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Analysis of Some Finite Elements for the Stokes Problem

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Abstract. We study some finite elements which are used in the approximation of the Stokes problem, so as to obtain error estimates of optimal order.

Résumé. Nous étudions deux éléments finis utilisés pour l'approximation du problème de Stokes et obtenons des estimations d'erreur d'ordre optimal.

I. Introduction. Let Ω be a bounded polyhedral domain in $\mathbf{R}^d$, $d = 2$ or 3. We consider the standard variational formulation of the stationary Stokes equations: for $\mathbf{f}$ given in $H^{-1}(\Omega)^d$, find $(\mathbf{u}, p)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$(\mathrm{I.1})\qquad \begin{cases} \forall \mathbf{v} \in H_0^1(\Omega)^d, & \nu(\mathbf{grad}\,\mathbf{u}, \mathbf{grad}\,\mathbf{v}) - (p, \operatorname{div} \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ \forall q \in L_0^2(\Omega), & (q, \operatorname{div} \mathbf{u}) = 0, \end{cases}$$

where we denote by $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$ (or $L^2(\Omega)^d$ or $L^2(\Omega)^{d^2}$). Hereafter $L_0^2(\Omega)$ is the space $\{q \in L^2(\Omega); \int_\Omega q\,dx = 0\}$. Now let h be a real positive parameter tending to zero. We introduce two finite-dimensional subspaces X_h and M_h of $H_0^1(\Omega)^d$ and $L_0^2(\Omega)$ respectively, satisfying the usual condition: for any q_h in M_h, $q_h \neq 0$, there exists $\mathbf{v}_h$ in X_h such that $(q_h, \operatorname{div} \mathbf{v}_h) \neq 0$. We consider the discretized problem: find $(\mathbf{u}_h, p_h)$ in $X_h \times M_h$ such that

$$(\mathrm{I.2})\qquad \begin{cases} \forall \mathbf{v}_h \in X_h, & \nu(\mathbf{grad}\,\mathbf{u}_h, \mathbf{grad}\,\mathbf{v}_h) - (p_h, \operatorname{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ \forall q_h \in M_h, & (q_h, \operatorname{div} \mathbf{u}_h) = 0. \end{cases}$$

We recall that problem (I.1) (respectively problem (I.2)) has a unique solution $(\mathbf{u}, p)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$ (respectively $(\mathbf{u}_h, p_h)$ in $X_h \times M_h$). Moreover, when $(\mathbf{u}, p)$ belongs to the space $H^{m+1}(\Omega)^d \times H^m(\Omega)$, it is well-known (see [7]) that the error estimate

$$(\mathrm{I.3})\qquad \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leqslant Ch^m\left(\|\mathbf{u}\|_{m+1,\Omega} + \|p\|_{m,\Omega}\right)$$

holds whenever the following additional hypotheses are satisfied:

(H1) for any q in $H^m(\Omega) \cap L_0^2(\Omega)$, one has

$$\operatorname*{Inf}_{q_h \in M_h} \|q - q_h\|_{0,\Omega} \leqslant Ch^m\|q\|_{m,\Omega};$$

(H2) there exists a linear operator Π_h from $H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d$ into X_h such that

$$\forall \mathbf{v} \in H^{m+1}(\Omega)^d \cap H_0^1(\Omega)^d, \quad \begin{cases} \forall q_h \in M_h, & (q_h, \operatorname{div}(\mathbf{v} - \Pi_h\mathbf{v})) = 0, \\ \|\mathbf{v} - \Pi_h\mathbf{v}\|_{1,\Omega} \leqslant Ch^m\|\mathbf{v}\|_{m+1,\Omega}; \end{cases}$$

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(H3) for each q_h in M_h, there exists a function $\mathbf{v}_h$ in X_h such that

$$(\operatorname{div} \mathbf{v}_h, q_h) \geqslant \beta \|q_h\|_{0,\Omega} \|\mathbf{v}_h\|_{1,\Omega},$$

where $\beta > 0$ is a constant independent of h.

Our aim is to give some examples of finite-element spaces such that hypotheses (H1), (H2) and (H3) are satisfied. To this end, we introduce a family $(\mathcal{T}_h)_h$ of triangulations of $\bar{\Omega}$, where $\mathcal{T}_h$ is made of d-simplices with diameters bounded by h.

For any integer k, $P_k(K)$ denotes the space of polynomials of degree $\leqslant k$ on K. We set

$$M_h^{(m)} = \left\{ q_h \in L_0^2(\Omega);\ \forall K \in \mathcal{T}_h,\ q_{h/K} \in P_{m-1}(K) \right\}.$$

Then hypothesis (H1) is satisfied (see [2] for instance). Finally, we set

$$X_h = \left\{ \mathbf{v}_h \in \mathscr{C}^0(\bar{\Omega})^d \cap H_0^1(\Omega)^d;\ \forall K \in \mathcal{T}_h,\ \mathbf{v}_{h/K} \in P_K \right\};$$

hereafter we study some examples of spaces P_K introduced by Fortin [6] such that hypotheses (H2) and (H3) are satisfied.

More precisely, we give in Section II an example of a simplicial element of order $m = 1$ and, in Section III, an example of a three-dimensional tetrahedral element of order $m = 2$.

From now on we denote by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ the usual norm and seminorm on the Sobolev space $H^m(\Omega)$.

II. **A Simplicial Element of Order** 1 ($d = 2$ **or** 3). Let us consider a d-simplex K with vertices $a_1, \ldots$ and a_{d+1}. For $1 \leqslant i \leqslant d+1$, we denote by λ_i the barycentric coordinate associated with a_i, by F_i the face which does not contain a_i, and by $\mathbf{n}_i$ the unit outward normal to F_i, and we set

$$\mathbf{p}_i = \mathbf{n}_i \prod_{j=1,\, j \neq i}^{d+1} \lambda_j .$$

Then, we consider

$$P_K = P_1(K)^d \oplus \operatorname{Span}\{\mathbf{p}_i,\ 1 \leqslant i \leqslant d+1\}. \tag{II.1}$$

(Note that $\dim P_K = (d+1)^2$.) As far as the degrees of freedom are concerned, we can choose the values at the vertices a_i, $1 \leqslant i \leqslant d+1$, and the flux through the faces F_i, $1 \leqslant i \leqslant d+1$.

LEMMA II.1. *For any* $\mathbf{v}$ *in* $\mathscr{C}^0(K)^d$, *there exists a unique* $\Pi_K \mathbf{v}$ *in* P_K *such that*

$$\begin{cases} \Pi_K \mathbf{v}(a_i) = \mathbf{v}(a_i), & \\ \displaystyle\int_{F_i} (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \mathbf{n}_i \, d\sigma = 0, & \end{cases} \quad 1 \leqslant i \leqslant d+1. \tag{II.2}$$

Moreover, $\Pi_K \mathbf{v}_{/F_i}$ *depends only on* $\mathbf{v}_{/F_i}$, $1 \leqslant i \leqslant d+1$.

Proof. Let us denote by $\tilde{\Pi}_K \mathbf{v}$ the classical Lagrange interpolate of $\mathbf{v}$ in $P_1(K)^d$, i.e.,

$$\tilde{\Pi}_K \mathbf{v} = \sum_{i=1}^{d+1} \mathbf{v}(a_i) \lambda_i .$$

Then, as the $\mathbf{p}_i$'s are equal to 0 at any vertex, one has

$$\text{(II.3)}\qquad \begin{cases} \Pi_K\mathbf{v} = \tilde{\Pi}_K\mathbf{v} + \sum\limits_{i=1}^{d+1} \alpha_i \mathbf{p}_i, \\ \qquad\qquad \text{with } \alpha_i = \left(\displaystyle\int_{F_i} (\mathbf{v} - \tilde{\Pi}_K\mathbf{v}) \cdot \mathbf{n}_i \, d\sigma \right) \Big/ \displaystyle\int_{F_i} \prod_{j=1,\, j\neq i}^{d+1} \lambda_j \, d\sigma. \end{cases}$$

Moreover, on F_i,

$$\Pi_K\mathbf{v}_{/F_i} = \sum_{j=1,\, j\neq i}^{d+1} \mathbf{v}(a_j)\lambda_j + \alpha_i \mathbf{p}_i,$$

so that $\Pi_K\mathbf{v}_{/F_i}$ depends only on $\mathbf{v}(a_j), j \neq i$, and on $\int_{F_i} \mathbf{v} \cdot \mathbf{n}_i \, d\sigma$.

Now, for each h, we consider a triangulation $\mathscr{T}_h$ of $\bar{\Omega}$ made of d-simplices with diameters bounded by h and we assume that the family $(\mathscr{T}_h)_h$ is regular, i.e., (see [2]) there exists a constant σ such that

$$\text{(II.4)}\qquad \forall h, \forall K \in \mathscr{T}_h, \quad h_K \leqslant \sigma \rho_K,$$

where h_K is the diameter of K, and ρ_K the diameter of the sphere inscribed in K.

With each K in $\mathscr{T}_h$, we associate the space P_K defined by (II.1); then Lemma II.1 allows us to define an operator Π_h from $\mathscr{C}^0(\bar{\Omega})^d \cap H_0^1(\Omega)^d$ into X_h by

$$\text{(II.5)}\qquad \forall K \in \mathscr{T}_h, \quad \Pi_h\mathbf{v}_{/K} = \Pi_K\mathbf{v}.$$

LEMMA II.2. *The operator* Π_h *satisfies* (H2) *for* $m = 1$.

Proof. Clearly, one has

$$\int_K \operatorname{div}(\mathbf{v} - \Pi_K\mathbf{v}) \, dx = \sum_{i=1}^{d+1} \int_{F_i} (\mathbf{v} - \Pi_K\mathbf{v}) \cdot \mathbf{n}_i \, d\sigma = 0,$$

so that $\forall q_h \in M_h^{(1)}, (q_h, \operatorname{div}(\mathbf{v} - \Pi_h\mathbf{v})) = 0$.

Moreover, we know that (see [2], for instance), for $k = 0$ and 1,

$$|\mathbf{v} - \tilde{\Pi}_K\mathbf{v}|_{k,K} \leqslant Ch^{2-k}|\mathbf{v}|_{2,K}.$$

Let us compute $\Pi_K\mathbf{v} - \tilde{\Pi}_K\mathbf{v} = \sum_{i=1}^{d+1} \alpha_i \mathbf{p}_i$. We consider an affine invertible mapping F_K: $\hat{x} \mapsto x = B_K\hat{x} + b_K$ which maps the d-simplex $\hat{K} = \{\hat{x} \in \mathbf{R}^d;\ \forall i,\ 1 \leqslant i \leqslant d,\ \hat{x}_i \geqslant 0 \text{ and } \sum_{i=1}^d \hat{x}_i \leqslant 1\}$ onto K, and use the notations $x = F_K(\hat{x})$, $v = \hat{v} \circ F_K^{-1}$. Clearly, one has

$$|\mathbf{p}_i|_{k,K}^2 = \int_K \left\| D^k \left(\prod_{j=1,\, j\neq i}^{d+1} \lambda_j \right) \right\|^2 dx$$

$$\leqslant C \int_{\hat{K}} \left\| D^k \left(\prod_{j=1,\, j\neq i}^{d+1} \hat{\lambda}_j \right) \right\|^2 \|B_K^{-1}\|^{2k} |\det B_K| d\hat{x} \leqslant C|\det B_K| \, \|B_K\|^{-2k}$$

so that, by the regularity of the family $(\mathscr{T}_h)_h$,

$$\text{(II.6)}\qquad |\mathbf{p}_i|_{k,K} \leqslant Ch_K^{d/2-k}.$$

But, since

$$\int_{F_i} \prod_{j=1,\, j\neq i}^{d+1} \lambda_j \, d\sigma = |\det B_{K/\hat{F}_i}| \int_{\hat{F}_i} \prod_{j=1,\, j\neq i}^{d+1} \hat{\lambda}_j \, d\hat{\sigma},$$

we obtain by (II.3)

$$|\alpha_i| \leqslant C\left|\det B_{K/\hat{F}_i}\right|^{-1}\int_{F_i}|\mathbf{v}-\tilde{\Pi}_K\mathbf{v}|\,d\sigma \leqslant C\int_{\hat{F}_i}|\hat{\mathbf{v}}-\hat{\tilde{\Pi}}_{\hat{K}}\hat{\mathbf{v}}|\,d\hat{\sigma};$$

therefore, as $P_1(\hat{K})^d$ is invariant under $\hat{\tilde{\Pi}}_{\hat{K}}$,

$$|\alpha_i| \leqslant C|\hat{\mathbf{v}}|_{2,\hat{K}} \leqslant C|\det B_K|^{-1/2}\|B_K\|^2|\mathbf{v}|_{2,K} \leqslant Ch_K^{2-d/2}|\mathbf{v}|_{2,K}.$$

The previous inequalities yield, for $k = 0$ and 1,

$$|\mathbf{v}-\Pi_K\mathbf{v}|_{k,K} \leqslant Ch_K^{2-k}|\mathbf{v}|_{2,K},$$

so that

$$\|\mathbf{v}-\Pi_K\mathbf{v}\|_{1,\Omega} \leqslant Ch|\mathbf{v}|_{2,\Omega}.$$

We recall the proof of the following inequality only for the reader's convenience.

LEMMA II.3. *For any v in $H^1(K)$, we have*

$$\|v\|_{0,F_i} \leqslant C|\text{mes } F_i|^{1/2}h_K^{-d/2}\left\{\|v\|_{0,K}+h_K|v|_{1,K}\right\}. \tag{II.7}$$

Proof. As the trace mapping is continuous from $H^1(\hat{K})$ into $L^2(\hat{F}_i)$,

$$\begin{aligned}\|v\|_{0,F_i}^2 &= \left|\det B_{K/\hat{F}_i}\right|\int_{\hat{F}_i}\hat{v}^2\,d\hat{\sigma} \leqslant C\left|\det B_{K/\hat{F}_i}\right|\left\{\|\hat{v}\|_{0,\hat{K}}^2+|\hat{v}|_{1,\hat{K}}^2\right\}\\ &\leqslant C|\text{mes } F_i|\,h_K^{-d}\left\{\|v\|_{0,K}^2+h_K^2|v|_{1,K}^2\right\}.\end{aligned}$$

Let us now study the hypothesis (H3). We know (see [7, Chapter I, Lemma 3.2]) that, for each q_h in $M_h^{(1)}$, there exists $\mathbf{v}$ in $H_0^1(\Omega)^d$ such that

$$\operatorname{div}\mathbf{v} = q_h \quad \text{and} \quad \|\mathbf{v}\|_{1,\Omega} \leqslant C\|q_h\|_{0,\Omega}. \tag{II.8}$$

Hence, the hypothesis (H3) is an immediate consequence of the following

LEMMA II.4. *For any $\mathbf{v}$ in $H_0^1(\Omega)^d$, there exists $\mathbf{v}_h$ in X_h such that*

$$\forall q_h \in M_h^{(1)}, \quad \begin{cases}(q_h, \operatorname{div}(\mathbf{v}-\mathbf{v}_h)) = 0\\ \textit{and } \|\mathbf{v}_h\|_{1,\Omega} \leqslant C\|\mathbf{v}\|_{1,\Omega}.\end{cases} \tag{II.9}$$

Proof. Let us denote by $\mathbf{w}_h$ the interpolate of $\mathbf{v}$ in the space

$$\left\{u_h \in \mathscr{C}^0(\bar{\Omega}) \cap H_0^1(\Omega);\ \forall K \in \mathscr{T}_h,\ u_{h/K} \in P_1(K)\right\}^d,$$

defined by local regularization as in [4] (see [1] for an explicit generalization to the case $d = 3$). By the regularity of the family $(\mathscr{T}_h)_h$, we know that the following local interpolation error holds

$$\|\mathbf{v}-\mathbf{w}_h\|_{0,K}+h_K|\mathbf{w}_h|_{1,K} \leqslant Ch_K\|\mathbf{v}\|_{1,\Delta_K}, \tag{II.10}$$

where Δ_K is the union of all K' in $\mathscr{T}_h$ such that $K \cap K' \neq \varnothing$; moreover, each element of $\mathscr{T}_h$ is contained in at most M subsets Δ_K, where M is an integer independent of h.

Then, we consider the element $\mathbf{v}_h$ in V_h defined by

$$\begin{cases}\mathbf{v}_h(a_i) = \mathbf{w}_h(a_i),\\ \displaystyle\int_{F_i}(\mathbf{v}-\mathbf{v}_h)\cdot\mathbf{n}_i\,d\sigma = 0,\end{cases} \qquad 1 \leqslant i \leqslant d+1,$$

or, in other words, equal on K to

$$\begin{cases} \mathbf{v}_{h/K} = \mathbf{w}_h + \displaystyle\sum_{i=1}^{d+1} \alpha_i \mathbf{p}_i \\ \qquad\qquad \text{with } \alpha_i = \left(\displaystyle\int_{F_i} (\mathbf{v} - \mathbf{w}_h)\cdot \mathbf{n}_i\, d\sigma\right) \Big/ \displaystyle\int_{F_i} \prod_{j=1, j\neq i}^{d+1} \lambda_j\, d\sigma. \end{cases}$$

Clearly, one has $\forall q_h \in M_h^{(1)}$, $(q_h, \operatorname{div}(\mathbf{v} - \mathbf{v}_h)) = 0$. Moreover, by (II.6),

$$\|\mathbf{v}_h\|_{1,K} \leqslant \|\mathbf{w}_h\|_{1,K} + \sum_{i=1}^{d+1} |\alpha_i| \|\mathbf{p}_i\|_{1,K} \leqslant \|\mathbf{w}_h\|_{1,K} + Ch_K^{d/2-1} \sum_{i=1}^{d+1} |\alpha_i|.$$

But, we also have

$$|\alpha_i| \leqslant C|\det B_{K/\hat{F}_i}|^{-1} \int_{F_i} (\mathbf{v} - \mathbf{w}_h)\cdot \mathbf{n}_i\, d\sigma \leqslant C|\operatorname{mes} F_i|^{-1/2} \|\mathbf{v} - \mathbf{w}_h\|_{0,F_i}.$$

Lemma II.3 implies

(II.11) $$|\alpha_i| \leqslant Ch_K^{-d/2}\left\{\|\mathbf{v} - \mathbf{w}_h\|_{0,K} + h_K|\mathbf{v} - \mathbf{w}_h|_{1,K}\right\}.$$

Finally, we obtain

$$\|\mathbf{v}_h\|_{1,K} \leqslant \|\mathbf{w}_h\|_{1,K} + h_K^{-1}\left\{\|\mathbf{v} - \mathbf{w}_h\|_{0,K} + h_K|\mathbf{v} - \mathbf{w}_h|_{1,K}\right\},$$

which, together with (II.10), yields $\|\mathbf{v}_h\|_{1,\Omega} \leqslant C\|\mathbf{v}\|_{1,\Omega}$.

As assumptions (H1) to (H3) are satisfied with $m = 1$, this element can be used to solve the Stokes problem with an $O(h)$-error estimate.

Remark II.1. In the two-dimensional case, we can also consider a triangulation $\mathscr{T}_h$ of $\bar{\Omega}$ made of triangles and convex quadrilaterals. Then, if K is a triangle, the space P_K is defined by (II.1). If K is a convex quadrilateral with vertices $a_1, \ldots$ and a_4, there exists an invertible mapping F_K in $\hat{Q}_1^2$ which maps the unit square $\hat{K} = [0,1]^2$ onto K ($\hat{Q}_1$ is the space of polynomials spanned by $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3 = 1 - \hat{x}_1$ and $\hat{x}_4 = 1 - \hat{x}_2$); for $1 \leqslant i \leqslant 4$, we denote by F_i the edge with vertices a_{i-1} and a_i (of course, $a_0 = a_4$) and by $\mathbf{n}_i$ the unit outward normal to F_i, and we set

$$\mathbf{p}_i = \mathbf{n}_i(\hat{q}_i \circ F_K^{-1}), \qquad \hat{q}_i = \prod_{j=1, j\neq i}^{4} \hat{x}_j.$$

Then, we consider

(II.12) $$P_K = Q_1(K)^2 \oplus \operatorname{Span}\{\mathbf{p}_i, 1 \leqslant i \leqslant 4\},$$

where $Q_1(K) = \{\hat{p} \circ F_K^{-1}, \hat{p} \in \hat{Q}_1\}$. (Note that $\dim P_K = 12$.) The degrees of freedom can be chosen as previously. If the family $(\mathscr{T}_h)_h$ is regular (see [3] for instance), the previous results are still valid.

III. A Tetrahedral Element of Order 2 ($d = 3$). Let us consider a tetrahedron K with vertices $a_1, \ldots$ and a_4. We use the same notations as in Section II, in particular, we set

$$\mathbf{p}_i = \mathbf{n}_i \prod_{j=1, j\neq i}^{4} \lambda_j, \qquad 1 \leqslant i \leqslant 4;$$

we also introduce the points $a_{ij} = \frac{1}{2}(a_i + a_j)$, $1 \leqslant i < j \leqslant 4$. Then, we consider

(III.1) $$P_K = P_2(K)^3 \oplus \operatorname{Span}\{\mathbf{p}_i, 1 \leqslant i \leqslant 4\} \oplus (\operatorname{Span}\{\lambda_1\lambda_2\lambda_3\lambda_4\})^3.$$

(Note that $\dim P_K = 37$.) Let us remark that this space generalizes in the three-dimensional case the space studied in [5] for $d = 2$. As far as the degrees of freedom are concerned, we choose the values at the vertices a_i, $1 \leqslant i \leqslant 4$, and at the midpoints a_{ij}, $1 \leqslant i < j \leqslant 4$, the flux through the faces F_i, $1 \leqslant i \leqslant 4$, and the moments $\int_K x_l \operatorname{div}(\cdot)\, dx$, $1 \leqslant l \leqslant 3$.

LEMMA III.1. *For any* $\mathbf{v}$ *in* $\mathscr{C}^0(K)^3 \cap H^1(K)^3$, *there exists a unique* $\Pi_K \mathbf{v}$ *in* P_K *such that*

$$\text{(III.2)} \qquad \begin{cases} \Pi_K \mathbf{v}(a_i) = \mathbf{v}(a_i), & 1 \leqslant i \leqslant 4, \\ \Pi_K \mathbf{v}(a_{ij}) = \mathbf{v}(a_{ij}), & 1 \leqslant i < j \leqslant 4, \\ \displaystyle\int_{F_i} (\mathbf{v} - \Pi_K \mathbf{v}) \cdot \mathbf{n}_i \, d\sigma = 0, & 1 \leqslant i \leqslant 4, \\ \displaystyle\int_K x_l \operatorname{div}(\mathbf{v} - \Pi_K \mathbf{v})\, dx = 0, & 1 \leqslant l \leqslant 3. \end{cases}$$

Moreover, $\Pi_K \mathbf{v}_{/F_i}$ *depends only on* $\mathbf{v}_{/F_i}$, $1 \leqslant i \leqslant 4$.

Proof. Let us denote by $\tilde{\Pi}_K \mathbf{v}$ the classical Lagrange interpolate of $\mathbf{v}$ in $P_2(K)^3$, i.e.,

$$\tilde{\Pi}_K \mathbf{v} = \sum_{i=1}^{4} \mathbf{v}(a_i) \lambda_i (2\lambda_i - 1) + \sum_{1 \leqslant i < j \leqslant 4} \mathbf{v}(a_{ij}) 4 \lambda_i \lambda_j .$$

Then, as the $\mathbf{p}_i$'s and $\lambda_1\lambda_2\lambda_3\lambda_4$ are equal to 0 on any edge, $\Pi_K \mathbf{v}$ can be written

$$\text{(III.3)} \qquad \Pi_K \mathbf{v} = \tilde{\Pi}_K \mathbf{v} + \sum_{i=1}^{4} \alpha_i \mathbf{p}_i + \boldsymbol{\beta} \lambda_1\lambda_2\lambda_3\lambda_4 .$$

Since $\lambda_1\lambda_2\lambda_3\lambda_4$ is equal to 0 on ∂K, we have

$$\text{(III.4)} \qquad \alpha_i = \left(\int_{F_i} (\mathbf{v} - \tilde{\Pi}_K \mathbf{v}) \cdot \mathbf{n}_i \, d\sigma \right) \Big/ \int_{F_i} \prod_{j=1,\, j \neq i}^{4} \lambda_j \, d\sigma, \qquad 1 \leqslant i \leqslant 4.$$

Then, setting

$$\text{(III.5)} \qquad \bar{\Pi}_K \mathbf{v} = \tilde{\Pi}_K \mathbf{v} + \sum_{i=1}^{4} \alpha_i \mathbf{p}_i ,$$

and using the Green's formula, we obtain

$$\text{(III.6)} \qquad \beta_l = -\left(\int_K x_l \operatorname{div}\left(\mathbf{v} - \bar{\Pi}_K \mathbf{v}\right) dx \right) \Big/ \int_K \lambda_1\lambda_2\lambda_3\lambda_4 \, dx, \qquad 1 \leqslant l \leqslant 3.$$

Moreover, on F_i, one has

$$\Pi_K \mathbf{v}_{/F_i} = \tilde{\Pi}_K \mathbf{v}_{/F_i} + \alpha_i \mathbf{p}_i ,$$

so that $\Pi_K \mathbf{v}_{/F_i}$ depends only on $\mathbf{v}_{/F_i}$.

Now, for each h, we consider a triangulation $\mathscr{T}_h$ of $\bar{\Omega}$ made of tetrahedra with diameters bounded by h and we assume that the family $(\mathscr{T}_h)_h$ is regular.

With each K in $\mathscr{T}_h$, we associate the space P_K defined by (III.1); then Lemma III.1 allows us to define an operator Π_h from $\mathscr{C}^0(\bar{\Omega})^3 \cap H_0^1(\Omega)^3$ into X_h by (II.5).

LEMMA III. 2. *The operator* Π_h *satisfies* (H2) *for* $m = 2$.

Proof. Clearly, one has

$$\int_K \operatorname{div}(\mathbf{v}-\Pi_K\mathbf{v})\,dx=\int_K x_l \operatorname{div}(\mathbf{v}-\Pi_K\mathbf{v})\,dx=0,\qquad 1\leqslant l\leqslant 3,$$

so that $\forall q_h\in M_h^{(2)}$, $(q_h,\operatorname{div}(\mathbf{v}-\Pi_h\mathbf{v}))=0$.

Moreover, we know that (see [2]), for $k=0$ and 1,

$$|\mathbf{v}-\tilde{\Pi}_K\mathbf{v}|_{k,K}\leqslant Ch_K^{3-k}|\mathbf{v}|_{3,K}.$$

Let us compute $\overline{\Pi}_K\mathbf{v}-\tilde{\Pi}_K\mathbf{v}=\sum_{i=1}^4\alpha_i\mathbf{p}_i$. As in Section II,

$$|\alpha_i|\leqslant C\int_{\hat{F}_i}|\hat{\mathbf{v}}-\hat{\tilde{\Pi}}_{\hat{K}}\hat{\mathbf{v}}|\,d\hat{\sigma};$$

therefore, as $P_2(\hat{K})^3$ is invariant under $\hat{\tilde{\Pi}}_{\hat{K}}$,

$$|\alpha_i|\leqslant C|\hat{\mathbf{v}}|_{3,\hat{K}}\leqslant Ch_K^{3/2}|\mathbf{v}|_{3,K}.$$

The previous inequalities, together with (II.6), yield

$$\left|\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right|_{k,K}\leqslant Ch_K^{3-k}|\mathbf{v}|_{3,K}.$$

Finally, we compute $\Pi_K\mathbf{v}-\overline{\Pi}_K\mathbf{v}=\boldsymbol{\beta}\lambda_1\lambda_2\lambda_3\lambda_4$. Clearly, one has

$$\text{(III.7)}\qquad |\lambda_1\lambda_2\lambda_3\lambda_4|_{k,K}\leqslant C\left(\int_{\hat{K}}\left\|D^k(\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4)\right\|^2\|B_K\|^{2k}|\det B_K|\,d\hat{x}\right)^{1/2}$$
$$\leqslant Ch_K^{3/2-k},$$

and, by (III.6),

$$|\beta_l|\leqslant C|\det B_K|^{-1}\left|\int_K x_l\operatorname{div}\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)dx\right|.$$

We use Green's formula

$$|\beta_l|\leqslant C|\det B_K|^{-1}\left\{\left|\int_K\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)_{,l}dx\right|+\left|\int_{\partial K}x_l\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)\cdot\mathbf{n}\,d\sigma\right|\right\}$$
$$\leqslant C\left\{|\det B_K|^{-1/2}\|\mathbf{v}-\overline{\Pi}_K\mathbf{v}\|_{0,K}+|\det B_K|^{-1}\left|\int_{\partial K}x_l\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)\cdot\mathbf{n}\,d\sigma\right|\right\}.$$

But we remark that, since $x=B_K\hat{x}+b_K$,

$$\int_{\partial K}x_l\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)\cdot\mathbf{n}\,d\sigma=\int_{\partial\hat{K}}(B_K\hat{x})_l\left(\widehat{\mathbf{v}-\overline{\Pi}_K\mathbf{v}}\right)\cdot\hat{\mathbf{n}}|\det B_{K/\partial\hat{K}}|\,d\hat{\sigma}$$
$$+b_{Kl}\int_{\partial\hat{K}}\left(\widehat{\mathbf{v}-\overline{\Pi}_K\mathbf{v}}\right)\cdot\hat{\mathbf{n}}|\det B_{K/\partial\hat{K}}|\,d\hat{\sigma}.$$

Therefore,

$$\left|\int_{\partial K}x_l\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)\cdot\mathbf{n}\,d\sigma\right|\leqslant\|B_K\|\int_{\partial K}|\mathbf{v}-\overline{\Pi}_{\dot{K}}\mathbf{v}|\,d\sigma$$
$$+|b_K|\left|\int_{\partial K}\left(\mathbf{v}-\overline{\Pi}_K\mathbf{v}\right)\cdot\mathbf{n}\,d\sigma\right|.$$

Since the last term is equal to 0, we obtain

$$|\beta_l| \leqslant C\left\{|\det B_K|^{-1/2}\|\mathbf{v}-\overline{\Pi}_K\mathbf{v}\|_{0,K}\right.$$
$$\left. + |\det B_K|^{-1}\|B_K\|\sum_{i=1}^{4}|\operatorname{mes} F_i|^{1/2}\|\mathbf{v}-\overline{\Pi}_K\mathbf{v}\|_{0,F_i}\right\}$$

so that, by Lemma II.3,

$$|\beta_l| \leqslant \left\{h_K^{-3/2}h_K^3 + h_K^{-3}h_K h_K^{7/2}\right\}|\mathbf{v}|_{3,K} \leqslant Ch_K^{3/2}|\mathbf{v}|_{3,K}.$$

The previous inequalities yield, for $k = 0$ and 1,

$$|\mathbf{v} - \Pi_K\mathbf{v}|_{k,K} \leqslant Ch^{3-k}|\mathbf{v}|_{3,K}.$$

By (II.8), the hypothesis (H3) is an immediate consequence of

LEMMA III.3. *For any* $\mathbf{v}$ *in* $H_0^1(\Omega)^3$, *there exists* $\mathbf{v}_h$ *in* X_h *such that*

$$\text{(III.8)} \qquad \forall q_h \in M_h^{(2)}, \quad \begin{cases} (q_h, \operatorname{div}(\mathbf{v}-\mathbf{v}_h)) = 0 \\ \textit{and } \|\mathbf{v}_h\|_{1,\Omega} \leqslant C\|\mathbf{v}\|_{1,\Omega}. \end{cases}$$

Proof. Let us denote by $\mathbf{w}_h$ the interpolate of $\mathbf{v}$ in the space

$$\left\{u_h \in \mathscr{C}^0(\overline{\Omega}) \cap H_0^1(\Omega);\ \forall K \in \mathscr{T}_h,\ u_{h/K} \in P_2(K)\right\}^3,$$

defined by local regularization as in [1], so that (II.10) is still satisfied.

Then, we consider the element $\mathbf{v}_h$ in V_h equal on K to

$$\mathbf{v}_h = \mathbf{w}_h + \sum_{i=1}^{4}\alpha_i\mathbf{p}_i + \boldsymbol{\beta}\lambda_1\lambda_2\lambda_3\lambda_4$$

with

$$\alpha_i = \left(\int_{F_i}(\mathbf{v}-\mathbf{w}_h)\cdot\mathbf{n}_i\,d\sigma\right)\Big/\int_{F_i}\prod_{j=1,\,j\neq i}^{4}\lambda_j\,d\sigma,$$

$$\beta_l = -\int_K x_l \operatorname{div}\left(\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\right)dx\Big/\int_K\lambda_1\lambda_2\lambda_3\lambda_4\,dx.$$

Clearly, one has $\forall q_h \in M_h^{(2)}$, $(q_h, \operatorname{div}(\mathbf{v}-\mathbf{v}_h)) = 0$. Moreover, by (II.6) and (III.7),

$$\|\mathbf{v}_h\|_{1,K} \leqslant \|\mathbf{w}_h\|_{1,K} + Ch_K^{1/2}\left(\sum_{i=1}^{4}|\alpha_i| + |\boldsymbol{\beta}|\right).$$

The α_i's still satisfy (II.11). We also have

$$|\beta_l| \leqslant C|\det B_K|^{-1}\left\{\left|\int_K\left(\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\right)_l dx\right|\right.$$
$$\left. + \left|\int_{\partial K}x_l\left(\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\right)\cdot\mathbf{n}\,d\sigma\right|\right\}.$$

By the same way as in the proof of Lemma III.2,

$$\begin{aligned}|\beta_l| \leqslant C\Bigg[&|\det B_K|^{-1/2}\Big\|\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\Big\|_{0,K}+|\det B_K|^{-1}\|B_K\|\\&\times\sum_{i=1}^{4}|\mathrm{mes}\,F_i|h_K^{-3/2}\Big\{\Big\|\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\Big\|_{0,K}+h_K\Big|\mathbf{v}-\mathbf{w}_h-\sum_{i=1}^{4}\alpha_i\mathbf{p}_i\Big|_{1,K}\Big\}\Bigg]\\ \leqslant C\Big\{&h_K^{-3/2}\|\mathbf{v}-\mathbf{w}_h\|_{0,K}+h_K^{-1/2}|\mathbf{v}-\mathbf{w}_h|_{1,K}+\sum_{i=1}^{4}|\alpha_i|\Big\}.\end{aligned}$$

Finally, we obtain

$$\|\mathbf{v}_h\|_{1,K}\leqslant\|\mathbf{w}_h\|_{1,K}+Ch_K^{-1}\big\{\|\mathbf{v}-\mathbf{w}_h\|_{0,K}+h_K|\mathbf{v}-\mathbf{w}_h|_{1,K}\big\},$$

which, together with (II.10), yields $\|\mathbf{v}_h\|_{1,\Omega}\leqslant C\|\mathbf{v}\|_{1,\Omega}$.

Consequently, this element can be used to solve the Stokes problem in the three-dimensional case with an $O(h^2)$-error estimate.

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