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# Analysis of resources distribution in economics based on entropy

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**Abstract**

We propose a new approach to the problem of efficient resources distribution in different types of economic systems. We also propose to use entropy as an indicator of the efficiency of resources distribution. Our approach is based on methods of statistical physics in which the states of economic systems are described in terms of the density functions $\rho(g, \alpha)$ of the variable $g$ parametrized by $\alpha$. The parameter $\alpha$ plays a role of the integral characteristic of the state of the economic system. Having the density function $\rho(g, \alpha)$ we can use the corresponding entropy to evaluate the efficiency of the resources distribution. Our theoretical study have been tested on real data related to the portfolio investment. 

## 1. Introduction

A key issue in economics consists of the efficient distribution of resources [1]. Usually resource distribution problems are solved for economic systems consisting of a large number of components. Such macroeconomic systems has been recently studied using the well-developed and efficient methods of statistical physics [2–5]. In this approach the state of economic system is described by a probability distribution [6–8].

Using the density function one can calculate the corresponding entropy functional. Conservation of entropy in time may indicate the absence of macroscopic changes

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Table 1
General scheme of the decision making problem. The quantity $G_n$ is the size of resource corresponding to the component $n$, $n = 1, \dots, N$ and $S_N = \sum_{n=1}^{N} G_n$

| Component | Quantity |
|---|---|
| 1 | $G_1$ |
| 2 | $G_2$ |
| — | — |
| $n$ | $G_n$ |
| — | — |
| $N$ | $G_N$ |

in redistribution of resources. Assuming the absence of macro-changes in economic systems and in related additional expenses of resources, we may consider the entropy as an indicator of the efficiency of the resources management.

It is generally accepted that entropy can be used for the study of economic systems consisting of large number of components [2]. Economic systems with small number of components were not studied with the statistical approach. The traditional statistical approach requires that either the sample is big enough so that the reasoning based on limit theorems can be applied or the distribution of the sample is known a priori. However, there are also common situations where neither of these two conditions are satisfied. In order to fill this gap we have developed a new approach for the analysis of economic systems that includes systems with small number of components.

The paper is organized as follows. We introduce in Section 2 a general scheme for the analysis of economic systems with a small number of components, based on the Lorenz diagram and its interpolation by a continuous function. We discuss the motivation and the form of the interpolation function in Section 3. In Section 4 we use the interpolation function for the determination of probability densities and discuss their properties. Based on these probability densities we calculate in Section 5 the entropy of the investment portfolio for some leading Russian companies and scientific projects in the frame of the Russian National Program on High Temperature Superconductivity.

## 2. General scheme

Let us consider the resource distribution problem (management decision taking) for economic systems with relatively small number of components $N$: see Table 1.

In order to extend the statistical methods for the analysis of the efficiency of management decisions, we use the Lorenz diagram technique. This technique is widely applied in economics and social science for the description of irregularity of the resource distribution among different objects. For instance, this method is used in studying the irregularities in income distribution among population [1].

The procedure of the Lorenz diagram technique for solving problems consists of the following. First, we construct the Lorenz diagram using data set related to the problem: see, for example, Table 1. Then, this diagram is interpolated by a continuous function,

Table 2

| Year | $N$ | $S_N$ (mln. rubles) |
|---|---|---|
| 1988 | 273 | 143.1 |
| 1989 | 362 | 137.6 |
| 1990 | 432 | 136.9 |
| 1991 | 553 | 109.4 |
| 1992 | 345 | 411.2 |
| 1993 | 353 | 930 |

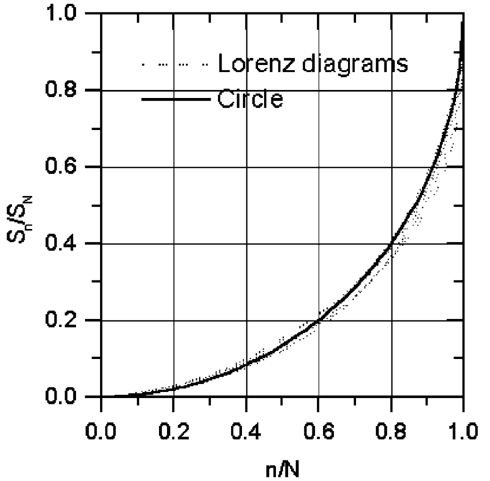


Fig. 1. Lorenz diagram of the financial support of projects over 1988–1993.

which we use to obtain the probability $\rho(G)\,\mathrm{d}G$ of the decision $\mathrm{d}n$ corresponding to the interval $(G, G+\mathrm{d}G)$ while $N \to \infty$ (see details below).

## 3. General form of the interpolation function

Before identifying the general form of the approximation function, let us consider a specific example—the empirical distributions of the financial support of projects in frame of the Russian National Program on High Temperature Superconductivity. In spite of the drastic changes of sizes of total supports $S_N$ and the number of projects $N$ (see Table 2) over 1988–1993 the degree of irregularity in distribution of finance had very stable form (see Fig. 1) [9,10].

In this example, the most suitable approximation of the Lorenz diagram over all empirical data can be presented by a circle

$$\left(1 - \frac{S_n}{S_N}\right)^2 + \left(\frac{n}{N}\right)^2 = 1\,, \tag{1}$$

where $S_n = \sum_{k=1}^{n} G_k$ and $G_k$ is the size of the grant for the project $k$.

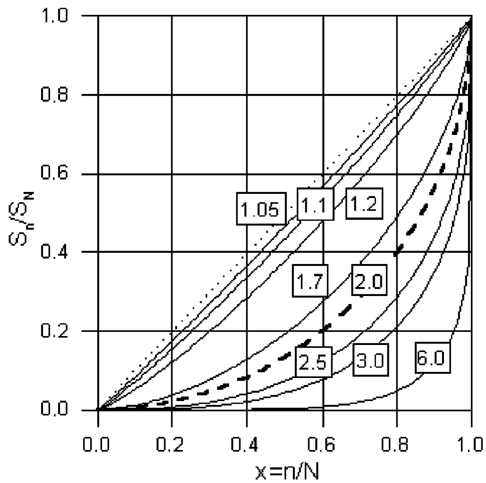


Fig. 2. Interpolation curves for different values of the parameter $\alpha$.

On the other hand a general form of the production function is [11,12]

$$V = (AK^{-\alpha} + BL^{-\alpha})^{-1/\alpha} , \tag{2}$$

where $V$ is the volume of production, $K$ and $L$ are, correspondingly, the expenditure of capital and labor, $A$ and $B$ are the interpolation coefficients.

Eq. (2) can be transformed to the form

$$\left(A^{1/\alpha}\frac{V}{K}\right)^{\alpha} + \left(B^{1/\alpha}\frac{V}{L}\right)^{\alpha} = 1 . \tag{3}$$

By comparing Eqs. (1) and (3) we conjecture the following general form of the interpolation function

$$\left(1 - \frac{S_n}{S_N}\right)^{\alpha} + \left(\frac{n}{N}\right)^{\alpha} = 1 . \tag{4}$$

Eq. (4) can also be represented as

$$[1 - y(x,\alpha)]^{\alpha} + x^{\alpha} = 1 , \tag{5}$$

where $y(x,\alpha) = S_n/S_N$ and $x = n/N$. The interpolation functions for different values of the parameter $\alpha$ are shown in Fig. 2.

Evidently, the parameter $\alpha$ determines the nonuniformity of the resource distribution. In particular, the equality $\alpha = 1$ corresponds to the uniform distribution. When $\alpha \to \infty$ we have distribution of resources concentrated to one component only.

## 4. Properties of density functions

The probability to pick up $G_n$, corresponding to the interval $(G, G + \Delta G)$, is approximated by

$$\Delta w = \frac{\Delta n}{N}\,, \tag{6}$$

where $\Delta n$ is the number of components corresponding to the interval $(G, G+\Delta G)$, $N$ is the total number of components in the system [13].

Assuming $N \to \infty$, we may rewrite (6) as

$$\mathrm{d}w = \frac{\mathrm{d}n}{N}\frac{\mathrm{d}G}{\mathrm{d}G} = \left(\frac{1}{N}\frac{\mathrm{d}n}{\mathrm{d}G}\right)\mathrm{d}G = \rho(G)\,\mathrm{d}G\,. \tag{7}$$

Thus, in order to find the distribution function $\rho(G)$, we have to determine $\mathrm{d}n/\mathrm{d}G$. This can be accomplished by following the following four steps:

1. Replace the cumulative sums $S_n$ in the Lorenz diagram by the integral $S_n = \int_0^n G_n\,\mathrm{d}n$ (assuming $N \to \infty$).
2. Using (5), calculate the dependencies $G = \mathrm{d}S_n/\mathrm{d}n = F(n)$.
3. Find the inverse function $n = F^{-1}(G)$.
4. Determine $\mathrm{d}n/\mathrm{d}G$.

Following steps 1–4, we rewrite (5) in the more convenient form:

$$y(x,\alpha) = 1 - (1 - x^{\alpha})^{1/\alpha}, \quad 0 \leqslant x \leqslant 1\,. \tag{8}$$

Then

$$\frac{\mathrm{d}y(x,\alpha)}{\mathrm{d}x} = \frac{x^{\alpha-1}}{(1-x^{\alpha})^{(\alpha-1)/\alpha}} = F(x) \quad \text{for } \alpha > 1\,. \tag{9}$$

Replacing now $x$ and $y$ by their initial values, we get

$$g = \frac{\mathrm{d}y(x,\alpha)}{\mathrm{d}x} = \frac{\mathrm{d}S_n}{S_N}\frac{N}{\mathrm{d}n} = \frac{G_n}{\bar{G}}\,. \tag{10}$$

Eq. (10) shows that the term $g$ is equal to $G$ normalized by the average value of allocation $\bar{G}$: $\bar{G} = S_N/N$.

Due to (10), $g = F(x)$ is an increasing function, its inverse function $x = F^{-1}(g)$ exists and is also increasing. Moreover, $F^{-1}(0) = 0$, and

$$x = F^{-1}(g,\alpha) = \frac{g^{1/(\alpha-1)}}{(1 + g^{\alpha/(\alpha-1)})^{1/\alpha}}\,. \tag{11}$$

It follows from (11) that $F^{-1}(g,\alpha) \to 1$ while $g \to \infty$. Thus, $F^{-1}(g,\alpha)$ is the distribution function of the positive random variable $G$.

Then, the corresponding density function $\rho(g,\alpha)$ has the form

$$\rho(g,\alpha) = \frac{\mathrm{d}F^{-1}(g,\alpha)}{\mathrm{d}g} = \frac{1}{\alpha-1}\,\frac{g^{(2-\alpha)/(\alpha-1)}}{(1 + g^{\alpha/(\alpha-1)})^{(\alpha+1)/\alpha}}\,. \tag{12}$$

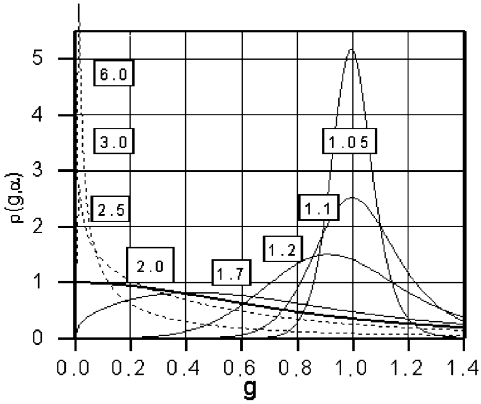


Fig. 3. Density functions $\rho(g,\alpha)$ for different values of the parameter $\alpha$.

The forms of the density function $\rho(g,\alpha)$ for different values of parameter $\alpha > 1$ are presented in Fig. 3.

Depending on the value of parameter $\alpha$ (which shows the nonuniformity of the distribution), all these curves can be subdivided into three classes with different properties.

**Class 1.** If $1 < \alpha < 2$ then $\rho(0,\alpha) = 0$ and $\rho(0,\alpha)$ is unimodal restricted function. Let $\alpha = 1 + \varepsilon$, where $\varepsilon \to +0$. The following asymptotic formula is valid:

$$\rho(g,\alpha) \sim \frac{1}{\varepsilon}\frac{g^{1/\varepsilon}}{(1+g^{1/\varepsilon})^2} \to \delta(g-1), \quad \text{while } \varepsilon \to +0\,.$$

**Class 2.** If $\alpha = 2$ then $\rho(0,2) = 1$ and $\rho(g,2)$ is decreasing and has a single point where the derivative changes sign

$$\rho(g,2) = \frac{1}{(1+g^2)^{3/2}}\,.$$

**Class 3.** If $\alpha > 2$ then $\rho(g,\alpha)$ is strictly decreasing such that $\rho(g,\alpha) \to \infty$ while $g \to +0$. When $\alpha \to \infty$, we can use, instead of (13), the asymptotic formula

$$\rho(g,\alpha) \sim \frac{1}{\alpha}\frac{1}{g^{1-1/\alpha}(1+g^{1+1/\alpha})^{1+1/\alpha}}$$

from which follows that $\rho(g,\alpha) \to \delta_+(g)$ while $\alpha \to +\infty$ and where $\delta_+(g) = 0$ for $g > 0$ and $\int_0^\infty \delta_+(g)\,\mathrm{d}g = 1$.

Thus, the density function $\rho(g,2)$ with $\alpha = 2$ plays the role of a boundary between two different classes of density functions $\rho(g,\alpha)$: with $\alpha < 2$ and $\alpha > 2$. It must also be noted that $\rho(g,2)$ is the unique density function from the family of functions $\rho(g,\alpha)$, $\alpha > 1$, which has a finite and nonzero value at $g = 0$.

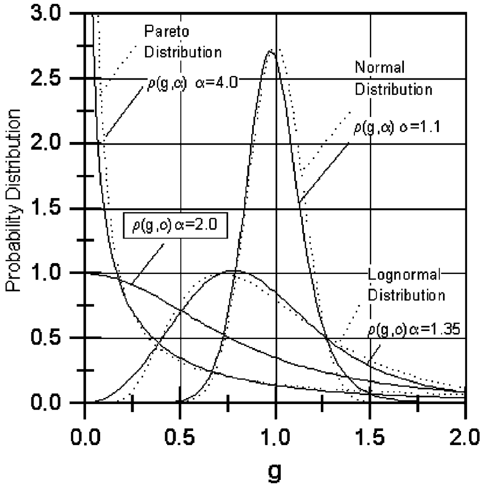


Fig. 4. Comparison of distributions (12) with well-known distributions.

It is interesting to compare the obtained density functions for different values of the parameter $\alpha$ with well-known distributions: Pareto and Gaussian. Fig. 4 demonstrates the results of such comparison.

Based on the above results, we can do the following conclusions. For different degrees of nonuniformity of distribution in economic systems we have to use different forms of the density function $\rho(g,\alpha)$. For "anarchic" (statistically independent) distribution of resources ($\alpha \to 1$) in economic systems the normal distribution is preferable. For the "dictatorship" case ($\alpha \geqslant 2$) we must use the Pareto distribution.

## 5. Entropy as indicator of efficiency of resource distribution

Applying the methods of statistical physics in studies of economic systems [2], we propose to use the entropy as an indicator of the efficiency of distribution of resources.

Let us calculate the entropy for a system with small number of elements using the family of density functions $\rho(g,\alpha)$. For the entropy calculation we use the usual formula [14]

$$S = -\int \mathrm{d}g\, \rho(g,\alpha) \ln \rho(g,\alpha) ,$$

where $S$ is the entropy and $\rho$ is the density function.

Fig. 5 shows the dependence of the normalized entropy $S$ (the maximal value of entropy was assumed to be 1) versus the parameter $\alpha$, which characterizes the nonuniformity in the resource distribution. The curve demonstrates the presence of maximum in the region of $\alpha$-parameter values, close to 1.84. Below we present the results of $\alpha$-parameter calculations for the High Temperature Super-Conductivity program.

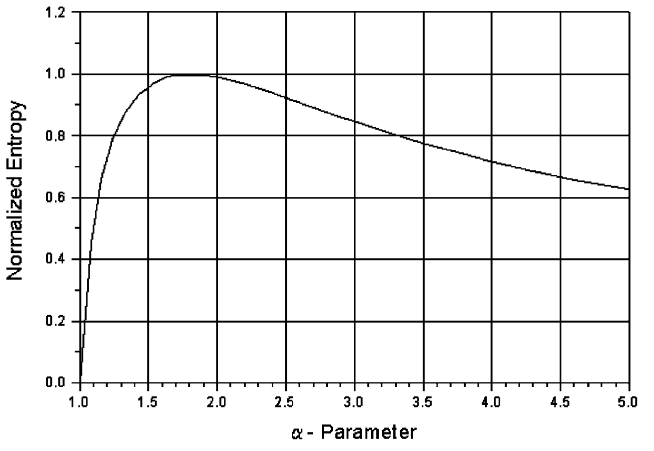


Fig. 5. Dependence of entropy versus α-parameter.

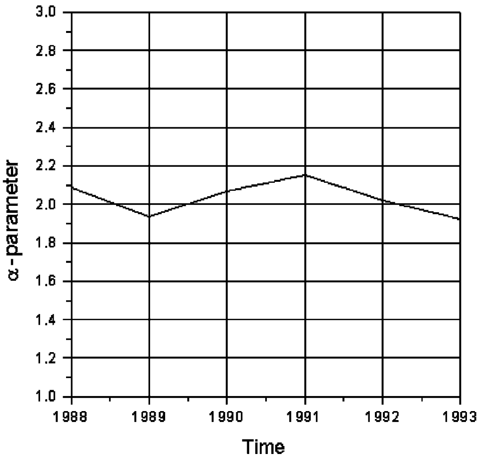


Fig. 6. Dependence of α-parameter versus of time for the High Temperature Super-Conductivity program.

Fig. 6 presents the dependence of α-parameter versus time. We see that the changes of the α-parameter value in time are quite small, which suggests their relative independence and shows the minimum of the entropy production and the absence of possible expenses of resources related to the general change in the distribution character.

The stable maximal value of entropy in Fig. 5 indicates stability and that the High Temperature Superconductivity program considered as economic system is closed. A possible explanation of the state of this program can be illustrated by the short-term character of its creation (this program was prepared in 1 year). This accounts for its relative independence from external conditions because of the small ratio time scales system versus environment.

Table 3
Characteristics of investment portfolios

| Investment company | April 2000 | | | June 2000 | | |
|---|---|---|---|---|---|---|
| | $N$ | $\alpha$ | $S$ (%) | $N$ | $\alpha$ | $S$ (%) |
| ATON | 16 | 1.74 | 99.5 | 12 | 1.99 | 99.0 |
| Alpha-Capital | 12 | 1.97 | 99.0 | 12 | 1.74 | 99.5 |

The approach, developed in our paper, is applicable not only for closed systems, but also for open systems, which are exchanging with the surrounding environment different resources: information, finance, materials. During the evolution of an economic system, the initial resources are transformed into the definite result. The degree of stability of the system performance can be estimated by the entropy as a function of time.

In this connection, our scheme has been applied to the analysis of the investment portfolios of some well-known Russian companies (Alpha-Capital, ATON) for two time periods: before and after Presidential election in Russia (March 26, 2000). These companies demonstrated emerging stability in an unstable political situation in Russia (see Table 3).

In Table 3 $N$ is the number of components in the investment portfolio, $S$ is the normalized entropy of the portfolio. We see that these companies were working in the region close to the maximum of entropy versus the $\alpha$-parameter (see Fig. 5).

Relatively small changes of entropy (see Table 3) demonstrate the efficient management by these companies in conditions of instability and extreme political changes in Russia related to Presidential elections.

At the same time, for investment portfolios of some other well-known Russian investment companies the parameter $\alpha$ varied from 1.52 to 1.81.

## 6. Conclusion

We introduced a new approach for the presentation of economic systems with a small number of components as a statistical system described by density functions and entropy interpreted as an indicator of efficiency of the resources distribution.

The developed approach is not limited by the number of components of the economic system and can be applied to wide class of economic problems.

We think that the bridge between distribution of resources and proposed probability distributions may permit us to use the methods of nonequilibrium statistical mechanics [6–8] for the study and forecast of the dynamics of complex economic systems and to make correct management decisions.

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