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# Analysis of resonant structures of 4D symplectic mappings, using normal forms

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**Abstract**

The geometry of the resonant orbits of symplectic 4D mappings in the neighbourhood of an elliptic fixed point is analysed; a perturbative approach based on the construction of the resonant normal forms and interpolating Hamiltonians is proposed. The classification of the different types of normal forms and related resonant structures in phase space is given; the analysis of the truncated interpolating Hamiltonian provides an analytical hint on a conjecture to generalize the Poincarè-Birkhoff theorem to the 4D case.

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Four-dimensional symplectic mappings are a model of a wide class of problems in different fields of physics, such as celestial mechanics or beam dynamics. Contrary to the 2D case, in 4D the structures of the phase space of a nonintegrable map are not yet completely understood, even if several results have been obtained in the last decades [1,2]. In this communication we outline a method, based on perturbative tools, in order to analyse the resonant structures of the orbits of a 4D symplectic map in the neighbourhood of a bi-elliptic fixed point.

Let us first consider a symplectic 2D integrable map in the form

$$z' = T(z, z^*) = e^{i\Omega(r)} z \qquad r \equiv zz^* \qquad z \in \mathbf{C}, \tag{1}$$

i.e. a twist mapping. Let $\Omega(r) = \omega + \Omega_2 r + O(r^2)$ be a real function of $r$ with $\Omega_2 \neq 0$. If the amplitude $r$ is such that the frequency $\Omega(r)/2\pi$ is irrational, the orbits are dense on the 1D torus, i.e. on the circle $z = \sqrt{r}e^{i\vartheta}$, $\vartheta \in [0, 2\pi[$. On the other hand, if the frequency is rational $\Omega(r)/2\pi = p/q$[1], then $T$ has an infinity of parabolic fixed points $z = \sqrt{r}e^{i\vartheta}$, $\vartheta \in [0, 2\pi[$ of period $q$.

If one considers a small perturbation which preserves the symplectic conditions

$$z' = F(z, z^*) = T(z, z^*) + \mu G(z, z^*), \tag{2}$$

[where $G(z, z^*) = O(|z^2|)$], the KAM theorem [3] ensures that for small perturbations $\mu \ll 1$ there exist invariant curves of $F$, with 'strongly irrational' (i.e., diophantine) frequency, that are deformed circles. On the other hand, the Poincarè–Birkhoff theorem [3–5] states that among the infinite set of parabolic fixed points of period $q$ of the map $T$, only $2jq$ fixed points survive under perturbation: one has $jq$ hyperbolic fixed points and $jq$ elliptic fixed points of period $q$.

Whilst the KAM theorem is valid for mappings with higher dimensionality, the Poincarè–Birkhoff theorem is valid only for 2D symplectic mappings. In fact the proof is based on a method which fully exploits the reduced dimensionality of the phase space [3–5]. In order to analyse the topology of the orbits of 4D mappings, we propose an heuristic method based on the perturbative tools of resonant normal forms. This approach suggests a conjecture to generalize the Poincarè–Birkhoff theorem to the 4D case, which has been successfully tested using computer simulations.

The normal form approach [6–8] is the natural generalization of the canonical perturbation theory for hamiltonian flows to symplectic mappings: given a symplectic map $\mathbf{F}$ in a $2n$-dimensional phase space, having a fixed point in the origin, one looks for a nonlinear

[1] Throughout the paper $p, q$ will denote integers without common divisors.

transformation $\boldsymbol{\Phi}$ such that $\mathbf{F}$ is transformed to a new map $\mathbf{U}$ that is 'particularly simple', i.e. that has explicit invariants and symmetries. The map $\mathbf{U}$ is the normal form. The conjugating equation of the map to its normal form reads

$$\boldsymbol{\Phi}^{-1} \circ \mathbf{F} \circ \boldsymbol{\Phi}(\zeta) = \mathbf{U}(\zeta), \tag{3}$$

where $\zeta$ are the new variables in phase space, called normal coordinates. $\mathbf{U}$ is invariant under a symmetry group generated by a linear transformation $\boldsymbol{\Lambda}_\alpha$, i.e. it commutes with $\boldsymbol{\Lambda}_\alpha$: this symmetry condition defines the normal form $\mathbf{U}$ and the conjugating function $\boldsymbol{\Phi}$ (up to a gauge group).

The existence of a formal solution is guaranteed by theorems that state that one can build a normal form $\mathbf{U}$ with respect to the symmetry group generated by the linear part of the map $\boldsymbol{\Lambda}_\omega$, or subgroups of it. Analytic solutions to the functional equation (3) in open neighbourhoods of an elliptic fixed point do not exist in the generic case: the series are divergent. Indeed, one can prove that the perturbative series are asymptotic, and therefore optimal truncation can provide very accurate approximation of the dynamics of the nonlinear map: this has allowed applications to numerous problems of celestial mechanics [9,10] and accelerator physics [11–13].

In order to analyse the geometry of the orbits of the normal forms, one can build an interpolating Hamiltonian $H$ whose orbits interpolate the orbits of $\mathbf{U}$; since $\mathbf{U}$ commutes with the symmetry group, one has

$$H(\boldsymbol{\Lambda}_\alpha \mathbf{z}) = H(\mathbf{z}). \tag{4}$$

The analysis of the interpolating Hamiltonian allows one to determine perturbative expansions for a wide class of nonlinear quantities that characterize the dynamics, such as the frequencies, the location and stability of the fixed points, and the topology of resonant orbits.

In the 2D case, in the neighbourhood of an elliptic fixed point, one can build normal forms defined by the symmetry groups generated by the linear matrix $\boldsymbol{\Lambda}_\alpha = \mathrm{Diag}\,(e^{i\alpha}, e^{-i\alpha})$. We shall express the normal forms and the interpolating Hamiltonians in the variables $(\rho, \theta)$, related to the normal coordinates by $\zeta = \sqrt{\rho}e^{i\theta}$. One can have two different types of normal forms: nonresonant normal forms $[\alpha/(2\pi)$ irrational$]$, which are invariant under the group of continuous rotations, and resonant normal forms $[\alpha/(2\pi) = p/q]$, which are invariant under the group of discrete rotations by an angle of $2\pi/q$. In the first case the normal form is an amplitude–dependent rotation (i.e. a twist mapping), and the interpolating Hamiltonian is a function of the amplitude $\rho$; in the second case the interpolating Hamiltonian has the form $h(\rho, \theta) \equiv H(\zeta)$:

$$h(\rho, \theta) = \sum_{k,l} h_{k,l}\, \rho^{k+lq/2} \cos(lq\theta + \varphi_{k,l}). \tag{5}$$

We consider a generic mapping with $h_{1,0} \neq 0$ and a resonance of order $q \geq 5$. The analysis of the interpolating Hamiltonian of the resonant normal form gives the topology of the resonant orbits: if we truncate the Hamiltonian at the first significative resonant term [i.e. neglecting $O(\rho^{(q+1)/2})$], we obtain a pendulum Hamiltonian which has $q$ hyperbolic and $q$ elliptic fixed points, in agreement with the Poincarè–Birkhoff theorem. If the first order resonant coefficient $h_{0,1}$ is zero, one has to consider the higher orders: therefore it is possible to find cases where one has $2kq$ fixed points, with $k \geq 1$. The above-described approach is complementary to the theorem: in fact, even if it does not prove the existence of the fixed points[2], it gives an estimate of the position and of the eigenvalues of the fixed points, which are relevant quantities of the dynamics [8].

We classify different types of normal forms and interpolating Hamiltonians in order to have an analytical hint on the structure of resonant orbits: in the 4D case, symmetry groups that define the normal form (in the neighbourhood of a bi-elliptic fixed point) are generated by the linear matrix $\Lambda_\alpha = \text{Diag}\,(e^{i\alpha_1}, e^{-i\alpha_1}, e^{i\alpha_2}, e^{-i\alpha_2})$. According to the different values of the frequencies $\alpha_1$, $\alpha_2$, the linear part of the map generates different subgroups of the group of continuous rotations, and therefore one can define four types of normal forms. Interpolating Hamiltonians will be expressed in the coordinates $(\rho_1, \rho_2, \theta_1, \theta_2)$, related to the normal coordinates by

$$\zeta_1 = \sqrt{\rho_1} e^{i\theta_1} \qquad \zeta_2 = \sqrt{\rho_2} e^{i\theta_2}. \tag{6}$$

Nonresonant case: if $\alpha_1/2\pi, \alpha_2/2\pi, \alpha_1/\alpha_2 \in \mathbf{R} \setminus \mathbf{Q}$, the symmetry group is a two-parameter compact group, direct product of 2D continuous rotations. The normal form is an amplitude-dependent rotation and the interpolating Hamiltonian is a power series in the amplitudes $\rho_1, \rho_2$, which are the independent integrals of motion. The 2D tori $[0, 2\pi[\times[0, 2\pi[$ are the invariant surfaces.

Single resonance; if $\alpha_1 = 2\pi p/q$, $\alpha_2/2\pi \in \mathbf{R} \setminus \mathbf{Q}$, the symmetry group generated by the linear part is a two-parameter compact group, given by the direct product of 2D continuous rotations times 2D discrete rotations by an angle $2\pi/q$. The interpolating Hamiltonian has the form

$$h(\rho_1, \rho_2, \theta_1) = \sum_{k_1,k_2,l} h_{k_1,k_2,l}\, (\rho_1)^{k_1+lq/2} (\rho_2)^{k_2} \cos(lq\theta_1 + \varphi_{k_1,k_2,l}). \tag{7}$$

[2] In fact, the perturbative series are asymptotic, and therefore one should check the effect of the neglected remainder of the series in order to have a rigorous proof.

One has two independent integrals of motion: $h$ and $\rho_2$.

Coupled resonance; if $\alpha_1 = \alpha p$, $\alpha_2 = \alpha q$, $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, the symmetry group generated by the linear part is a compact group that depends on one continuous parameter. The interpolating Hamiltonian has the form

$$\begin{aligned} h(\rho_1, \rho_2, \theta_1, \theta_2) &= \sum_{k_1, k_2, l} h_{k_1, k_2, l} \, (\rho_1)^{k_1 + lq/2} (\rho_2)^{k_2 + lp/2} \times \\ &\times \cos(l(q\theta_1 - p\theta_2) + \varphi_{k_1, k_2, l}). \end{aligned} \tag{8}$$

The independent integrals of motion are $p\rho_1 + q\rho_2$ and $h$.

Double resonance; if $\alpha_1 = 2\pi p_1/q_1$, $\alpha_2 = 2\pi p_2/q_2$, the symmetry group generated by the linear part is a two-parameter compact group, given by the direct product of discrete 2D rotations by an angle $2\pi/q_1$ times discrete 2D rotations by an angle $2\pi/q_2$. The interpolating Hamiltonian has the form

$$\begin{aligned} h(\rho_1, \rho_2) &= \sum_{k_1, k_2, l_1, l_2} (\rho_1)^{k_1 + l_1 q_1/2} (\rho_2)^{k_2 + l_1 q_2/2} \times \\ &\times \ [\ h^{+}_{k_1, k_2, l_1, l_2} \cos(l_1 q_1 \theta_1 + l_2 q_2 \theta_2 + \varphi^{+}_{k_1, k_2, l_1, l_2}) + \\ &+ \ h^{-}_{k_1, k_2, l_1, l_2} \cos(l_1 q_1 \theta_1 - l_2 q_2 \theta_2 + \varphi^{-}_{k_1, k_2, l_1, l_2})]. \end{aligned} \tag{9}$$

In this case $h$ is the only explicit integral of motion, and therefore the interpolating Hamiltonian is not trivially integrable such as in the previous cases.

We first analyse the resonant orbits of the 4D twist mapping

$$\begin{aligned} z_1' &= T_1(\mathbf{z}) = \exp\left(i\Omega_1(\rho_1, \rho_2)\right) z_1 \qquad & \rho_1 = z_1 z_1^* \\ z_2' &= T_2(\mathbf{z}) = \exp\left(i\Omega_2(\rho_1, \rho_2)\right) z_2 \qquad & \rho_2 = z_2 z_2^*, \end{aligned} \tag{10}$$

since the map is integrable, one can compute all the relevant quantities of its orbits; the iterates always lies on the 2D torus, but according to the different values of the nonlinear frequencies, one finds different topologies.

Nonresonant case. Having fixed $(\rho_1, \rho_2)$ such that $\Omega_1(\rho_1, \rho_2)/(2\pi)$, $\Omega_2(\rho_1, \rho_2)/(2\pi)$, $\Omega_1(\rho_1, \rho_2)/\Omega_2(\rho_1, \rho_2)$ are irrational, one obtains an orbit that is dense on the torus; its closure has dimension two, and it is connected.

Single resonance. We fix $(\rho_1, \rho_2)$ such that $\Omega_1(\rho_1, \rho_2) = 2\pi p/q$ and $\Omega_2(\rho_1, \rho_2)/(2\pi)$ is irrational: one obtains an orbit that is dense on the direct product of a 1D torus times $q$ parabolic 2D fixed points. We call these structures 'single-resonance parabolic fixed lines of period $q$'. The closure of the orbit has dimension one, and is made up of $q$ pieces connected. For each initial condition chosen in $\theta_1 \in [0, 2\pi/q[$, $\theta_2 = 0$, one obtains an infinity of different parabolic fixed lines of the same period, having the same geometry.

Coupled resonance. We fix $(\rho_1, \rho_2)$ such that $\Omega_1(\rho_1, \rho_2) = 2\pi p\nu$ and $\Omega_2(\rho_1, \rho_2) = 2\pi q\nu$, where $\nu \in \mathbf{R} \setminus \mathbf{Q}$: one obtains an orbit, which lies on the 2D torus, that is dense on the 1D curve of equation

$$\theta_1(t) = \theta_1 + tp\nu \qquad \theta_2(t) = \theta_2 + tq\nu \qquad t \in [0, 2\pi[. \tag{11}$$

We call this structure 'coupled-resonance parabolic fixed line'; the closure of the orbit is connected. For each initial condition chosen in $\theta_1 \in [0, 2\pi/p[$, $\theta_2 = 0$, one obtains an infinity of different parabolic fixed lines of the same type, having the same geometry.

Double resonance. We fix $(\rho_1, \rho_2)$ such that $\Omega_1(\rho_1, \rho_2) = 2\pi p_1/q_1$ and $\Omega_2(\rho_1, \rho_2) = 2\pi p_2/q_2$: one obtains an orbit, which lies on the 2D torus, that is made up of $q_1 q_2$ parabolic fixed points. The dimension of the orbit is zero, and is made up of $q_1 q_2$ components trivially connected. For each initial condition chosen in $\theta_1 \in [0, 2\pi/q_1[$, $\theta_2 \in [0, 2\pi/q_2[$, one obtains an infinity of families of fixed points.

We have considered the first order resonant truncation of the interpolating Hamiltonian in order to analyse the resonant structures of a symplectic 4D map. We consider a 4D twist mapping $\mathbf{T}$ [see Eq. (10)] having a family of resonant orbits in the neighbourhood of the origin. We restrict ourselves to the analysis of single or double resonances with $q \geq 5$, and coupled resonances with $|p| + |q| \geq 5$. We conjecture that a small symplectic perturbation changes the topology of these orbits according to the following cases.

Single resonance. Let $\rho_1, \rho_2$ be positive amplitudes such that the single-resonance condition is satisfied, i.e. there exist an infinity of single-resonance parabolic fixed lines of period $q$. A generic perturbation[3] preserves only two single-resonance fixed lines of period $q$: one is elliptic and one is hyperbolic.

Coupled resonance. Let $\rho_1, \rho_2$ be positive amplitudes such that the coupled-resonance condition is satisfied, i.e. there exist an infinity of coupled-resonance parabolic fixed lines. A generic symplectic perturbation (see previous footnote) preserves only two fixed lines: one is elliptic and one is hyperbolic.

Double resonance. Let $\rho_1, \rho_2$ be positive amplitudes such that the double resonance condition is satisfied, i.e. there exists an infinity of

[3]This means a perturbation which produces a generic resonant interpolating Hamiltonian, i.e. whose first resonant coefficient is different from zero. Non-generic cases give rise to a situation analogous to the 2D case with $j > 1$.

parabolic fixed points of period $q_1 q_2$. A generic symplectic perturbation (see previous footnote) preserves $4q_1 q_2$ fixed points, which can be split in four families: fixed points which are obtained by iteration of the map belong to the same family. We denote by $(\rho_1^{\pm}, \rho_2^{\pm})$ the amplitudes of the four families of fixed points, and we define the quantities

$$
\begin{aligned}
\alpha &= 2h_{2,0,0,0} \qquad \beta = h_{1,1,0,0} \qquad \gamma = 2h_{0,2,0,0} \\
\delta &= \mp (q_1)^2 h_{0,0,1,0} (\rho_1^{\pm +})^{\frac{q_1}{2}} \quad \delta = \mp (q_1)^2 h_{0,0,1,0} (\rho_1^{\pm -})^{\frac{q_1}{2}} \\
\eta &= (q_2)^2 h_{0,0,0,1} (\rho_2^{\pm -})^{\frac{q_2}{2}} \quad \eta = -(q_2)^2 h_{0,0,0,1} (\rho_2^{\pm +})^{\frac{q_2}{2}}
\end{aligned}
\tag{12}
$$

and the signs

$$
s_1 \equiv \mathrm{sgn}[(|\delta|\alpha + |\eta|\gamma)^2 - 4|\delta\eta|\beta^2] \qquad s_2 \equiv \mathrm{sgn}[\alpha\gamma - \beta^2]. \tag{13}
$$

Then, one can prove that, according to the different values of these signs, the stability of the four families is given by

$$
\begin{aligned}
s_1 &> 0 \Longrightarrow 2\mathrm{EH} + \mathrm{EE} + \mathrm{HH} \\
s_1 &< 0, s_2 > 0 \Longrightarrow 2\ \mathrm{CI} + \mathrm{EE} + \mathrm{HH} \\
s_1 &< 0, s_2 < 0 \Longrightarrow 2\ \mathrm{CI} + 2\ \mathrm{EH}
\end{aligned}
\tag{14}
$$

where EE = bi-elliptic fixed points, EH = elliptic-hyperbolic fixed point, HH = bi-hyperbolic fixed points, and CI = complex instability. Computations are rather lengthy but straightforward, and are not presented for sake of brevity.

The numerical check of this conjecture has been performed for different models using a code for the visualization and animation of projections and sections of 4D orbits [14], a code for computing the coefficients of the interpolating Hamiltonian [15], and a code which computes fixed points [16, 17]. Elliptic fixed lines relative to both single and coupled resonances have been found for different models. More explicitly, we have considered the 4D Hénon map:

$$
\begin{aligned}
z_1' &= e^{i\omega_1} \left( z_1 - \frac{i}{4} \left[ (z_1 + z_1^*)^2 - (z_2 + z_2^*)^2 \right] \right) \\
z_2' &= e^{i\omega_2} \left( z_2 + \frac{i}{2} (z_1 + z_1^*)(z_2 + z_2^*) \right)
\end{aligned}
\tag{15}
$$

In Fig. 1a we display the projection on a 3D space $(x, p_x, y)$ (where $z_1 = x - ip_x$ and $z_2 = y - ip_y$) of the stable neighbourhood of a single-resonance elliptic fixed line of period $q = 5$. The linear frequencies were fixed at $\omega_1/(2\pi) = 0.205$ and $\omega_2/(2\pi) = 0.6180$; 50 000 iterates of a suitable initial condition are plotted. In Fig. 1b we display the 3D projection of the stable neighbourhood of a coupled-resonance elliptic fixed line, $p = 2$ and $q = 3$, for the same model with $\omega_1/(2\pi) = 0.638$ and $\omega_2/(2\pi) = 0.412$. One can see that the topology of the orbits is consistent with the above-quoted scheme. The case of the double resonance has also been carefully analysed for different models: using the fixed point code we have found the four families of fixed points foreseen by the perturbative analysis and checked their stability; transitions of stability for different values of an external parameter have been analytically computed according to Eq. (14) and numerically verified.

A more detailed exposition of the analytical computations and of the numerical check are given in [14] and it will be described in a forthcoming paper. We would like to acknowledge Prof. Vrahatis and Dott. Bazzani for important contributions. We also want to thank Prof. Turchetti and Prof. Ramis for useful and stimulating discussions.

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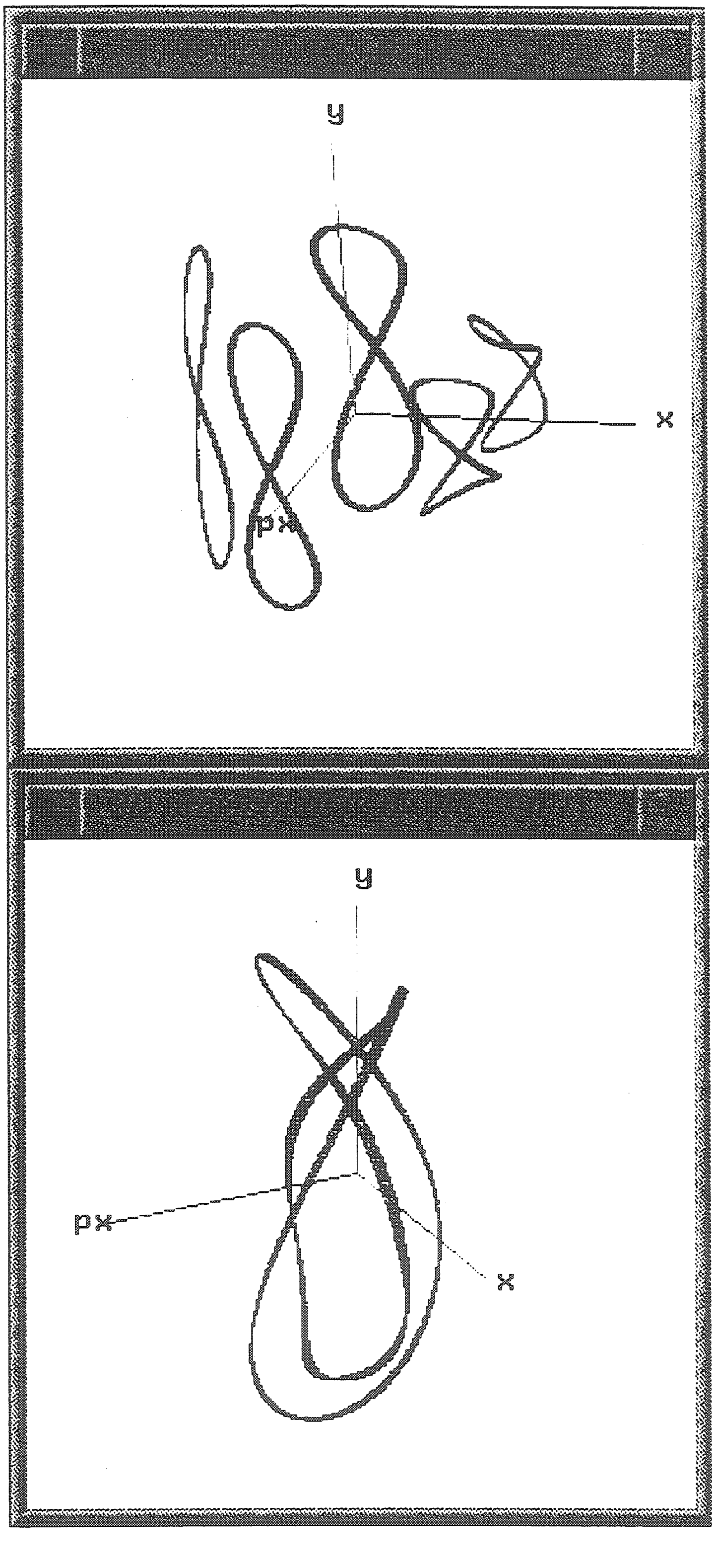


Figure 1. Projection on a 3D space of the iterates of the 4D Hénon mapping: neighbourhood of a single-resonance ($q = 5$) elliptic line (a) and neighbourhood of a coupled-resonance ($p = 2$, $q = 3$) elliptic line (b). 50 000 iterations are displayed.