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Abstract

Recursive algorithms, where random observations enter are studied in a fairly general framework. An important feature is that the observations may depend on previous "outputs" of the algorithm. The considered class of algorithms contains, e g, stochastic approximation algorithms, recursive identification algorithms and algorithms for adaptive control of linear systems.

It is shown how a deterministic differential equation can be associated with the algorithm. Problems like convergence with probability one, possible convergence points and asymptotic behaviour of the algorithm can all be studied in terms of this differential equation. Theorems stating the precise relationships between the differential equation and the algorithm are given as well as examples of applications of the results to problems in identification and adaptive control.

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1. INTRODUCTION

Recursive algorithms, where stochastic observations enter are common in many fields. In the control and estimation literature such algorithms are widely discussed, e g in connection with adaptive control, (adaptive) filtering and on-line identification. The convergence analysis of the algorithms is not seldom difficult. As a rule, special techniques for analysis are used for each type of application and often the convergence properties have to be studied only by simulation.

In this paper a general approach to the analysis of the asymptotic behaviour of recursive algorithms is described. In effect, the convergence analysis is reduced to stability analysis of a deterministic, ordinary differential equation. This technique is believed to be a fairly general tool and to have a wide applicability. Applications to various problems have been published in [1], [2], [3], [4] and some theory was presented in [5].

The objective of the present paper is to give a comprehensive presentation of formal results and useful techniques for the convergence analysis, as well as to illustrate with several examples how the techniques can be applied.

In Section 2 a general recursive algorithm is described and discussed. A heuristic treatment of the convergence problem is given in Section 3 and this leads to the basic ideas of the present approach. Section 4 contains a discussion of the conditions which are imposed on the algorithm in order to prove the formal results. These theorems are given in Sections 5 and 6. The theorems suggest certain techniques for the convergence analysis, and these aspects are treated in Section 7. Several examples of how the theorems may be used, some of them reviewing previous applications are given in Section 8.

2. THE ALGORITHM

A general recursive algorithm can be written

$$x(t) = x(t-1) + \gamma(t)Q\big(t;x(t-1),\varphi(t)\big) \qquad (1)$$

where $x(\cdot)$ is a sequence of n-dimensional column vectors, which are the objects

of our interest. We shall refer to $x(\cdot)$ as "the estimates", and they could, e g, be the current estimates of some unknown parameter vector. They could, however, also be parameters that determine a feedback law of an adaptive controller, etc, and we shall be precise about the character of $x(\cdot)$ only in the examples below. The sequence $\gamma(\cdot)$ is throughout the paper assumed to be a sequence of positive scalars. The m-dimensional vector $\varphi(t)$ is an observation obtained at time t, and these are the objects that cause $x(t-1)$ to be updated to take new information into account. (The notion "observation" does not have to be taken literally. The variable φ may very well be the result of certain treatment of actual measurements.) The observations are in general functions of the previous estimates $x(\cdot)$ and of a sequence of random vectors $e(\cdot)$. This means that the observation is a random variable, which may be affected by previous estimates. This is the case, e g, for adaptive systems, when the input signal is determined on the basis of previous estimates. If the experiment designer has some test signal at his disposal, this may be included in $e(\cdot)$.

The function $Q(\cdot;\cdot,\cdot)$ from $R\times R^n\times R^m$ into R^n is a deterministic function with some regularity conditions to be specified below. This function, together with the choice of the scalar "gain" sequence $\gamma(\cdot)$ determine entirely the algorithm.

We shall not work with completely general dependence of $\varphi(t)$ on $x(\cdot)$; some results for this case are given in [6], but the following structure for the generation of $\varphi(\cdot)$ will be used:

$$\varphi(t) = A\big(x(t-1)\big)\varphi(t-1) + B\big(x(t-1)\big)e(t) \qquad (2)$$

Here $A(\cdot)$ and $B(\cdot)$ are $m|m$ and $m|r$ matrix functions.

Remark: It is perhaps more natural to think of an observation $\tilde{\varphi}(t)$ as the (lower-dimensional) output of a dynamical system like (2), $\tilde{\varphi}(t) = C\big(x(t-1)\big)\varphi(t)$. However, this case is naturally subsumed in the present one, since $\varphi(t)$ may enter in Q only as the combination $\tilde{\varphi}(t)$.

The assumption (2) seems to be appropriate for many applications. The same results as those below can be obtained also for non linear dynamics.

$$\varphi(t) = g\big(t;\varphi(t-1),x(t-1),e(t)\big) \qquad (3)$$

and the proofs for this case are given in [7].

Throughout this paper it is assumed that the estimates are desired to converge to some "true" or "optimal" value(s). Since $Q(t,x(t-1),\varphi(t))$ is a random variable, with, in general, non zero variance, convergence can take place only if the noise is rejected by paying less and less attention to the noisy observations, i e by letting

$$\gamma(t) \to 0 \quad \text{as} \quad t \to \infty \tag{4}$$

In tracking problems, when a time-varying parameter is to be tracked using algorithm (1), this condition is not feasible. Then $\gamma(t)$ usually tends to some small, non zero value, the size of which depends on what is known about the variability of the tracked parameters and about the noise characteristics. This case is not treated here, but it is reasonable to assume that analysis under the condition (4) also will have some relevance for the case of tracking slowly varying parameters.

Suppose we have a linear, stochastic, discrete time system, governed by a linear output feedback law, which at time t is determined by x(t-1). Then the behaviour of this overall system can be described by (2), with $\varphi(t)$ consisting of lagged inputs and outputs. Therefore the algorithms (1), (2) can be understood as archetypical for adaptive control of a linear system. Indeed this set-up is useful for analysis of certain adaptive controllers, as will be examplified below, but the basic algorithm (1), (2) covers also several other cases of interest.

3. HEURISTIC ANALYSIS

The algorithm (1), (2) is fairly complex to analyse, being a time-variant, stochastic, non linear difference equation. Notice also that the correction $x(t) - x(t-1)$ depends in general via $\varphi(t)$ implicitly on all old x(s). Therefore, while (1) certainly is recursive from the user's point of view, it is not so for analysis purposes.

In this section we shall illustrate heuristically how a differential equation can be associated with (1), and how it seems reasonable that asymptotic properties of (1) may be studied in terms of this differential equation. The formal analysis and results follow in the next two sections.

Consider

$$x(t) = x(t-1) + \gamma(t)Q\big(x(t-1),\varphi(t)\big) \tag{5}$$

where for simplicity we let Q be time independent. As remarked before, $\varphi(t)$ depends on all previous estimates:

$$\varphi(t) = \sum_{j=1}^{t}\left[\prod_{i=j+1}^{t} A\big(x(i-1)\big)\right]B\big(x(j-1)\big)e(j) \tag{6}$$

Now, if (2) is exponentially stable, then the first terms in (6) will be very small and, for some M,

$$\varphi(t) \approx \sum_{j=t-M}^{t}\left[\prod_{i=j+1}^{t} A\big(x(i-1)\big)\right]B\big(x(j-1)\big)e(j)$$

Moreover, it follows from (5) and (4) that the difference $x(t) - x(t-1)$ becomes smaller as t increases. Therefore, for sufficiently large t, we have $x(k) \approx x(t)$; $t \geq k \geq t-2M$. Hence

$$\varphi(k) \approx \sum_{j=k-M}^{k}\big[A\big(x(t)\big)\big]^{k-j}B\big(x(t)\big)e(j) \approx$$

$$\approx \sum_{j=1}^{k}\big[A\big(x(t)\big)\big]^{k-j}B\big(x(t)\big)e(j) \triangleq \bar{\varphi}\big(k;x(t)\big) \tag{7}$$

for $t \geq k \geq t-M$. Furthermore,

$$Q\big(x(k-1),\varphi(k)\big) \approx Q\Big(x(t),\bar{\varphi}\big(k;x(t)\big)\Big) = f\big(x(t)\big) + w(k) \tag{8}$$

where

$$f(x) = E\,Q\big(x,\bar{\varphi}(k;x)\big)$$

and hence w(k) is a random variable with zero mean.

Using (8), we can approximatively evaluate

$$x(t+s) = x(t) + \sum_{k=t+1}^{t+s} \gamma(k)Q\big(x(k-1),\varphi(k)\big) \approx$$

$$\approx x(t) + f\big(x(t)\big) \sum_{k=t+1}^{t+s} \gamma(k) + \sum_{k=t+1}^{t+s} \gamma(k)w(k) \approx$$

$$\approx x(t) + f\big(x(t)\big) \sum_{t+1}^{t+s} \gamma(k) \qquad (9)$$

where the last step should follow since the last term is a zero mean random variable which is dominated by the second term. Expression (9) suggests that the sequence of estimates more or less follows the difference equation

$$x^D(\tau+\Delta\tau) = x^D(\tau) + \Delta\tau f\big(x^D(\tau)\big) \qquad (10)$$

where $\Delta\tau$ corresponds to $\sum_{t+1}^{t+s} \gamma(k)$. It is useful to interpret (10) as a way of solving the differential equation ($\Delta\tau$ small),

$$\frac{d}{d\tau} x^D(\tau) = f\big(x^D(\tau)\big) \qquad (11)$$

where the (fictitious) time τ relates to the original time t in (5) by

$$\tau_t = \sum_{k=1}^{t} \gamma(k) \qquad (12)$$

We therefore have some reason to believe that the sequence of estimates $x(\cdot)$ asymptotically should follow the trajectories $x^D(\cdot)$ of (11).

(We could also have related (9) to the difference equation

$$x^D(t) = x^D(t-1) + \gamma(t)f\big(x^D(t-1)\big) \qquad (13)$$

but the differential equation is easier to handle since it is time-invariant.)

It now seems reasonable that asymptotic properties of (1), (2) may be studied in terms of the differential equation (11). This heuristic discussion is perhaps not very convincing, but along a similar path, though with far more technical labour, formal results to this effect can be proven. These are given below.

4. ASSUMPTIONS ON THE ALGORITHM

In order to prove the formal results, certain regularity conditions on the functions Q, A and B and on the driving "noise" term e, have to be introduced. Some of these are fairly technical, but it is believed that none is very restrictive. Several sets of assumptions are possible, and we shall give a few. In particular, there is a possibility to treat the sequence $e(\cdot)$ either in a stochastic or in a deterministic framework.

We shall start by giving a formal definition of $\bar{\varphi}$ used in the previous section. Let

$$D_S = \{x \mid A(x) \text{ has all eigenvalues strictly inside the unit circle}\}$$

Then for each $x \in D_S$, there exists a $\lambda = \lambda(x)$, such that

$$\left|A(x)^k\right| < C\,\lambda(x)^k;\ \lambda(x) < 1 \tag{14}$$

Take $\bar{x} \in D_S$ and define the random variables $\bar{\varphi}(t,\bar{x})$ and $v(t,\bar{x})$ by

$$\bar{\varphi}(t,\bar{x}) = A(\bar{x})\bar{\varphi}(t-1,\bar{x}) + B(\bar{x})e(t);\ \bar{\varphi}(0,\bar{x}) = 0 \tag{15}$$

$$v(t,\bar{x}) = \lambda(x)v(t-1,\bar{x}) + |B(\bar{x})|\,|e(t)|;\ v(0,\bar{x}) = 0 \tag{16}$$

Let D_R be an open, connected subset of D_S. The regularity conditions will be assumed to be valid in D_R. Now, the first set of assumptions is the following:

A.1 $e(\cdot)$ is a sequence of independent random variables (not necessarily stationary or with zero means).

A.2 $|e(t)| < C$ with probability one (w.p.1) all t.

A.3 The function $Q(t,x,\varphi)$ is continuously differentiable w.r.t x and φ for $x \in D_R$. The derivatives are, for fixed x and φ, bounded in t.

A.4 The matrix functions $A(\cdot)$ and $B(\cdot)$ are Lipschitz continuous in D_R.

A.5 $\lim_{t\to\infty} E\,Q\left(t,\bar{x},\bar{\varphi}(t,\bar{x})\right)$ exists for $\bar{x} \in D_R$ and is denoted by $f(\bar{x})$. The expectation is over $e(\cdot)$.

A.6 $\sum_{1}^{\infty} \gamma(t) = \infty$

A.7 $\sum_{1}^{\infty} \gamma(t)^p < \infty$ for some p.

A.8 $\gamma(\cdot)$ is a decreasing sequence.

A.9 $\limsup_{t\to\infty} [1/\gamma(t) - 1/\gamma(t-1)] < \infty$

These conditions will be referred to as "assumptions A". A.1 introduces the stochastic structure into the set-up. While A.2 certainly is most reasonable for all practical purposes, it is somewhat unattractive from a theoretical point of view, since it excludes e g the common gaussian models for noise. Below (assumptions B) are given conditions which allow more general noise. Conditions A.3 and A.4 are reasonable regularity properties, and A.5 is the basic assumption that makes it possible to associate (1), (2) with a differential equation. Condition A.6 makes it possible for the algorithm (1) to move the estimate to the desired limit, regardless of the initial value, and it is thus obviously necessary. A.7 gives a condition on how fast $\gamma(t)$ must tend to zero. This is considerably less restrictive than the usually given condition

$$\sum_{1}^{\infty} \gamma^2(t) < \infty \qquad (17)$$

Conditions A.8 and A.9 are motivated by technical arguments in the proofs, but they have so far not appeared to be restrictive. For example, it is easy to see that the sequence $\gamma(t) = C\, t^{-\alpha}$ satisfies A.6 - A.9 for $0 < \alpha \leq 1$.

If we would like to alleviate A.2, further regularity conditions on Q are required. This gives us our second set of assumptions. ($B(\bar{x},\rho)$ denotes a ρ-neighbourhood of $\bar{x}$, i e $B(\bar{x},\rho) = \{x \mid |\bar{x}-x| < \rho\}$.)

B.1 $e(\cdot)$ is a sequence of independent random variables (not necessarily stationary or with zero means).

B.2 $E|e(t)|^p$ exists and is bounded in t for each $p > 1$.

B.3 The function $Q(t,x,\varphi)$ is Lipschitz continuous in x and φ: $|Q(t,x_1,\varphi_1) - Q(t,x_2,\varphi_2)| < K_1(x,\varphi,\rho,v)\{|x_1-x_2| + |\varphi_1-\varphi_2|\}$ for $x_i \in B(x,\rho)$ for some $\rho = \rho(x) > 0$ where $x \in D_R$; $\varphi_i \in B(\varphi,v)$, $v \geq 0$.

B.4 $|K_1(x,\varphi_1,\rho,v_1) - K_1(x,\varphi_2,\rho,v_2)| \leq K_2(x,\varphi,\rho,v,w)\{|\varphi_1-\varphi_2| + |v_1-v_2|\}$ for $\varphi_i \in B(\varphi,w)$ and $v_i \in B(v,w)$

B.5 $A(\cdot)$ and $B(\cdot)$ are Lipschitz continuous in D_R.

B.6 $\lim_{t\to\infty} E\, Q\left(t,\bar{x},\bar{\varphi}(t,\bar{x})\right)$ exists for $x \in D_R$ and is denoted by $f(\bar{x})$. The expectation is over $e(\cdot)$.

B.7 For $x \in D_R$, the random variables $Q\left(t,x,\bar{\varphi}(t,x)\right)$, $K_1\left(x,\bar{\varphi}(t,x),\rho(x),v(t,x)\right)$, and $K_2\left(x,\bar{\varphi}(t,x),\rho(x),v(t,x),v(t,x)\right)$ have bounded p-moments for all $p > 1$. Here $\bar{\varphi}(\cdot,x)$ and $v(\cdot,x)$ are the random variables defined by (15) and (16).

B.8 $\sum_{1}^{\infty} \gamma(t) = \infty$

B.9 $\sum_{1}^{\infty} \gamma(t)^p < \infty$ for some p.

B.10 $\gamma(\cdot)$ is a decreasing sequence.

B.11 $\lim_{t\to\infty} \sup\left[1/\gamma(t) - 1/\gamma(t-1)\right] < \infty$

Conditions B.4, B.3 and B.7 admittedly look somewhat complex, but they are as a rule easy to check in a given situation, especially since $Q(t,x,\varphi)$ is a simple function of x and φ in most applications. The conditions B.3 and B.4 essentially require that $Q(t,x,\varphi)$ is twice continuously differentiable and B.7 implies that Q and its derivatives must not increase too rapidly with φ and v.

In these two cases the algorithm (1), (2) is treated directly in a stochastic framework, due to assumption A.1 = B.1. In certain cases it may not be suitable to treat $e(\cdot)$ in (2) as a sequence of random variables. Naturally the algorithm (1), (2) still makes sense, even if $e(\cdot)$ is a given, deterministic sequence. Convergence of (1) will then depend, among other things, on the properties of this sequence $e(\cdot)$. In such a case we may work with the following assumptions. Let K_1 be defined as in B.3 and let $\varphi(\cdot,\bar{x})$ and $v(\cdot,\bar{x})$ be given by (15), (16). Introduce the quantities $z(\cdot,\bar{x})$, $k(\cdot,\bar{x})$ and $k_v(\cdot,\bar{x})$ by

$$z(t,\bar{x}) = z(t-1,\bar{x}) + \gamma(t)\left[Q\left(t,\bar{x},\bar{\varphi}(t,\bar{x})\right) - z(t-1,\bar{x})\right]$$

$$z(0,\bar{x}) = 0 \qquad (18a)$$

$$k(t,\bar{x}) = k(t-1,\bar{x}) + \gamma(t)\left[K_1\left(\bar{x},\bar{\varphi}(t,\bar{x}),\rho(\bar{x}),v(t,\bar{x})\right) - k(t-1,\bar{x})\right]$$

$$k(0,\bar{x}) = 0 \qquad (18b)$$

$$k_v(t,\bar{x}) = k_v(t-1,\bar{x}) + \gamma(t)\Big[K_1\big(\bar{x},\bar{\varphi}(t,\bar{x}),\rho(\bar{x}),v(t,\bar{x})\big)v(t,\bar{x}) - k_v(t-1,\bar{x})\Big]$$

$$k_v(0,\bar{x}) = 0 \qquad (18c)$$

Notice that for the common choice $\gamma(t) = 1/t$, (18a) implies that

$$z(t,\bar{x}) = \frac{1}{t}\sum_{k=1}^{t} Q\big(k;\bar{x},\bar{\varphi}(k,\bar{x})\big) \quad \text{etc}$$

The assumptions then are:

C.1 The function $Q(t,x,\varphi)$ is Lipschitz continuous in x and φ: $|Q(t,x_1,\varphi_1) - Q(t,x_2,\varphi_2)| < K_1(x,\varphi,\rho,v)\{|x_1-x_2| + |\varphi_1-\varphi_2|\}$ for $x_i \in B(x,\rho)$ for some $\rho = \rho(x) > 0$ where $x \in D_R$; $\varphi_i \in B(\varphi,v)$, $v \geq 0$.

C.2 The matrix functions $A(\cdot)$ and $B(\cdot)$ are Lipschitz continuous in D_R.

C.3 $z(t,\bar{x})$ as defined by (18a) converges for all $\bar{x} \in D_R$. Denote the limit by $f(\bar{x})$.

C.4 $k(t,\bar{x})$ and $k_v(t,\bar{x})$, defined by (18bc) are bounded in t for all $\bar{x} \in D_R$.

C.5 $\sum_{1}^{\infty} \gamma(t) = \infty$

C.6 $\gamma(t) \to 0$ as $t \to \infty$.

When these assumptions are used, no stochastic framework has to be introduced. The statements about the behaviour of $x(\cdot)$ to be given below are true as long as $e(\cdot)$ is such that C.3 and C.4 hold. If a stochastic framework is imposed and C.3, C.4 hold with probability one, then the statements about $x(\cdot)$ will be true w.p.1. This is, essentially, the approach taken in [5] and [8], which also contain a detailed study of algorithms like (18) (esp. [8], Ch.4). There several different sets of conditions implying convergence of (18) are given. In fact, conditions B imply that C.3 and C.4 hold w.p.1. It may in this context be remarked that there is actually a trade-off between condition A.7 = B.9 and conditions B.2 and B.7: The largest p for which B.2 and B.7 need to hold is twice the p for which A.7 holds. Therefore, if $\gamma(\cdot)$ is subject to (17), then only the fourth moments of e, Q and K_i have to be bounded. This is discussed further in [5] and [8], and we shall not pursue it here.

5. MAIN THEOREMS.

The function $f(\cdot)$ defined in A.5, B.6 or C.3 is the basic object of interest. As the heuristic discussion in Section 3 indicated the differential equation

$$\frac{d}{d\tau} x^D(\tau) = f\left(x^D(\tau)\right) \tag{19}$$

will be relevant for the asymptotic behaviour of the algorithm (1), (2). The exact relationships between (19) and (1), (2) are given in three theorems. The first one concerns convergence of (1).

Theorem 1. Consider the algorithm (1), (2) subject to assumptions A, B or C. Let $\bar{D}$ be a compact subset of D_R such that the trajectories of (19) that start in $\bar{D}$ remain in there for $\tau > 0$. Assume that

1) there is a random variable C such that

$$x(t) \in \bar{D} \text{ and } |\varphi(t)| < C \text{ infinitely often (i.o.) w.p.1} \tag{20}$$

2) the differential equation (19) has an invariant set D_c with domain of attraction $D_A \supset \bar{D}$. (21)

Then $x(t) \to D_c$ w.p.1 as $t \to \infty$. □

Remarks. By (20) is meant that there exists w.p.1 a subsequence t_k, possibly depending on the realization ω, such that $x(t_k) \in \bar{D}$ and $|\varphi(t_k)| < C(\omega)$, $k = 1, 2, \ldots$ This condition, which we may call the "boundedness condition", is further discussed in Section 6.

An invariant set of a differential equation is a set such that the trajectories remain in there for $-\infty < \tau < \infty$. The domain of attraction of an invariant set D_c consists of those points from which the trajectories converge into D_c as τ tends to infinity. It is obviously an open set. See e g [9]. An interesting special case is when the invariant set D_c is just a stationary point of (19) say x^*, with $f(x^*) = 0$. Then the theorem proves convergence of $x(t)$ to x^*. By $x(t) \to D_c$ is meant that $\inf_{x \in D_c} |x(t) - x| \to 0$.

Our second theorem concerns the set of possible convergence points. It can be used to prove failure of convergence by showing that the "desired" or "true" parameter value does not belong to this set.

Theorem 2. Consider algorithm (1), (2) subject to assumptions A or B. Suppose that $x^* \in D_R$ has the property that

$$P\left(x(t) \to B(x^*,\rho)\right) > 0 \quad \text{for all } \rho > 0 \tag{22}$$ †

Further suppose that

$$Q\left(t,x^*,\bar{\varphi}(t,x^*)\right) \text{ has a covariance matrix bounded from below by a strictly positive definite matrix,} \tag{23}$$

and that

$$E\,Q\left(t,x,\bar{\varphi}(t,x)\right) \text{ is continuously differentiable w.r.t } x \text{ in a neighbourhood of } x^* \text{ and the derivatives converge uniformly in this neighbourhood as } t \text{ tends to infinity.}$$

Then

$$f(x^*) = 0 \tag{24}$$

and

$$H(x^*) = \frac{d}{dx} f(x)\Big|_{x=x^*} \text{ has all eigenvalues in the LHP } (\mathrm{Re}\ z \leq 0). \tag{25}$$

□

The matrix $H(x^*)$ defines, of course, the linear differential equation obtained from (19) by linearization around x^*. Therefore this theorem essentially states that the algorithm can converge only to stable stationary points of the differential equation (19).

If $f(x) = -\frac{d}{dx} V(x)$, which might be the case if the algorithm is based on criterion-minimization, then $V(x)$ can be chosen as a Lyapunov function for the differential equation (19). Since $\frac{d}{d\tau} V\left(x(\tau)\right) = -\left|f\left(x(\tau)\right)\right|^2$, we see that the stationary points of (19), together with the point $\{\infty\}$, form an invariant set with global domain of attraction. Moreover, if the stationary points are isolated, it follows from Theorem 2 that only stable ones, i e local minima, are possible convergence points. It also follows from Theorem 1 that the estimates

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† P(A) = The probability of the event A.

cannot oscillate between different minima. Collecting all this we obtain a corollary to Theorems 1 and 2.

Corollary. Suppose that $D_R = R^n$, that $f(x) = -\frac{d}{dx} V(x)$ and that $V(x)$ has isolated stationary points. Assume that $|\varphi(t)| < C$ i.o. w.p.1. Then w.p.1., $x(t)$ tends either to a local minimum of $V(x)$ (i.e. $V''(x)$ positive semidefinite) or to infinity as t tends to infinity.

Finally, our third theorem relates the trajectories of the differential equation (19) to the paths of the algorithm (1), (2). The result is formulated as follows. Let $x(t)$, $t = t_0, \ldots$, be generated by (1), (2). The values can be plotted with the sample numbers t as the abscissa. It is also possible to introduce as before a fictitious time τ by

$$\tau_t = \sum_{k=1}^{t} \gamma(k) \qquad (26)$$

Suppose that the estimates $x(t)$ are plotted against this time τ:

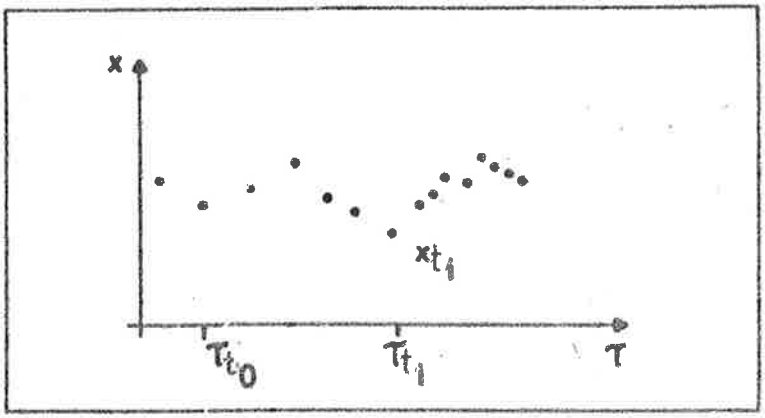

Fig. 1a.

Let $x^D(\tau, \tau_{t_0}, x(t_0))$ be the solution of (19) with initial value $x(t_0)$ at time τ_{t_0}. Plot also this solution in the same diagram:

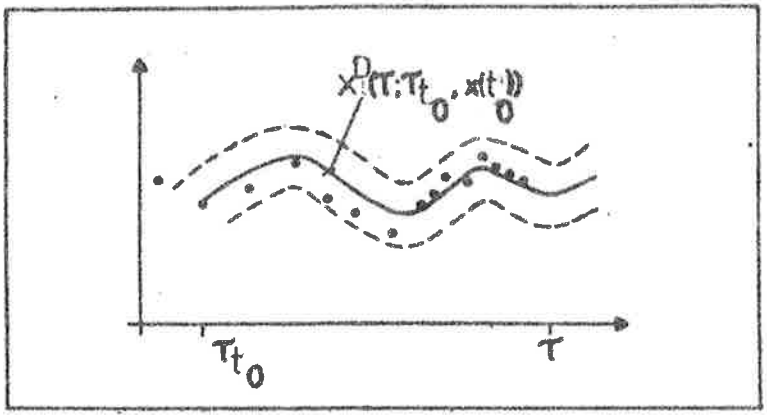

Fig. 1b - To illustrate Theorem 4.

Let I be a set of integers. The probability that all points $x(t)$, $t \in I$, simultaneously are within a certain distance ε from the trajectory is estimated in the following theorem.

Theorem 3. Consider algorithm (1), (2) under assumptions A or B. Assume that $f(x)$ is continuously differentiable and that $E\, Q(t;x,\bar{\varphi}(t,x))$ $(= f(x))$ does not depend on t. (We may here disregard a possible transient in $\bar{\varphi}(t,x)$ and assume that $\bar{\varphi}(\cdot,x)$ has reached stationarity.) Assume for some t_0 that $x(t_0) \in D_0 \subset D_R$ and $|\varphi(t_0)| < K_0$, where D_0 is compact. Assume that the solution $x^D(\tau,\tau_{t_0};x(t_0))$ to (19) is exponentially stable, and let I be a set of integers, such that $\inf|\tau_i-\tau_j| = \delta_0 > 0$ where $i \neq j$ and $i,j \in I$. Then for any $r > 1$ there exist constants K and ε_0 that depend on r, D_0, K_0 and δ_0, but not on t_0, $x(t_0)$ or $\varphi(t_0)$ such that for $\varepsilon < \varepsilon_0$,

$$P\{\sup_{\substack{t\in I\\ t\geq t_0}} |x(t) - x^D(\tau_t;\tau_{t_0},x(t_0))| > \varepsilon\} \leq \frac{K}{\varepsilon^{4r}} \sum_{j=t_0}^{N} \gamma(j)^r \qquad \text{all } r > 1 \qquad (27)$$

where $N = \sup i$; $i \in I$, which may be ∞.

Remark. In the proof of Theorem 3 it is assumed that the exponential stability of the solution $x^D(\tau,\tau_{t_0};x(t_0))$ is ensured by a quadratic Lyapunov function for the (linear and time-varying) variational equation around this solution, cf, e g [10].

Although the proof of Theorem 3 provides an estimate of K from given constants, we do not intend to use (27) to obtain numerical bounds for the probability. The point of the theorem is that a connection between the differential equation (19) and the algorithm (1), (2) is established. In particular, we notice that, due to A.7 there is an r such that the RHS of (27) becomes arbitrarily small when t_0 increases, and ε, D_0 and K_0 are fixed. This means that the estimates stay close to the corresponding trajectory with higher and higher probability as t_0 increases. Another way of interpreting (27) is that the gain sequence $\gamma(\cdot)$ can be scaled so that $x(\cdot)$ stays arbitrarily close to $x^D(\cdot)$, with an arbitrary high degree of probability.

The proofs of Theorems 1, 2 and 3 are long and technical and the space here does not permit to include them. Full proofs are given in [7]. An outline of the proof of Theorem 2 was given also in [1]. The idea of the proofs of Theorems 1 and 3 follows the discussion in Section 3. However, a considerable amount of technica-

lities are required to justify rigorously the "approximatively equal"-signs.

6. THE BOUNDEDNESS CONDITION

In this section we shall discuss condition (20). The reason why it is required is twofold. Firstly, obviously $x(t)$ must be inside D_R (with $\varphi(t)$ not too large to prevent an immediate jump) for the differential equation to be valid at all, and also inside D_A to get "caught" by a trajectory converging to D_c. Secondly, and perhaps less obviously, even if $D_R = D_A = R^n$ it may happen that $x(t)$ tends to infinity. The reason for divergence is that if $Q(t,x,\varphi)$ increases rapidly with $|x|$ it may happen that the correction $\gamma(t)Q\big(t,x(t-1),\varphi(t)\big)$ always is too large even though $\gamma(t)$ tends to zero. Another reason is that the variance of the "noise" $Q(t,x,\varphi) - f(x)$ may increase so fast with $|x|$ that a "random walk" effect becomes predominating close to infinity.

From a practical point of view, the question of boundedness of the estimates may seem uninteresting, since no implementation of (1) will allow that $x(t)$ tends to infinity. It will be kept bounded either by deliberate measures or due to e g overflow in the computer. Now the measures to keep $x(t)$ in a bounded area may not be completely arbitrary to obtain convergence.

A feature that can be used when $D_R = D_A = R^n$, is to introduce a saturation in $Q(\cdot;\cdot,\cdot)$ so that $|Q(t;x,\varphi)| < K$. This is further discussed in [7].

Another possibility of preventing $x(t)$ from tending to infinity is to project $x(t)$ into a bounded area if $|x(t)|$ is too large or if $x(t)$ does not belong to a desired area, say D_S. In fact, if $A(x)$ is a known function of x, it is common, and often even necessary to test if $x(t) \in D_S$ and project it into D_S otherwise. We then have an algorithm of the following type:

$$x(t) = \Big[x(t-1) + \gamma(t)Q\big(t,x(t-1),\varphi(t)\big)\Big]_{D_1,D_2} \tag{28}$$

$$\varphi(t) = \begin{cases} A\big(x(t-1)\big)\varphi(t-1) + B\big(x(t-1)\big)e(t) & \text{if } x(t-1) \in D_1 \\ \text{a value } \hat{} \text{ in a given compact subset of } R^m & \text{if } x(t-1) \notin D_1 \end{cases} \tag{29}$$

where, for $D_1 \supset D_2$

$$[f]_{D_1,D_2} = \begin{cases} f \text{ if } f \in D_1 \\ \text{some value in } D_2 \text{ if } f \notin D_1 \end{cases}$$

It should be clear that D_1, D_2 cannot be chosen arbitrarily. Loosely speaking, the trajectories of (19) that start in D_2 must not leave the area D_1. Otherwise there may be an undesired cluster point on the boundary of D_1. This may be formalized as follows.

Theorem 4. Consider the algorithm (28), (29) subject to assumptions A, B or C. Let $D_1 \subset D_R$ be an open bounded set containing the compact set D_2. Let $\tilde{D} = D_1 \setminus D_2$ (D_1 "minus" D_2). Assume that $D_2 \subset D_A$, with D_A defined as in Theorem 1. Suppose that there exists a twice differentiable function $U(x) \geq 0$, defined in a neighbourhood of $\tilde{D}$ with properties

$$\sup_{x \in \tilde{D}} U'(x)f(x) < 0 \tag{30}$$

$$\begin{aligned} U(x) &\geq C_1 \quad \text{for } x \notin D_1 \\ U(x) &\leq C_2 < C_1 \quad \text{for } x \in D_2 \end{aligned} \tag{31}$$

Then Theorem 1 holds without assumption (20).

The proof of Theorem 4 is given in [7].

Assumption (30) clearly makes $U(\cdot)$ a Lyapunov function in $\tilde{D}$, while (31) formalizes the intuitive notion of trajectories from D_2 never leaving D_1. We may remark that (30), (31) hold, e g if the trajectories of (19) do not intersect the boundary of D_1 "outwards" and D_2 is sufficiently close to D_1.

7. HOW TO USE THE THEOREMS

The intuitive content of the theorems of Section 5 is that the algorithm (1), (2)

$$x(t) = x(t-1) + \gamma(t)Q\big(t,x(t-1),\varphi(t)\big) \tag{32a}$$

$$\varphi(t) = A\big(x(t-1)\big)\varphi(t-1) + B\big(x(t-1)\big)e(t) \tag{32b}$$

can be studied and analysed in terms of the differential equation

$$\frac{d}{d\tau} x^D(\tau) = f\left(x^D(\tau)\right) \tag{33}$$

where

$$f(x) = \lim_{t\to\infty} E\, Q\left(t,x,\bar{\varphi}(t,x)\right) \tag{34}$$

The precise statements about the relations between (32) and (33) of Theorems 1 - 3 may be summarized in a somewhat looser language as follows.

a) $x(t)$ can converge only to stable stationary points of (33).

b) If $x(\cdot)$ belongs to the domain of attraction of a stable stationary point x^* of (33) i.o. w.p.1, then $x(t)$ converges w.p.1 to x^* as t tends to infinity.

c) The trajectories of (33) are "the asymptotic paths" of the estimates $x(\cdot)$, generated by (32).

These statements are fairly attractive intuitively, and they suggest certain unified techniques to analyse recursive algorithms. We shall illustrate this below, but let us here point out some aspects.

By the result a) the possible convergence points of (32) may be determined and studied. That a possible convergence point must be a zero of (34) is fairly obvious and it may be derived without reference to any differential equation. However, the observation that among these stationary points only stable ones are candidates for being limit points of (32) is a most important complement and it is probably less obvious without the present interpretation in terms of the differential equation. Perhaps the main use of result a) is to prove failure of convergence. It may be remarked that usually an algorithm is constructed so that the desired limit indeed is a stationary point. Consequently the possible lack of convergence is then due to the unstable character of the stationary point, so it is the complement (25) that is the key result for proving divergence.

Result b) is the result by which convergence can be proved. In many cases it is not easy to find a proper Lyapunov function to prove global stability of (33), and sometimes the right hand side of (33) is quite complex. For certain algorithms, though, in particular those arising from criteria-minimization, it is

possible to do this analytically, and some examples will be given below.

While analytic treatment of (33) may be difficult, it is always possible to solve it numerically when the dimension of x is not too large. In that way insight can be gained into the global stability properties of the differential equation, the stationary points and their character. In view of result c) the trajectories thus obtained are also relevant for the asymptotic behaviour of the algorithm. Therefore, numerical solution of (33) is a valuable complement to simulation of (32). Due to the time scaling (26) in the differential equation, this reveals more rapidly the asymptotic properties and the stationary points of the algorithm. Since the estimates change more and more slowly, due to (4), it is not seldom difficult to decide from simulations only whether the estimates have settled around a limit value or are just converging slowly. In addition, it might be difficult to tell from a simulation if a certain effect is an inherent feature of the algorithm or just depends on random influence. Numerical solution of (33) may resolve such questions.

In the next section we shall apply the method to a few examples and illustrate how the techniques of the items above may be used.

8. EXAMPLES

Example 1:- Stochastic approximation algorithms.

Consider the problem of solving

$$E_\varphi Q(x,\varphi) = 0 \tag{35}$$

for x. Here "E_φ" denotes the expectation with respect to φ, while the vector x is considered as a fixed parameter. Quantities $Q\big(x,\varphi(t)\big)$, t = 1, 2, ..., are available for any x, where the distribution of the random vector $\varphi(\cdot)$ does not depend on x. Robbins-Monro [11] proved that under certain assumptions the scheme ("the Robbins-Monro scheme")

$$x(t) = x(t-1) + \gamma(t)Q\big(x(t-1),\varphi(t)\big) \tag{36}$$

gives a sequence of estimates that converges to the (a) solution of (35) in the mean square sense. Convergence w.p.1 of (36) has then been studied in several papers, e.g. [12] - [14], and the theorems of these studies do not differ very much conceptually from Theorem 1. The functions used in the convergence theorems of e g [12] or [13] can be interpreted as Lyapunov functions for the differential equation (33) and condition (20) of Theorem 1 is ensured by further conditions on this function, rather than by the more practically oriented theorem 4, see e g [12], condition A, or [13], condition B, p. 184. It can also be remarked that the condition

$$E(x-x^*)^T Q(x,\varphi) < 0 \tag{37}$$

frequently used in Tsypkin's work, see e g [15], [16], clearly can be understood as a stability condition for (33) with $V(x) = \|x-x^*\|^2$ as the Lyapunov function. Tsypkin's condition

$$E(x-x^*)^T Q(x,\varphi)(x-x^*) \leq C(1 + \|x\|^2) \tag{38}$$

is then a variant of the "boundedness condition".

The convergence results thus obtained are, however, essentially restricted to the case $\varphi(\cdot)$ being independent random variables $(A(\cdot) = 0)$ and $\gamma(\cdot)$ satisfying (17) which is quite restrictive for control and estimation application. These conditions are inherently tied to the use of martingale theory in the proofs and cannot easily be dispensed with. Our Theorem 1 when applied to (36) is thus more general in that $\varphi(\cdot)$ may be dependent (generated as white noise through a linear filter) and $\gamma(\cdot)$ has only to satisfy A.7. This is satisfied e g for $\gamma(t) = C\, t^{-\alpha}$ $0 < \alpha \leq 1$, while (17) admits only $1/2 < \alpha \leq 1$. Notice that slowly decreasing gain sequences may be of interest in practice to achieve fast convergence of the sequence of estimates. We must, however, admit that we in return require more regularity of Q and of $e(\cdot)$. On the other hand, non-smoothness of the involved functions is seldom a problem in applications, and we believe that our version of the convergence theorem is more widely applicable.

In addition, Theorems 2 and 3 are important results for convergence analysis, and we are not aware of similar previous results for the Robbins-Monro scheme.

In many applications it is of interest to minimize a function $E_\varphi J(x,\varphi) = P(x)$

with respect to x. If the derivative of J with respect to x can be calculated, the stationary points of P(x) can be found as solutions of

$$E_\varphi\left[\frac{\partial}{\partial x} J(x,\varphi)\right] = 0$$

This is a problem that can be solved using the Robbins-Monro scheme and then the corollary of Theorems 1 and 2 is quite useful.

If the derivative of J cannot be calculated it seems natural to replace it with some difference approximation. This was suggested by Kiefer and Wolfowitz [17] and their procedure has also been used for various control and estimation problems. Kushner has in several recent papers discussed interesting variants of this procedure, see e g [18], [19]. Our theorems are not directly applicable to the Kiefer-Wolfowitz scheme as they stand, since condition A.3 (or B.3) is not valid. The reason is that the function Q in this case increases to infinity with t. For the case of additive noise to the function to be minimized, however, it can readily be shown that Theorems 1 - 4 hold anyway. Details are given in [5] and [7].

Stochastic approximation algorithms have been applied to a broad variety of problems in control theory, see e g Tsypkin [15], [16], Fu [20], and Saridis et al [21]. The approach is known as "learning systems", and in this framework estimation and identification problems, adaptive control, supervised and unsupervised pattern recognition etc can be treated.

An approach that is related to stochastic approximation is suggested by Aizerman et al [13]. Their "Potential Function Method" can be applied to various problems in machine learning.

Therefore the Robbins-Monro scheme appears in various disguises in many control and estimation algorithms, and consequently the described techniques can be applied to these. A particular example is given below. □

Example 2 - An automatic classifier.

A classifier receives scalar valued signals $\varphi(t)$ which may belong to either of two a priori unknown classes A and B. The classifier must find a classification rule, i e a number $c(t)$ such that $\varphi(t)$ is classified as A if $\varphi(t) \leq c(t)$ and B

otherwise. The number c(t) can e g be determined as follows:

$$c(t) = \left(x^A(t) + x^B(t)\right)/2$$

where

$$x^A(t) = \begin{cases} x^A(t-1) + \gamma(t)[\varphi(t) - x^A(t-1)] & \text{if } \varphi(t) \text{ is classified as A} \\ x^A(t-1) & \text{otherwise} \end{cases} \tag{39}$$

$x^B(t)$ is defined analogously. Clearly, $x^A(t)$ is the mean value of the outcomes classified as A. This scheme is discussed by Tsypkin [22] and Braverman [23].

Let $\varphi(t)$ have the distribution shown in Fig. 2 consisting of two triangular distributions. The probability of outcomes in the left triangle is λ. We assume that $\varphi(\cdot)$ is a sequence of independent random variables. Clearly, it is desirable that the classification rule, the number c(t), should converge to some value between -1 and +1. Introduce

$$x(t) = \begin{pmatrix} x^A(t) \\ x^B(t) \end{pmatrix}$$

Then (39) can be written

$$x(t) = x(t-1) + \gamma(t)Q\left(x(t-1),\varphi(t)\right) \tag{40}$$

where

$$Q(x,\varphi) = \begin{pmatrix} Q^A(x^A,x^B,\varphi) \\ Q^B(x^A,x^B,\varphi) \end{pmatrix}$$

and

$$Q^A(x^A,x^B,\varphi) = \begin{cases} \varphi - x^A & \text{if } \varphi < \frac{1}{2}(x^A+x^B) - \delta \\ 0 & \text{if } \varphi > \frac{1}{2}(x^A+x^B) + \delta \end{cases} \quad \text{and } Q^B \text{ analogously}$$

where the values for $\frac{1}{2}(x^A+x^B) - \delta \leq \varphi \leq \frac{1}{2}(x^A+x^B) + \delta$ are such that Q^A is a continuously differentiable function of φ and x. Here δ is some small positive

number.

Clearly, the algorithm (40) together with the observation equation

$$\varphi(t) = e(t) \tag{41}$$

is a simple case of (32). Since $\varphi(\cdot) = e(\cdot)$ is bounded we may use assumptions A. Obviously A.1 to A.5 are satisfied, and let us assume that $\gamma(\cdot)$ is such that A.6 - A.9 hold. (Here A.3 holds in virtue of our somewhat artificial modification of Q^A; but this example will illustrate that a heuristic use of the present convergence results will reveal important features of the algorithm.)

$E_\varphi Q(x,\varphi) = f(x)$ is readily computed as follows. For a given x the corresponding classification point is $c(x) = (x^A+x^B)/2$. $f^A(x)$ is then the mean value of the distribution left of the point c(x), minus x^A. $f^B(x)$ is found correspondingly. The algebraic expression for f(x) as a function of x and λ is lengthy and is omitted.

We first note that by construction, the estimates are confined to the area $\bar{D}$: $3 > x^B \geq x^A > -3$. Therefore condition (20) of Theorem 1 is trivially satisfied. Analytical treatment of the differential equation $\dot{x} = f(x)$ is not easy, but its trajectories can easily be determined by numerical solution and they are shown in Fig. 3 for two choices of λ. For the case $\lambda = 0.5$, (Fig. 3a) there is convincing evidence that the point $x^* = (-2,2)$ is a stable stationary point with global domain of attraction. Therefore, for $\lambda = 0.5$ it follows from Theorem 1 that $x(t) \to x^*$ w.p.1 as $t \to \infty$, which gives a correct classification rule $c^* = 0$. The case $\lambda = 0.99$ (Fig. 3b) corresponds to a common situation where errors that occur rather seldom (1%), "outliers", shall be detected. In this case there are two stable stationary points of the differential equation, $x^* = (-2,2)$ and $x^{**} = (-2.3, -1.4)$. There is obviously a non zero probability that x(t) belongs to the domain of attraction of x^{**} i.o. Therefore Theorem 1 shows that for $\lambda = 0.99$, and for any starting value x(0) there is a non zero probability (that depends on x(0)) that $x(t) \to x^{**}$ as $t \to \infty$. This gives an asymptotic classification rule $c^{**} = -1.8$, that classifies 39% of the "correct values" as outliers. For this case simulations of the classifier are shown in Fig. 4. In fact, the simulation leading to the undesired value c^{**} appeared only after several (257) attempts and from simulations only it might have been tempting to conclude general convergence to c^*.

In this example it is cumbersome to find a suitable Lyapunov function for the stability problem. However, as seen in Fig. 3 numerical solution of the ODE yields sufficient insight into the stability properties. Such detailed information can naturally be obtained only if the dimensionality of the problem is small.

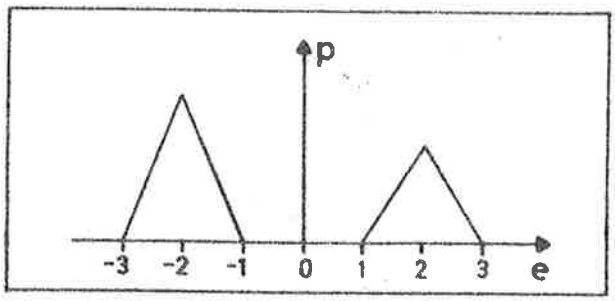

Fig. 2 - Probability density function of the random variable to be classified by the automatic classifier.

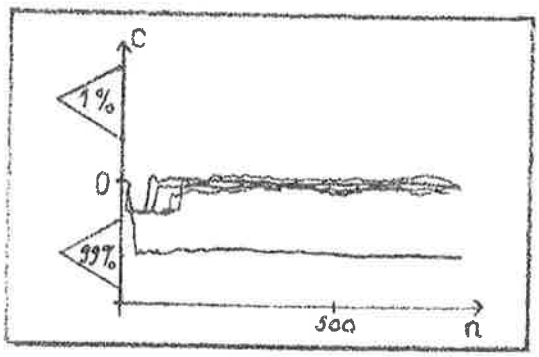

Fig. 4 - Simulations of the classifier (40) for the case $\lambda = 0.99$.

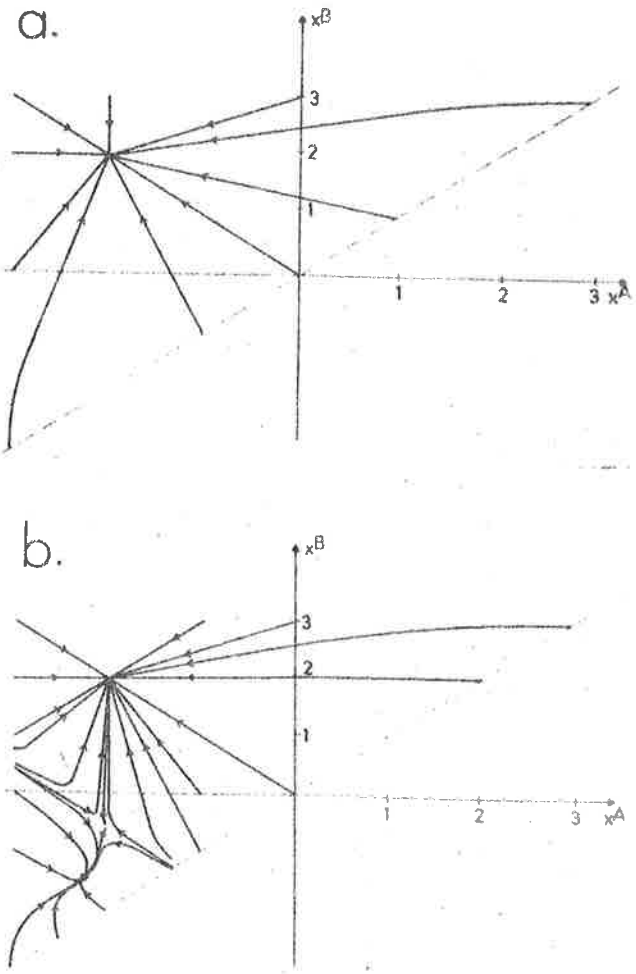

Fig. 3 - Trajectories for the ODE that is associated with the self-learning classifier (40).

a) $\lambda = 0.5$ b) $\lambda = 0.99$

Example 3 - Equation error identification methods.

A common way of modelling dynamic systems is as a vector difference equation (VDE),

$$y(t) + A_1 y(t-1) + \ldots + A_n y(t-n) = B_1 u(t-1) + \ldots + B_n u(t-n) \tag{42}$$

where $y(t)$ and $u(t)$ are column vectors and A_i and B_i are matrices of appropriate dimensions. Introduce

$$\theta = (A_1 \ \ldots \ A_n \ B_1 \ \ldots \ B_n)^T \tag{43}$$

$$\psi(t) = \left(-y(t-1)^T \ \ldots \ -y(t-n)^T \ u(t-1) \ \ldots \ u(t-n)^T\right)^T$$

Then (42) can be written

$$y(t) = \theta^T \psi(t) \tag{44}$$

We may remark that (44) also covers several other interesting estimation problems, not necessarily related to system identification.

Usually the true system cannot be described exactly in the form (44). Suppose that it can be described as

$$y(t) = \theta_0^T \psi(t) + v(t) \tag{45}$$

where $v(\cdot)$ is a disturbance that can be modelled as

$$v(t) = D(q^{-1}) e_1(t) \tag{46}$$

Here $D(q^{-1})$ is a matrix with rational functions of the backward shift operator q^{-1} as entries and $e_1(\cdot)$ is a stationary sequence of independent random vectors with finite moments. It is assumed that the denominator polynomials in $D(z)$ (z replacing q^{-1}) have all roots outside the unit circle, i e $D(q^{-1})$ is an exponentially stable filter.

Even if an exact description of the system is impossible, a θ can be determined that gives a model (44) which describes the recorded data as well as

possible. Often θ is determined by minimizing a criterion based on the equation error

$$\| y(t) - \theta^T \psi(t) \|^2 \tag{47}$$

Several algorithms based on the idea of somehow minimizing (47) have been suggested in the literature, see e g [24] and also [25] for a comprehensive treatment. The probably best known method of this type is the least squares algorithms, see e g [24]. Then the sum

$$\sum_{t=1}^{N} \| y(t) - \theta^T \psi(t) \|^2 \tag{48}$$

is minimized w.r.t θ to obtain the estimate $\theta(N)$ based on measurement up to time N. An important and well known feature of this method is that the sequence of estimates can be obtained recursively as

$$\theta(t) = \theta(t-1) + \gamma(t)K(t)[y(t) - \theta(t-1)^T \psi(t)]^T \tag{49a}$$

$$K(t) = R^{-1}(t)\psi(t) / \left[1 + \gamma(t)\left(\psi(t)^T R^{-1}(t)\psi(t) - 1\right)\right] \tag{49b}$$

$$R(t) = R(t-1) + \gamma(t)[\psi(t)\psi(t)^T - R(t-1)] \tag{49c}$$

(usually (49c) is written in terms of $R^{-1}(t)$, which makes it of "Riccati type"). For the minimization of (48) $\gamma(t)$ has to be taken as $1/t$. Other sequences $\gamma(\cdot)$ correspond to criteria where old measurements are discounted, which often is relevant in practice.

Let us assume that the input to the process is determined as

$$u(t) = F(q^{-1})e_2(t) + H(q^{-1};\theta(t-1))y(t) \tag{50}$$

where $F(q^{-1})$ and $H(q^{-1},\theta)$ are matrices with rational functions of the backward shift operator q^{-1} as entries. Let $e_2(\cdot)$ be a stationary sequence of random vectors with finite moments, that are mutually independent and also independent of $e_1(\cdot)$. Moreover, $H(q^{-1},\theta)$ is a causal operator that allows output feedback terms in the input. This feedback law may depend on the current parameter estimate as is further discussed in Example 5.

It is clear that the rational filters in (46) and (50) can be represented in a state space form,

$$z_v(t+1) = A_v z_v(t) + B_v e_1(t+1); \qquad v(t) = (I\ 0\ \dots\ 0) z_v(t) \tag{51}$$

$$z_u(t+1) = A_u(\theta(t)) z_u(t) + z_u + B_{ue} e_2(t+1) + B_{uy} y(t+1)$$

$$u(t) = (I\ 0\ \dots\ 0) z_u(t) \tag{52}$$

where $z_v(\cdot)$ and $z_u(\cdot)$ are the corresponding state vectors of appropriate dimensions. We may now form the "observation vector",

$$\varphi(t) = [y(t)^T,\ \psi(t)^T,\ z_v(t)^T,\ z_u(t)^T]^T \tag{53}$$

which obeys

$$\varphi(t) = A_\varphi(\theta(t-1))\varphi(t-1) + B_\varphi \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \tag{54}$$

where the matrix $A_\varphi(\cdot)$ is formed from (45), (43), (51), (52) in an obvious manner. Its eigenvalues are the poles of the filters $D(q^{-1})$, $F(q^{-1})$ and of the closed loop system which is obtained for (42) with a constant feedback (50) using $\theta(t-1)$. There are also a number of eigenvalues in the origin, arising from the shifting of the vector $\psi(t)$. Notice that $A_\varphi(\theta)$ depends on θ only since the feedback filter $H(q^{-1};\theta)$ does. Let us take

$$x(t) = (\theta(t)^T \mathrm{col}^T\ R(t))^T \tag{55}$$

Then eq (49) takes the form

$$x(t) = x(t-1) + \gamma(t) Q(t; x(t-1); \varphi(t)) \tag{56}$$

with an obvious definition of $Q(t;x,\varphi)$ from (49). Therefore the algorithm (49) together with (54) is of the general form (32). Let us check if assumptions B of Section 4 are satisfied. Conditions B.1 and B.2 are satisfied due to our assumptions. By straightforward calculations it is readily shown that B.3 is satisfied in the open area $D_R = \{x | R > 0\}$, (cf (55)) e g with

$$K_1(x,\varphi,\rho,v) = (|\theta| + \rho)(1 + |\varphi| + v)^2/(1 - \rho|R^{-1}|)^2 \tag{57}$$

for $\rho = \rho(x) < 1/|R^{-1}|$. Then B.4 will be satisfied with

$$K_2(x,\varphi,\rho,v,w) = (|\theta| + \rho)(|\varphi| + 2w + v)/(1 - \rho|R^{-1}|)^2 \tag{58}$$

Condition B.5 is satisfied if the matrix $H(q^{-1};\theta)$ is Lipschitz continuous in θ. For condition B.6 we define

$$\bar{\varphi}(t,\bar{x}) = A_\varphi(\bar{\theta})\bar{\varphi}(t-1,\bar{x}) + B_\varphi\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} \qquad \left(\bar{x} = (\bar{\theta}^T \operatorname{col}^T \bar{R})^T\right)$$

Since $e_i(\cdot)$ are stationary, $\bar{\varphi}(t,\bar{x})$ will approach stationarity exponentially, for all $\bar{x}$, such that $\bar{\theta}$ makes the closed loop system stable. Therefore the limits

$$f(\bar{\theta}) = \lim_{t\to\infty} E\ \bar{\psi}(t,\bar{x})[\bar{y}(t,\bar{x}) - \bar{\theta}^T\bar{\psi}(t,\bar{x})]^T \tag{59a}$$

$$G(\bar{\theta}) = \lim_{t\to\infty} E\ \bar{\psi}(t,\bar{x})\bar{\psi}(t,\bar{x})^T \tag{59b}$$

are well defined where $\bar{y}(t,\bar{x})$ and $\bar{\psi}(t,\bar{x})$ are the corresponding parts of $\bar{\varphi}(t,\bar{x})$, and

$$\lim_{t\to\infty} E\ Q\left(t;\bar{x},\bar{\varphi}(t,\bar{x})\right) = \begin{bmatrix} \bar{R}^{-1}f(\bar{\theta}) \\ \operatorname{col}\left(G(\bar{\theta}) - \bar{R}\right) \end{bmatrix} \tag{60}$$

so B.6 is satisfied. Moreover, from (57) and (58) it follows that B.7 holds, since all moments of $\bar{\varphi}(t,x)$ and $v(t,x)$ exist. Conditions B.8 - B.11 about the sequence $\gamma(\cdot)$ are assumed to be satisfied.

The conclusion therefore is that the differential equation

$$\frac{d}{d\tau}\theta(\tau) = R^{-1}(\tau)f\left(\theta(\tau)\right) \tag{61a}$$

$$\frac{d}{d\tau}R(\tau) = G\left(\theta(\tau)\right) - R(\tau) \tag{61b}$$

can be associated with the algorithm (49). In the remaining part of this example, we shall assume that the feedback matrix H does not depend on θ

(i e there is no adaptive feedback), that the matrix F(z) has full rank a e z and that $e_i(\cdot)$ are full rank processes. (Adaptive feedback is further discussed in Example 5.) This means that the matrix $A_\varphi(\cdot)$ does not depend on θ, $\bar{\varphi}(t,\bar{x}) = \varphi(t)$, and $\bar{y}(t,\bar{x}) = y(t)$, so the values in (59) are directly defined in terms of input-output covariances. In particular, the matrix G is independent of θ ; $G(\theta) = G$.

Introduce

$$r = E\,\varphi(t)v(t)^T \tag{62}$$

and we have, using (45),

$$f(\theta) = G\cdot(\theta_0 - \theta) + r \tag{63}$$

Hence, (61) can be rewritten as

$$\frac{d}{d\tau}\theta(\tau) = R^{-1}(\tau)G[(\theta_0+G^{-1}r) - \theta(\tau)] \tag{64a}$$

$$\frac{d}{d\tau}R(\tau) = G - R(\tau) \tag{64b}$$

With

$$\tilde{\theta}(\tau) = \theta(\tau) - (\theta_0+G^{-1}r)$$

and

$$V(\tilde{\theta},R) = \tilde{\theta}^T R\tilde{\theta}$$

we have

$$\frac{d}{d\tau}V\big(\tilde{\theta}(\tau),R(\tau)\big) = -2\tilde{\theta}^T G\tilde{\theta} + \tilde{\theta}^T\big(G - R(\tau)\big)\tilde{\theta} = -\tilde{\theta}^T(\tau)\big(G + R(\tau)\big)\tilde{\theta}(\tau) \tag{65}$$

so that V is a Lyapunov function for equation (64) or (61) that assures that the stationary point

$$\theta^* = \theta_0 + G^{-1}r \tag{66}$$

has a domain of attraction equal to D_R. Therefore condition (21) of Theorem 1 is satisfied with $D_A = D_R$. To check condition (20) we note first that the assumption on full rank and finite moments of $e_i(\cdot)$ implies that $CI > G > \delta I$ for some $\delta > 0$, $C < \infty$. Therefore also $\frac{\delta}{2} I < R(t) < 2CI$ and $|\varphi(t)| < 2C$ i.o. w.p.1. We also note that (49a) can for large $\theta(t-1)$ be written

$$\theta(t) \approx \left(I - R^{-1}(t)\varphi(t)\varphi(t)^T\right)\theta(t-1)$$

which shows that $\theta(t)$ can, w.p.1, not tend to infinity. Hence $x(t)$ belongs to a compact subset of D_R i.o. w.p.1 and condition (20) is satisfied. Theorem 1 now implies that

$$\theta(t) \to \theta^* \text{ w.p.1 as } t \to \infty$$

In particular, we see that the least squares estimate is consistent only if $r = 0$, which essentially is the same as requiring that $v(\cdot)$ is a sequence of uncorrelated random variables, and that the current $u(t)$ is uncorrelated with future $v(s)$, $s \geq t$.

Other variants of equation error methods are treated analogously.

These facts are, of course, well-known, [24], [25], at least for the case $\gamma(t) = 1/t$, but one reason for this example is that the analysis extends into less trivial problems. □

These three examples have all been for the case where $A(\cdot)$ in (32b) actually does not depend on x. The convergence part in this case can, at least under further assumptions, notseldom be treated by more conventional statistical methods. When $A(\cdot)$ does depend on x, conventional approaches become much more difficult, and in fact, also in the proof of Theorem 1, a major burden is to keep control over the coupled stability questions in (32a) and (32b). We shall now conclude with two examples where the inclusion of x-dependent A-matrices is necessary.

Example 4 - Recursive identification algorithms.

The only disadvantage of the least squares algorithm (49) is that it in general gives biased estimates, and several recursive algorithms have been suggested to overcome this problem, see e g [24], [26] - [28]. Most of them have the common feature that the disturbance $v(t)$ in (45) is further modelled. We shall in this example assume that the system (42) is single input-single output. In the recur-

sive generalized least squares algorithm, [29], $v(\cdot)$ is modelled as an AR process. In the "extended least squares method", [30], [31], see also [1], $v(\cdot)$ is modelled as an MA process

$$v(t) = C(q^{-1})e(t) = e(t) + c_1 e(t-1) + \ldots + c_n e(t-n) \qquad (67)$$

where $e(\cdot)$ is a sequence of independent random variables. The same is true for the "recursive maximum likelihood method", cf [1] and [28]. The point is that when the c-parameters of (67) are included in the θ-vector, the "residuals" $e(t-i)$ should enter in the $\psi(t)$-vector. But since they are not measurable, they are replaced by the estimated residuals,

$$\varepsilon(t) = y(t) - \theta(t-1)^T \psi_E(t)$$

where

$$\psi_E(t) = \left(-y(t-1) \ldots -y(t-n)\ u(t-1) \ldots u(t-n)\ \varepsilon(t-1) \ldots \varepsilon(t-n)\right)^T$$

Clearly, here the generation of $\varepsilon(t)$ and hence of $\psi_E(t)$ depends on $\theta(t-1)$. Therefore in eq (54), i e the generation of the observation vector $\varphi(t)$ of Example 3 (with $\psi_E(t)$ replacing $\psi(t)$ in (53)) the matrix A_φ will always depend on $\theta(t-1)$ even if there is no adaptive feedback present ($H(q^{-1};\theta)$ in (50) independent of θ).

The extended least squares method, e g, can be formally described by (49) with $\psi_E(t)$ replacing $\psi(t)$ and the differential equation (61) is still relevant for analysis of this method, see [1]. Notice, however, that since $A(\cdot)$ in (54) depends on θ the variables $\bar{\psi}(t;\bar{x})$ and $\bar{y}(t,\bar{x})$ in (59) indeed depend on θ and hence so does G. Therefore (63) does not hold and the convergence proof of Example 3 cannot be applied. In fact, using Theorem 2, it can be shown that there exist systems for which the algorithm does not converge, [1].

In [28] a comprehensive study of some recursive identification methods is made, partly based on Theorems 1, 2 and 3. In particular, it can be shown that for a recursive maximum likelihood method, the expected value of the log likelihood function can be chosen as Lyapunov function for the associated differential equation. The convergence properties are therefore according to the corollary as good as those for off-line maximum likelihood identification, cf also [1].

For the case (67) the stability region D_S for $A_\varphi(\cdot)$ is given by

$$D_S = \{\theta | C(z) = 0 \Rightarrow |z| > 1\}$$

Since this region is known to the user, it is useful, and often necessary to project $\theta(t)$ into D_S using e g the projection algorithm (28), (29). □

Example 5 - Self-tuning regulators.

As remarked in Section 2 the structure (32) can be understood as typical for adaptive control of linear systems. We shall in this example discuss an application to the self-tuning regulator, described in [32]; see also [2] and [4]. This regulator is based on least squares identification, (49), and the output feedback law is determined from the current parameter estimates. Usually the feedback law is chosen to be a minimum variance regulator, [32], but here it could be a general linear regulator as in (50), where perhaps in most cases F is zero.

In this case the matrix $A_\varphi(\theta)$ in (54) does depend on θ since the feedback term does. However, the point now is that, in contrast to conventional analysis of the least squares algorithm, most of what was said in Example 3 still holds. Up to equation (61) the development was quite general. This differential equation is valid also in the case of adaptive feedback, although G and r now are functions of θ. If $v(\cdot)$ is a sequence of independent random variables, then $r = 0$ and (64) and (65) hold. ((65) does not hold if $r \neq 0$ depends on θ.) We therefore still find that the points defined by

$$D_c = \{\theta | f(\theta) = 0\}$$

form an invariant set with global domain of attraction. Clearly $\theta_0 \in D_c$, and whether D_c contains more points depends on the choice of feedback law and model order. There is a further complication before Theorem 1 can be applied. In this case the area D_S is unknown, i e the area of such θ that inserted in a constant feedback law (50) make the closed loop system stable ($A_\varphi(\theta)$ in (54) has all eigenvalues inside the unit circle). Therefore we cannot guarantee stability by projecting θ into D_S as in Example 4. Hence condition (20) of Theorem 1 has to be verified by other considerations, e g, by showing that the over-all system has a certain stability property as in [33]. But when this is shown, Theorem 1 proves convergence of $\theta(t)$ into D_c w.p.1. Let us repeat that this holds for the case of arbitrary feedback law, but under the assumption that $v(\cdot)$ is white noise. For general noise $v(\cdot)$ the convergence analysis is more

cumbersome, but it can be performed in certain special cases, [2], [4]. We refer also to these papers for more details how Theorems 1, 2 and 3 can be used in the analysis of self-tuning regulators. Numerical solution of the associated differential equation has turned out to be a valuable tool here, and it has been used in [34] as well as in the references above.

9. CONCLUSIONS

Recursive algorithms like (1) have been analysed in various contexts. However, we would like to stress again that when φ in (1) is generated as in (2), the analysis becomes drastically more difficult. The reason is that (1) no longer is recursive in x for analysis purposes: the whole history of $x(\cdot)$ enters in each step of (1). Moreover, the coupled stability problems between (1) and (2) are intricate. But the structure (1), (2) is nonetheless common in estimation and control problems; a typical example is adaptive control of linear stochastic systems. The analysis of this case is also known to be usually very difficult.

With the present approach we are able to give a general treatment of (1), (2) under assumptions that do not appear to be restrictive. The examples indicate that the theorems may be applied to rather diverse problems, and perhaps the technique also may serve as a basis for a unified approach to the analysis of adaptive controllers. In addition, an extension is obtained for the conventional convergence results in the simple case where $A(\cdot)$ in (2) is independent of x. We may remark that the analysis is restricted to the asymptotic behaviour, convergence, possible convergence points etc of the algorithm. Two related algorithms which are associated with the same differential equation may differ noticably in transient behaviour and convergence rate.

In the described theory, we would like to stress the intuitive content of the theorems and the methodology of analysis as outlined in Section 7. It is no doubt important to appreciate the exact formulations of the theorems and to know the exact conditions under which they are valid. But it is perhaps equally rewarding to use the properly defined differential equation as a general instrument for analysis in a more heuristic fashion. This may be examplified in Theorem 3, which has a fairly technical formulation and is probably more valuable as a "moral support" for studying the trajectories of the differential equation, than in its literal content.

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11. REFERENCES

[1] L Ljung, T Söderström and I Gustavsson, "Counterexamples to general convergence of a commonly used recursive identification method", IEEE Trans. Automatic Control, Vol. AC-20 No. 5, Oct 1975, pp. 643-652.

[2] K J Åström, U Borisson, L Ljung and B. Wittenmark, "Theory and applications of adaptive regulators based on recursive identification", Proc. 6th IFAC Congress, Boston, Mass 1975. An extended version of this paper has been submitted to Automatica.

[3] L Ljung and S Lindahl, "Convergence properties of a method for state estimation in powers systems", Int. J. Control, Vol. 22, No. 1, July 1975, pp. 113-118.

[4] L Ljung and B Wittenmark, "Analysis of a class of adaptive regulators", Proc. IFAC Symp. on Stochastic Control, Budapest, Sept. 1974.

[5] L Ljung, "Convergence of recursive, stochastic algorithms", Proc. IFAC Symp. on Stochastic Control, Budapest, Sept. 1974.

[6] L Ljung, "On the convergence of certain recursive algorithms", Report 7505(C), Department of Automatic Control, Lund Institute of Technology, Feb. 1975.

[7] L Ljung, "Theorems for the asymptotic analysis of recursive, stochastic algorithms", Report 7522, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

[8] L Ljung, "Convergence of recursive stochastic algorithms", Report 7403, Division of Automatic Control, Lund Institute of Technology, Lund, Sweden, Feb. 1974.

[9] W Hahn, "Stability of Motion", Springer-Verlag, Berlin 1967.

[10] R W Brockett, "Finite dimensional linear systems", Wiley, N.Y. 1970.

[11] H Robbins and S Monro, "A stochastic approximation method", Ann.Math.Stat., Vol. 22, pp. 400-407, 1951.

[12] J Blum, "Multidimensional stochastic approximation methods", Ann. Math. Stat., Vol. 25, pp. 737-744, 1954.

[13] M A Aizermann, E M Braverman and L I Rozonoer, "Metod Potentsialnykh funktsij v teorii obuchenija mashin" (The method of potential functions in the theory of machine learning), Izd. Nauka, Moscow 1970 (in Russian).

[14] M T Wasan, "Stochastic approximation", Cambridge, University Press, 1969.

[15] Ya Z Tsypkin, "Adaption and learning in automatic systems", Academic Press, New York, 1971.

[16] Ya Z Tsypkin, "Foundations of the theory of learning systems", Academic Press, New York, 1973.

[17] J Kiefer and J Wolfowitz, "Stochastic estimation of the maximum of a regression function", Ann. Math. Stat., Vol. 23, pp. 462-466, 1952.

[18] H J Kushner, "Stochastic approximation algorithms for the local optimization of functions with non-unique stationary points", IEEE Trans. Automatic Control, Vol. AC-17, pp. 646-655, 1972.

[19] H J Kushner and T Gavin, "Stochastic approximation type methods for constrained systems: algorithms and numerical results", IEEE Trans. Automatic Control, Vol. AC-19, pp. 349-358, 1974.

[20] K S Fu, "Learning system theory", In L A Zadeh and E Polak (Ed.): System Theory, McGraw-Hill, New York, pp. 425-466, 1969.

[21] G N Saridis, Z J Nikolic and K S Fu, "Stochastic approximation algorithms for systems identification, estimation and decomposition of mixtures", IEEE Trans. Systems Science and Cybernetics, Vol. SSC-5, pp. 8-15, 1969.

[22] Ya Z Tsypkin, "Self-learning - what is it?", IEEE Trans. Automatic Control, Vol. AC-13, No. 6, pp. 608-612, 1968.

[23] E M Braverman, "The method of potential functions in the problem of training machines to recognize patterns without a teacher", Autm. and Remote Control, Vol. 27, No. 10, pp. 1748-1770, 1966.

[24] K J Åström and P Eykhoff, "System identification - a survey", Automatica, Vol. 7, pp. 123-164, 1971.

[25] J M Mendel, "Discrete techniques of parameter estimation", Marcel Dekker Inc., New York, 1973.

[26] G N Saridis, "Comparison of six on-line identification algorithms", Automatica, Vol. 10, pp. 69-79, 1974.

[27] R R Iserman, U Baur, W Bamberger, P Knepo and R Siebert, "Comparison of six on-line identification and parameter estimation methods", Automatica, Vol. 10, pp. 81-103, 1974.

[28] T Söderström, L Ljung and I Gustavsson, "A comparative study of recursive identification methods", Report 7427, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden, 1974.

[29] R Hastings-James and M W Sage, "Recursive generalized least squares procedure for on-line identification of process parameters", Proc. IEE, Vol. 116, pp. 2057-2062, 1969.

[30] P C Young, "The use of linear regression and related procedures for the identification of dynamic processes", Proc. 7th IEEE Symposium on Adaptive Processes, UCLA 1968.

[31] V Panuska, "A stochastic approximation method for identification of linear systems using adaptive filtering", Proc. JACC 1968.

[32] K J Åström and B Wittenmark, "On self-tuning regulators", Automatica, Vol. 9, pp. 185-199, 1973.

[33] L Ljung and B Wittenmark, "On a stabilizing property of adaptive regulators", submitted to the 4th IFAC Symposium on Identification to be held in Tbilisi, USSR, 1976.

[34] K J Åström and B Wittenmark, "Analysis of a self-tuning regulator for nonminimum phase systems", Proc. IFAC Symposium on Stochastic Control, Budapest, 1974.