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ANALYSIS OF PERIODIC AND APERIODIC COUPLED NONUNIFORM TRANSMISSION LINES USING THE FOURIER SERIES EXPANSION

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Abstract—A general method is proposed to analyze periodic or aperiodic Coupled Nonuniform Transmission Lines (CNTLs). In this method, the per-unit-length matrices are expanded in the Fourier series. Then, the eigenvalues of periodic CNTLs and so the S parameters of aperiodic CNTLs are obtained. The validity of the method is studied using a comprehensive example.

1. INTRODUCTION

Single and coupled Nonuniform Transmission Lines (NTLs) are widely used in microwave circuits as resonators [1], impedance matching [1, 2], delay equalizers [3], filters [4], wave shaping [5], analog signal processing [6], VLSI interconnect [7] and etc.. The differential equations describing these structures have non-constant coefficients because the per-unit-length parameters or matrices vary along the lines. So, except for a few special cases, no analytical solution exists for NTLs. Coupled NTLs (CNTLs) with exponential variation is an example for these special cases [8]. Although, the method of using power series expansion directly [8, 9] or indirectly [10–13] has been utilized to solve many types of CNTLs. Of course, the conventional and most straightforward method to analyze arbitrary CNTLs is subdividing them into many short uniform sections [14, 15].

In this paper, a new method is introduced to analyze CNTLs. First, the periodic CNTLs are analyzed using the Fourier series expansion of the per-unit-length matrices to find their propagation constant and voltage and current eigenvectors. Then, the found parameters of periodic CNTLs are used to determine the $ABCD$ parameters of the aperiodic CNTLs. The validity of the method is verified using a comprehensive example. This method is applicable to all arbitrary coupled and single NTLs.

2. THE EQUATIONS OF CNTLS

In this section, the equations related to the CNTLs in the frequency domain are reviewed. It is assumed that the principal propagation mode of the lines is TEM or quasi-TEM. This assumption is valid when the lengths in the cross section are being small enough compared to the wavelength. Figure 1 shows a typical aperiodic CNTL consisting of M lines with length d along with its equivalent $2M$-port circuit. Also, Figure 2 shows a typical periodic CNTL made by cascading infinite number of an aperiodic CNTL with each other.

The differential equations describing lossy and dispersive periodic

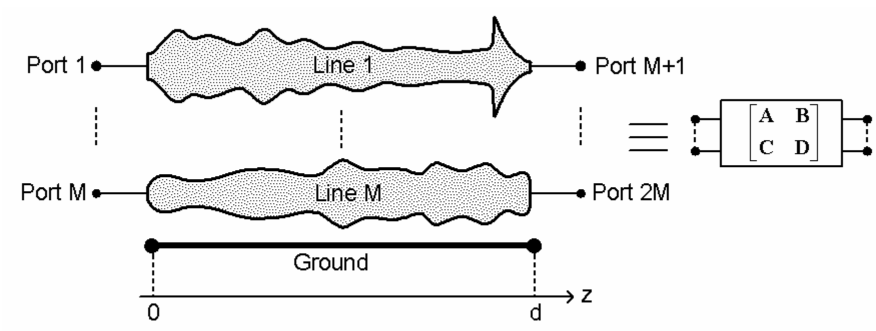

Figure 1. Typical aperiodic coupled nonuniform transmission line consisting of M lines with the length of d, as a $2M$-port circuit.

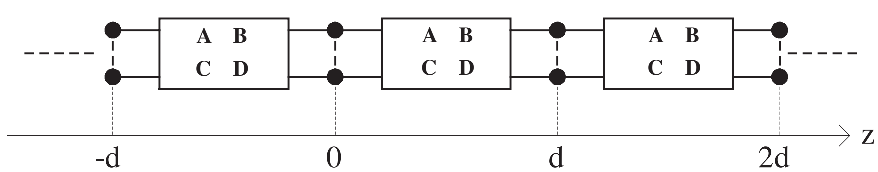

Figure 2. Typical periodic CNTL, made by cascading infinite number of aperiodic CNTL.

or aperiodic CNTLs are given by

$$\frac{d\mathbf{V}(z)}{dz} = -\mathbf{Z}(z)\mathbf{I}(z) \tag{1}$$

$$\frac{d\mathbf{I}(z)}{dz} = -\mathbf{Y}(z)\mathbf{V}(z) \tag{2}$$

in which $\mathbf{V}$ and $\mathbf{I}$ are $M \times 1$ voltage and current vectors, respectively. Also we have

$$\mathbf{Z}(z) = \mathbf{R}(z) + j\omega\mathbf{L}(z) \tag{3}$$

$$\mathbf{Y}(z) = \mathbf{G}(z) + j\omega\mathbf{C}(z) \tag{4}$$

where $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$ and $\mathbf{C}$ are the per-unit-length matrices of CNTL. Combining (1) and (2), gives the following differential equations

$$\frac{d^2\mathbf{V}(z)}{dz^2} - \frac{d\mathbf{Z}(z)}{dz}\mathbf{Z}^{-1}(z)\frac{d\mathbf{V}(z)}{dz} - \mathbf{Z}(z)\mathbf{Y}(z)\mathbf{V}(z) = \mathbf{0} \tag{5}$$

$$\mathbf{I}(z) = -\mathbf{Z}^{-1}(z)\frac{d\mathbf{V}(z)}{dz} \tag{6}$$

It is seen that these differential equations are quite difficult to solve analytically. Knowing the voltage and current vectors at $z = 0$ and $z = d$, the $ABCD$ matrix of aperiodic CNTL can be defined as follows,

$$\begin{bmatrix} \mathbf{V}(0) \\ \mathbf{I}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}(d) \\ \mathbf{I}(d) \end{bmatrix} \tag{7}$$

3. GENERAL SOLUTION OF CNTLS

In this section, the general solution of periodic and aperiodic CNTLs is presented. It is known from Floquet's theorem (an M dimensional one in here) that the voltage and current of periodic CNTLs can be expandable into an infinite set of spatial harmonics [2], as follows

$$\mathbf{V}(z) = \exp(-\gamma_0 z) \sum_{n=-\infty}^{\infty} \mathbf{V}_n \exp(-j2\pi nz/d) \tag{8}$$

$$\mathbf{I}(z) = \exp(-\gamma_0 z) \sum_{n=-\infty}^{\infty} \mathbf{I}_n \exp(-j2\pi nz/d) \tag{9}$$

in which the frequency dependent vectors $\mathbf{V}_n$ and $\mathbf{I}_n$ are unknown coefficients and γ_0, which its imaginary part is between $-\pi/d$ and

$+\pi/d$, is the principal value of the propagation constant. One sees from (7) and (8)–(9) that

$$\begin{bmatrix} \mathbf{V}((k-1)d) \\ \mathbf{I}((k-1)d) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{V}(kd) \\ \mathbf{I}(kd) \end{bmatrix} = \exp(\gamma_0 d) \begin{bmatrix} \mathbf{V}(kd) \\ \mathbf{I}(kd) \end{bmatrix} \tag{10}$$

in which k is an integer number. The equation (10) is a $2M$ dimensional eigenvalue problem, which has nontrivial vectorial solutions $[\mathbf{V}^{m+}\,\mathbf{I}^{m+}]^T$ and $[\mathbf{V}^{m-}\,\mathbf{I}^{m-}]^T$ for the m-th ($m=1, 2, \ldots, M$) solution set of $+\gamma_0$ and $-\gamma_0$, respectively. In fact, there are $2M$ waves propagating in $+z$ and $-z$ directions, in two equal groups. Knowing all nontrivial solutions of (10), the $ABCD$ matrix of aperiodic CNTL can be determined, as follows

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{1+} & \mathbf{V}^{1-} & \cdots & \mathbf{V}^{M+} & \mathbf{V}^{M-} \\ \mathbf{I}^{1+} & \mathbf{I}^{1-} & \cdots & \mathbf{I}^{M+} & \mathbf{I}^{M-} \end{bmatrix} \exp(\boldsymbol{\gamma}_0 d) \begin{bmatrix} \mathbf{V}^{1+} & \mathbf{V}^{1-} & \cdots & \mathbf{V}^{M+} & \mathbf{V}^{M-} \\ \mathbf{I}^{1+} & \mathbf{I}^{1-} & \cdots & \mathbf{I}^{M+} & \mathbf{I}^{M-} \end{bmatrix}^{-1} \tag{11}$$

in which

$$\boldsymbol{\gamma}_0 = \mathrm{diag}\left([\gamma_1,\, -\gamma_1,\, \ldots,\, \gamma_m,\, -\gamma_m,\, \ldots,\, \gamma_M,\, -\gamma_M]\right) \tag{12}$$

is a diagonal matrix containing the propagation constants. Each propagation constant γ_m ($m = 1, 2, \ldots, M$) is determined in one of three forms $j\beta_m$, α_m or $\alpha_m + j\pi/d$. It is seen from (10) and (11) that the $ABCD$ matrix of aperiodic CNTLs and the eigen-parameters of periodic CNTLs can be determined from the each other.

4. ANALYSIS OF CNTLS

In this section, the analysis of CNTLs using Fourier series expansion is presented. First, the periodic CNTLs are analyzed and then the aperiodic ones. It is assumed that each of four per-unit-length matrices of the periodic CNTLs can be expressed by a Fourier series as follows

$$\mathbf{L}(z) = \sum_{n=-\infty}^{\infty} \mathbf{L}_n \exp(-j2\pi nz/d) \tag{13}$$

$$\mathbf{C}(z) = \sum_{n=-\infty}^{\infty} \mathbf{C}_n \exp(-j2\pi nz/d) \tag{14}$$

$$\mathbf{R}(z) = \sum_{n=-\infty}^{\infty} \mathbf{R}_n \exp(-j2\pi nz/d) \tag{15}$$

$$\mathbf{G}(z) = \sum_{n=-\infty}^{\infty} \mathbf{G}_n \exp(-j2\pi nz/d) \tag{16}$$

The frequency dependent matrices $\mathbf{L}_n$, $\mathbf{C}_n$, $\mathbf{R}_n$ and $\mathbf{G}_n$ ($\mathbf{P}_n$ in general) are obtained using the following integral over an aperiodic CNTL.

$$\mathbf{P}_n = \frac{1}{d}\int_0^d \mathbf{P}(z)\exp(j2\pi nz/d)dz \tag{17}$$

Using (8)–(9) and (13)–(16) in (1)–(4) and equating the coefficients of similar spatial harmonics, gives us the following recursive relations

$$\mathbf{V}_n = \frac{1}{\gamma_0 + j2\pi n/d}\sum_{k=-\infty}^{\infty} \mathbf{Z}_{n-k}\mathbf{I}_k \tag{18}$$

$$\mathbf{I}_n = \frac{1}{\gamma_0 + j2\pi n/d}\sum_{k=-\infty}^{\infty} \mathbf{Y}_{n-k}\mathbf{V}_k \tag{19}$$

where

$$\mathbf{Z}_m = \mathbf{R}_m + j\omega\mathbf{L}_m \tag{20}$$

$$\mathbf{Y}_m = \mathbf{G}_m + j\omega\mathbf{C}_m \tag{21}$$

Now, to find the unknown coefficients $\mathbf{V}_n$ and $\mathbf{I}_n$, we truncate the (8)–(9) to $2N+1$ spatial harmonics, i.e., $-N \leq n \leq N$, first. Consequently, there will be two sets of $2N+1$ equations in (18)–(19), which can be expressed as two matrix equations given by

$$\tilde{\mathbf{V}} = \mathbf{H}^{-1}\tilde{\mathbf{Z}}\tilde{\mathbf{I}} \tag{22}$$

$$\tilde{\mathbf{I}} = \mathbf{H}^{-1}\tilde{\mathbf{Y}}\tilde{\mathbf{V}} \tag{23}$$

where $\tilde{\mathbf{V}} = [\mathbf{V}_{-N} \ldots \mathbf{V}_0 \ldots \mathbf{V}_N]^T$, $\tilde{\mathbf{I}} = [\mathbf{I}_{-N} \ldots \mathbf{I}_0 \ldots \mathbf{I}_N]^T$ are the voltage and current harmonic vectors, respectively and $\mathbf{H}$ is a diagonal matrix given by

$$\mathbf{H} = \mathrm{diag}\big([(\gamma_0 - j2\pi N/d)\mathbf{I}_d \ldots \gamma_0\mathbf{I}_d \ldots (\gamma_0 + j2\pi N/d)\mathbf{I}_d]\big) \tag{24}$$

in which $\mathbf{I}_d$ is an identity matrix. Also, $\tilde{\mathbf{Z}}$ and $\tilde{\mathbf{Y}}$ are the convolution matrices containing $4N+1$ spatial harmonics, given by

$$\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z}_0 & \mathbf{Z}_{-1} & \cdots & \mathbf{Z}_{-2N} \\ \mathbf{Z}_1 & \mathbf{Z}_0 & \cdots & \mathbf{Z}_{-2N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_{2N} & \mathbf{Z}_{2N-1} & \cdots & \mathbf{Z}_0 \end{bmatrix} \tag{25}$$

$$\tilde{\mathbf{Y}} = \begin{bmatrix} \mathbf{Y}_0 & \mathbf{Y}_{-1} & \cdots & \mathbf{Y}_{-2N} \\ \mathbf{Y}_1 & \mathbf{Y}_0 & \cdots & \mathbf{Y}_{-2N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}_{2N} & \mathbf{Y}_{2N-1} & \cdots & \mathbf{Y}_0 \end{bmatrix} \tag{26}$$

Combining two matrix equations (22)–(23), the following matrix equation for the voltage vector is obtained.

$$\mathbf{A}\tilde{\mathbf{V}} = \mathbf{0} \tag{27}$$

where $\mathbf{A}$ is a matrix given by

$$\mathbf{A} = \mathbf{H}^{-1}\tilde{\mathbf{Z}}\mathbf{H}^{-1}\tilde{\mathbf{Y}} - \mathbf{I}_d \tag{28}$$

The equation (27) is an eigenvalue problem, which has nontrivial solutions $\pm\gamma_m (m = 1, 2, \ldots, M)$, if

$$\det(\mathbf{A}) = 0 \quad \text{or} \quad \text{eig}(\mathbf{A}) = 0 \tag{29}$$

Thus (29), (27) and (23) give us the propagation constant, the voltage harmonic vector (as an eigenvector) and the current harmonic vector, respectively. To solve (29), one can use an optimization approach, in which the following defined error has to become minimum and near to zero.

$$\text{Error} = |\det(\mathbf{A})|^2 \tag{30}$$

It is notable that, some of solutions γ_m for (10) or (27) may be equal to each other, in some cases. In an especial case, for which $\mathbf{Z}(z)\mathbf{Y}(z)$ is proportional to an identity matrix, all of solutions γ_m are equal to each other. In these cases, there will be several independent eigenvectors for each solution.

From definitions in (10)–(11) for the voltage and current vectors at the terminals and also using (8)–(9), we will have

$$\mathbf{V}^{m\pm} = \mathbf{V}(kd) = \exp(\mp k\gamma_m d) \sum_{n=-\infty}^{\infty} \mathbf{V}_n \tag{31}$$

$$\mathbf{I}^{m\pm} = \mathbf{I}(kd) = \exp(\mp k\gamma_m d) \sum_{n=-\infty}^{\infty} \mathbf{I}_n \tag{32}$$

Now, the $ABCD$ matrix of aperiodic CNTLs are obtained using (11). Of course, one can determine the S matrix from the $ABCD$ matrix as follows

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \tag{33}$$

in which

$$\begin{aligned} \mathbf{S}_{11} &= \left[(\mathbf{A}+\mathbf{B}/Z_0)^{-1} + (\mathbf{C}Z_0+\mathbf{D})^{-1}\right]^{-1} \\ &\quad \times \left[-(\mathbf{A}+\mathbf{B}/Z_0)^{-1} + (\mathbf{C}Z_0+\mathbf{D})^{-1}\right] && (34) \\ \mathbf{S}_{12} &= \left[(\mathbf{A}+\mathbf{B}/Z_0)^{-1} + (\mathbf{C}Z_0+\mathbf{D})^{-1}\right]^{-1} \\ &\quad \times \left[(\mathbf{A}+\mathbf{B}/Z_0)^{-1}(\mathbf{A}-\mathbf{B}/Z_0) - (\mathbf{C}Z_0+\mathbf{D})^{-1}(\mathbf{C}Z_0-\mathbf{D})\right] && (35) \\ \mathbf{S}_{21} &= 2[\mathbf{A}+\mathbf{B}/Z_0+\mathbf{C}Z_0+\mathbf{D}]^{-1} && (36) \\ \mathbf{S}_{22} &= [\mathbf{A}+\mathbf{B}/Z_0+\mathbf{C}Z_0+\mathbf{D}]^{-1}[-\mathbf{A}+\mathbf{B}/Z_0-\mathbf{C}Z_0+\mathbf{D}] && (37) \end{aligned}$$

where Z_0 is the assumed characteristic impedance.

To obtain a criterion for the choice of the order of necessary number of spatial harmonics, consider the voltage and current distribution in Figure 2 as a repeated pulse function with period of d and with duration of a fraction of the wavelength (λ). With this assumption, the necessary number of spatial harmonics in (8) and (9) is obtained as follows

$$N \gg d/\lambda \tag{38}$$

Moreover, it is expected from (13)–(16) that increasing the amount of variations of the per-unit-length matrices increase the necessary number of spatial harmonics.

5. EXAMPLE AND RESULTS

In this section, a comprehensive example is presented to study the validity of the introduced method. Consider a lossless microstrip coupled NTL with $M = 2$ strips, whose length is $d = 10\,\text{cm}$. The width of strips and the gap between them are equal to the thickness of the substrate and the substrate relative permittivity is $\varepsilon_r = 10$ at $z=0$. This inhomogeneous structure has the following per-unit-length matrices.

$$\begin{aligned} \mathbf{L}(z) &= \mathbf{L}(0)\exp(kz/d) && (39) \\ \mathbf{C}(z) &= \mathbf{C}(0)\exp(-kz/d) && (40) \\ \mathbf{R}(z) &= \mathbf{G}(z) = 0 && (41) \end{aligned}$$

in which

$$\mathbf{L}(0) = \begin{bmatrix} 425.6 & 74.83 \\ 74.83 & 425.6 \end{bmatrix} \text{nH/m} \tag{42}$$

$$\mathbf{C}(0) = \begin{bmatrix} 174.9 & -14.25 \\ -14.25 & 174.9 \end{bmatrix} \text{pF/m} \tag{43}$$

The Fourier coefficients of the periodic model of this CNTL will be as follows

$$\mathbf{L}_n = \frac{1}{k + j2\pi n}(\exp(k) - 1)\mathbf{L}(0) \tag{44}$$

$$\mathbf{C}_n = \frac{1}{k - j2\pi n}(1 - \exp(-k))\mathbf{C}(0) \tag{45}$$

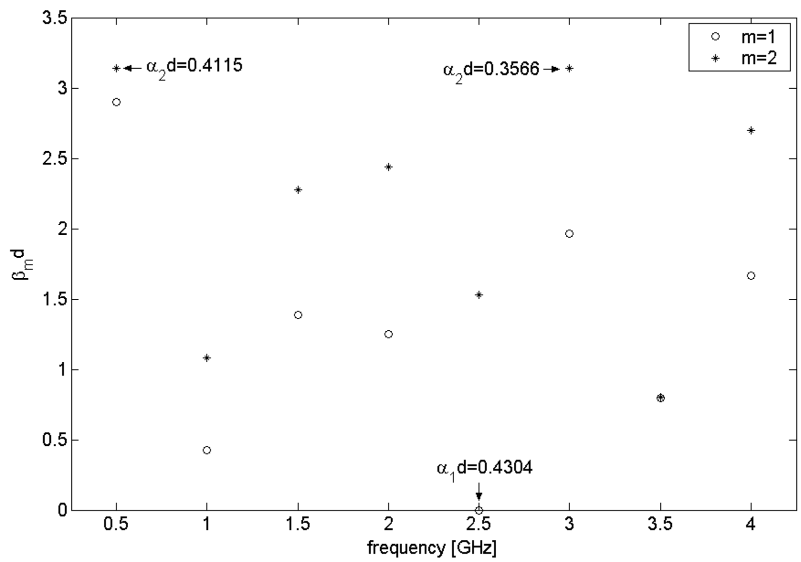

Figure 3. The real and imaginary parts of $\gamma_0 d$ (Brillouin diagram), with $N = 5$.

Figure 3 shows the principal values of propagation constant at some frequencies, assuming $k = 1$ and considering $N = 5$. All three forms of γ_0, which are due to the existing of passbands and stopbands, are being observed in this figure. Also, the nontrivial vectorial solutions at frequency 1.0 GHz are as follows:

$$\begin{aligned}
\mathbf{V}^{1+} &= \mathbf{V}^{1-} = [1\ 1]^T \text{ V},\\
\mathbf{V}^{2+} &= \mathbf{V}^{2-} = [1\ {-1}]^T \text{ V},\\
\mathbf{I}^{1+} &= 10.86 \exp(j135^\circ)[1\quad 1]^T \text{ mA},\\
\mathbf{I}^{1-} &= 10.86 \exp(j44.5^\circ)[1\quad 1]^T \text{ mA},\\
\mathbf{I}^{2+} &= 14.08\ [\exp(j166.9^\circ)\quad \exp(-j13.1^\circ)]^T \text{ mA and}\\
\mathbf{I}^{2-} &= 14.08\ [\exp(j13.1^\circ)\quad \exp(-j166.9^\circ)]^T \text{ mA}.
\end{aligned}$$

Figures 4–5, compare the S parameters of the CNTL assuming $Z_0 = 50\,\Omega$, obtained from the conventional method, i.e., subdividing to many ($K = 20000$) uniform sections [14] (as the exact solutions), and from the introduced method with $N = 5$ and 10, at frequencies 1.0 and 2.0 GHz, respectively. One sees an excellent agreement between the exact solutions and the results from the introduced method. It is seen and also evident that, as the number of spatial harmonics, $2N+1$, increases the accuracy of the obtained solutions increases. Also, one sees that, as the excitation frequency increases the amount of error and so the necessary number of spatial harmonics increases.

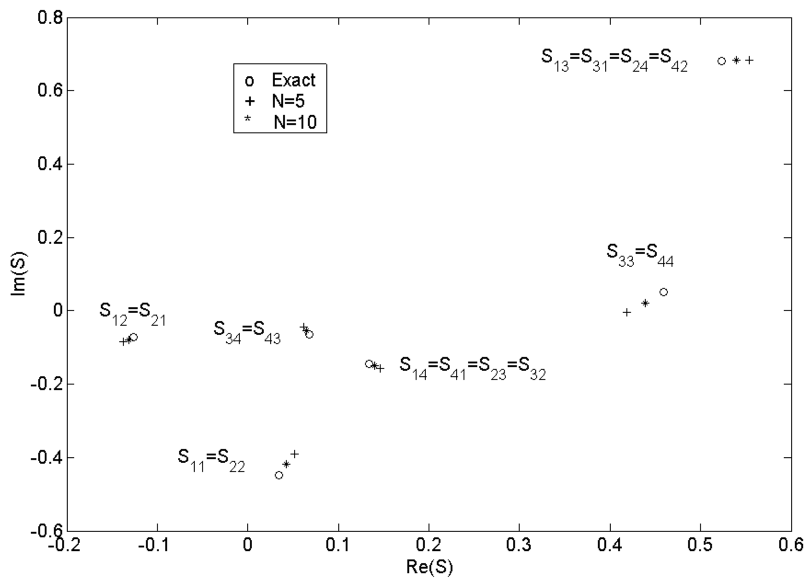

Figure 4. The S parameters of the exponential CNTL at frequency 1.0 GHz.

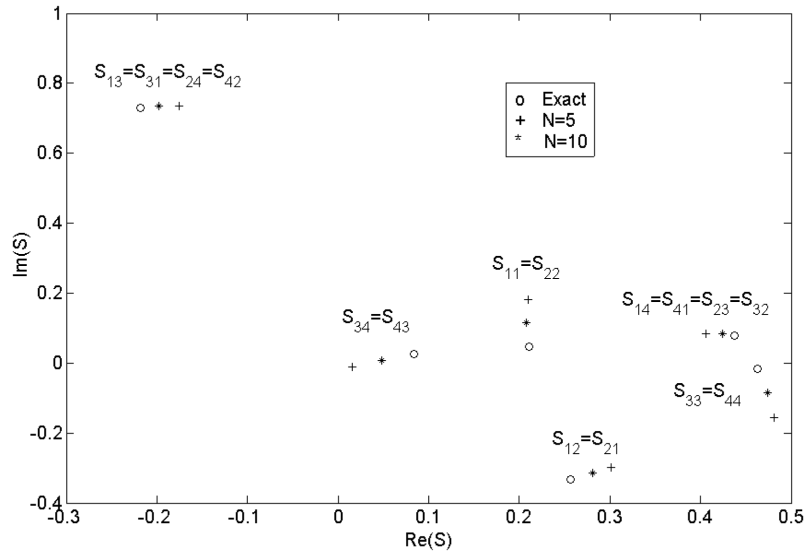

Figure 5. The S parameters of the exponential CNTL at frequency 2.0 GHz.

6. CONCLUSIONS

A general method was introduced to analyze aperiodic or periodic Coupled Nonuniform Transmission Lines (NTLs). The periodic CNTLs are analyzed using the Fourier series expansion of the per-unit-length matrices to find their propagation constant and voltage and current eigenvectors. The found parameters of periodic CNTLs are used to determine the $ABCD$ and so the S parameters of aperiodic CNTLs. The validity of the method was verified using a comprehensive example. It was seen that, as the number of spatial harmonics increases the accuracy of the obtained solution increases. Also, as the length of the lines with respect to the wavelength or the variations of the per-unit-length matrices increases, the necessary number of special harmonics increases. The required time and memory of the proposed method are less than those of the other numeric or full-wave methods, which give the distribution of voltages and currents along the length of CNTLs in addition to their $ABCD$ parameters. In fact, this method is very simple and fast and can be used for all lossy and dispersive CNTLs, whose per-unit-length matrices can be expressed by a converged Fourier series.

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