}{\rho_0^2} O(\bar{h}^2) + \frac{1}{\rho_0^2} O(\bar{h}^{2\epsilon}) + \frac{1}{\rho_0} O(\bar{h}) + \frac{1}{\rho_0} O(\bar{h}^{\epsilon}) + \frac{1}{\rho_0^2} O(\bar{h}^{3/2})\right]$$

which is (4.46b).

It may be noted that in [3] we proved an upper bound for $_I\|e\|_E$ of the form (4.46a) with 1/12 in (4.44) replaced by the larger factor $1/\pi^2$.

Theorem 4.8 states that

$$\theta(\Delta) = \frac{_I\|e\|_E}{\epsilon(\Delta)} = 1 + O(\bar{h}(\Delta)) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0. \tag{4.48}$$

In other words, the effectivity quotient θ tends to one with $\bar{h} \longrightarrow 0$. In contrast, the corresponding estimates in [3] only provided for $\theta^2 \leqslant 12/\pi^2$.

We expect the error indicators to be asymptotically equal to the quantities μ_j of (4.24). Theorem 4.9 below shows that this is indeed correct for regular partitions. Hence, our aim is to construct such partitions for which all ϵ_j are asymptotically equal. It turns out that—as before—these partitions are automatically regular.

THEOREM 4.9. (a) *Let Δ be a (λ, κ)-regular partition with $1 \leqslant \kappa < 2$. Then*

$$_{I_j}\|z_j\|_E^2 = \epsilon_j^2(1 + O(\bar{h}^{\epsilon})) \quad \textit{as } \bar{h}(\Delta) \longrightarrow 0, \tag{4.49}$$

with $\epsilon = 1 - \kappa/2$.

(b) *All partitions Δ for which*

$$\epsilon_j = \mu(1 + o(1)) \quad \textit{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \dots, m, \tag{4.50}$$

with μ independent of j, are $(\lambda, 1)$-regular.

Proof. (a) As in the case of (4.39), it follows from (3.16) and the assumed regularity of Δ that $_{I_j}\|\psi_j\|_E^2 \leqslant C\eta_j^2\bar{h}^2 \leqslant C\eta_j^2 h_j^3 \bar{h}^{2\epsilon}$ and, thus, $_{I_j}\|\psi_j\|_E = \eta_j h_j^{3/2} O(\bar{h}^{\epsilon})$ as $\bar{h}(\Delta) \longrightarrow 0$. Now (4.31) shows that

$$\begin{aligned} _{I_j}\|z_j\|_E^2 &= \eta_j^2 h_j^3(1 + O(\bar{h})) + 2\alpha\eta_j^2 h_j^3 O(\bar{h}^{\epsilon})\,(1 + O(\bar{h})) + \eta_j^2 h_j^3 O(\bar{h}^{2\epsilon}) \\ &= \eta_j^2 h_j^3(1 + O(\bar{h}^{\epsilon})). \end{aligned}$$

Since by Theorem 4.8(b)

$$\eta_j^2 h_j^3 = \frac{1}{12}\frac{\bar{\rho}_j^2 h_j^3}{a_{j-1/2}} = \epsilon_j^2(1 + O(\bar{h}^{\epsilon})), \tag{4.51}$$

this proves (4.49).

(b) Because, generally, $_{I_j}\|\tau_j\|_E \leqslant C\, _{I_j}\|e\|_E \leqslant C\bar{h}$, it follows from (4.47) that

$$\nu_j^2 = \bar{\rho}_j^2 h_j(1 + O(h_j)) + O(\bar{h}^{3/2}),$$

whence

$$\mu^2(1 + o(1)) = \epsilon_j^2 = \eta_j^2 h_j^3 + h_j^2\bar{h}O(\bar{h}^{1/2}).$$

Now suppose that (4.50) holds for Δ and that $h_{j_0} = \overline{h}$; then $\mu^2(1 + o(1)) = \overline{h}^3(\eta_{j_0}^2 + O(\overline{h}^{1/2}))$. Similarly, for $h_{j_1} = \underline{h}$ we obtain

$$\mu^2(1 + o(1)) = \eta_{j_1}^2 \underline{h}^3 + \underline{h}^2 \overline{h} O(\overline{h}^{1/2})$$

and, hence, $z = \overline{h}/\underline{h}$ satisfies

$$[\eta_{j_0}^2 + O(\overline{h}^{1/2})] z^3 = \eta_{j_1}^2 + O(\overline{h}^{1/2}) z.$$

By comparing this with the polynomial for z in which the last term on the right has been dropped, it follows by Rouche's theorem that $|z| \leqslant 2(\eta_{j_1}/\eta_{j_0})^{2/3}$ for sufficiently small $\overline{h}$ and, therefore, that Δ is $(\lambda, 1)$-regular.

Theorems 4.9 and 4.7 confirm that, as expected, our aim should be to construct partitions for which all ϵ_j are asymptotically equal. Then the error of the partition will be close to the asymptotically optimal error (4.20). A natural approach for this construction is the use of an adaptive mesh refinement algorithm of the form discussed, for instance, in [4]. We shall not repeat the details.

5. Optimal Partitions—Part II. In the previous section we assumed that $u_0''(x) \neq 0$ for $x \in I$. Clearly, this represents a severe restriction. Actually, the results are largely valid also when u_0'' has zeros in I, but the proofs become more delicate. We illustrate the approach for the frequent case when $u_0'' \in C^1(I)$ has finitely many simple roots in I, say

$$(5.1)\quad u_0''(\xi_k) = 0, \quad u_0'''(\xi_k) \neq 0, \qquad k = 1, \ldots, q, \ 0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_q \leqslant 1.$$

LEMMA 5.1. *Under the stated assumptions we have for any* (λ, κ)*-regular partition* Δ *with* $1 \leqslant \kappa < 2$,

$$(5.2)\qquad Q(\Delta)^2 = \frac{1}{12}\left[\sum_{j=1}^{m} \frac{\overline{\rho}_j^2}{a_{j-1/2}} h_j^3\right] (1 + O(\overline{h}(\Delta)^\epsilon)) \quad \text{as } \overline{h}(\Delta) \to 0,$$

where $\epsilon = 1 - \kappa/2$ *and the constant in the bound of the O-term depends on a, b and f.*

Proof. Because of (5.1), we may choose $c_2 \geqslant c_1 > 0$, $\delta_0 > 0$, such that

$$(5.3)\qquad c_2|x - \xi_k| \geqslant |\rho(x)| \geqslant c_1|x - \xi_k| \qquad \forall |x - \xi_k| \leqslant \delta_0, k = 1, \ldots, q.$$

For any $\delta > 0$ we introduce the sets

$$(5.4)\qquad \begin{aligned} I_\delta &= \{x \in \overline{I} \mid |x - \xi_k| < \delta \text{ for some } \xi_k\}, \qquad I_\delta^c = \overline{I}\backslash I_\delta, \\ J_\delta &= \{j = 1, \ldots, m; I_j \cap I_\delta \neq \varnothing\}, \qquad J_\delta^c = \{1, \ldots, m\}\backslash J_\delta. \end{aligned}$$

We assume that $\delta_0 \leqslant (8q)^{-1}$ and, hence, that

$$\sum_{j \in J_{\delta_0}} h_j \leqslant 2(\delta_0 + \overline{h})q \leqslant 4\delta_0 q \leqslant \frac{1}{2} \quad \text{for } \overline{h} \leqslant \delta_0.$$

Since $\min\{|\rho(x)|, x \in I_{\delta_0}^c\} = \rho_0 > 0$, the estimate (4.9) of the proof of Lemma 4.1 holds for the subintervals I_j with $j \in J_{\delta_0}^c$; that is

$$_{I_j}\|\varphi_j\|_E^2 = \eta_j^2 h_j^3(1 + O(\overline{h})) \quad \text{as } \overline{h}(\Delta) \to 0, \ j \in J_{\delta_0}^c.$$

Hence, for $\bar{h} \leqslant \delta_0$ we have

$$(5.5) \qquad Q(\Delta)^2 \geqslant \left[\sum_{j\in J^c_{\delta_0}} \eta_j^2 h_j^3\right](1 + O(\bar{h})) \geqslant C\rho_0^2\lambda^2\bar{h}^{2\kappa} \sum_{j\in J^c_{\delta_0}} h_j \geqslant C\bar{h}^{2\kappa}.$$

Now consider the sets (5.4) with $\delta = \bar{h}^{\kappa/2} \leqslant \delta_0$. Then (5.3) implies that $|\bar{\rho}_j| \geqslant c_1\bar{h}^{\kappa/2}$, $j \in J^c_\delta$, and, hence, (4.8) modifies to

$$(5.6) \qquad {}_{I_j}\|\varphi_{2,j}\|_E^2 \leqslant C\frac{1}{\bar{h}^\kappa}\eta_j^2 h_j^5(1 + O(\bar{h})) \leqslant C\eta_j^2 h_j^3 \bar{h}^{2\epsilon} \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j \in J^c_\delta.$$

On the other hand, we have

$$|\sigma_j(x)| \leqslant 2\max_{x\in I_j}|\rho(x)| \leqslant C\bar{h}^{\kappa/2}, \qquad j \in J_\delta,$$

whence by (4.7) and (5.5)

$$(5.7) \qquad {}_{I_j}\|\varphi_{2,j}\|_E^2 \leqslant Ch_j^3\bar{h}^\kappa(1 + O(\bar{h})) \leqslant CQ(\Delta)^2\bar{h}^{2\epsilon}h_j \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j \in J_\delta.$$

For ease of notation set

$$S(\Delta)^2 = \sum_{j=1}^m \eta_j^2 h_j^3, \qquad R(\Delta)^2 = \sum_{j=1}^m {}_{I_j}\|\varphi_{2,j}\|_E^2.$$

Then it follows from (5.6) and (5.7) that

$$(5.8) \qquad \begin{aligned} R(\Delta)^2 &\leqslant C\left[\sum_{j\in J^c_\delta}\eta_j^2 h_j^3\right]\bar{h}^{2\epsilon} + CQ(\Delta)^2\bar{h}^{2\epsilon}\sum_{j\in J_\delta} h_j \\ &\leqslant C[S(\Delta)^2 + Q(\Delta)^2]\,\bar{h}^{2\epsilon}. \end{aligned}$$

In the case of Lemma 4.1 the assumption $u_0'' = 0$ does not enter into the proof of (4.6); and thus, we have also in the present case

$$(5.9) \qquad {}_{I_j}\|\varphi_{1,j}\|_E^2 = \eta_j^2 h_j^3(1 + O(\bar{h})) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m,$$

and, therefore,

$$(5.10) \qquad \sum_{j=1}^m {}_{I_j}\|\varphi_{1,j}\|_E^2 = S(\Delta)^2(1 + O(\bar{h})) \quad \text{as } \bar{h} \longrightarrow 0.$$

Altogether, with some suitable constants $\alpha, \beta \in [-1, 1]$ it follows from (5.8) and (5.10) that

$$\begin{aligned} Q(\Delta)^2 &= \sum_{j=1}^m {}_{I_j}\|\varphi_{1,j}\|_E^2 + 2\alpha\left(\sum_{j=1}^m {}_{I_j}\|\varphi_{1,j}\|_E^2\right)^{1/2}\left(\sum_{j=1}^m {}_{I_j}\|\varphi_{2,j}\|_E^2\right)^{1/2} + R(\Delta)^2 \\ &= S(\Delta)^2(1 + O(\bar{h})) + \alpha CS(\Delta)(1 + O(\bar{h}))(S(\Delta)^2 + Q(\Delta)^2)^{1/2}\bar{h}^\epsilon \\ &\quad + \beta C(S(\Delta)^2 + Q(\Delta)^2)\bar{h}^{2\epsilon}. \end{aligned}$$

After separating the middle term on the right and squaring, we obtain the equation

$$\begin{aligned} &\alpha^2 CS(\Delta)^2(S(\Delta)^2 + Q(\Delta)^2)\bar{h}^{2\epsilon} \\ &\quad = Q(\Delta)^4(1 + O(\bar{h}^{2\epsilon})) - 2Q(\Delta)^2 S(\Delta)^2(1 + O(\bar{h}^{2\epsilon})) + S(\Delta)^4(1 + O(\bar{h}^{2\epsilon})), \end{aligned}$$

which has the solutions

$$Q(\Delta)^2 = S(\Delta)^2(1 + O(\bar{h}^{2\epsilon})) \pm [S(\Delta)^4 O(\bar{h}^{2\epsilon})]^{1/2} \quad \text{as } h(\Delta) \longrightarrow 0,$$

and, hence, proves (5.2).

Now the theory of Section 4 can be carried over to the present case. As before, we consider partition functions ξ, but here we need to weaken (4.11) by requiring instead of $\xi'(x) \geq \delta > 0$ on each I_j that $\xi'(x)$ and $\rho(x)/\xi'(x)$ are Hölderian functions on $\bar{I}$. Moreover, we assume that for given ξ and $m \longrightarrow \infty$ the resulting (ξ, m)-partitions are (λ, κ)-regular with $1 \leq \kappa < 2$.

Then as in the case of Theorem 4.2 it follows that

$$(5.11) \quad Q(\Delta)^2 = \frac{1}{12m^2}\left[\int_0^1 \left(\frac{\rho(x)}{\xi'(x)}\right)^2 \frac{1}{a(x)}\,dx\right](1 + o(\bar{h})) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,$$

where the constant in the bound of the O-term depends on a, b, f and ξ but not on m. This suggests again consideration of the variational problems (4.16a/b) and, hence, of the optimal partition function ξ_0 of Theorem 4.4. Clearly,

$$\frac{\rho(x)}{\xi_0'(x)} = \frac{1}{\gamma_0}(\rho(x)a(x))^{1/3}$$

and $\xi_0'(x)$ are Hölderian functions on $\bar{I}$.

We show first that the (ξ_0, m)-partitions are $(\lambda, 5/3)$-regular. Since

$$(5.12) \qquad |\rho(x)| \geq C \min\left\{1, \min_k |x - \xi_k|\right\} \quad \forall x \in I,$$

with some $C > 0$, it follows that

$$C_1 h_j \geq \int_{I_j} \xi_0'(x)\,dx = \frac{1}{m} \geq C_2 \int_{I_j} \rho(x)^{2/3}\,dx \geq C_3 h_j^{5/3},$$

where again $C_3 > 0$. This implies that $\bar{h} \leq Cm^{-3/5}$ as well as $1/m \leq C\underline{h}$; and hence, the partition is indeed $(\lambda, 5/3)$-regular. Now Theorem 4.5 is easily shown to hold with $O(\bar{h})$ replaced by $O(\bar{h}^{1/6})$ because by using (5.5) we need to sum (5.2) only over J_δ^c and integrate (5.11) only over I_δ^c, $\delta = \bar{h}^{\kappa/2}$.

For any I_j which intersects I_δ^c with $\delta = \bar{h}^{5/6}$ we can use (5.6) in the estimates leading to Theorem 4.6 to obtain

$$(5.13) \quad \mu_j(\xi_0, m)^2 = \frac{1}{12m^3\gamma_0^3}(1 + O(\bar{h}^{1/6})) \quad \text{as } \bar{h} \longrightarrow 0,\ j \in J_\delta^c,\ \delta = \bar{h}^{5/6}.$$

In other words, for the optimal partition the μ_j are asymptotically equal for all intervals which are not too close to a root of u_0''. As the numerical examples of Section 6 show, the $\mu_j(\xi_0, m)$ for the intervals close to roots are generally larger; and the ratio of the largest to the smallest of these values does not tend to one for $m \longrightarrow \infty$.

The analog of Theorem 4.7 is somewhat more complicated. We formulate it as the following theorem.

THEOREM 5.2. *For the partition Δ suppose that*

$$(5.14) \qquad {}_{I_j}\|z_j(\Delta)\|_E = \mu(1 + O(\bar{h}^{\epsilon})) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m,$$

where μ does not depend on j and $\epsilon = 1/12$. Then Δ is $(\lambda, 5/3)$-regular and

$$ {}_I\|e(\Delta)\|_E \leqslant {}_I\|e(\Delta_0)\|_E(1 + O(\bar{h}^{\epsilon})) \quad \text{as } \bar{h}(\Delta) \to 0. \tag{5.15}$$

Proof. (1) We show first that

$$ {}_{I_j}\|\varphi_j\|_E \geqslant C h_j^{5/2}, \qquad j = 1, \ldots, m, \tag{5.16}$$

with some $C > 0$ which depends only on ρ. From (3.13) it follows that

$$ {}_{I_j}\|\varphi_j\|_E = \sup\left\{\frac{1}{{}_{I_j}\|v\|_E}\left|\int_{I_j}\rho(x)v(x)\,dx\right| \ \forall v \in E_0(I_j)\right\}.$$

Clearly, for given γ, δ the function

$$v_j(x) = \left(\frac{1}{2}\gamma h_j + \frac{1}{6}\delta h_j^2\right)(x - x_{j-1}) - \frac{1}{2}\gamma(x - x_{j-1})^2 - \frac{1}{6}\delta(x - x_{j-1})^3 \quad \forall x \in I_j$$

belongs to $E_0(I_j)$ and a short calculation shows that

$$C_1\alpha_j^2 h_j^3 \leqslant \int_{I_j} v'(x)^2\,dx \leqslant C_2\alpha_j^2 h_j^3, \qquad \int_{I_j} v(x)^2\,dx \leqslant C_3\alpha_j^2 h_j^5,$$

where $\alpha_j = (\gamma^2 + (\delta h_j)^2)^{1/2}$ and all constants are positive. Hence, also ${}_{I_j}\|v_j\|_E^2 \leqslant C\alpha_j^2 h_j^3$. Now with

$$\gamma = \rho(x_{j-1}), \quad \delta = \rho'(x_{j-1}), \quad \rho(x) = -v_j''(x) + \tilde{\rho}_j(x) \quad \forall x \in I_j,$$

$$|\tilde{\rho}_j(x)| = o(h_j) \quad \text{as } h_j \to 0,$$

we obtain

$$\left|\int_{I_j}\rho(x)v(x)\,dx\right| \geqslant \int_{I_j} v'(x)^2\,dx - \left(\int_{I_j}\tilde{\rho}_j(x)^2\,dx\right)^{1/2}\left(\int_{I_j} v(x)^2\,dx\right)^{1/2},$$

whence—for sufficiently small $\bar{h}$—

$$ {}_{I_j}\|\varphi_j\|_E \geqslant C\frac{1}{\alpha_j h_j^{3/2}}[\alpha_j^2 h_j^3 - o(h_j)h_j^{1/2}\alpha_j h_j^{5/2}] \geqslant C\alpha_j h_j^{3/2}\left[1 - \frac{1}{\alpha_j}o(h_j)\right]. \tag{5.17}$$

By assumption we have

$$u_0''(x)^2 + u_0'''(x)^2 \geqslant C > 0 \quad \forall x \in I; \tag{5.18}$$

and hence, also, with some $C > 0$

$$\alpha_j^2 = \gamma^2 + (\delta h_j)^2 \geqslant h_j^2(\rho(x_{j-1})^2 + \rho'(x_{j-1})^2) \geqslant C h_j^2 > 0,$$

which together with (5.17) proves (5.16).

(2) Next we show that for sufficiently small h_j

$$ {}_I\|e\|_E \geqslant {}_{I_j}\|e\|_E \geqslant C h_j^{5/2}, \qquad C > 0. \tag{5.19}$$

Obviously, we have

$$ {}_{I_j}\|e\|_E^2 \geqslant \inf_v \int_{I_j} a(x)[u_0'(x) - v'(x)]^2\,dx,$$

where the infimum is taken over all linear functions on I_j. Thus, $v'(x)$ is constant; and it follows that

$$_{I_j}\|e\|_E^2 \geqslant C\left[\int_{I_j} u_0'(x)^2\,dx - \frac{1}{h_j}\left(\int_{I_j} u_0'(x)\,dx\right)^2\right].$$

Since

$$u_0'(x) = u_0'(x_{j-1}) + u_0''(x_{j-1})(x - x_{j-1}) + \tfrac{1}{2}u_0'''(x_{j-1})(x - x_j)^2$$
$$+ o(h_j^2) \quad \text{as } h_j \longrightarrow 0,$$

a simple calculation shows that

$$_{I_j}\|e\|_E^2 \geqslant C[u_0''(x_{j-1})^2 h_j^3 + u_0'''(x_{j-1})^2 h_j^5] - o(1)h_j^5.$$

Using (5.18), we then obtain (5.19) for sufficiently small h_j.

(3) Now we can show that Δ is a $(\lambda, 5/3)$-regular partition. By Theorem 3.1 and (5.14) it follows that

$$_I\|e\|_E^2 = m\mu^2(1 + (\overline{h}^{1/12})), \tag{5.20}$$

and, thus, by (5.19) that

$$\overline{h} \leqslant Cm^{1/5}\mu^{2/5}. \tag{5.21}$$

On the other hand, we have by (5.9), (4.7) and (3.16)

$$\begin{aligned} \mu(1 + O(\overline{h}^{1/12})) &= {}_{I_j}\|\varphi_{1,j} + \varphi_{2,j} + \psi_j\|_E \\ &\leqslant C[\eta_j h_j^{3/2} + h_j^{5/2} + h_j\, {}_{I_j}\|e\|_E] \leqslant C[h_j^{3/2} + m^{1/2}\mu h_j]. \end{aligned} \tag{5.22}$$

Let $h_{j_0} = \underline{h}$ and note that $m \leqslant 1/\underline{h}$, then $\mu \leqslant C(\underline{h}^{3/2} + \underline{h}^{1/2}\mu)$; and hence, for sufficiently small $\overline{h}$,

$$\underline{h} \geqslant C\mu^{2/3}. \tag{5.23}$$

On the other hand, (5.16) and (3.16) give

$$\mu(1 + O(\overline{h}^{1/12})) \geqslant {}_{I_j}\|\varphi_j + \psi_j\|_E \geqslant {}_{I_j}\|\varphi_j\|_E - {}_{I_j}\|\psi_j\|_E \geqslant C(h_j^{5/2} - h_j\, {}_{I_j}\|e\|_E);$$

and therefore, by (5.21) and (5.23), for $h_{j_0} = \overline{h}$,

$$\mu \geqslant C[\overline{h}^{5/2} - \overline{h}m^{1/2}\mu] \geqslant C[\overline{h}^{5/2} - \overline{h}\mu^{2/3}]. \tag{5.24}$$

If the term on the right is positive then, with some positive constant (independent of h), $z = \overline{h}^{1/2}$ must satisfy $z^5 - \mu^{2/3}z^2 = C\mu$. By Cauchy's rule this implies that

$$\overline{h}^{1/2} \leqslant \max((2C\mu)^{1/5},\ (2\mu^{2/3})^{1/3}) \leqslant C\mu^{1/5},$$

since by (5.23) we have $\mu \leqslant 1$ for sufficiently small $\overline{h}$. On the other hand, if the term on the right of (5.24) is nonpositive, then we have immediately $\overline{h} \leqslant \mu^{4/9} \leqslant \mu^{2/5}$. Thus with (5.23) this gives indeed $\overline{h}^{5/3} \leqslant C\underline{h}$.

(4) For the proof of (5.15) let now $\delta = \overline{h}^\epsilon$. By (5.12) we have $\eta_j \geqslant C\delta$ for $j \in J_\delta^c$ and, hence,

$$\sum_{i=1}^m \eta_i^2 h_j \leqslant C\left(\frac{\eta_j}{\delta}\right)^2 \quad \forall j \in J_\delta^c.$$

Thus, from Lemma 5.1 and Theorems 3.1, 3.2 it follows that

$$(5.25)\qquad {}_I\|e\|_E^2 \leqslant C\frac{1}{\delta^2}\eta_j^2\bar{h}^2 \quad \forall j \in J_\delta^c,$$

and, therefore, (3.16) gives

$$(5.26)\qquad {}_{I_j}\|\psi_j\|_E \leqslant C\frac{\eta_j}{\delta}h_j\bar{h} < C\frac{\eta_j}{\delta}h_j^{3/2}\bar{h}^{2\epsilon} \leqslant C\eta_j h_j^{3/2}\bar{h}^{\epsilon}, \quad j \in J_\delta^c,$$

respectively. Note that in the second inequality the $(\lambda, 5/3)$-regularity of Δ was used. Together with (5.6) and (5.9), (5.26) leads to

$$ {}_{I_j}\|z_j\|_E^2 = {}_{I_j}\|\varphi_{1,j} + \varphi_{2,j} + \psi_j\|_E^2 = \eta_j^2 h_j^3(1 + O(\bar{h}^{2\epsilon})), \quad j \in J_\delta^c,$$

and, therefore, by (5.14) to

$$(5.27)\qquad \mu^2 = \eta_j^2 h_j^3(1 + O(\bar{h}^{\epsilon})), \quad j \in J_\delta^c.$$

Hence, analogous to (4.41), we have

$$(5.28)\qquad \int_{I_j}\left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt = (12\mu^2)^{1/3}(1 + O(\bar{h}^{\epsilon})), \quad j \in J_\delta^c.$$

Let m_1 and m_2 denote the cardinalities of J_δ^c and J_δ, respectively, that is, $m = m_1 + m_2$. We want to show that

$$(5.29)\qquad m_2/m_1 = O(\delta) = O(\bar{h}^{\epsilon}).$$

For this note first that because of $\eta_j \leqslant C\delta$ for $j \in J_\delta$ we have by (5.22)

$$(5.30)\qquad \mu \leqslant C(\delta h_j^{3/2} + h_j^{5/2} + h_j m^{1/2}\mu), \quad j \in J_\delta,$$

and, thus,

$$(5.31)\qquad h_j \geqslant C\min((\mu/\delta)^{2/3}, m^{-1/2}, \mu^{2/5}), \quad j \in J_\delta.$$

Suppose first that

$$h_j \geqslant C(\mu/\delta)^{2/3}, \quad j \in J_\delta;$$

then

$$(5.32)\qquad m_2 \leqslant \delta\left(\min_{j\in J_\delta} h_j\right)^{-1} \leqslant C\frac{\delta^{5/3}}{\mu^{2/3}}, \quad j \in J_\delta.$$

Because of $\eta_j \geqslant C\delta$ for $j \in J_\delta^c$, it follows from (5.27) that $h_j \leqslant C(\mu/\delta)^{2/3}$, $j \in J_\delta^c$, and, hence,

$$m_1 \geqslant (1-\delta)\left(\max_{j\in J_\delta^c} h_j\right)^{-1} \geqslant C\left(\frac{\delta}{\mu}\right)^{2/3}.$$

Together with (5.32) this proves (5.29) in this case.

Now suppose that in (5.31) we have $h_j \geqslant C\mu^{2/5}$. Then $m_2 \leqslant C\delta\mu^{-2/5}$, while (5.24) implies that $m_1 \geqslant C\mu^{-2/5}$. Together these estimates show once more that (5.29) holds.

Finally, consider the case $h_j \geqslant Cm^{-1/2}$, $j \in J_\delta$. Then $m_2 \leqslant C\delta m^{1/2} = C\delta(m_1 + m_2)^{1/2}$ or $m_2^2 - C^2\delta^2 m_2 \leqslant C^2\delta^2 m_1$. By applying Rouche's theorem to the pair of polynomials $z^2 - az - am_1 = 0$, $z^2 - am_1 = 0$, $a = C^2\delta^2$ it follows readily that $m_2 \leqslant C\delta m_1^{1/2} \leqslant C\delta m_1$. Thus (5.29) is valid, and we have

$$m_1 = m(1 + O(\bar{h}^{1/12})). \tag{5.33}$$

By definition of ξ_0 it follows now from (5.28) and (5.33) that

$$\frac{1}{\gamma_0} = \int_0^1 \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt = \sum_{j \in J_\delta^c} \int_{I_j} \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt + \sum_{j \in J_\delta} \int_{I_j} \left[\frac{\rho(t)^2}{a(t)}\right]^{1/3} dt$$

$$\geqslant m_1(12\mu^2)^{1/3}(1 + O(\bar{h}^\epsilon)) = m(12\mu^2)^{1/3}(1 + O(\bar{h}^\epsilon))$$

or

$$m\mu^2 \leqslant \frac{1}{12}\frac{1}{\gamma_0^3 m^2}(1 + O(\bar{h}^\epsilon)).$$

From (5.20) and Theorem 4.5 (modified to the present case) this implies (5.15).

It may be noted that the analogous relation to (4.30), namely, $|x_j^\Delta - x_j^{\Delta_0}| = O(\bar{h}^\epsilon)$ is easily proved when there is only one root of u_0'' in $\bar{I}$. In general, the situation appears to be more complicated.

Now we turn to the analog of Theorem 4.8.

THEOREM 5.3. *For any* (λ, κ)*-regular partition with* $1 \leqslant \kappa < 2$,

$$_I\|e\|_E^2 = \frac{1}{12}\left[\sum_{j=1}^m \frac{\nu_j^2 h_j^2}{a_{j-1/2}}\right](1 + O(\bar{h}^\epsilon)), \tag{5.34}$$

where ν_j *is given by* (4.44) *and* $\epsilon = 1 - \kappa/2$.

Proof. Recall that

$$\epsilon(\Delta)^2 = \frac{1}{12}\sum_{j=1}^m \frac{\nu_j^2 h_j^2}{a_{j-1/2}}.$$

Using

$$\nu_j^2 = \int_{I_j} (\bar{\rho}_j + \sigma_j(t) + \tau_j(t))^2\, dt, \quad |\sigma_j(x)| \leqslant Ch_j, \quad {}_{I_j}\|\tau_j\|_0 \leqslant C\, {}_{I_j}\|e\|_E, \tag{5.35}$$

we obtain, with certain $|C_i| \leqslant C$, $i = 1, \ldots, 5$,

$$12\epsilon(\Delta)^2 = \sum_{j=1}^m \frac{\bar{\rho}_j^2 h_j^3}{a_{j-1/2}} + C_1 \sum_{j=1}^m h_j^5 + C_2 \sum_{j=1}^m {}_{I_j}\|e\|_E^2 h_j^2$$

$$+ C_3 \sum_{j=1}^m \bar{\rho}_j h_j^4 + C_4 \sum_{j=1}^m {}_{I_j}\|e\|_E \bar{\rho}_j h_j^{5/2} + C_5 \sum_{j=1}^m {}_{I_j}\|e\|_E\, h_j^{7/2}.$$

By Lemma 5.1 and Theorems 3.1 and 3.2 the first term is asymptotically equal to $12\, {}_I\|e\|_E^2$ and from (5.5) it follows that ${}_I\|e\|_E \geqslant C\bar{h}^\kappa$. The other terms are then easily estimated to give—with different constants C_i—

$$\begin{aligned}\epsilon(\Delta)^2 &= {}_I\|e\|_E^2(1 + O(\bar{h}^\epsilon)) + C_1\, {}_I\|e\|_E^2\bar{h}^{4\epsilon} + C_2\, {}_I\|e\|_E^2\bar{h}^2 \\ &\quad + C_3\left[\sum_{j=1}^m \frac{\bar{\rho}_j^2 h_j^3}{a_{j-1/2}}\right]^{1/2}[\bar{h}^2 + {}_I\|e\|_E\bar{h}^{1/2}] + C_4\, {}_I\|e\|_E^2\bar{h}^{2\epsilon+1/2} \\ &= {}_I\|e\|_E^2(1 + O(\bar{h}^\epsilon)),\end{aligned}$$

which proves (5.34).

Finally, we show that also Theorem 4.9 carries over to this case.

THEOREM 5.4. *For the partition Δ suppose that*

$$\epsilon_j = \mu(1 + O(\bar{h}^\epsilon)) \quad \text{as } \bar{h}(\Delta) \longrightarrow 0,\ j = 1, \ldots, m, \tag{5.36}$$

where μ does not depend on j and $\epsilon = 1/12$. Then Δ is $(\lambda, 5/3)$-regular and

$$ {}_I\|e(\Delta)\|_E \leqslant {}_I\|e(\Delta_0)\|_E(1 + O(\bar{h}^\epsilon)) \quad \text{as } m \longrightarrow \infty. \tag{5.37}$$

Proof. We show first that Δ is $(\lambda, 5/3)$-regular. For this, note that generally

$$ {}_{I_j}\|z_j\|_E^2 \leqslant \frac{12}{\pi^2}\epsilon_j^2(1 + O(h_j)) \quad \text{as } h_j \longrightarrow 0,\ j = 1, \ldots, m.$$

This follows from (3.8) in the same manner as (3.14) follows from (3.13b). Thus by Theorem 3.1 and (5.36) we have ${}_I\|e\|_E^2 \leqslant Cm\mu^2$, whence by (5.19) $\bar{h} \leqslant Cm^{1/5}\mu^{2/5}$. Now (5.35) gives

$$\nu_j = {}_{I_j}\|\bar{\rho}_j + \sigma_j + \tau_j\|_0 \leqslant C(\bar{\rho}_j h_j^{1/2} + h_j^{3/2} + {}_{I_j}\|e\|_E) \tag{5.38}$$

and thus, as in the case of (5.23), for sufficiently small $\bar{h}$, $\underline{h} \geqslant C\mu^{2/3}$. By (5.12) we have

$$\int_{I_j} \rho(x)^2\, dx \geqslant Ch_j^3;$$

and thus,

$$\mu \geqslant Ch_j({}_{I_j}\|\rho\|_0 - {}_{I_j}\|\tau_j\|_0) \geqslant C(h_j^{5/2} - h_j\, {}_{I_j}\|e\|_E)$$

leads to (5.24). The remainder of the proof of the regularity of Δ now proceeds exactly as part (3) of the proof of Theorem 5.3.

Similarly, the proof of (5.37) follows that of (5.15). In fact, for $j \in J_\delta^c$ we have

$$ {}_{I_j}\|\sigma_j\|_0 = \frac{\bar{\rho}_j h_j^{1/2}}{\delta} O(h_j) = \bar{\rho}_j h_j^{1/2} O(\bar{h}^{1-\epsilon}),$$

$$ {}_{I_j}\|\tau_j\|_0 = O(\bar{h}) = \frac{\bar{\rho}_j}{\delta} h_j^{1/2} O(\bar{h}^{2\epsilon}) = \bar{\rho}_j h_j^{1/2} O(\bar{h}^\epsilon),$$

which as in (4.47) leads to $\nu_j^2 = \bar{\rho}_j^2 h_j^3(1 + O(\bar{h}^\epsilon))$, $j \in J_\delta^c$, that is, $\mu^2 = \eta_j^2 h_j^3(1 + O(\bar{h}^\epsilon))$, $j \in J_\delta^c$. Therefore, (5.28) holds again. Moreover, because of $\bar{\rho}_j \leqslant C\eta_j$ we obtain from (5.38) the estimates (5.30) and (5.31) which in turn imply (5.29). Now the remaining conclusions of the proof of Theorem 5.4 apply verbatim.

6. Numerical Examples. We illustrate the theoretical results with some computational results for the following two sample problems:

Sample Problem A.

$$-\frac{d}{dx}(x+\alpha)^p\frac{du}{dx}+(x+\alpha)^q u=f,\qquad 0<x<1,\ \alpha>0, \tag{6.1a}$$

$$u(0)=u(1)=0, \tag{6.1b}$$

where f is chosen such that the solution of (6.1a/b) is

$$u_0(x)=(x+\alpha)^r-[\alpha^r(1-x)+(1+\alpha)^r x],\qquad x\in\bar{I}. \tag{6.1c}$$

Here the coefficient functions and f are analytic in $\bar{I}$, and we have $u_0''(x)\neq 0$ for $x\in\bar{I}$. Hence, the theory of Section 4 applies. Note that for small α and negative r we can create severe near-singularities.

Sample Problem B.

$$-u''+u=f,\qquad 0<x<1, \tag{6.2a}$$

$$u(0)=u(1)=0, \tag{6.2b}$$

where f is chosen such that the true solution is

$$u_0(x)=e^{\alpha x}(x-\beta)+[\beta(1-x)-e^{\alpha}(1-\beta)x],\qquad \alpha\neq 0,\ \beta=1/2+2/\alpha. \tag{6.2c}$$

Here f is analytic on I and $u_0''(x)$ has a simple root at $x=1/2$, and hence we can apply the theory of Section 5.

The tables of computational results given below include the following data:

m	number of intervals used in the partition
$E=100_I\Vert e\Vert_E/_I\Vert u_0\Vert_E$	relative error in the energy norm expressed in percent
$E_0=\dfrac{1}{12m^3\gamma_0^2}$	asymptotically smallest error achievable with meshes of m intervals
$\theta={}_I\Vert e\Vert_E/\epsilon(\Delta)$	effectivity quotient (4.39)
$\omega=\left(\max\limits_{j=1,\ldots,m}\epsilon_j\right)\Big/\left(\min\limits_{j=1,\ldots,m}\epsilon_j\right)$	ratio between the largest and smallest value of the error indicators $\epsilon_i(\Delta)$ of (4.38a)
x_j	partition points of the particular mesh

Tables 1 through 6 concern two cases of Sample Problem A. Each time the left endpoint $x=0$ of the interval is near a singularity of u_0; and this is reflected in the fact that the largest and smallest error indicators $\epsilon_j(\Delta)$ always occur on the first and last subinterval I_1 and I_m, respectively. But in the second case the energy expression includes a weight $(x+\alpha)^2$ which goes strongly down near $x=0$. Thus in this case the near-singularity of the solution shows up more weakly under the energy norm.

In all cases the effectivity quotient is less than one; and hence, the estimate $\epsilon(\Delta)$ turns out to be an upper bound of ${}_I\Vert e\Vert_E$. Of course, the theory is only asymptotic in nature and, thus, $\epsilon(\Delta)$ could be smaller than ${}_I\|e\|_E$. Note that for relative accuracies better than 10% the estimate never overshoots the true error by more than 10%. In fact, for higher accuracies θ equals one for all practical purposes. This is in complete agreement with the theory and shows that the a posteriori error estimate is very reliable and not at all pessimistic.

In the presence of the near-singularity, the use of nonuniform meshes is very advantageous (see, e.g., Table 6), and the approximately optimal meshes produce errors close to the optimal values which by Theorem 4.5 decrease with $1/m$. Here the "weaker" singularity of the second case is rather noticeable. The nonuniform meshes are only approximately optimal as the ratio ω shows which in each case is reasonably close to one but certainly not equal to it. Nevertheless, as expected, the corresponding errors are clearly not very sensitive to such changes of the mesh except for low accuracies.

Tables 7 through 11 contain results for two cases of Sample Problem B. Essentially, all aspects are the same as for problem A. However, in all cases the maximal ϵ_j occurs in the neighborhood of the root $x_0 = 1/2$ of $u_0''(x)$, and, as expected, the ratio ω does not converge to one. However, if ω is computed only for all intervals outside a small neighborhood of x_0, then we have again the desired convergence of ω to one.

TABLE 1

Problem A with $p = 0, q = 1, r = -1/4, \alpha = 1/100$

Uniform mesh, ${}_I\|u_0\|_E = 6.09811$

m	E	E_0	θ	ω
5	85.301	22.613	.1706	8.84(+6)
10	73.768	11.306	.2950	2.34(+7)
20	58.784	5.653	.4702	5.31(+7)
40	41.933	2.827	.6708	1.11(+8)
80	26.316	1.413	.8419	2.19(+8)

TABLE 2

Problem A with $p = 0, q = 1, r = -1/4, \alpha = 1/100$

Asymptotically optimal mesh, ${}_I\|u_0\|_E = 6.09811$

m	E	E_0	θ	ω
5	22.243	22.613*	.6524	5.854
10	11.289	11.306	.9025	2.274
20	5.652	5.653	.9757	1.372
40	2.826	2.827	.9940	1.111
80	1.413	1.413	.9984	1.031

*Note here the asymptotic nature of the estimate E_0 of the lowest achievable error.

TABLE 3

Problem A with $p = 2, q = 1, r = -1, \alpha = 1/100$
Uniform mesh, ${}_I\|u_0\|_E = 0.28678$

m	E	E_0	θ	ω
5	25.215	9.619	.3270	3.227(+3)
10	13.696	4.809	.4467	4.640(+3)
20	7.296	2.404	.5966	8.316(+3)
40	3.813	1.202	.7610	1.055(+4)
80	1.959	0.601	.8947	1.308(+4)

TABLE 4

Problem A with $p = 2, q = 1, r = -1, \alpha = 1/100$
Asymptotically optimal mesh, ${}_I\|u_0\|_E = 0.28678$

m	E	E_0	θ	ω
5	9.994	9.619	.7679	3.355
10	4.809	4.809	.9331	1.666
20	2.409	2.404	.9828	1.184
40	1.203	1.202	.9956	1.049
80	0.601	0.601	.9989	1.012

TABLE 5

Problem A: Asymptotically optimal mesh for $m = 10$
Case A1: $p = 0, q = 1, r = -1/4, \alpha = 1/100$
Case A2: $p = 2, q = 1, r = -1, \alpha = 1/100$

	x_j	
j	Case A1	Case A2
0	.0000	.0000
1	.00207	.0127
2	.00487	.0340
3	.00877	.0676
4	.01443	.1170
5	.02308	.1862
6	.03732	.2798
7	.06318	.4025
8	.11781	.5598
9	.26831	.7569
10	1.00000	1.0000

TABLE 6

Problem A with $p = 0, q = 1, r = -1/4, \alpha = 1/100$
Partitions obtained from the uniform mesh
by successively subdividing the first interval into half

m	E	E_0	θ	ω
10	73.768	11.306	.2950	2.34(+7)
11	58.853	10.278	.4705	5.85(+6)
12	42.286	9.421	.6922	1.45(+6)
13	27.357	8.696	.8457	3.45(+5)
14	17.634	8.075	.9340	7.49(+4)
15	13.580	7.530	.9447	1.41(+4)

TABLE 7

Problem B with $\alpha = 1, \beta = 5/2$
Uniform mesh, ${}_I\|u_0\|_E = .071070$

m	E	E_0	θ	ω
5	43.462	32.317	.9759	1.126(+2)
10	22.080	16.158	.9939	1.757(+2)
20	11.083	8.079	.9984	7.568(+2)
40	5.547	4.039	.99924	3.142(+3)
80	2.774	2.019	.99990	1.281(+4)

TABLE 8

Problem B with $\alpha = 1, \beta = 5/2$
Asymptotically optimal mesh, ${}_I\|u_0\|_E = .071070$

m	E	E_0	θ	ω
5	33.869	32.317	.9466	1.577
10	16.519	16.158	.9694	1.676
20	8.153	8.079	.9823	1.755
40	4.049	4.039	.9894	1.788
80	2.018	2.019	.9933	2.437

TABLE 9

Problem B with $\alpha = 5$, $\beta = 9/10$

Uniform mesh

m	E	E_0	θ	ω
5	49.477	18.174	.9059	4.049(+3)
10	26.554	9.087	.9742	1.229(+4)
20	13.530	4.543	.9934	4.621(+4)
40	6.797	2.271	.9983	1.808(+5)
80	3.403	1.135	.9995	7.173(+5)

TABLE 10

Problem B with $\alpha = 5$, $\beta = 9/10$

Asymptotically optimal mesh

m	E	E_0	θ	ω
5	17.021	18.174	.7988	2.617
10	9.181	9.087	.9217	2.822
20	4.521	4.543	.9595	2.324
40	2.254	2.271	.9820	1.661
80	1.138	1.135	.9958	1.614

TABLE 11

Problem B: Asymptotically optimal mesh for $m = 10$

Case B1: $\alpha = 1$, $\beta = 5/2$

Case B2: $\alpha = 5$, $\beta = 9/10$

	x_j	
j	Case B1	Case B2
0	.0000	.0000
1	.0887	.4192
2	.1859	.6918
3	.3001	.7715
4	.5218	.8255
5	.6872	.8673
6	.7754	.9016
7	.8442	.9309
8	.9025	.9565
9	.9538	.9794
10	1.0000	1.0000

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