MASSACHUSETTS INSTITUTE OF TECHNOLOGY

RE-35

ANALYSIS OF SPACE STABILIZED INERTIAL NAVIGATION SYSTEMS

by

Kenneth R. Britting

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ABSTRACT

This report is a tutorial exposition of the class of inertial navigation systems which instrument an inertially nonrotating coordinate frame. A detailed discussion of the possible types of system mechanizations is followed by a discussion and error analysis of self-contained alignment schemes. Stabilization of the vertical error is investigated.

Perturbation methods are used to derive linear error equations which apply to a high performance vehicle such as the Supersonic Transport. The error sources treated consist of:

1. Altimeter uncertainty
2. Deflection of the vertical
3. Accelerometer uncertainty and scale factor error
4. Gyro drift
5. Initial misalignment error
6. Initial condition errors

The error equations are solved for the case of constant gyro drift and altimeter uncertainty.

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SPACE STABILIZED SYSTEM

I. Introduction

The space stabilized inertial navigator is a semi-analytic system instrumenting the inertial frame, which in practical systems can be taken to be earth centered and nonrotating with respect to inertial space.

The three types of system mechanizations, geometric, semi-analytic, and strapdown all mechanize the same equations and basically exhibit the same modes of cscillation if excited. They have in common the property that they:

1. Measure $\underline{f}$ (specific force)
2. Instrument a reference frame
3. Have some knowledge of gravity
4. Integrate in some fashion the specific force components.

The choice of which mechanization to use, is strongly influenced by the availability of the equipment with which to implement the design.

The geometric system was the first practical system because the navigational information was available in analog fashion directly from the gimbals. The local vertical system, or semi-analytic system which instruments the geographic frame appears next. The necessary computations were originally performed by an analog computer and calculations involved in computing $\underline{G}$, the gravitational field, were "transformed away" by aligning the accelerometer axes in the local horizontal plane.

The development of high speed digital computers, which could be packaged in a reasonably sized space, has opened the door to the other type of semi-analytic system which is the topic of this paper. Further down the road is the analytic or strapdown system with its very large capacity computer. The strapdown system, because it imposes a very large dynamic range on the instruments, does not yet challenge the two types of semi-analytic systems from an accuracy standpoint. Rapid advances in component development may soon overcome this limitation.

An inertial frame of reference is one in which Newton's Laws, expressed in their simplest form, can be used to describe the dynamic behavior of a body. Any frame which is nonrotating and unaccelerated with respect to the "fixed" stars qualifies as an inertial frame

Consider the vector output of an ideal set of accelerations whose sensitive axes are mutually orthogonal. From Newton's Second Law the output from such an instrument package is given by the difference between the inertially referenced acceleration and the net gravitational accelerations at the instrument's location:

$$\underline{f}^a = \underline{C}_i^a (\ddot{\underline{R}} - \sum_k \underline{G}_k)^i \qquad (1)$$

where:

- $\underline{C}_i^a$ ∿ Coordinate transformation matrix relating the inertial axes to accelerometer axes.
- $\ddot{\underline{R}}$ ∿ Inertially referenced acceleration.
- $\underline{G}_k$ ∿ Gravitational acceleration at instrument location due to the $k^{\underline{th}}$ body in the universe.
- $\underline{f}$ ∿ Non field specific forces exerted on the instruments.

and the sub/superscripts denote

- a ∿ Indicates resolution or coordinatization in accelerometer axes.
- i ∿ Indicates coordinatization in inertial axes.

Because the inertially referenced position vector, $\underline{R}$, involves galactic distances, it is convenient to refer the accelerometer outputs to an earth centered frame which is nonrotating relative to the fixed stars. Note that the origin of this frame, which is located at the earth's center of gravity, is in free fall and that the output of the accelerometer triad would be zero at this point. The vector substitution

$$\underline{R} = \underline{r} + \underline{\rho}$$

where:

$\underline{R}$ ∿ Vector from hypothetical inertial frame origin to instrument location.

$\underline{r}$ ∿ Vector from earth centered frame origin to instrument location.

$\underline{\rho}$ ∿ Vector from hypothetical inertial frame origin to the earth's center of gravity.

is made in Eq (1) yielding:

$$\underline{f}^a = \underline{C}_i^a(\ddot{\underline{r}} + \ddot{\underline{\rho}} - \sum_k \underline{G}_k)^i \qquad (2)$$

Since the earth is in free fall, however,

$$\ddot{\underline{\rho}} - \sum_k \underline{G}'_k = 0$$

where:

$\underline{G}'_k$ ∿ Gravitational acceleration at earth's center of mass due to all of the k bodies excluding that of the earth itself.

Thus:

$$\underline{f}^a = \underline{C}_i^a[\ddot{\underline{r}} + \sum_k (\underline{G}'_k - \underline{G}_k)]. \qquad (3)$$

Since $\underline{G}'_k$ excludes the effect of the earth,

$$\underline{f}^a = \underline{C}_i^a[\ddot{\underline{r}} - \underline{G} + \sum_{k'} (\underline{G}'_{k'} - \underline{G}_{k'})] \qquad (4)$$

where:

k^* ∿ summation over all bodies of universe except the earth.

$\underline{g}$ ∿ gravitational acceleration at the instrument location due to the earth alone.

Equation (4) shows that the effect of all of the other bodies in the universe on the instrument output appears as a term which is the difference between the gravitational acceleration at the center of the earth and that at the instrument location. Fortunately, these difference terms are on the order of $10^{-7}\underline{g}$ for the bodies in the universe causing the largest effect, namely the moon and the sun. Thus, for instruments whose resolution does not extend down to 10^{-7} earth G's, which is the case for practical instruments, the output of the accelerometer triad can be approximated as:

$$\underline{f}^a = \underline{C}_i^a(\ddot{\underline{r}} - \underline{G})^i \qquad (5)$$

Equation (5) points out the important fact that the vector output of an accelerometer triad will be proportional to the nonfield specific force, coordinatized in the particular frame that happens to be mechanized.

Since the manipulation of $\underline{f}$, the specific force, is crucial in distinguishing between the various system mechanizations, it will be well to review its form in the various coordinate frames that will be used. Figure (1) illustrates the coordinate frames. If the instrumented frame is nonrotating relative to inertial space, $\underline{C}_i^a$ is a constant matrix and

$$\underline{C}_a^i\,\underline{f}^a = \underline{f}^i = \ddot{\underline{r}}^i - \underline{G}^i$$

Since it is seen from Fig. 1 that, in x-y-z inertial coordinates,

$$\underline{r}^i = \begin{bmatrix} r\cos L_g \cos\lambda \\ r\cos L_g \sin\lambda \\ r\sin L_g \end{bmatrix}$$

where:

L_g = geocentric latitude

λ = celestial longitude

it is evident that two integrations of the gravity-compensated accelerometer signals will yield the position vector, $\underline{r}$, in the inertial frame. If, however, the geographic frame is instrumented, a more complicated expression results:
The local geographic frame is also shown in Fig. 1.

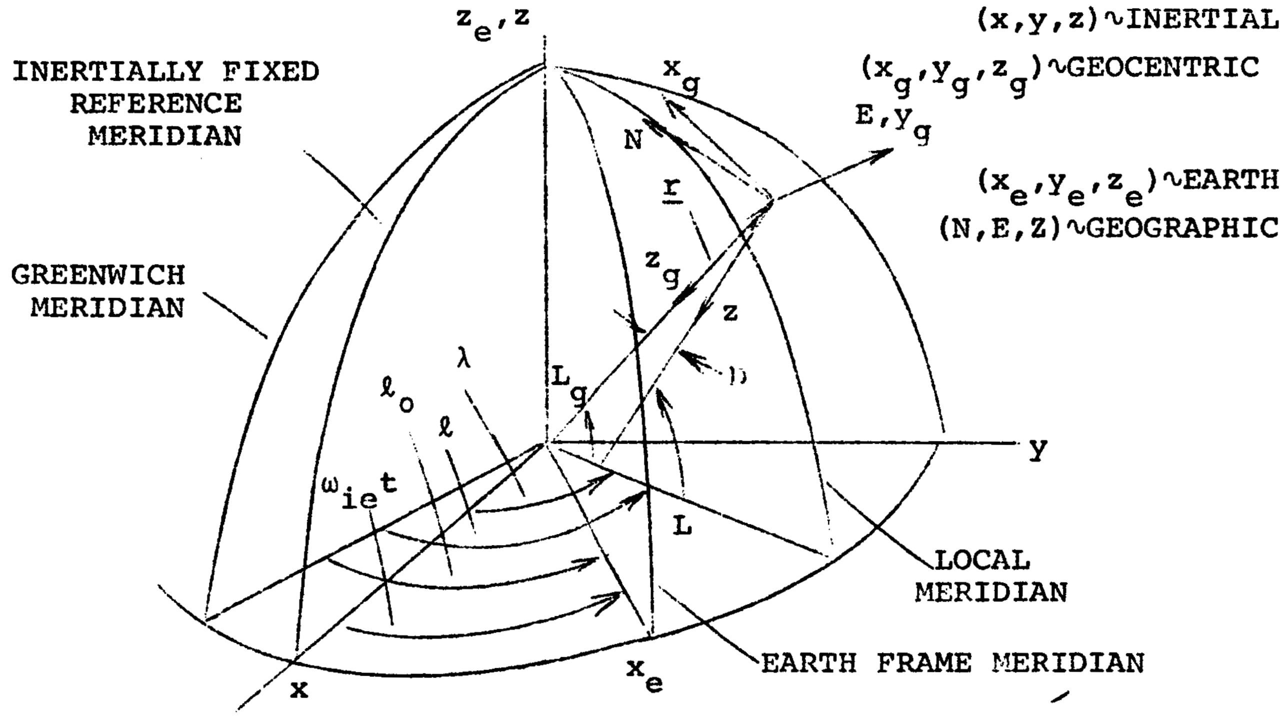


Fig. 1 ~ Coordinate Frame Geometry

Now the relationship between the inertial and geographic frames is given by:

$$\underline{r}^i = \underline{C}_n^i \, \underline{r}^n$$

Differentiating with respect to time:

$$\dot{\underline{r}}^i = \underline{C}_n^i \, \dot{\underline{r}}^n + \dot{\underline{C}}_n^i \, \underline{r}^n = \underline{C}_n^i (\dot{\underline{r}}^n + \underline{C}_i^n \, \dot{\underline{C}}_n^i \, \underline{r}^n)$$

But it can be shown that:

$$\underline{C}_i^n \, \dot{\underline{C}}_n^i = \underline{\Omega}_{in}^n = \begin{bmatrix} 0 & -\omega_z & \omega_E \\ \omega_z & 0 & -\omega_N \\ -\omega_E & \omega_N & 0 \end{bmatrix}$$

where the elements of the skew-symmetric matrix, $\underline{\Omega}_{in}^n$, are composed of elements of the angular velocity of the geographic frame relative to the inertial frame, coordinatized in geographic axes.

Thus:

$$\dot{\underline{r}}^i = C_n^i(\dot{\underline{r}}^n + \underline{\Omega}_{in}^n\ r^n)$$

Differentiating again yields:

$$\ddot{\underline{r}}^i = \underline{C}_n^i(\ddot{\underline{r}}^n + 2\underline{\Omega}_{in}^n\ \dot{\underline{r}}^n + \dot{\underline{\Omega}}_{in}^n\ \underline{r}^n + \underline{\Omega}_{in}^n\ \underline{\Omega}_{in}^n\ \underline{r}^n)$$

Thus for case of geographic frame representation:

$$\underline{f}^a = \underline{f}^n = \underline{C}_i^n(\ddot{\underline{r}}^i - \underline{G}^i)$$

$$\underline{f}^n = \ddot{\underline{r}}^n + 2\underline{\Omega}_{in}^n\ \dot{\underline{r}}^n + \underline{\Omega}_{in}^n\ \underline{\Omega}_{in}^n\ \underline{r}^n - \underline{G}^n + \dot{\underline{\Omega}}_{in}^n\ \underline{r}^n$$

For a body mounted or strapdown system, this above equation is valid with b replacing n. The vector $\underline{r}^b$ is generally a complicated function of time, and thus the transformation is usually made to either the inertial or geographic frames before proceeding.

If, for instance, the accelerometer triad were mounted along the axes of a local geocentric frame:

$$\underline{r}^g = \begin{bmatrix} 0 \\ 0 \\ -r \end{bmatrix}, \quad \underline{\Omega}_{ig}^g = \begin{bmatrix} 0 & \dot{\lambda}\sin L_g & -\dot{L}_g \\ -\dot{\lambda}\sin L_g & 0 & -\dot{\lambda}\cos L_g \\ \dot{L}_g & \dot{\lambda}\cos L_g & 0 \end{bmatrix},$$

and

$$\underline{C}_i^g\ \ddot{\underline{r}}^i = \begin{bmatrix} 2\dot{r}\ \dot{L}_g + r\ \ddot{L}_g + r\ \dot{\lambda}^2 \sin L_g \cos L_g \\ -2\dot{r}\ \dot{L}_g\dot{\lambda} \sin L_g + 2\dot{r}\dot{\lambda} \cos L_g + r\ddot{\lambda} \cos L_g \\ -\ddot{r} + r\ \dot{L}_g^2 + r\ \dot{\lambda}^2 \cos^2 L_g \end{bmatrix}$$

Thus, the expression for specific force in geocentric axes would be given by:

$$\underline{f}^g = \underline{C}_i^g(\ddot{\underline{r}}^i - \underline{G}^i)$$

II. Description of System

All space stabilized systems have at least one feature in common, they instrument the inertial frame. Thus, the components of the specific force vector are available, but coordinatized in the instrumented inertial frame. The specific force is given by Eq (5) as

$$\underline{f}^a = \underline{C}_i^a(\ddot{\underline{r}}^i - \underline{G}^i)$$

One of the more obvious procedures one might try in order to mechanize a navigation system would consist of immediately transforming these specific force components into the local geographic frame and proceeding exactly as one would with a local vertical system, but with the platform torquing replaced by mathematical updating of the $\underline{C}_i^n$ matrix. The following sketch illustrates this mechanization.

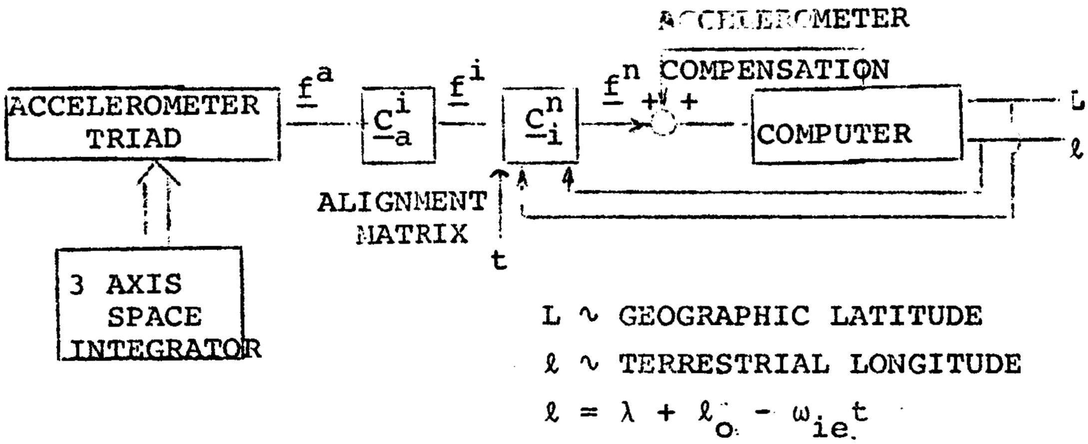


Fig. 2 Computation in geographic coordinates.

Analysis would proceed in a manner similar to that of the local vertical system developed in Ref. (1). Although the above procedure is the most obvious, it might be unnecessarily complex since Coriolis computations are involved in computing the system position. A simpler design can be conceived if one notes, as was done in the introduction, that the geocentric position vector, coordinatized in inertial coordinates, is given by:

$$\underline{r}^i = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}^i = \begin{bmatrix} r \cos L_g \cos \lambda \\ r \cos L_g \sin \lambda \\ r \sin L_g \end{bmatrix}^i \tag{6}$$

Thus, it is seen that if position is computed in inertial coordinates, then position information in latitude and longitude is readily avaialble.

Of course, if one needs ground speed information, Coriolis must be brought into the calculations, but at the velocity level. The functional block diagram for this system is shown in Fig. 3.

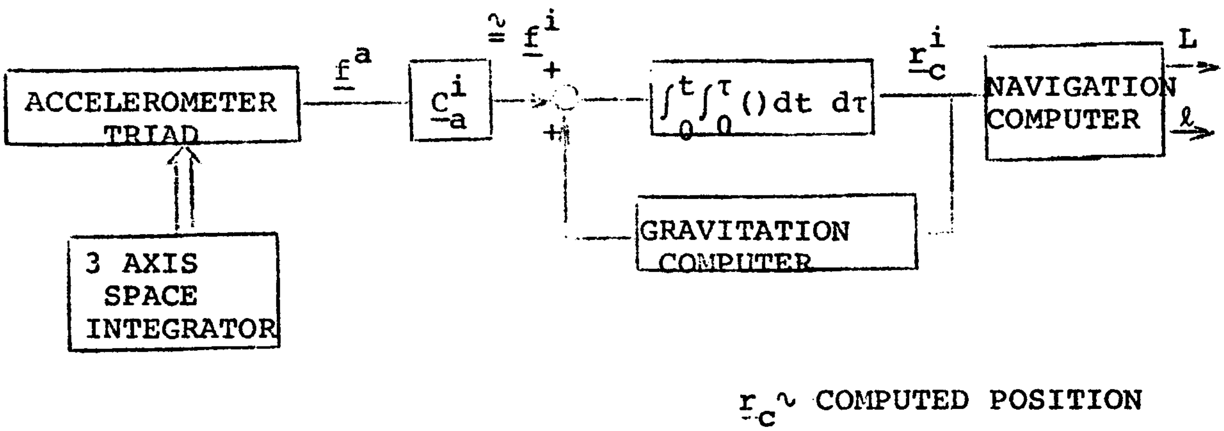


Fig. 3 Computation in inertial coordinates.

Because navigational information is obtained at the position level, no Coriolis compensation is necessary. However, the penalty is paid of having to compute the gravitational field vector explicitly.

A third variation would be available if the accelerometer outputs were proportional to velocity rather than acceleration. This is the case for an important class of accelerometers called P.I.G.A.'s (pendulous integrating gyro accelerometers).

The functional block diagram for this system is shown in Fig. 4.

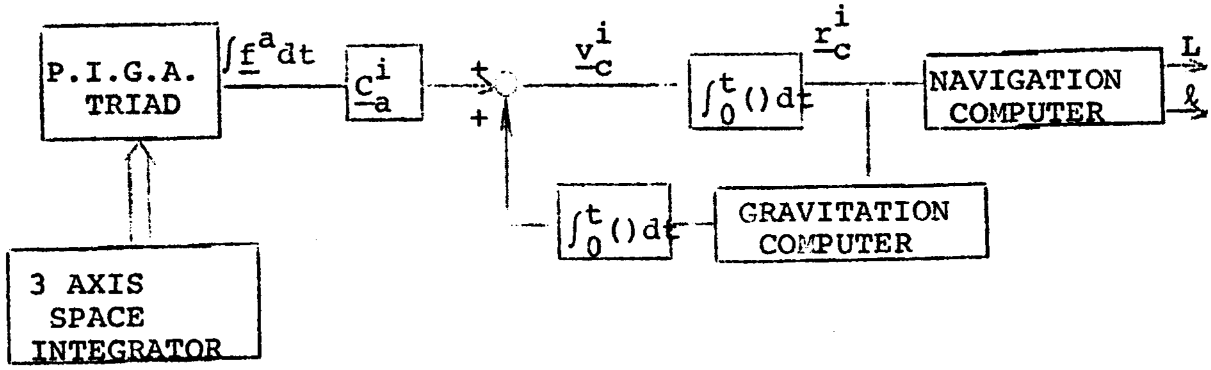


Fig. 4 Computation in inertial coordinates using velocity data.

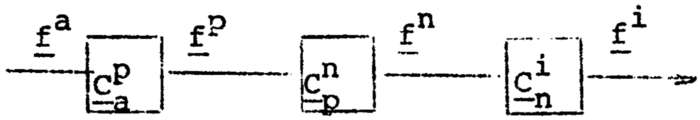
Except for possible differences in performance between integrating and non-integrating accelerometers, the systems shown in Figs. 3 and 4 behave identically. As will be shown in the error analysis, if the computation is performed in geographic coordinates (Fig. 2), certain aspects of the error response are significantly different. Other variations and schemes are, of course, also conceivable.

III. Alignment

The alignment procedures for the space stabilized system are somewhat different from that of the local vertical system. The problem boils down to finding the transformation between the accelerometer frame and the inertial frame. This transformation is ideally time invariant (zero gyro drift). We can visualize the transformation as taking place in three steps:

1. The first transformation is from accelerometer axes to platform axes. $\underline{C}_a^p$
2. The second transformation goes from platform axes to local geographic axes. $\underline{C}_p^n$
3. The final transformation is from geographic axes to inertial axes. $\underline{C}_n^i$

which can be visualized as follows:



Once determined, the matrix product never changes, since the accelerometer and inertial axes are assumed to be nonrotating. The concern herein will be directed toward exposition of means of self alignment. A summary and error analysis of other alignment methods is found in Ref. 2.

A. Analytic Alignment

It is possible to align mathematically (analytic gyrocompassing) via simple measurement of two vectors; namely, the vehicle angular rate vector, $\underline{\omega}$, and the specific force vector, $\underline{f}$.

Vehicle angular rate can be measured by monitoring the gimbal angle signal generators and specific force is, of course, measured by the accelerometers. Thus, because of the invariance of vector lengths through linear coordinate transformations,

$$|\underline{f}^i| = |\underline{f}^a| \text{ and } |\underline{\omega}^i| = |\underline{\omega}^a|$$

or in matrix notation, the square of the length is given by:

$$(\underline{f}^i)^T\underline{f}^i = (\underline{f}^a)^T\underline{f}^a$$

$$(\underline{\omega}^i)^T\underline{\omega}^i = (\underline{\omega}^a)^T\underline{\omega}^a$$

The above two equations implicitly contain the six direction cosines which specify the transformation. For symmetry we can define a third vector $\underline{\nu} = \underline{f} \times \underline{\omega}$.

We then have relationships:

$$(\underline{f}^i)^T\underline{f}^i = (\underline{f}^a)^T\underline{f}^a$$

$$(\underline{\omega}^i)^T\underline{\omega}^i = (\underline{\omega}^a)^T\underline{\omega}^a$$

$$(\underline{\nu}^i)^T\underline{\nu}^i = (\underline{\nu}^a)^T\underline{\nu}^a$$

But

$$\underline{\omega}^i = \underline{C}_a^i\,\underline{\omega}^a \;;\quad \underline{f}^i = \underline{C}_a^i\,\underline{f}^a \;:\; \underline{\nu}^i = \underline{C}_a^i\,\underline{\nu}^a$$

Therefore

$$[(\underline{f}^i)^T\underline{C}_a^i - (\underline{f}^a)^T]\underline{f}^a = 0$$

$$[(\underline{\omega}^i)^T\underline{C}_a^i - (\underline{\omega}^a)^T]\underline{\omega}^a = 0$$

$$[(\underline{\nu}^i)^T\underline{C}_a^i - (\underline{\nu}^a)^T]\underline{\nu}^a = 0$$

Thus,

$$(\underline{f}^i)^T \underline{C}_a^i = (\underline{f}^a)^T$$

$$(\underline{\omega}^i)^T \underline{C}_a^i = (\underline{\omega}^a)^T$$

$$(\underline{\nu}^i)^T \underline{C}_a^i = (\underline{\nu}^a)^T$$

Without any loss of generality the left hand matrix can be partitioned yielding:

$$\begin{bmatrix} (\underline{f}^i)^T \\ (\underline{\omega}^i)^T \\ (\underline{\nu}^i)^T \end{bmatrix} \underline{C}_a^i = \begin{bmatrix} (\underline{f}^a)^T \\ (\underline{\omega}^a)^T \\ (\underline{\nu}^a)^T \end{bmatrix}$$

Premultiplying by the inverse of the left hand bracketed term yields,

$$\underline{C}_a^i = \begin{bmatrix} (\underline{f}^i)^T \\ (\underline{\omega}^i)^T \\ (\underline{\nu}^i)^T \end{bmatrix}^{-1} \begin{bmatrix} (\underline{f}^a)^T \\ (\underline{\omega}^a)^T \\ (\underline{\nu}^a)^T \end{bmatrix} \qquad (7)$$

Now $(\underline{f}^i)^T$, $(\underline{\omega}^i)^T$ and $(\underline{\nu}^i)^T$ are known a priori while $(\underline{f}^a)^T$, $(\underline{\omega}^a)^T$, and $(\underline{\nu}^a)^T$ are measured quantities. Thus, the alignment matrix is uniquely defined provided the inverse indicated above exists. The inverse will exist providing that no two rows are equal or in constant ratio. This constraint rules out the use of the analytic alignment procedure at the earth's poles since $\bar{\omega}_{ie}$ and $\bar{g}$ are in constant ratio at that point.

Kasper (Ref. 2) shows that for fixed base alignment, the analytic scheme can compare favorably with the existing optical alignment methods. Because the presence of angular vibrations and accelerations that one would encounter in a practical alignment application would yield an instantaneous alignment matrix which could differ considerably from the average alignment matrix, a scheme which would average the present determination of the matrix with past determinations is called for. If the statistics of the base motion are known, then an optimum filtering scheme can be devised to determine the alignment matrix. This filtering could nevertheless result in an initial misalignment error when the system is switched to the navigation mode. Nevertheless, the analytic scheme provides a rapid means of obtaining a crude alignment matrix.

An error analysis for this alignment scheme which takes into account the effects of instrument uncertainties and base motion uncertainties is not readily amenable to analytic methods. An analysis amenable to computer solution is developed as follows: Equation (7) is written in this form:

$$\underline{C}_a^i = \underline{M}\,\underline{N} \qquad (8')$$

where

$$\underline{M} = \{(\underline{f}^i)^T,\ (\underline{\omega}^i)^T,\ (\underline{\nu}^i)^T\}^{-1};\ \underline{N} = \{(\underline{f}^a)^T,\ (\underline{\omega}^a)^T,\ (\underline{\nu}^a)^T\}$$

The elements of this $\underline{M}$ matrix depend on the computational frame only (in this case, the inertial frame) and are thus free of error. The elements of the $\underline{N}$ matrix, on the other hand, contain the measurement and instrument uncertainties. Thus, Eq (8') is rewritten:

$$\underline{C}_{a'}^i = \underline{M}(\underline{N} + \delta\underline{N})$$

where

a' ∿ the computed accelerometer frame

$\delta\underline{N}$ ∿ the 3x3 measurement uncertainty matrix ∿ $\begin{bmatrix} \delta f_x & \delta f_y & \delta f_z \\ \delta\omega_x & \delta\omega_y & \delta\omega_z \\ \delta\nu_x & \delta\nu_y & \delta\nu_z \end{bmatrix}$

Thus:

$$\underline{C}_{a'}^{i} = \underline{M}\,\underline{N}[\underline{I} + (\underline{M}\,\underline{N})^{-1}\,\underline{M}\,\delta\underline{N}]$$

$$= \underline{C}_{a}^{i}[\underline{I} + \underline{N}^{-1}\,\delta\underline{N}]$$

$$= \underline{C}_{a}^{i}\,\underline{C}_{a'}^{a} \qquad (8)$$

The matrix $\underline{N}^{-1}\,\delta\underline{N}$ will necessarily be of the skew symmetric form if the lengths of the measured vectors and the angles between them are required to be constant,

$$\underline{N}^{-1}\,\delta\underline{N} = \begin{bmatrix} 0 & -\mu_z & \mu_y \\ \mu_z & 0 & -\mu_x \\ -\mu_y & \mu_x & 0 \end{bmatrix}$$

where the μ_k, k = x, y, z misalignment angles are readily calculable functions of the measurement uncertainties.

B. Physical Gyrocompass Alignment

The coordinate transformation between local geographic axes and platform axes can be found by physical gyrocompassing, in which case the $\underline{C}_{a}^{n}$ transformation matrix is physically driven to be the unit matrix. The physical instrumentation is motivated by the following observations:

1. An ideal uncommanded space integrator (servo-driven gyro stabilized platform) will remain nonrotating with respect to inertial space.

2. If the gyro input axes are physically aligned with geographic axes (north, east, and down), then the system will remain aligned if the gyros are torqued at a rate proportional to Earth rate.

3. If the platform is level (north and east gyro axes lying in the local horizontal plane) but not quite stabilized in azimuth, the platform will rotate about the east axis with a rate:

$$\dot{\varepsilon}_E = \varepsilon_Z \, \omega_{ie} \cos L$$

where

$\dot{\varepsilon}_E$ ∿ angular rate about east axis

ε_Z ∿ azimuth misalignment

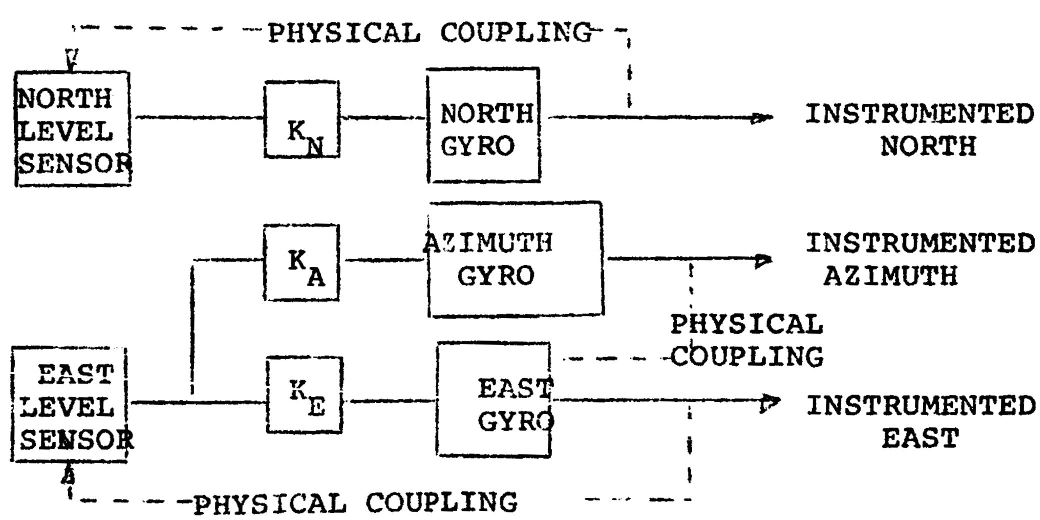
ω_{ie} ∿ earth rate

4. Since the signal from the east level sensor is proportional to the above azimuth misalignment, then that signal can theoretically be used to drive the azimuth error to zero.

Conceptually then, one could mechanize a gyrocompass by:

1. Supplying earth rate commands to the space integrator.
2. Providing tight level control.
3. Providing azimuth nulling via the east level sensor.

Consider the following form for an accelerometer coupled gyrocompass.



Gyrocompassing schemes of the above type are capable of very high accuracies.

To analyze the effect of component uncertainties on system performance, assume that the platform axes are almost coincident with the local geographic axes. If we define error angles ε_N, ε_E, and ε_Z resulting from positive rotations of the platform axes relative to geographic axes, then the transformation between geographic and platform axes is given by:

$$\underline{C}^{n}_{n'} = \begin{bmatrix} 1 & -\varepsilon_Z & \varepsilon_E \\ \varepsilon_Z & 1 & -\varepsilon_N \\ -\varepsilon_E & \varepsilon_N & 1 \end{bmatrix} = (\underline{I} + \underline{E})$$

where

$\underline{E}$ ∿ skew symmetric matrix of misalignment angles

$\underline{I}$ ∿ identity matrix

n' ∿ instrumented geographic axes

n ∿ true navigational axes.

Now the angular velocity of the platform axes with respect to inertial space is given by:

$$\underline{\omega}_{in'} = \underline{\omega}_{in} + \underline{\omega}_{nn'}$$

Coordinatizing in platform axes:

$$\underline{\omega}^{n'}_{in'} = \underline{C}^{n'}_{n}\underline{\omega}^{n}_{in} + \underline{\omega}^{n'}_{nn'}$$

$$\begin{bmatrix} \omega_{N'} \\ \omega_{E'} \\ \omega_{Z'} \end{bmatrix} = (\underline{I} - \underline{E}) \begin{bmatrix} \omega_{ie} \cos L \\ 0 \\ -\omega_{ie} \sin L \end{bmatrix} + \begin{bmatrix} \dot{\varepsilon}_N \\ \dot{\varepsilon}_E \\ \dot{\varepsilon}_Z \end{bmatrix}$$

where L is geographic latitude.

$$\begin{bmatrix} \omega_N' \\ \omega_E' \\ \omega_Z' \end{bmatrix} = \begin{bmatrix} \omega_{ie} \cos L + \varepsilon_E \, \omega_{ie} \sin L + \dot{\varepsilon}_N \\ -\varepsilon_Z \, \omega_{ie} \cos L - \varepsilon_N \, \omega_{ie} \sin L + \dot{\varepsilon}_E \\ \varepsilon_E \, \omega_{ie} \cos L - \omega_{ie} \sin L + \dot{\varepsilon}_Z \end{bmatrix}$$

The above expression must be equal to the commanded angular velocity which consists of the processed level sensor outputs and earth rate commands. Thus, the commanded angular velocity is given by:

$$\begin{bmatrix} \omega_{N_C} \\ \omega_{E_C} \\ \omega_{Z_C} \end{bmatrix} = \begin{bmatrix} \omega_{ie} \cos L - K_N \, \varepsilon_N + \delta\omega_N \\ -K_E \, \varepsilon_E + \delta\omega_E \\ -\omega_{ie} \sin L - K_A \, \varepsilon_E + \delta\omega_Z \end{bmatrix}$$

where $\underline{\delta\omega}^{n'}$ ~ error in the command signal. ~ $\{\delta\omega_N, \delta\omega_E, \delta\omega_Z\}$

Note that it is being assumed that the earth rate terms are being supplied exactly. Note also that the signs of K_N, K_E, and K_A have been chosen to drive the $\dot{\varepsilon}$ term to zero. Equating:

$$\begin{bmatrix} \dot{\varepsilon}_N \\ \dot{\varepsilon}_E \\ \dot{\varepsilon}_Z \end{bmatrix} = \begin{bmatrix} -K_N \, \varepsilon_N - \varepsilon_E \, \omega_{ie} \sin L + \delta\omega_N \\ -K_E \, \varepsilon_E + \varepsilon_N \, \omega_{ie} \sin L + \varepsilon_Z \, \omega_{ie} \cos L + \delta\omega_E \\ -K_A \, \varepsilon_E - \varepsilon_E \, \omega_{ie} \cos L + \delta\omega_Z \end{bmatrix}$$

Taking the Laplace transformation and arranging in matrix form:

$$\begin{bmatrix} S + K_N & \omega_{ie} \sin L & 0 \\ -\omega_{ie} \sin L & S + K_E & -\omega_{ie} \cos L \\ 0 & K_A + \omega_{ie} \cos L & S \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_N \\ \bar{\varepsilon}_E \\ \bar{\varepsilon}_Z \end{bmatrix} = \begin{bmatrix} \varepsilon_N(0) + \overline{\delta\omega}_N \\ \varepsilon_E(0) + \overline{\delta\omega}_E \\ \varepsilon_Z(0) + \overline{\delta\omega}_Z \end{bmatrix}$$

or $$\underline{M} \, \underline{\bar{\varepsilon}} = \underline{U} \qquad (9)$$

where S denotes the Laplace operator and the super bar denotes a Laplace transformed variable.

Now the error angular velocity command rate is caused by gyro and level sensor uncertainties. Call the gyro drift uncertainties for N, E, and Z gyros: $(u)\omega_N$, $(u)\omega_E$, and $(u)\omega_Z$ respectively; and the level sensor uncertainties for the north and east level sensors $(u)\varepsilon_N$ and $(u)\varepsilon_E$ respectively.

Then:

$$\begin{bmatrix} U_N \\ U_E \\ U_Z \end{bmatrix} = \begin{bmatrix} \varepsilon_N(0) \\ \varepsilon_E(0) \\ \varepsilon_Z(0) \end{bmatrix} + \begin{bmatrix} \delta\bar{\omega}_N \\ \delta\bar{\omega}_E \\ \delta\bar{\omega}_Z \end{bmatrix} = \begin{bmatrix} (u)\bar{\omega}_N \\ (u)\bar{\omega}_E \\ (u)\bar{\omega}_Z \end{bmatrix} - \begin{bmatrix} K_N(u)\bar{\varepsilon}_N \\ K_E(u)\bar{\varepsilon}_E \\ K_A(u)\bar{\varepsilon}_E \end{bmatrix} + \begin{bmatrix} \varepsilon_N(0) \\ \varepsilon_E(0) \\ \varepsilon_Z(0) \end{bmatrix}$$

The signal flow diagram for this system is drawn:

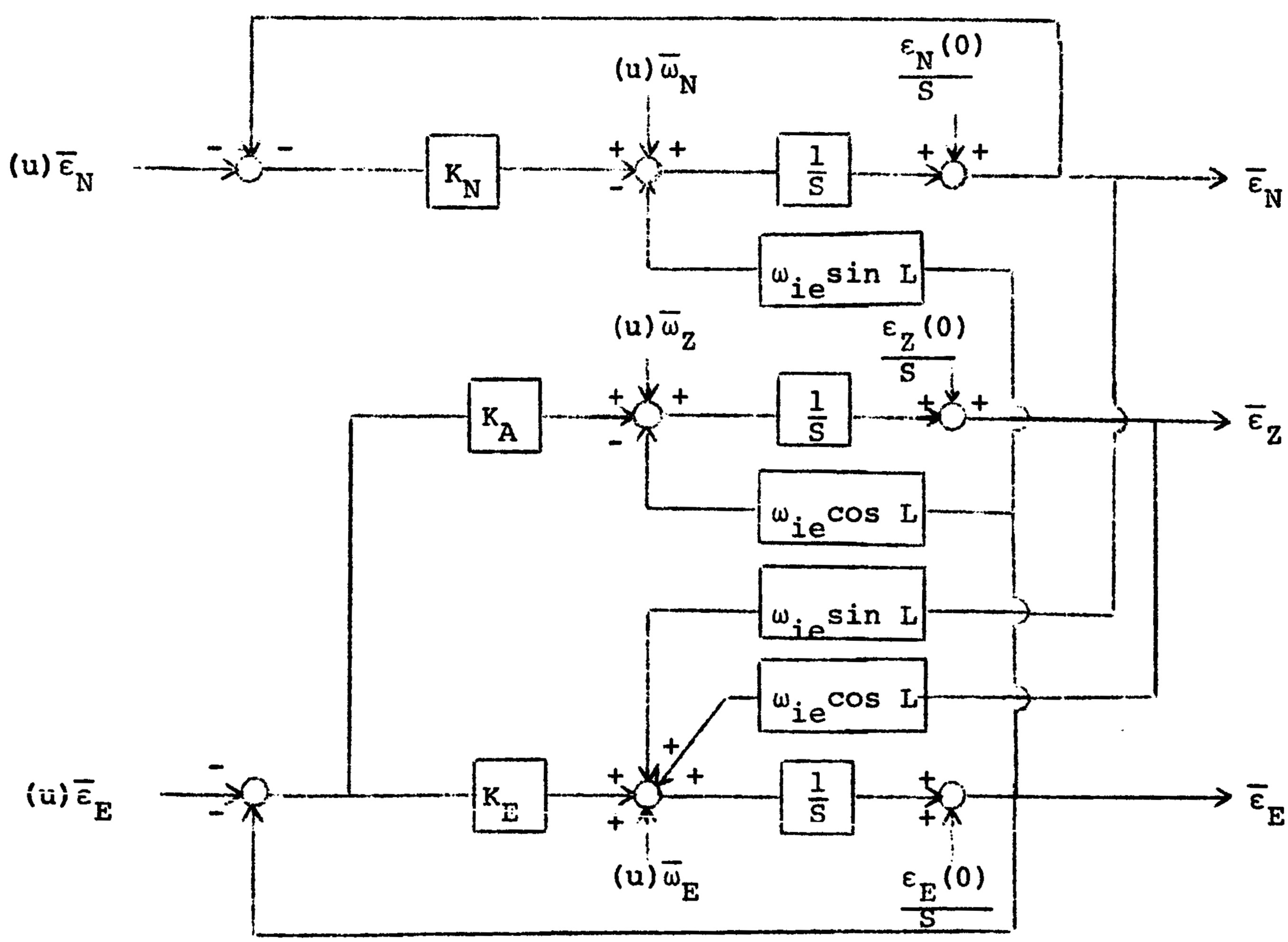


Fig. 5 Signal Flow Diagram ∿ Acceleration Coupled Gyrocompass

...ice the determinant associated with the above characteristics matrix is given by:

$$\Delta = S^3 + (K_E + K_N)S^2 + (K_E K_N + K_A \omega_{ie} \cos L + \omega_{ie}^2)S + K_N \omega_{ie} \cos L(K_A + \omega_{ie} \cos L),$$

the matrix equation (9) is readily solved using Cramer's rule. The equations are first solved for the steady state errors assuming constant drift rates and level sensor bias by invoking the final value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{S\to 0} SF(s)$$

Solution yields:

$$\varepsilon_{N_{ss}} = \frac{1}{K_N} [(u)\omega_N - K_N(u)\varepsilon_N] - \frac{\omega_{ie} \sin L}{K_N(K_A + \omega_{ie} \cos L)} [(u)\omega_Z - K_A(u)\varepsilon_E] \quad (10a)$$

$$\varepsilon_{E_{ss}} = \frac{(u)\omega_Z - K_A(u)\varepsilon_E}{K_A + \omega_{ie} \cos L} \quad (10b)$$

$$\varepsilon_{Z_{ss}} = -\frac{\tan L}{K_N} [(u)\omega_N - K_N(u)\varepsilon_N] - \frac{1}{\omega_{ie} \cos L} [(u)\omega_E - K_E(u)\varepsilon_E] + \frac{K_E K_N + \omega_{ie}^2 \sin^2 L}{K_N \omega_{ie} \cos L\ (K_A + \omega_{ie} \cos L)} [(u)\omega_Z - K_A(u)\varepsilon_E] \quad (10c)$$

These equations are summarized in the following chart:

Fig. 6 Acceleration coupled gyrocompass steady state error coefficients.

	$\varepsilon_N/$	$\varepsilon_E/$	$\varepsilon_Z/$
$/(u)\,\omega_N$	$\frac{1}{K_N}$	0	$-\frac{\tan L}{K_N}$
$/(u)\,\omega_E$	θ	0	$-\frac{1}{\omega_{ie}\cos L}$
$/(u)\,\omega_Z$	$\frac{-\omega_{ie}\sin L}{K_N(K_A+\omega_{ie}\cos L)}$	$\frac{1}{K_A+\omega_{ie}\cos L}$	$\frac{K_E K_N+\omega_{ie}^2\sin^2 L}{K_N\omega_{ie}\cos L(K_A+\omega_{ie}\cos L)}$
$/(u)\,\varepsilon_N$	-1	0	tan L
$/(u)\,\varepsilon_E$	$\frac{K_A\omega_{ie}\sin L}{K_N(K_A+\omega_{ie}\cos L)}$	$\frac{-K_A}{K_A+\omega_{ie}\cos L}$	$\frac{K_E K_N\omega_{ie}\cos L-K_A\omega_{ie}^2\sin^2 L}{K_N\omega_{ie}\cos L(K_A+\omega_{ie}\cos L)}$

The constants K_N, K_E, and K_A are chosen to satisfy a specified error budget and to provide adequate response time. A reasonable design emerges from Figure 6 if the gains are chosen as follows:

$$K_A >> K_N, \; K_N >> K_E, \; K_A >> \omega_{ie}$$

Routh's criteria shows that, in addition, there is no stability problem for this system. Note, however, that level sensor dynamics, electronic component dynamics, and gyro dynamics have been neglected. The combination of high gain in the north loop and close coupling in the east-azimuth loop tends to decouple the north loop which basically operates in a leveling mode.

Figure 6 points out the important fact that the sensitivity $\varepsilon_Z/(u)\varepsilon_E$ is independent of system gain. Its magnitude of

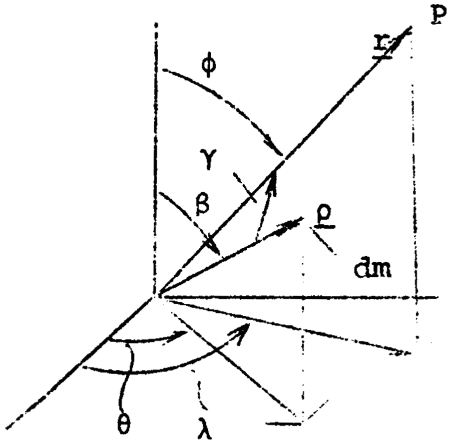
$$- \frac{3.4 \text{ min}}{\cos L} \text{ / meru drift}$$

shows that east gyro uncertainty is the limiting factor in gyrocompass performance. Because of the magnitude of this error coefficient, special calibration techniques, known as wheel speed modulation and east-west averaging, have been developed and are discussed in Refs. 3 and 4.

IV. Gravitational Field Computation

Since it is necessary to compute gravity explicitly in the space stabilized navigation system, it is well to investigate the form of the expression. This derivation, which was adapted from Ref. 5, is fairly complex and will only be sketched in a brief form.

The gravitational force field, $\underline{G}(x,y,z)$ is a vector field which is derivable from a scalar function called a scalar potential, $V(x,y,z)$. The potential at the point $P(r,\phi,\lambda)$ which is external to the mass due to the distributed mass of density $D(\rho,\beta,\theta)$ is given by the equation:



$$V(r,\phi,\lambda) = G\iiint \frac{dm}{|\underline{r}-\underline{\rho}|}$$

where

G ∿ Gravitational Constant

dm ∿ Differential Mass

∿ $\rho^2 \sin\beta \, d\rho \, d\beta \, d\theta$

The distance between $\underline{r}$ and dm, $|\underline{r} - \underline{\rho}|$, is shown by the law of cosines to be given by:

$$|\underline{r} - \underline{\rho}| = (r^2 + \rho^2 - 2r\rho\cos\gamma)^{1/2}$$

The right hand side of which is recognized as the Legendre generating function. If the potential is evaluated outside of the mass dm, i.e., $r > \rho$, $|\underline{r} - \underline{\rho}|^{-1}$ can be expanded in power series:

$$|\underline{r} - \underline{\rho}|^{-1} = r^{-1}\left(1 + \frac{\rho^2}{r^2} - 2\frac{\rho}{r}\cos\gamma\right)^{-1/2}$$

$$= r^{-1}\left[1 - \frac{1}{2}\left(\frac{\rho^2}{r^2} - 2\frac{\rho}{r}\cos\gamma\right) + \frac{3}{8}\left(\frac{\rho^2}{r^2} - 2\frac{\rho}{r}\cos\gamma\right)^2 - \frac{5}{16}\left(\frac{\rho^2}{r^2} - 2\frac{\rho}{r}\cos\gamma\right)^3 + \ldots\ldots\right]$$

$$= r^{-1}\left[1 + \frac{\rho}{r}\cos\gamma - \frac{1}{2}\frac{\rho^2}{r^2} + \frac{3}{8}\left(\frac{\rho^4}{r^4} - 4\frac{\rho^3}{r^3}\cos\gamma + 4\frac{\rho^2}{r^2}\cos^2\gamma\right)\right.$$

$$\left. - \frac{5}{16}\left(\frac{\rho^6}{r^6} - 6\frac{\rho^5}{r^5}\cos\gamma + 12\frac{\rho^4}{r^4}\cos^2\gamma - 8\frac{\rho^3}{r^3}\cos^3\gamma\right) + \ldots\right]$$

$$= r^{-1}[1 + \frac{\rho}{r}\cos\gamma + \frac{1}{2}\frac{\rho^2}{r^2}(3\cos^2\gamma - 1) + \frac{1}{2}\frac{\rho^3}{r^3}\cos\gamma\,(5\cos^2\gamma - 3)$$

$$+ \ldots]$$

which can be written as a series of Legendre polynomials:

$$|\underline{r} - \underline{\rho}|^{-1} = r^{-1}\sum_{k=0}^{\infty} P_k(\cos\gamma)\left(\frac{\rho}{r}\right)^k$$

Thus

$$V(r,\phi,\lambda) = \frac{G}{r}\iiint \sum_{k=0}^{\infty} P_k(\cos\gamma)\left(\frac{\rho}{r}\right)^k dm$$

Now $P_k(\cos\gamma)$ can be expanded in terms of ϕ, β, θ, and λ in accordance with the addition theorem for spherical harmonics, which yields a potential function of the form:

$$V(r,\phi,\lambda) = \sum_{k=0}^{\infty}\frac{A_k}{r^{k+1}} P_k(\cos\phi) + f\left[\int D(\rho,\beta,\theta)\cos n\theta\, d\theta, \int D(\rho,\beta,\theta)\sin n\theta\, d\theta\right]$$

where $n = 1,2,\ldots,$

$$A_k = G\iiint \rho^k P_k(\cos\beta)\, dm$$

<u>If symmetry exists about the polar axis, z,</u> which is the situation for the reference ellipsoid model,

$$D(\rho,\beta,\theta) = D(\rho,\beta)$$

and the periodic functions in θ drop out of the potential function, which can now be written:

$$V(r,\phi,\lambda) = \frac{G}{r}\iiint dm + \frac{G}{r^2}\cos\phi \iiint \rho\cos\beta\, dm + \sum_{k=2}^{\infty} \frac{A_k}{r^{k+1}} P_k(\cos\phi)$$

The first integral in the above expression is just the total attracting mass, while the quantity $\rho \cos \beta$ in the second expression is recognized as the distance from the equatorial plane to dm. Consequently, if the center of attracting mass coincides with the center of coordinates, as it does for the reference ellipsoid, then

$$\iiint \rho \cos \beta \, dm = 0$$

and

$$V(r,\phi,\lambda) = \frac{Gm}{r} + \sum_{k=2}^{\infty} \frac{A_k}{r^{k+1}} P_k(\cos\phi)$$

A further manipulation is made by multiplying numerator and denominator by r_e^k

$$\therefore \quad \frac{r_e^k}{r_e^k \; r^{k+1}} = \left(\frac{r_e}{r}\right)^k \frac{1}{r_e^k \; r}$$

$$\therefore \; V(r,\phi) = \frac{Gm}{r}\left[1 - \sum_{k=2}^{\infty} \left(\frac{r_e}{r}\right)^k J_k \, P_k(\cos\phi)\right]$$

where

$$J_k = -\frac{A_k}{r_e^k \, Gm}$$

The coefficients J_k are determined experimentally by observing satellite orbital deviations, for instance. Note that since

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1) = \frac{1}{4}(3\cos 2\phi + 1)$$

$$P_3(\cos\phi) = \frac{1}{2}(5\cos^3\phi - 3\cos\phi) = \frac{1}{8}(5\cos 3\phi + 3\cos\phi)$$

$$P_4(\cos\phi) = \frac{1}{8}(35\cos^4\phi - 3\cos^2\phi + 3) = \frac{1}{64}(35\cos 4\phi + 20\cos 2\phi + 9),$$

the even harmonics are symmetric about the pole giving rise to the oblate terms, while the odd harmonics are anti-symmetric giving rise to the so-called pear-shape terms. Writing out the expression for three terms:

$$V(r,\phi) = Gm\left[\frac{1}{r} - \frac{J_2}{2}\frac{r_e^2}{r^3}(3\cos^2\phi - 1) - \frac{J_3}{2}\frac{r_e^3}{r^4}(5\cos^3\phi - 3\cos\phi) - \frac{J_4}{8}\frac{r_e^4}{r^5}(35\cos^4\phi - 30\cos^2\phi + 3) - \ldots\right]$$

It is instructive to take the gradient in spherical coordinates:

$$\underline{G} = +\underline{\nabla}V$$

where

$$\underline{\nabla} = \frac{\partial}{\partial r}\underline{i}_r + \frac{1}{r}\frac{\partial}{\partial\phi}\underline{i}_\phi + \frac{1}{r\sin\phi}\frac{\partial}{\partial\lambda}\underline{i}_\lambda$$

One finds the following maximum values associated with the various terms:

$G_{r_s} = -g$ $\qquad G_{\phi_o} = \pm 1.5$ mg $\qquad G_{\lambda} = 0$

$G_{r_o} = +3$ mg $\qquad G_{\phi_p} = \pm 5$ μg

$G_{r_p} = \pm 10$ μg

The gravitation vector, $\underline{G}$, can be resolved into the geocentric inertial frame by noting that $\cos\phi = \frac{r_z}{r}$ and operating with:

$$\underline{\nabla} = \frac{\partial}{\partial x}\underline{i}_x + \frac{\partial}{\partial y}\underline{i}_y + \frac{\partial}{\partial z}\underline{i}_z$$

The following is obtained:

$$G_x = -\frac{Gm}{r^2}\left[1 + \frac{3}{2}J_2\left(\frac{r_e}{r}\right)^2\left[1 - 5\left(\frac{r_z}{r}\right)^2\right]\right]\frac{r_x}{r}, \quad \bar{e} < 1.3\mathrm{x}10^{-5}g \qquad (11a)$$

$$G_y = -\frac{Gm}{r^2}\left[1 + \frac{3}{2}J_2\left(\frac{r_e}{r}\right)^2\left[1 - 5\left(\frac{r_z}{r}\right)^2\right]\right]\frac{r_y}{r}, \quad \bar{e} < 1.3\mathrm{x}10^{-5}g \qquad (11b)$$

$$G_z = -\frac{Gm}{r^2}\left[1 + \frac{3}{2}J_2\left(\frac{r_e}{r}\right)^2\left[3 - 5\left(\frac{r_z}{r}\right)^2\right]\right]\frac{r_z}{r}, \quad \bar{e} < 2 \times 10^{-5}g \qquad (11c)$$

where

$$Gm \approx 1.4 \times 10^{16}\ \frac{\mathrm{ft}^3}{\mathrm{sec}^2}$$

and

$$\frac{3}{2}J_2 \approx 1.6 \times 10^{-3}$$

The values for $\bar{e}$ following Eqs (11) represent the maximum value of the error that is incurred in neglecting the higher order terms. Note that these expressions do not include the higher order oblate and pear shaped terms. In light of the fact that these terms have typical mean values of about 7 μg (4.85 μg corresponds to a 1 sec uncertainty in the vertical); the error incurred in the above expression for G is beyond the resolution of most instruments.

It is important not to use the expression

$$r = (\underline{r}^T \underline{r})^{1/2}$$

in calculating r in the gravity expressions. As will be shown in Section V, such a procedure will result in unstable behavior. Instead, r is calculated as the sum of the elliptic geocentric radius and externally supplied altitude, h_a.

$$r = r_o + h_a$$

where

$$r_o = r_e[1 - \frac{e}{2}(1 - \cos 2L) + \frac{5}{16} e^2(1 - \cos 4L)]$$

and e = earth ellipticity = $\frac{\text{semimajor axis - semiminor axis}}{\text{semimajor axis}}$

$$\cong \frac{1}{297}$$

The above expression for r_o is accurate to better than one foot, an accuracy somewhat in excess of what is needed.

V. Instability along the Vertical and its Avoidance

A. Unstable Mechanization

This section serves to point out the necessity of externally supplied altitude information in the navigation computation. If this navigation computation is done in geocentric inertial coordinates (see Fig. 3) and if it is further assumed that specific force can be measured and transformed to the computed inertial frame without error, then the differential equation describing the system behavior is given by:

$$\underline{f} + \underline{G}_c(\underline{r}_c) = \ddot{\underline{r}}_c \tag{12}$$

where all vectors are assumed to be coordinatized in the inertial frame and:

$\underline{f}$ ∿ true specific force

$\underline{G}_c$ ∿ computed gravitational field vector

$\underline{r}_c$ ∿ computed position vector.

Now if no altitude information is available, the gravitational field vector is computed from:

$$\underline{G}_c(\underline{r}_c) \overset{\sim}{=} - \frac{Gm}{r_c^3} \underline{r}_c = - \frac{E}{r_c^3} \underline{r}_c \tag{13}$$

Equation (13) is the vector form of Eq (11) with the higher order terms neglected. Since it is assumed that specific force can be measured and transformed to the computed inertial frame without error,

$$\underline{f} = \ddot{\underline{r}} - \underline{G} \tag{14}$$

where

$\underline{r}$ ∿ actual system position vector

$\underline{G}$ ∿ actual gravitational field vector

Substituting (13) and (14) into (12) yields:

$$\ddot{\underline{r}} - \underline{G} - \frac{E}{r_c^3}\underline{r}_c = \ddot{\underline{r}}_c \tag{15}$$

Next define the variational parameter

$$\underline{\delta r} = \underline{r}_c - \underline{r} \tag{16}$$

which can be interpreted as the system position error, i.e., the difference between the computed position vector and the actual position vector.

If (16) is substituted into (15) the following is obtained:

$$\ddot{\underline{r}} - \underline{G} - \frac{E(\underline{r} + \underline{\delta r})}{[(\underline{r} + \underline{\delta r})\cdot(\underline{r} + \underline{\delta r})]^{3/2}} = \ddot{\underline{r}} + \underline{\delta \ddot{r}}$$

But

$$[(\underline{r} + \underline{\delta r})\cdot(\underline{r} + \underline{\delta r})]^{-3/2} = (r^2 + 2\,\underline{\delta r}\cdot\underline{r} + \delta r^2)^{-3/2}$$

$$\cong r^{-3}\left[1 + \frac{2\,\underline{\delta r}\cdot\underline{r}}{r^2}\right]^{-3/2}$$

$$\cong r^{-3}\left[1 - 3\,\frac{\underline{\delta r}\cdot\underline{r}}{r^2}\right]$$

Thus: $$\underline{\delta \ddot{r}} = -\underline{G} - \frac{E}{r^3}\underline{r} + 3\frac{E}{r^3}\underline{r}\frac{\underline{\delta r}\cdot\underline{r}}{r^2} - \frac{E}{r^3}\underline{\delta r}$$

Since $\underline{G} = -\frac{E}{r^3}\underline{r}$, then:

$$\underline{\delta \ddot{r}} + \omega_s^2 \, \underline{\delta r} = 3 \, \omega_s^2 \, (\underline{i}_r \cdot \underline{\delta r}) \underline{i}_r \qquad (17)$$

where

$\omega_s^2 \triangleq \frac{E}{r^3} = \frac{|\underline{G}|}{r}$ ∿ square of the system natural frequency which can be associated with the Schuler frequency.

$\underline{i}_r = \frac{\underline{r}}{r}$

Now:

$$\underline{i}_r \times (\underline{i}_r \times \underline{\delta r}) = \underline{i}_r(\underline{i}_r \cdot \underline{\delta r}) - \underline{\delta r}(\underline{i}_r \cdot \underline{i}_r)$$

or

$$\underline{i}_r(\underline{i}_r \cdot \underline{\delta r}) = \underline{\delta r} - \underline{A} \, \underline{\delta r} \qquad (18)$$

where $\underline{A}$ is the 3 x 3 singular matrix:

$$\underline{A} = \frac{1}{r^2} \begin{bmatrix} r_y^2 + r_z^2 & - r_y r_x & - r_z r_x \\ - r_x r_y & r_x^2 + r_z^2 & - r_z r_y \\ - r_x r_z & - r_y r_z & r_x^2 + r_y^2 \end{bmatrix}$$

Thus Eq (17) becomes:

$$\underline{\delta \ddot{r}} + \omega_s^2 \, \underline{\delta r} - 3 \, \omega_s^2 (\underline{\delta r} - \underline{A} \, \underline{\delta r}) = 0$$

or

$$\underline{\delta \ddot{r}} - \omega_s^2 (2\underline{I} - 3 \, \underline{A}) \, \underline{\delta r} = 0 \qquad (19)$$

where $\underline{I}$ is the identity matrix.

Introducing the operator notation $p = \frac{d}{dt}$, Eq (19) can be written as:

$$\underline{B}\ \underline{\delta r} = \underline{0} \tag{20}$$

where:

$$\underline{B} = \begin{bmatrix} p^2 + \omega_s^2 - 3\omega_s^2 \frac{r_x^2}{r^2} & -3\omega_s^2 \frac{r_x r_y}{r^2} & -3\omega_s^2 \frac{r_x r_z}{r^2} \\ -3\omega_s^2 \frac{r_x r_y}{r^2} & p^2 + \omega_s^2 - 3\omega_s^2 \frac{r_y^2}{r^2} & -3\omega_s^2 \frac{r_y r_z}{r^2} \\ -3\omega_s^2 \frac{r_x r_z}{r^2} & -3\omega_s^2 \frac{r_y r_z}{r^2} & p^2 + \omega_s^2 - 3\omega_s^2 \frac{r_z^2}{r^2} \end{bmatrix}$$

Because the elements of $\underline{B}$ are time varying, a closed form solution of (20) is not readily obtained. It is possible, however, to observe that the system is unstable by examining the determinant of the coefficient matrix:

$$|\underline{B}| = (p^2 + \omega_s^2)^2 (p^2 - 2\omega_s^2)$$

We see that the system characteristic equation has poles in the right half plane, giving rise to exponential growth. To estimate the magnitude of error growth, consider the system to be operating at the equator with:

$$\underline{r} = \{r, 0, 0\}$$

Then (20) becomes:

$$\begin{bmatrix} p^2 - 2\omega_s^2 & 0 & 0 \\ 0 & p^2 + \omega_s^2 & 0 \\ 0 & 0 & p^2 + \omega_s^2 \end{bmatrix} \underline{\delta r} = \underline{0}$$

and the x channel behavior is characterized by:

$$(p^2 - 2\omega_s^2)\,\delta r_x = 0$$

Taking the Laplace transformation:

$$(S^2 - 2\omega_s^2)\,\delta\bar{r}_x = S\,\delta r_x(0) + \delta\dot{r}_x(0)$$

yielding a time response of:

$$\delta r_x = \delta r_x(0)\cosh\sqrt{2}\,\omega_s t + \frac{1}{2\omega_s}\sinh\sqrt{2}\,\omega_s t\;\delta\dot{r}_x(0)$$

after $t = \frac{2\pi}{\omega_s}$, $\cosh\sqrt{2}\,\omega_s t = \cosh 2\sqrt{2}\,\pi$.

Thus $\frac{\delta r_x}{\delta r_x(0)} \approx \cosh(8.9) \approx 3666$.

Thus it is seen that the position error due to $\delta r_x(0)$ grows by a factor of 3666 in one Schuler period. Clearly, this type of behavior cannot be tolerated, and schemes have been developed to circumvent this instability.

B. Stabilization via use of an Altimeter

The most obvious solution to the divergence demonstration in the previous section is to avoid using the inertially computed position vector in calculating the gravity magnitude. This is accomplished either by calculating the gravitational magnitude at a mominal estimated altitude or by using continuous, externally obtained, altitude information in the calculation. In either case, Eq (13) takes the form:

$$\underline{G}_c(\underline{r}_c, h_a) \overset{\sim}{=} - \frac{E}{(r_o + h_a)^3}\,\underline{r}_c \qquad (21)$$

where

r_o ∿ Local geocentric earth radius magnitude

h_a ∿ Estimated height above the reference earth model's surface.

Substituting Eq (21) into (12), and defining as above:

$$\underline{\delta r} = \underline{r}_c - \underline{r}$$

and in addition, the altitude estimation error:

$$\delta h = h_a - h \qquad (22)$$

where

h ∿ True altitude above the reference surface

yields, after series expansion where appropriate:

$$\underline{\delta \ddot{r}} + \omega_s^2\,\underline{\delta r} = + 3\omega_s^2\,\delta h\,\underline{i}_r \qquad (23)$$

Equation (23) reveals that the navigation system has been stabilized by the calculation of $\underline{G}_c$ in the manner indicated by Eq (21)

C. Extraction of Altitude Information for Stable System

Define the inertially computed altitude as:

$$h_c = r_c - r_o$$

and the error in inertially computed altitude as:

$$\delta h_c = h_c - h$$

$$= r_c - r_o - h$$

but:

$$r_c = (\underline{r}_c \cdot \underline{r}_c)^{1/2} = [(\underline{r} + \underline{\delta r}) \cdot (\underline{r} + \underline{\delta r})]^{1/2}$$

$$= (r^2 + 2\ \underline{\delta r} \cdot \underline{r} + \delta r^2)^{1/2}$$

$$= r(1 + \frac{2\ \underline{\delta r} \cdot \underline{r}}{r^2} + \frac{\delta r^2}{r^2})^{1/2}$$

$$\cong r(1 + \frac{\underline{\delta r} \cdot \underline{i}_r}{r})$$

Substituting into δh_c:

$$\delta h_c = r + \underline{\delta r} \cdot \underline{i}_r - (r_o + h)$$

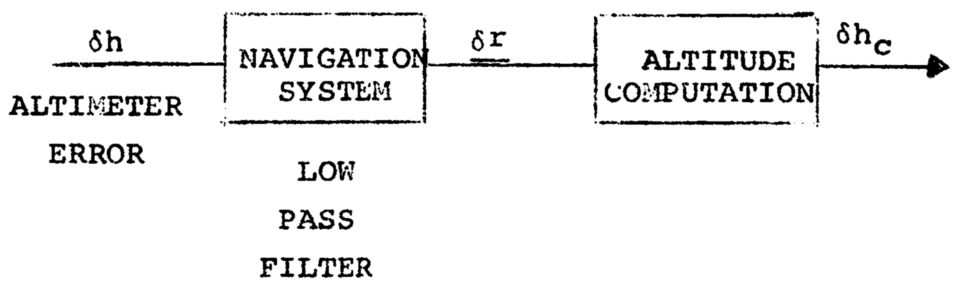
but

$$r_o + h \cong r$$

thus

$$\delta h_c = \underline{\delta r} \cdot \underline{i}_r \qquad (24)$$

The computed rate error is found by differentiating the above equation with respect to time. It is noted that the inertially derived altitude error depends directly on $\underline{\delta r}$ and not on h. Thus, short term filtering is obtained as illustrated in the following sketch:



To determine how effective this short term filtering is, Eq (23) is solved for both a constant bias in altimeter error, δh = constant, and a ramp error in altimeter error, $\delta \dot{h}$ = constant. The ramp error would correspond to the situation where an aircraft is ascending since the altimeter error will normally increase with altitude. The bias error would be the likely situation during the aircraft cruise regime.

In Eq (23), the unit vector in the direction of the geocentric radius, coordinatized in inertial coordinates is given by:

$$\underline{i}_r = \begin{bmatrix} \cos L_g \cos \lambda \\ \cos L_g \sin \lambda \\ \sin L_g \end{bmatrix}$$

If we define our reference frames such that the local meridian is coincident with both the inertially fixed reference meridian and the earth frame meridian at t = 0, then $\lambda(t=0) = \ell(t=0) - \ell_o = 0$ (see Fig. 1). Thus for the case of constant velocity motion at constant latitude, $\dot{\ell} = \dot{\ell}_o$ = constant for t > 0, and the celestial longitude is given by:

$$\lambda = (\dot{\ell}_o + \omega_{ie})t$$

For the bias altimeter error, $\delta h = \delta h_o$ = constant, solution of Eq (23) yields:

$$\underline{\delta r} = \begin{bmatrix} \dfrac{\omega_s^2 \cos L_g}{\omega_s^2 - (\dot{\ell}_o + \omega_{ie})^2} [\cos(\dot{\ell}_o + \omega_{ie})t - \cos \omega_s t] \\ \dfrac{\omega_s^2 \cos L_g}{\omega_s^2 - (\dot{\ell}_o + \omega_{ie})^2} [\sin(\dot{\ell}_o + \omega_{ie})t - \dfrac{(\dot{\ell}_o + \omega_{ie})}{\omega_s} \sin \omega_s t] \\ \sin L_g (1 - \cos \omega_s t) \end{bmatrix} 3\delta h_o \tag{25}$$

Substitution of Eq (25) into Eq (24) and noting that

$$\omega_s^2 - (\dot{\ell}_o + \omega_{ie})^2 = \omega_s^2 \left[1 - \frac{(\dot{\ell}_o + \omega_{ie})^2}{\omega_s^2}\right] \cong \omega_s^2 ,$$

yields:

$$\delta h_c = 3\delta h_o \{1 - [\sin^2 L_g + \cos^2 L_g \cos(\dot{\ell}_o + \omega_{ie})t] \cos \omega_s t - \frac{(\dot{\ell}_o + \omega_{ie})}{\omega_s} \cos^2 L_g \sin(\dot{\ell}_o + \omega_{ie})t \sin \omega_s t\}$$

Now for periods of time on the order of 1/2 hour the small angle assumption can be invoked in the above trigonometric functions without incurring an error of more than about ten percent. Thus:

$$\delta h_c \cong \frac{3}{2} \omega_s^2 t^2 \ \delta h_o \qquad (26)$$

It is seen that the coefficient in the above expression reaches unity at $t \cong 11$ min., showing that the inertially derived indication of altitude is more accurate than the altimeter indication for $t \leq 11$ min. As time increases, however, δh_c increases to a value which is about six times the altimeter error. For this case the inertially derived altitude rate, $\delta \dot{h}_c$, is of course always greater than the altimeter estimate of $\dot{h}$.

For the ramp altimeter error, $\delta h = \delta \dot{h}_o t$, solution of Eq (23), noting as before that

$$\omega_s^2 - (\dot{\ell}_o + \omega_{ie})^2 \cong \omega_s^2$$

yields:

$$\underline{\delta r} = \begin{bmatrix} \cos L_g[t \cos (\dot{\ell}_o+\omega_{ie})t - \frac{1}{\omega_s} \sin \omega_s t] \\ \cos L_g\{t \sin(\dot{\ell}_o+\omega_{ie})t + \frac{2(\dot{\ell}_o+\omega_{ie})}{\omega_s^2} [\cos \omega_s t-\cos(\dot{\ell}_o+\omega_{ie})t]\} \\ \sin L_g(t - \frac{1}{\omega_s}\sin \omega_s t) \end{bmatrix} 3\delta\dot{h}_c$$

Substituting the above expression into Eq (24) yields:

$$\delta h_c = 3\delta\dot{h}_o\{t- \frac{1}{\omega_s} [\cos^2 L_g \cos(\dot{\ell}_o+\omega_{ie})t + \sin^2 L_g]\sin \omega_s t$$

$$+ 2\frac{(\dot{\ell}_o+\omega_{ie})}{\omega_s^2} \cos^2 L_g \sin(\dot{\ell}_o+\omega_{ie})t[\cos \omega_s t$$

$$- \cos(\dot{\ell}_o+\omega_{ie}t]\}$$

Making the small angle assumptions as before gives:

$$\delta h_c = \frac{1}{2} \omega_s^2 t^3 \delta\dot{h}_o \qquad (27)$$

Now the inertially derived altitude error will be less than the altimeter error until

$$\frac{\delta h_c}{\delta\dot{h}_o t} = \frac{\omega_s^2 t^2}{2} = 1$$

or for periods of time less than about t = 19 minutes. The inertially derived altitude rate error is given by the time rate of change of Eq (27):

$$\delta\dot{h}_c = \frac{3}{2} \omega_s^2 t^2 \delta\dot{h}_o$$

Thus, the inertially derived altitude rate is more accurate than the altimeter estimate of $\dot{h}$ for t <11 minutes.

VI. Error Analysis

In this section, the error equations will be derived for the following error sources:

1. Altimeter uncertainty
2. Deflection of the vertical
3. Accelerometer uncertainty and scale factor error
4. Gyro drift
5. Initial misalignment error
6. Initial condition errors

The conspicuous absence of torquing uncertainty as an error source in the space stabilized mechanization is due to the fact that, except for the low level gyro compensation torques, the gyroscopes are free of the Earth and vehicle rate torquing commands present in local vertical mechanizations. Reference 6 documents the analysis showing that torquing uncertainty is not an important source of error for the space stabilized system.

The analysis which follows utilizes perturbation methods to linearize the nonlinear system differential equations. Perturbation analysis of this type involves the substitution

$$\underline{\omega}_c = \underline{\omega} + \underline{\delta\omega}$$

where

$\underline{\omega}_c$ = computed dependent variable
$\underline{\omega}$ = errorless dependent variable
$\underline{\delta\omega}$ = error in computed dependent variable

When the above substitution is made, linear differential equations involving only the error quantities emerge. These error equations, which are simpler in form than the original differential equations, are analytically tractable. It is within the framework of this philosophy that products of the error variables with other "small" quantities such as the earth's ellipticity and higher order terms in the gravitational field equations will be negligibly small and consequently will not appear in final error equations. It is the author's opinion that this method is to be preferred over straight computer simulation because of the insight gained into the system behavior by examining only the error response. Computer simulation of the nonlinear system equations for selected error sources has confirmed the validity of the linearized approach.

A. Error Source Evaluation

To analyze the effect of these various error sources, one proceeds with an evaluation of the basic equation:

$$\ddot{\underline{r}}_c^{i'} = \underline{f}_c^{i'} + \underline{G}_c^{i'}(\underline{r}_c^{i'}, h_a) \qquad (28)$$

where the computation is being performed in computed geocentric inertial coordinates (i') and:

$\underline{r}_c^{i'}$ = computed position vector

$\underline{f}_c^{i'}$ = computed specific force

$\underline{G}_c^{i'}$ = computed gravitational field vector

h_a = estimated height above reference earth model's surface.

In Eq (28) the following substitution is made:

$$\underline{r}_c^{i'} = \underline{r}^i + \underline{\delta r}^i \qquad (29)$$

Note that within the framework of first order analysis the error vector, $\underline{\delta r}$, can be looked at as being coordinatized in either the inertial or computed inertial frames since:

$$\underline{\delta r}^i = \underline{C}_{i'}^{i}\ \underline{\delta r}^{i'} = (\underline{I} + \underline{\theta})\underline{\delta r}^{i'} \cong \underline{\delta r}^{i'}$$

where $\underline{I}$ is the identity matrix and $\underline{\theta}$ is a skew symmetric matrix, which, when multiplied by the error vector constitute second order terms. The convention will be adopted to treat the error quantities as being coordinatized in the inertial frame as is dome in Eq (29).

1. Altimeter Uncertainty

The computed gravitational field equations are given by Eqs (11) as:

$$\underline{G}_c^{i'}(\underline{r}_c, h_a) = -\frac{E}{(r_{o_c} + h_a)^3}\begin{bmatrix} \left[1 + J\left(\frac{r_e}{r_{o_c} + h_a}\right)^2\left[1 - 5\left(\frac{r_{z_c}}{r_{o_c} + h_a}\right)^2\right]\right] r_{x_c} \\ \left[1 + J\left(\frac{r_e}{r_{o_c} + h_a}\right)^2\left[1 - 5\left(\frac{r_{z_c}}{r_{o_c} + h_a}\right)^2\right]\right] r_{y_c} \\ \left[1 + J\left(\frac{r_e}{r_{o_c} + h_a}\right)^2\left[3 - 5\left(\frac{r_{z_c}}{r_{o_c} + h_a}\right)^2\right]\right] r_{z_c} \end{bmatrix} \qquad (30)$$

where

$$J = \frac{3}{2} J_2$$

$$E = Gm$$

r_{o_c} = calculated radius of elliptic earth model

$$\cong r_e\left[1 - e\left(\frac{r_{z_c}}{r}\right)^2\right]$$

Substituting the above expression for r_{o_c}, Eq (29), and Eq (22) into (30) there results after some manipulation:

$$\underline{G}_c^{i'}(\underline{r}_c, h_a) = \underline{G}_e^i - \omega_s^2\, \underline{\delta r}^i + 3\omega_s^2\, \delta h\, \underline{1}_r^i \tag{31}$$

where

$\underline{G}_e^i$ = gravitational field vector representing the elliptical earth model

$\omega_s^2 \triangleq \dfrac{E}{(r_o + h)^3}$, the square of the Schuler frequency

$\underline{1}_r^i$ = unit vector in the direction of the geocentric position vector.

$\delta h = h_a - h$

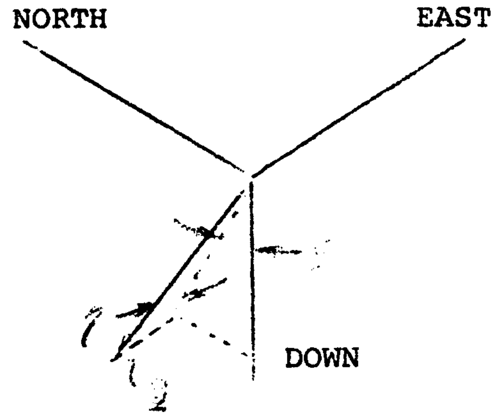
2. Deflection of the Vertical

If it were possible to specify completely the analytic form of the gravitational field equations, the deflection of the vertical would not affect the operation of the system in the sense of mode excitation. Since the reference maps of the earth use a projection based on a reference ellipsoid, the system readout would not have the desired correspondence with the map coordinates. The approach that is taken in this analysis is to take as the gravity model the best estimate of the reference ellipsoid, accepting as error sources the deflection of the vertical terms. If, in the future, it becomes possible to define the true gravitational field more precisely and to measure it more precisely with the accelerometers, a system could be built which is insensitive to deflections of the vertical. For the present, if the components of the deflection of the vertical are defined as:

ξ = meridian deflection of the vertical (positive about east)

η = prime deflection of the vertical (positive about north).

which are shown on the accompanying sketch, it is seen that



$$\underline{g}^n = \begin{bmatrix} \xi g \\ -\eta g \\ g \end{bmatrix} \qquad (32)$$

Since $\underline{g}^n \triangleq \underline{G}^n - \underline{\omega}_{ie}^n \times (\underline{\omega}_{ie}^n \times \underline{r}^n)$, (33)

$$\underline{G}^n = \underline{g}^n + \underline{\omega}_{ie}^n \times (\underline{\omega}_{ie}^n \times \underline{r}^n) = \underline{g}^n + \begin{bmatrix} r\,\omega_{ie}^2 \sin L \cos L_g \\ 0 \\ r\,\omega_{ie}^2 \cos L \cos L_g \end{bmatrix}$$

where L_g is the geocentric latitude
L is the geographic latitude

Thus:

$$\underline{G}^n = \begin{bmatrix} r\,\omega_{ie}^2 \sin L \cos L_g \\ 0 \\ g + r\,\omega_{ie}^2 \cos L \cos L_g \end{bmatrix} + \begin{bmatrix} \xi g \\ -\eta g \\ 0 \end{bmatrix} = \underline{G}_e^n + \underline{\alpha}^n \qquad (34)$$

where

$\underline{G}_e$ is the gravitational field vector associated with the reference ellipsoid

and

$$\underline{\alpha}^n = \begin{bmatrix} \xi g \\ -\eta g \\ 0 \end{bmatrix}$$

If (34) is transformed into the inertial frame, an expression for the gravitational field vector results which includes the deflection of the vertiaal terms:

$$\underline{G}^i = \underline{G}_e^i + \underline{C}_n^i \, \underline{\alpha}^n \qquad (35)$$

3. Accelerometer Uncertainty and Scale Factor Error

Because imperfections exist in the accelerometer, the output of the accelerometer triad is:

$$\underline{f}_c^a = \underline{f}^a + \underline{\delta f}^a + \underline{A}^a \underline{f}^a \qquad (36)$$

where

$\underline{f}_c$ is the measured specific force
$\underline{f}$ is the specific force
$\underline{\delta f}$ is the accelerometer uncertainty
$\underline{A}$ is the diagonal scale factor error matrix:

$$\underline{A}^a = \begin{bmatrix} a_x & 0 & 0 \\ 0 & a_y & 0 \\ 0 & 0 & a_z \end{bmatrix}$$

where a_k is the scale factor uncertainty associated with the k^{th} accelerometer, expressed as the ratio of two numbers.

4. Gyro Drift

If the platform is rotating due to the effect of gyro drift, the accelerometers will resolve the specific force data into a drifting accelerometer frame rather than into the desired nonrotating frame. The coordinate transformation relating the inertial frame to the accelerometer frame is:

$$\underline{C}_i^a = \underline{C}_p^a \, \underline{C}_{p_0}^p \, \underline{C}_i^{p_0}$$

where:

$\underline{C}_i^{p_0}$ ∿ transformation between inertial axes and platform axes at t = 0.

$\underline{C}_{p_0}^p$ ∿ transformation between platform axes at t=0 to platform axes (due to gyro drift).

$\underline{C}_p^a$ ∿ transformation between platform axes and accelerometer axes.

Since we will not be considering the effect of errors in the calibration matrix $\underline{C}_p^a$, it is notationally convenient to consider the gyro drift as occurring in accelerometer axes. That is:

$$\underline{C}_i^a = \underline{C}_{a'}^a \, \underline{C}_i^{a'} \tag{37}$$

where:

a ∿ accelerometer frame.

a' ∿ ideal (nonrotating) accelerometer frame.

The effect of gyro drift, i.e. $\underline{C}_{a'}^a$, can be written as a rotation matrix. If the drift is low, then:

$$\underline{C}_{a'}^a = [\underline{I} + \underline{D}^a]$$

where:

$\underline{D}^a$ ∿ skew symmetric matrix associated with the platform drift angle, coordinatized in accelerometer axes.

Thus, the expression for specific force in accelerometer axes is given by:

$$\underline{f}^a = [\underline{I}+\underline{D}^a] \, \underline{C}_i^{a'} \, \underline{f}^i \tag{38}$$

where $\underline{D}^a$ has the form:

$$\underline{D}^a = \begin{bmatrix} 0 & d_z & -d_y \\ -d_z & 0 & d_x \\ d_y & -d_x & 0 \end{bmatrix}$$

and the angles d_k, $k = x, y, z$ can be associated with the drift about the positive axes defined by the input axes of the $k^{\underline{th}}$ gyro.

It should be pointed out that gyro torquing uncertainty errors cannot be distinguished from the effects of gyro drift. Since gyro torquing in a space stabilized system is only used to provide low level compensation torques, the torquing induced errors are negligible.

5. Initial Misalignment Error

As pointed out previously, the alignment matrix resolves the accelerometer outputs into the computational frame which is hopefully coincident with the geocentric inertial frame.

In the notation of this section, the transformation is written, $\underline{C}_{a'}^{i'}$. If the matrix is not determined precisely then:

$$\underline{C}_{a'}^{i'} = \underline{C}_{i}^{i'}\ \underline{C}_{a'}^{i} = (\underline{I}+\underline{Z}^{i})\,\underline{C}_{a'}^{i} \tag{39}$$

where $\underline{Z}$ is the skew symmetric matrix associated with the misalignment error:

$$\underline{Z}^{i} = \begin{bmatrix} 0 & \zeta_z & -\zeta_y \\ -\zeta_z & 0 & \zeta_x \\ \zeta_y & -\zeta_x & 0 \end{bmatrix}$$

where ζ_k, $k = x, y, z$ results from a small angle rotation about the k^{th} positive inertial axis.

6. Initial Condition Errors

The initial condition errors are the initial position and velocity errors coordinatized in the computed geocentric inertial frame:

$$\underline{\delta r}(0)^{i'} = \left[\delta r_x(0),\ \delta r_y(0),\ \delta r_z(0)\right]^{i'} \tag{40}$$

$$\underline{\delta v}(0)^{i'} = \left[\delta v_x(0),\ \delta v_y(0),\ \delta v_z(0)\right]^{i'} \tag{41}$$

These errors will be incorporated into the differential equations in the derivation which follows.

B. Derivation of System Equations

1. Geocentric Inertial Computation Frame

The synthesis of the derived error source equations into one vector differential equation for the configurations of Figs. 3 and 4 can be visualized with the aid of the following vector-matrix signal flow diagram:

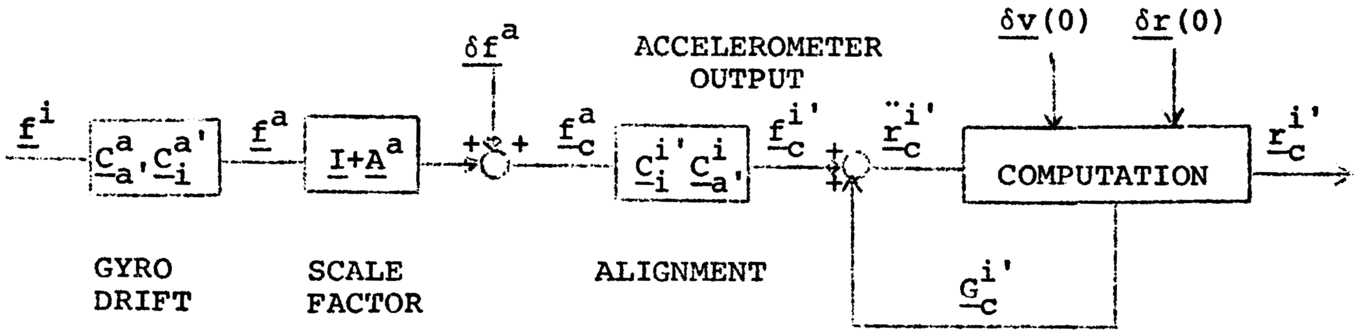


Fig. 7 Signal Flow Diagram.

Starting with Eq. (28), and following the signal path shown in the diagram, it is seen that:

$$\underline{r}_c^{i'} - \underline{G}_c^{i'} = \underline{f}_c^{i'} \tag{42}$$

$$= \underline{C}_i^{i'} \, C_{a'}^{i} \, [\underline{\delta f}^a + (\underline{I} + \underline{A}^a) \, \underline{C}_{a'}^{a} \, \underline{C}_i^{a'} \, \underline{f}^i]$$

But from (39), $\underline{C}_i^{i'} = (\underline{I} + \underline{Z}^i)$

and from (38), $\underline{C}_{a'}^{a} = (\underline{I} + \underline{D}^a)$

Thus if products of error quantities are neglected,

$$\underline{f}_c^{i'} = \ddot{\underline{r}}_c^{i'} - \underline{G}_c^{i'} = \underline{C}_{a'}^{i} \; \underline{\delta f}^a + [\underline{I} + \underline{Z}^i + \underline{C}_{a'}^{i} \; \underline{D}^a \underline{C}_i^{a'} + \underline{C}_{a'}^{i} \; \underline{A}^a \underline{C}_i^{a'}] f^i \quad (43)$$

Substitution of Eqs (29), (31), and (35) into (43) yields:

$$\underline{\delta \ddot{r}}^i + \omega_s^2 \, \underline{\delta r}^i = \underline{G}^i - \ddot{\underline{r}}^i - \underline{C}_n^i \, \underline{\alpha}^n + 3\omega_s^2 \, \delta h \, \underline{1}_r^i + \underline{C}_{a'}^i \, \underline{\delta f}^a$$

$$+ [\underline{I} + \underline{Z}^i + \underline{C}_{a'}^i (\underline{D}^a + \underline{A}^a) \underline{C}_i^{a'}] \underline{f}^i$$

Noticing that: $\underline{f}^i = -(\underline{G}^i - \ddot{\underline{r}}^i)$, the error differential equation is obtained.

$$\underline{\delta \ddot{r}}^i + \omega_s^2 \, \underline{\delta r}^i = 3\omega_s^2 \, \delta h \, \underline{1}_r^i - \underline{C}_n^i \, \underline{\alpha}^n + \underline{C}_{a'}^i \, \underline{\delta f}^a + \underline{Z}^i \underline{f}^i + \underline{C}_{a'}^i (\underline{D}^a + \underline{A}^a) \underline{C}_i^{a'} \, \underline{f}^i \quad (44)$$

One sees by inspection that the system will behave as a forced second order Schuler tuned oscillator. The similarity transformation is involved in the gyro drift and accelerometer scale factor terms since the elements of the $\underline{D}$ and $\underline{A}$ matrices are coordinatized in platform (accelerometer) axes. Note that since no assumptions have been made concerning the motion of the system, Eq (44) is valid for arbitrary vehicle motion.

The solution of Eq (44) will be an analytic expression for $\underline{\delta r}^i$. It is obviously still necessary to express the latitude, longitude, and earth referenced velocity errors in terms of $\underline{\delta r}^i$.

Geocentric latitude is computed from the relationship:

$$\sin L_{g_c} = \frac{r_{z_c}}{r_c}$$

where

L_{g_c} ∿ computed geocentric latitude

r_{z_c} ∿ computed position radius polar component

r_c ∿ computed position vector magnitude

Next define the error quantities:

$$L_{g_c} = L_g + \delta L_g$$

$$r_{z_c} = r_z + \delta r_z$$

$$r_c = r + \delta r$$

where

δL_g ∿ error in computed geocentric latitude.
δr ∿ error in computed position vector magnitude.

If the error quantities are substituted into the expression for sin L_{g_c}, there results after expansion and neglecting of higher order terms:

$$\delta L = \frac{\sec L}{r} (\delta r_z - \sin L \, \delta r)$$

where

$L = L_g + D$
$D \stackrel{\sim}{=} e \sin 2L$ ∿ deviation of normal
e ∿ earth's ellipticity

There are several ways of calculating r_c, the computed position vector magnitude. If one takes:

$$r_c = [r_{x_c}^2 + r_{y_c}^2 + r_{z_c}^2]^{1/2}$$

then expansion shows that:

$$\delta r = \cos L \cos \lambda \, \delta r_x + \cos L \sin \lambda \, \delta r_y + \sin L \, \delta r_z$$

Consequently,

$$\delta L = \frac{1}{r}(-\sin L \cos \lambda \, \delta r_x - \sin L \sin \lambda \, \delta r_y + \cos L \, \delta r_z).$$

The quantity in parenthesis is recognized as the north projection of the position error vector:

Thus:

$$\delta L = \frac{1}{r} \underline{q}_1^T \underline{C}_i^n \underline{\delta r}^i \qquad (44a)$$

where

$$\underline{q}_1 = \{1,0,0\}$$

If, on the other hand, one chooses to calculate r_c from:

$$r_c = r_{o_c} + h_a$$

where

$$r_{o_c} \cong r_e[1 - e\left(\frac{r_{z_c}}{r}\right)^2]$$

$\sim$ calculated local geocentric earth radius magnitude.

h_a $\sim$ estimated height above the reference earth model's surface.

r_e $\sim$ earth's equatorial radius.

Expansion shows that:

$$\delta r = \delta h.$$

Consequently,

$$\delta L^* = \frac{\delta r_z}{r \cos L} - \tan L \frac{\delta h}{r} \tag{44b}$$

Comparison of Eqs (55a) and (55b) reveals that the latitude error is free of celestial longitude rate modulation if the computed earth radius is calculated from:

$$r_c = r_{o_c} + h_a$$

Unless it is stated to the contrary, however, we will calculate the latitude error from Eq (55a) which implies:

$$r_c = [r_{x_c}^2 + r_{y_c}^2 + r_{z_c}^2]^{1/2}$$

Celestial longitude is computed from the relationship:

$$\tan \lambda_c = \frac{r_{y_c}}{r_{x_c}}$$

Defining the error quantities

$$\lambda_c = \lambda + \delta\lambda, \; r_{y_c} = r_y + \delta r_y, \; r_{x_c} = r_x + \delta r_x$$

where

$\delta\lambda \sim$ error in the computed celestial longitude,

and expanding yields:

$$\delta\lambda = \frac{1}{r \cos L} (-\sin \lambda \, \delta r_x + \cos \lambda \, \delta r_y)$$

But the quantity in parenthesis is recognized as the east projection of the position error vector:

Thus:

$$\delta\lambda = \frac{1}{r \cos L} \underline{q}_2^T \underline{C}_i^n \underline{\delta r}^i \tag{44c}$$

where:

$$\underline{q}_2 = \{0,1,0\}$$

Earth referenced velocity is found by differentiating the expression for the geocentric position vector in earth frame coordinates, and then coordinatizing in geographic coordinates:

$$\underline{v}^n = \underline{C}_e^n \dot{\underline{r}}^e = \underline{C}_i^n [\dot{\underline{r}}^i - \underline{\Omega}_{ie}^i \underline{r}^i]$$

where

$$\underline{\Omega}_{ie}^i = \begin{bmatrix} 0 & -\omega_{ie} & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sim$ angular velocity of the earth frame with respect to the inertial frame, coordinatized in inertial axes.

Thus the computed earth referenced velocity is given by:

$$\underline{v}_c^n = [\underline{C}_i^n]_c [\dot{\underline{r}}_c^i - \underline{\Omega}_{ie}^i \underline{r}_c^i]$$

Defining the velocity error, $\underline{\delta v}$, by:

$$\underline{v}_c = \underline{v} + \underline{\delta v}$$

and making use of Eqs (48) and (49) to find an expression for the computed coordinate transformation matrix yields:

$$\underline{\delta v}^n = \underline{C}_i^n (\underline{\delta \dot{r}}^i - \Omega_{ie}^i \underline{\delta r}^i) + \underline{\theta}^n \underline{v}^n \tag{44d}$$

2. Geographic Computation Frame

The derivation of the error equations for the configuration of Fig. 2 proceeds in a straightforward manner from the expression for specific force in geographic axes, $\underline{f}^n$. To obtain this expression, one starts with the expression on pg.6 for $\underline{C}_i^g \, \ddot{\underline{r}}^i$ and Eq (34) which is an expression for $\underline{G}^n$:

$$\underline{f}^n = \underline{C}_i^n \, \ddot{\underline{r}}^i - \underline{G}^n$$

$$= \underline{C}_g^n \, \underline{C}_i^g \, \ddot{\underline{r}}^i - \underline{G}^n$$

ince

$$\underline{C}_g^n = \begin{bmatrix} \cos D & 0 & \sin D \\ 0 & 1 & 0 \\ -\sin D & 0 & \cos D \end{bmatrix}$$

where

D $\sim$ deviation of the normal,

$$\underline{f}^n = \begin{bmatrix} (2\dot{r}\dot{L}_g + r\ddot{L}_g)\cos D + (r\dot{L}_g^2 - \ddot{r})\sin D + r\dot{\lambda}^2 \cos L_g \sin L \\ -2r\dot{L}_g\dot{\lambda}\sin L_g + 2\dot{r}\dot{\lambda}\cos L_g + r\ddot{\lambda}\cos L_g \\ -(2\dot{r}\dot{L}_g + r\ddot{L}_g)\sin D + (r\dot{L}_g^2 - \ddot{r})\cos D + r\dot{\lambda}^2 \cos L_g \cos L \end{bmatrix} - \underline{G}^n$$

It is desirable to write the above expression as a function of the components of $\underline{g}^n$. Thus, Eq (34) is substituted, noting that $\dot{\lambda} = \dot{\ell} + \omega_{ie}$, yielding:

$$\underline{f}^n = \begin{bmatrix} (2\dot{r}\dot{L}_g + r\ddot{L}_g)\cos D + (r\dot{L}_g^2 - \ddot{r})\sin D + r(\dot{\lambda}^2 - \omega_{ie}^2)\sin L \cos L_g - \xi g \\ -2r\dot{L}_g\dot{\lambda}\sin L_g + 2\dot{r}\dot{\lambda}\cos L_g + r\ddot{\lambda}\cos L_g + \eta g \\ -(2\dot{r}\dot{L}_g + r\ddot{L}_g)\sin D + (r\dot{L}_g^2 - \ddot{r})\cos D + r(\dot{\lambda}^2 - \omega_{ie}^2)\cos L \cos L_g - g \end{bmatrix}$$

Note carefully that the only approximation made so far has been the small angle assumptions involved in calculating the deflection of the vertical terms.

We next express the above relationship as a function of the geographic latitude, L. This is accomplished by series expansion with an accuracy consistent with the deflection of the vertical terms. We will therefore choose to retain terms with magnitude greater than 2×10^{-5}g. Since the magnitudes of the various terms in the expression for specific force depend on the motion of the vehicle, let us agree that the following data represent maximum values of vehicle motion:

$$r\ddot{L}_{max} = r\ddot{\lambda}_{max} = 0.5g$$

$$\dot{r}_{max} = 100 \text{ ft/sec}$$

$$\dot{L}_{max} = \dot{\lambda}_{max} = 1.6 \times 10^{-4} \text{ rad/sec}$$

$$\ddot{r}_{max} = 2g$$

These values correspond to those which one would expect to encounter in an aircraft application such as the supersonic transport, Obviously, if one has a different application, such as a ship navigation system, certain of the higher order terms in the expression for the specific force need not be retained.

It can be shown that the deviation of the normal can be calculated with an accuracy on the order of 2 $\widehat{sec}$ via the following equation:

$$D \cong e \sin 2L$$

Thus, it is immediately seen that the approximations:

$$\cos D \cong 1$$
$$\sin D \cong e \sin 2L$$
$$\sin L_g \cong (1 - 2\, e \cos^2 L)\sin L$$
$$\cos L_g \cong (1 + 2\, e \sin^2 L)\cos L$$

can be made in the expression for $\underline{f}^n$ without violating the aforementioned accuracy criteria. Thus:

$$\underline{f}^n = \begin{bmatrix} 2\dot{r}\dot{L}_g + r\ddot{L}_g + (r\dot{L}_g^2 - \ddot{r})e\sin 2L + \frac{1}{2} r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\sin 2L - \xi g \\ -2r\dot{L}_g\dot{\lambda}(1 - 2e\cos^2 L)\sin L + 2\dot{r}\dot{\lambda}(1 + 2e\sin^2 L)\cos L + r_\ell\ddot{\lambda}\cos L + \eta g \\ -(2\dot{r}\dot{L}_g + r\ddot{L}_g)e\sin 2L + r\dot{L}_g^2 - \ddot{r} + r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\cos^2 L - g \end{bmatrix}$$

where, in the above,

$$r_\ell \cong r(1 + 2\, e \sin^2 L)$$

$\sim$ radius of curvature in comeridian plane.

We next substitute the relationships:

$$L_g = L - D = L - e \sin 2\, L$$

$$\dot{L}_g = (1 - 2\, e \cos 2L)\dot{L}$$

$$\ddot{L}_g = (1 - 2\, e \cos 2L)\ddot{L} + 4\, e \sin 2L\, \dot{L}^2$$

Giving:

$$\underline{f}^n = \begin{bmatrix} r_L\ddot{L} + \frac{1}{2}r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\sin 2L + 2\dot{r}_L\dot{L} - \ddot{r}e\sin 2L - 3er\sin 2L\,\dot{L}^2 - \xi g \\ r_\ell\ddot{\lambda}\cos L - 2r_\ell\dot{L}\dot{\lambda}\sin L + 2\dot{r}_\ell\dot{\lambda}\cos L + \eta g \\ -g - \ddot{r} - r_L\ddot{L}e\sin 2L + r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\cos^2 L + \frac{r_L^2}{r}\dot{L}^2 \end{bmatrix} \quad (45)$$

where:

$$r_L \cong r(1 - 2\,e\,\cos 2L)$$

$\sim$ radius of curvature in meridian plane.

If compensation is provided for the Coriolis and cross coupling terms in Eq (45), then,

$$\underline{f}^n = \begin{bmatrix} r_L\ddot{L} - \xi g \\ r_\ell\ddot{\lambda}\cos L + \eta g \\ -\ddot{r} - g \end{bmatrix} \qquad (46)$$

and it is evident that L and λ can be found by double integration of the north and east specific force measurements, respectively. The computation scheme appears in Fig. 8.

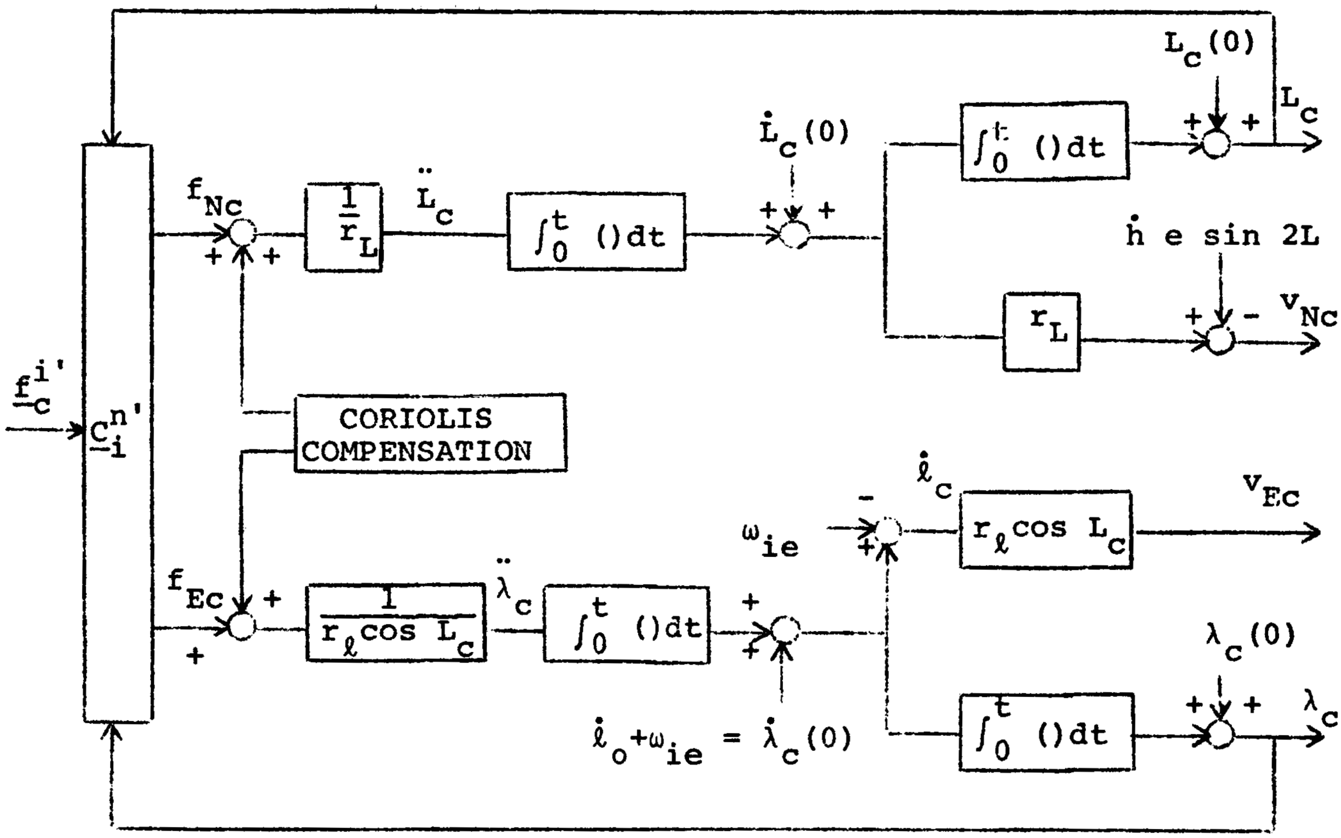


Fig. 8 Local Geographic Computation Scheme

In Fig. 8, it was noted that the Earth-referenced velocity, coordinatized in navigational axes is given by (to an accuracy of better than 0.1 ft/sec for aircraft altitudes):

$$\underline{v}^n \simeq \begin{bmatrix} r_L \dot{L} - \dot{h}\, e \sin 2L \\ r_\ell \dot{\ell} \cos L \\ -\dot{h} \end{bmatrix} \tag{47}$$

The computed specific force is transformed from the geocentric inertial to computed geographic coordinates via the transformation:

$$\underline{C}_i^{n'} = \underline{C}_n^{n'} \underline{C}_i^n = \begin{bmatrix} -\sin L_c \cos \lambda_c & -\sin L_c \sin \lambda_c & \cos L_c \\ -\sin \lambda_c & \cos \lambda_c & 0 \\ -\cos L_c \cos \lambda_c & -\cos L_c \sin \lambda_c & -\sin L_c \end{bmatrix} \tag{48}$$

Substitution of the error quantities

$$L_c = L + \delta L \text{ and } \lambda_c = \lambda + \delta\lambda$$

and expansion yields:

$$\underline{C}_i^{n'} = \underline{C}_n^{n'}\underline{C}_i^n = [\underline{I} + \underline{\theta}^n]\underline{C}_i^n \tag{49}$$

where

$$\underline{\theta}^n = \begin{bmatrix} 0 & -\delta\lambda \sin L & \delta L \\ \delta\lambda \sin L & 0 & \delta\lambda \cos L \\ -\delta L & -\delta\lambda \cos L & 0 \end{bmatrix}$$

Note that we could have just as well proceeded to expand the above expression in the form:

$$\underline{C}_{i'}^n = \underline{C}_i^n \underline{C}_{i'}^i = \underline{C}_i^n[\underline{I} + \underline{\theta}^i]$$

where

$$\underline{\theta}^i = \underline{C}_n^i \underline{\theta}^n \underline{C}_i^n$$

Thus the expression for computed specific force coordinatized in computed geographic axes is found by multiplying Eq (43) by Eq (49), yielding:

$$\underline{f}_c^{n'} = \underline{C}_{a'}^n \underline{\delta f}^a + [\underline{I} + \underline{\theta}^n + \underline{C}_i^n \underline{Z}^i \underline{C}_n^i + \underline{C}_{a'}^n \underline{D}^a \underline{C}_n^{a'} + \underline{C}_{a'}^n \underline{A}^a \underline{C}_n^{a'}]\underline{f}^n \tag{50}$$

Equating the appropriate components of Eq (50) to the computed north and east elements of Eq (45) gives:

$$\underline{q}_1^T \underline{f}_c^{n'} = r_{L_c}\ddot{L}_c + \frac{1}{2} r_{\ell_c}(\dot{\lambda}_c^2 - \omega_{ie}^2)\sin 2L_c + 2\dot{r}_{L_c}\dot{L}_c - \ddot{r}_c e \sin 2L_c - 3e r_c \sin 2L_c \dot{L}_c^2 \tag{51a}$$

$$\underline{q}_2^T \underline{f}_c^{n'} = r_{\ell_c}\ddot{\lambda}_c \cos L_c - 2r_{\ell_c}\dot{L}_c\dot{\lambda}_c \sin L_c + 2\dot{r}_{\ell_c}\dot{\lambda}_c \cos L_c \tag{51b}$$

Now the computed expressions for the radii of curvature are given by:

$$r_{L_c} = r_c(1 - 2e \cos 2L_c)$$

$$r_{\ell_c} = r_c(1 + 2e \sin^2 L_c)$$

But, from page 46b, and Eq (22),

$$r_c = r_{o_c} + h_a$$

$$= r_e(1 - e \sin^2 L_c) + h + \delta h$$

$$= r_e(1 - e \sin^2 L) + h + \delta h$$

$$= r + \delta h \qquad (51\ c)$$

Thus

$$r_{L_c} = r[1 - 2e \cos 2(L + \delta L)] + \delta h$$

$$r_{\ell_c} = r[1 + 2e \sin^2(L + \delta L)] + \delta h$$

or

$$r_{L_c} \cong r_L + \delta h \qquad (51\ d)$$

$$r_{\ell_c} \cong r_\ell + \delta h \qquad (51\ e)$$

If equations (51c), (51d) and (51e) and the error quantities:

$$L_c = L + \delta L \text{ and } \lambda_c = \lambda + \delta\lambda$$

are substituted into Eqs (51a) and (51b), there results:

$$\underline{q}_1^T \underline{f}_c^{n'} = \underline{q}_1^T \underline{f}^n + \xi g + r_L \delta\ddot{L} + r_\ell(\dot{\lambda}^2 - \omega_{ie}^2)\cos 2L\ \delta L + 2\dot{L}\ \delta\dot{h}$$
$$+ r_\ell\ \dot{\lambda} \sin 2L\ \delta\dot{\lambda} + [\ddot{L} + \tfrac{1}{2}(\dot{\lambda}^2 - \omega_{ie}^2) \sin 2L]\ \delta h \qquad (52\ a)$$

$$\underline{q}_2^T \underline{f}_c^{n'} = \underline{q}_2^T \underline{f}^n - \eta g + r_\ell \cos L\, \delta\ddot{\lambda} + 2[\dot{r}_\ell \cos L - r_\ell \dot{L} \sin L]\delta\dot{\lambda}$$

$$- 2\, r_\ell\, \dot{\lambda} \sin L\, \delta\dot{L} - [r_\ell\, \ddot{\lambda} \sin L + 2\, r_\ell\, \dot{L}\, \dot{\lambda} \cos L + 2\dot{r}_\ell\, \dot{\lambda} \sin L]\delta L$$

$$+ 2\, \dot{\lambda}\, \cos L\, \delta\dot{h} + [\ddot{\lambda} \cos L - 2\, \dot{L}\, \dot{\lambda} \sin L]\delta h \qquad (52b)$$

Certain of the terms in Eqs (52a) and (52b) are seen to have negligible magnitude ($< 2 \times 10^{-5} g$) if the vehicle motion can be described by the data on page 55 and if, in addition, the following error data is postulated:

$$\delta L_{max} = \delta\lambda_{max} = 10\ \widehat{min} = 2.9 \times 10^{-3}\ \text{rad}$$

$$\delta\dot{L}_{max} = \delta\dot{\lambda}_{max} = \delta L_{max}\omega_s = 3.6 \times 10^{-6}\ \text{rad/sec}$$

$$\delta\ddot{L}_{max} = \delta\ddot{\lambda}_{max} = \delta L_{max}\omega_s^2 = 4.5 \times 10^{-9}\ \text{rad/sec}^2$$

$$\delta h_{max} = 2000\ \text{ft.}$$

$$\delta\dot{h}_{max} = \delta h_{max}\omega_s = 2.5\ \text{ft/sec}$$

Thus:

$$\underline{q}_1^T \underline{f}_c^{n'} = \underline{q}_1^T \underline{f}^n + \xi g + r\, \delta\ddot{L} + 2\dot{L}\, \delta\dot{h} + r\, \dot{\lambda} \sin 2L\, \delta\dot{\lambda} + \ddot{L}\, \delta h \qquad (52c)$$

$$\underline{q}_2^T \underline{f}_c^{n'} = \underline{q}_2^T \underline{f}^n - \eta g + r \cos L\, \delta\ddot{\lambda} + 2[\dot{r} \cos L - r\, \dot{L} \sin L]\delta\dot{\lambda}$$

$$- 2r\, \dot{\lambda} \sin L\, \delta\dot{L} - r[\ddot{\lambda} \sin L + 2\dot{L}\, \dot{\lambda} \cos L]\delta L$$

$$+ 2\dot{\lambda} \cos L\, \delta\dot{h} + \ddot{\lambda} \cos L\, \delta h \qquad (52d)$$

Substituting Eq (50) into (52c) and (52d) and multiplication of the $\underline{\theta}^n\ \underline{f}^n$ product yields:

$$r\, \delta\ddot{L} + (\ddot{r} + g)\delta L + r\, \dot{\lambda} \sin 2L\, \delta\dot{\lambda} + \frac{1}{2} r(\ddot{\lambda} \sin 2L - 4\dot{L}\dot{\lambda} \sin^2 L)\delta\lambda =$$

$$- \xi g + \underline{q}_1^T \underline{C}_{a'}^n\, \delta\underline{f}^a + \underline{q}_1^T[\underline{C}_i^n\, \underline{Z}^i\, \underline{C}_n^i + \underline{C}_{a'}^n(\underline{D}^a + \underline{A}^a)\underline{C}_n^{a'}]\underline{f}^n - 2\dot{L}\delta\dot{h} - \ddot{L}\, \delta h \quad (53a)$$

$$r \cos L\, \delta\ddot{\lambda} + 2[\dot{r} \cos L - r\, \dot{L} \sin L]\, \delta\dot{\lambda} + [(\ddot{r}+g)\cos L - r\, \ddot{L} \sin L$$

$$+ 2\ddot{r}\, e \sin^2 L \cos L]\, \delta\lambda - 2r\, \dot{\lambda} \sin L\, \delta\dot{L} - r(\ddot{\lambda} \sin L + 2\dot{L}\, \dot{\lambda} \cos L)\delta L =$$

$$\eta g + \underline{q}_2^T\, \underline{C}_{a'}^n\, \delta\underline{f}^a + \underline{q}_2^T[\underline{C}_i^n\, \underline{Z}^i\, \underline{C}_n^i + \underline{C}_{a'}^n(\underline{D}^a + \underline{A}^a)\underline{C}_n^{a'}]\underline{f}^n - 2\dot{\lambda} \cos L\, \delta\dot{h} +$$

$$\ddot{\lambda} \cos L\, \delta h \qquad (53b)$$

The above equations give the latitude and longitude errors for arbitrary vehicle motion for the space stabilized system which computes in geographic coordinates. These coupled linear time varying equations are not as simple as those obtained for computation in inertial coordinates (Eq. 44).

For the case of a stationary system:

$$\ddot{L} = \dot{L} = \ddot{\lambda} = \ddot{r} = \dot{r} = 0,\ \dot{\lambda} = \omega_{ie}$$

and equations (53a) and (53b) simplify to the coupled linear equations:

$$\delta L + \omega_s^2\, \delta L + \omega_{ie} \sin 2L\, \delta\dot{\lambda} = -\omega_s^2\, \xi + \frac{1}{r}\{\underline{q}_1^T\, \underline{C}_{a'}^n\, \delta\underline{f}^a + \underline{q}_1^T[\underline{C}_i^n\, \underline{Z}^i\, \underline{C}_n^i + \underline{C}_{a'}^n$$

$$(\underline{D}^a + \underline{A}^a)\underline{C}_n^{a'}]\underline{f}^n\} \qquad (54a)$$

$$\delta\ddot{\lambda} + \omega_s^2\, \delta\lambda - 2\omega_{ie} \tan L\, \delta\dot{L} = \omega_s^2 \sec L\, \eta + \frac{\sec L}{r}\{\underline{q}_2^T\, \underline{C}_{a'}^n\, \delta\underline{f}^a +$$

$$\underline{q}_2^T\, [\underline{C}_i^n\, \underline{Z}^i\, \underline{C}_n^i + \underline{C}_{a'}^n\, (\underline{D}^a + \underline{A}^a)\underline{C}_n^{a'}]\underline{f}^n\} \qquad (54b)$$

where:

$$\frac{g}{r} \simeq \omega_s^2$$

Noting that the characteristic equation for equations (54) is given by:

$$p^4 + 2\omega_s^2[1 + 2(\frac{\omega_{ie}}{\omega_s})^2 \sin^2 L]\, p^2 + \omega_s^4 \simeq (p^2 + \omega_s^2)^2$$

Where $p = \frac{d}{dt}$, it is seen that the cross coupling between the latitude and longitude channels is on the order of the Earth's ellipticity--a rather weak effect. This coupling arises from the computation of the Coriolis terms for accelerometer compensation as can be seen if the equations are derived assuming perfect compensation. As a matter of fact, the above fourth order characteristic equation can be shown[10] to be the same as the characteristic equation for a Schuler-tuned Foucault pendulum. Thus we would expect that the error response resulting from the simultaneous solution of equations (54) would contain modulation at the Foucault frequency. That is, at a frequency given by $\omega_{ie} \sin L$, the vertical projection of Earth rate.

Since equations (54a) and (54b) are Laplace transformable, it is convenient to express the equations in signal flow diagram form. Taking the Laplace transformation:

$$s^2\,\delta\bar{L} - s\,\delta L(0) - \delta\dot{L}(0) + \omega_s^2\,\delta\bar{L} + \omega_{ie}\sin 2L[s\,\delta\bar{\lambda} - \delta\lambda(0)] =$$

$$-\omega_s^2\,\bar{\xi} + \frac{1}{r}\,\underline{q}_1^T[\overline{\underline{C}_a^n,\ \delta\underline{f}^a} + \overline{(\underline{Z}^n + \underline{D}^n + \underline{A}^n)\underline{f}^n}] \qquad (54c)$$

$$s^2\,\delta\bar{\lambda} - s\,\delta\lambda(0) - \delta\dot{\lambda}(0) + \omega_s^2\,\delta\bar{\lambda} - 2\,\omega_{ie}\tan L[s\,\delta\bar{L} - \delta L(0)] =$$

$$\omega_s^2\sec L\,\bar{\eta} + \frac{\sec L}{r}\,\underline{q}_2^T[\overline{\underline{C}_a^n,\ \delta\underline{f}^a} + \overline{(\underline{Z}^n + \underline{D}^n + \underline{A}^n)\underline{f}^n}] \qquad (54d)$$

Where it was noted that:

$$\underline{C}_i^n\,\underline{Z}^i\,\underline{C}_n^i = \underline{Z}^n$$

$$\underline{C}_{a'}^n(\underline{D}^a + \underline{A}^a)\underline{C}_n^{a'} = \underline{D}^n + \underline{A}^n$$

Equations (54c) and (54d) can be arranged in the signal flow diagram shown in Figure (9).

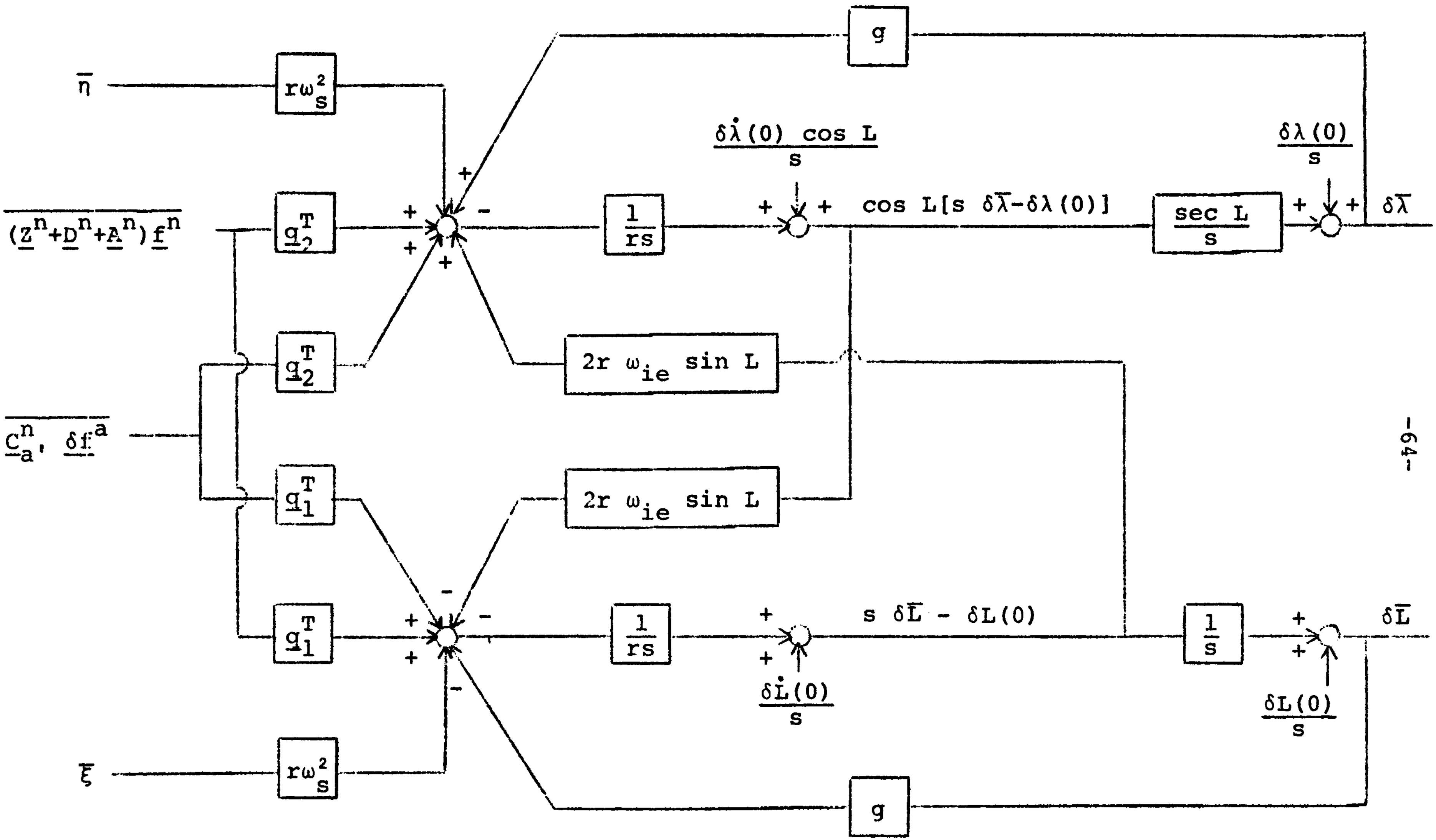


Figure 9 ~ Signal Flow for Space Stabilized Inertial Navigation System Which Computes in Geographic Coordinates

C. Solution of Differential Equations

1. Latitude and Longitude Errors for Constant Gyro Drift

Equation (44), which represents the error response for the space stabilized inertial navigation system which computes in geocentric inertial coordinates, is a linear, uncoupled vector differential equation with constant coefficients. Thus, any one of the familiar linear analysis techniques can be employed. To illustrate the computations involved, take a situation which will both illustrate the solution of the differential equation and will serve as a comparison between the two computational frames.

If gyro drift is taken to be the sole error source, Eq (44) becomes

$$\underline{\delta \ddot{r}}^i + \omega_s^2 \, \underline{\delta r}^i = \underline{C}_a^i \, \underline{D}^a \, \underline{C}_i^a \, \underline{f}^i$$

where the elements of $\underline{D}^a$ are given by

$$d_k = \omega_k t \quad , \quad k = x, y, z$$

Since skew symmetric matrices transform under a similarity transformation to another skew symmetric matrix whose elements are coordinatized in the new frame, the above equation can be written:

$$\underline{\delta \ddot{r}}^i + \omega_s^2 \, \underline{\delta r}^i = \underline{D}^i \, \underline{f}^i \tag{55}$$

For the stationary case, ($\ell = \ell_o$, $\lambda = \omega_{ie}t$)

$$\underline{f}^i = \underline{C}_n^i \, \underline{f}^n = \underline{C}_n^i \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} = g \begin{bmatrix} \cos L \cos \omega_{ie}t \\ \cos L \sin \omega_{ie}t \\ \sin L \end{bmatrix}$$

$$= g \frac{\underline{r}^i}{r} - gD \begin{bmatrix} \sin L \cos \omega_{ie}t \\ \sin L \sin \omega_{ie}t \\ -\cos L \end{bmatrix}$$

where D, the deviation of the normal, (not to be confused with the drift matrix, $\underline{D}$) is defined by

$$D = L - Lg \qquad \text{(see Fig. 1)}$$

Since the maximum value of the deviation of the normal is on the order of the earth's ellipticity or 1/297 the product of the deviation of the normal and the elements of the drift matrix are second order.

Thus, Eq (55) becomes:

$$\delta\ddot{\underline{r}}^i + \omega_s^2 \, \underline{\delta r}^i = \omega_s^2 \, \underline{D}^i \, \underline{r}^i \tag{56}$$

The solution of Eq (56) is of the form:

$$\underline{\delta r}^i \cong \underline{D}^i \, \underline{r}^i \tag{57}$$

This somewhat surprising result emerges because the frequency content of the forcing function of Eq (56) is at a much lower frequency than that of the systen mode (in fact $\frac{\omega_s}{\omega_{ie}} \cong 17$) and $\frac{\omega_s}{\omega_k} \cong 17{,}000$ for $\omega_k = 1$ meru).

The latitude error is given by north error component - earth radius ratio as:

$$\delta L = \underline{q}_1^T \frac{\delta \underline{r}^n}{r} \tag{58}$$

where $\underline{q}_1$ is the vector {1,0,0}.

Rewriting:

$$\delta L = \frac{1}{r} \underline{q}_1^T \underline{C}_i^n \underline{\delta r}^i = \frac{1}{r} \underline{q}_1^T \underline{C}_i^n \underline{D}^i \underline{r}^i = \frac{1}{r} \underline{q}_1^T \underline{C}_i^n \underline{D}^i \underline{C}_n^i \underline{r}^n$$

or

$$\delta L = \underline{q}_1^T \underline{D}^n \frac{\underline{r}^n}{r} \tag{59}$$

Longitude error is given by the east error component - equatorial earth radius projection ratio as:

$$\delta \lambda = \underline{q}_2^T \frac{\delta \underline{r}^n}{r \cos L} \tag{60}$$

where $\underline{q}_2$ is the vector {0,1,0}.

Rewriting:

$$\delta \lambda = \underline{q}_2^T \underline{C}_i^n \frac{\delta \underline{r}^i}{r} \sec L = \underline{q}_2^T \underline{C}_i^n \underline{D}^i \frac{\underline{r}^i}{r} \sec L = \underline{q}_2^T \underline{C}_i^n \underline{D}^i \underline{C}_n^i \frac{\underline{r}^n}{r} \sec L$$

or

$$\delta \lambda = \underline{q}_2^T \underline{D}^n \frac{\underline{r}^n}{r} \sec L \tag{61}$$

From Eqs. (38), (59), and (61), the latitude and longitude error for the case of constant gyro drift is given by:

$$\delta \dot{L} = dy_n \tag{62}$$

$$\delta \lambda = - dx_n \sec L \tag{63}$$

where dy_n and dx_n are the effective drift rates about the east and north axes. If one can assume that the gyro input axes are initially aligned with geocentric inertial axes,

$$\delta L = - \sin \omega_{ie}t\ \omega_x t + \cos \omega_{ie}t\ \omega_y t \tag{64}$$

$$\delta \lambda = \cos \omega_{ie}t \tan L\ \omega_x t + \sin \omega_{ie}t \tan L\ \omega_y t - \omega_z t \tag{65}$$

where ω_k, k = x,y,z, is the drift rate associated with k^{th} gyro. Equations (54a) and (54b) describe the error response for the case of computation in local geographic coordinates for a stationary system. The response to gyro drift is seen to be:

$$\ddot{\delta L} + \omega_s^2\ \delta L + \omega_{ie} \sin 2L\ \delta \dot{\lambda} = \frac{1}{r}\ \underline{q}_1^T\ \underline{C}_{a'}^n\ \underline{D}^a\ \underline{C}_n^{a'}\ \underline{f}^n$$

$$\ddot{\delta \lambda} + \omega_s^2\ \delta \lambda - 2\omega_{ie} \tan L\ \delta \dot{L} = \frac{\sec L}{r}\ \underline{q}_2^T\ \underline{C}_{a'}^n\ \underline{D}^a\ \underline{C}_n^{a'}\ \underline{f}^n$$

Since the above equations can be rewritten as:

$$\ddot{\delta L} + \omega_s^2\ \delta L + \omega_{ie} \sin 2L\ \delta \dot{\lambda} = - \underline{q}_1^T\ \underline{D}^n \begin{bmatrix} 0 \\ 0 \\ \omega_s^2 \end{bmatrix} = dy_n\ \omega_s^2 \tag{66}$$

$$\ddot{\delta \lambda} + \omega_s^2\ \delta \lambda - 2\omega_{ie} \tan L\ \delta \dot{L} = - \underline{q}_2^T\ \underline{D}^n \begin{bmatrix} 0 \\ 0 \\ \omega_s^2 \end{bmatrix} \sec L = - dx_n\ \omega_s^2 \sec L \tag{67}$$

Solution of Eqs. (66) and (67) yields Eqs. (64) and (65) as results (if Coriolis compensation is assumed to be perfect). Thus, the error response for gyro drift is seen to be the same, to at least first order accuracy, for computation in either the geocentric inertial or local geographic coordinate frames. Equations (64) and (65) are plotted in Figure 10 for the case of equal drift rates for each gyroscope. The reader is referred to Ref. (9), for a more comprehensive treatment of the effects of gyro drift.

2. Level and Azimuth Errors for Constant Gyro Drift

In a space stabilized system, components of the system geocentric position vector can be computed in geocentric inertial axes. From this knowledge and a clock, latitude and longitude are computed. Thus the coordinate transformation relating the geographic frame to the inertial frame is available via the transformation of Eq. (48). It can be inferred that knowledge of the vertical is implicitly contained in this transformation. One is, in fact, tempted to associate directly the appropriate elements of the $\underline{\theta}^n$ matrix of Eq. (49) with the level and azimuth errors. The inadequacy of this association is illustrated by an example.

To fix ideas, let us say that the platform is being used as a reference to measure some physical quantity such as specific force, which would be the case for an airborne gravimeter application[8]. From Eq. (50), the computed specific force, coordinatized in the computed geographic frame, is given by

$$\underline{f}_c^{n'} = \underline{f}^n + \underline{C}_a^n \, \delta\underline{f}^a + [\underline{\theta}^n + \underline{C}_i^n \, \underline{Z}^i \, \underline{C}_n^i + \underline{C}_a^n \, (\underline{D}^a + \underline{A}^a) \, \underline{C}_n^a] \, \underline{f}^n \qquad (68)$$

Note that the primed sub and superscripts have been deleted on the right-hand side of this equation since second order error quantities are involved.

There is a direct association between the level and azimuth errors and the terms of the bracketed expression of Eq (68). It is seen from Eq (49) that the $\underline{\theta}^n$ matrix is a function of the latitude and longitude errors which are, in turn, a function of the error sources. Thus, to evaluate the bracketed term in Eq (68), it is necessary to write $\underline{\theta}^n$ as an explicit function of the error sources. As can be seen from inspection of Eqs (44), (52), or (54), the solution of the appropriate equations and their substitution into Eq (68) is not likely to be analytically tractable. Furthermore, the presence of the diagonal scale factor error matrix, $\underline{A}^a$, shows that the bracketed term is not a skew symmetric matrix.

For the purposes of exposition, suppose that the predominant error source is gyro drift, which is generally found to be the case for a properly designed inertial system. In this case, Eq (68) becomes:

$$\underline{f}_c^{n'} = [\underline{I} + \underline{\theta}^n + \underline{D}^n]\ \underline{f}^n = \begin{bmatrix} 1 & \Psi_z & -\Psi_E \\ -\Psi_z & 1 & \Psi_N \\ \Psi_E & -\Psi_N & 1 \end{bmatrix} \underline{f}^n \qquad (69)$$

The off diagonal elements of $[\underline{\theta}^n + \underline{D}^n]$ can be directly associated with the level and azimuth errors.

That is:

$$\Psi_N = \text{north level error} = +\ \delta\lambda \cos L + dx_n$$
$$\Psi_E = \text{east level error} = -\ \delta L + dy_n$$
$$\Psi_z = \text{azimuth error} = -\ \delta\lambda \sin L + dz_n$$

But from Eqs (62) and (63)

$$\Psi_N \simeq 0 \qquad (70a)$$
$$\Psi_E \simeq 0 \qquad (70b)$$
$$\Psi_z = +\ dx_n \tan L + dz_n \qquad (70c)$$

If one can again assume that the gyro input axes are initially aligned with geocentric inertial axes, the azimuth error is given by:

$$\Psi_z = -\sec L \, (\omega_x t \cos \omega_{ie} t + \omega_y t \sin \omega_{ie} t) \qquad (71)$$

Equation (71) is plotted with the latitude and longitude errors in Fig. 10.

It is seen, therefore, that the space stabilized system yields latitude, longitude, and azimuth errors which are roughly proportional to the physical platform misalignment. The level errors, however, are seen to be free of this long term growth. A more detailed analysis would show that the level errors oscillate about the vertical with the Schuler period in an analogous fashion to a physical platform system.

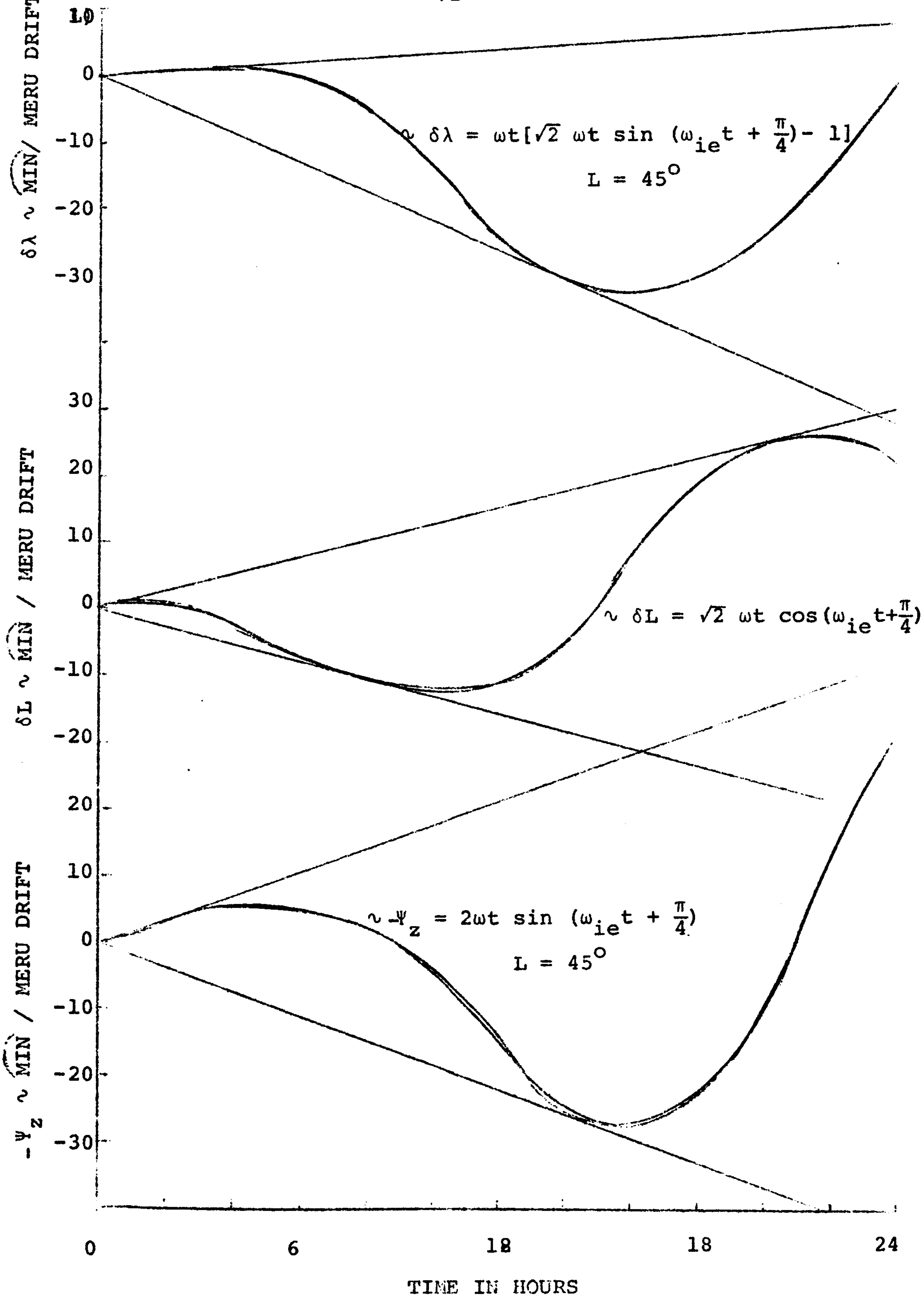


FIG. 10 GYRO DRIFT ERRORS

3. Altimeter Caused Errors

Comparison of Eq (54) with (44) reveals that the altimeter error response for a system computing in local geographic coordinates differs considerably from that of a system computing in geocentric inertial coordinates. Taking the case of a stationary system, it is seen that the geocentric inettial computational system is affected by altimeter uncertainty, while the local geographic computational system is not.

Treating the altitude uncertainty as the sole source of error, the differential equation describibg the error response for the system which computes in inertial axes is given by:

$$\underline{\delta \ddot{r}}^i + \omega_s^2 \, \underline{\delta r}^i = 3\omega_s^2 \, \delta h \, \underline{1}_r^i \qquad (72)$$

If we consider the case of altimeter bias for constant velocity motion at constant latitude as was done in Section VB, the solution to Eq (72) is given by Eq (25) as:

$$\underline{\delta r}^i \cong \begin{bmatrix} \cos L_g [\cos(\dot{\ell}_o + \omega_{ie})t - \cos \omega_s t] \\ \cos L_g [\sin(\dot{\ell}_o + \omega_{ie})t - \dfrac{(\dot{\ell}_o + \omega_{ie})}{\omega_s} \sin \omega_s t \\ \sin L_g (1 - \cos \omega_s t) \end{bmatrix} 3\delta h_o$$

where

$\dot{\ell}_o$ ∿ constant terrestrial longitude rate

δh_o ∿ constant altimeter error.

From Eqs. (44a) and (44c), the latitude and longitude errors are given by:

$$\delta L = \underline{q}_1^T \frac{\delta \underline{r}^n}{r}$$

and

$$\delta\lambda = \underline{q}_2^T \frac{\delta \underline{r}^n}{r \cos L}$$

Carrying out the required transformation, neglecting the higher order terms as usual, yields:

$$\delta L = -\frac{3}{2}\frac{\delta h_o}{r} \sin 2L\{[1-\cos(\dot{\ell}_o+\omega_{ie})t]\cos \omega_s t - \frac{\dot{\ell}_o+\omega_{ie}}{\omega_s}\sin(\dot{\ell}_o+\omega_{ie})t \sin \omega_s t\} \quad (73)$$

$$\delta\lambda = 3\frac{\delta h_o}{r}[\sin(\dot{\ell}_o+\omega_{ie})t \cos \omega_s t - \frac{\dot{\ell}_o+\omega_{ie}}{\omega_s}\cos(\dot{\ell}_o+\omega_{ie})t \sin \omega_s t] \quad (74)$$

The azimuth error is given by the z component of the $\underline{\theta}^n$ matrix of Eq. (49). Thus:

$$\psi_z = -\delta\lambda \sin L$$

$$= -3 \sin L\frac{\delta h_o}{r}[\sin(\dot{\ell}_o+\omega_{ie})t \cos \omega_s t - \frac{\dot{\ell}_o+\omega_{ie}}{\omega_s}\cos(\dot{\ell}_o+\omega_{ie})t \sin \omega_s t] \quad (75)$$

If we calculate the latitude error using Eq. (44b) which implies $r_c = r_{o_c} + h_a$, we get:

$$\delta L^* = 2\frac{\delta h_o}{r} \tan L(1 - \frac{3}{2} \cos \omega_s t) \quad (76)$$

Equations (73), (74), (75), and (76) are plotted in Figure 11.

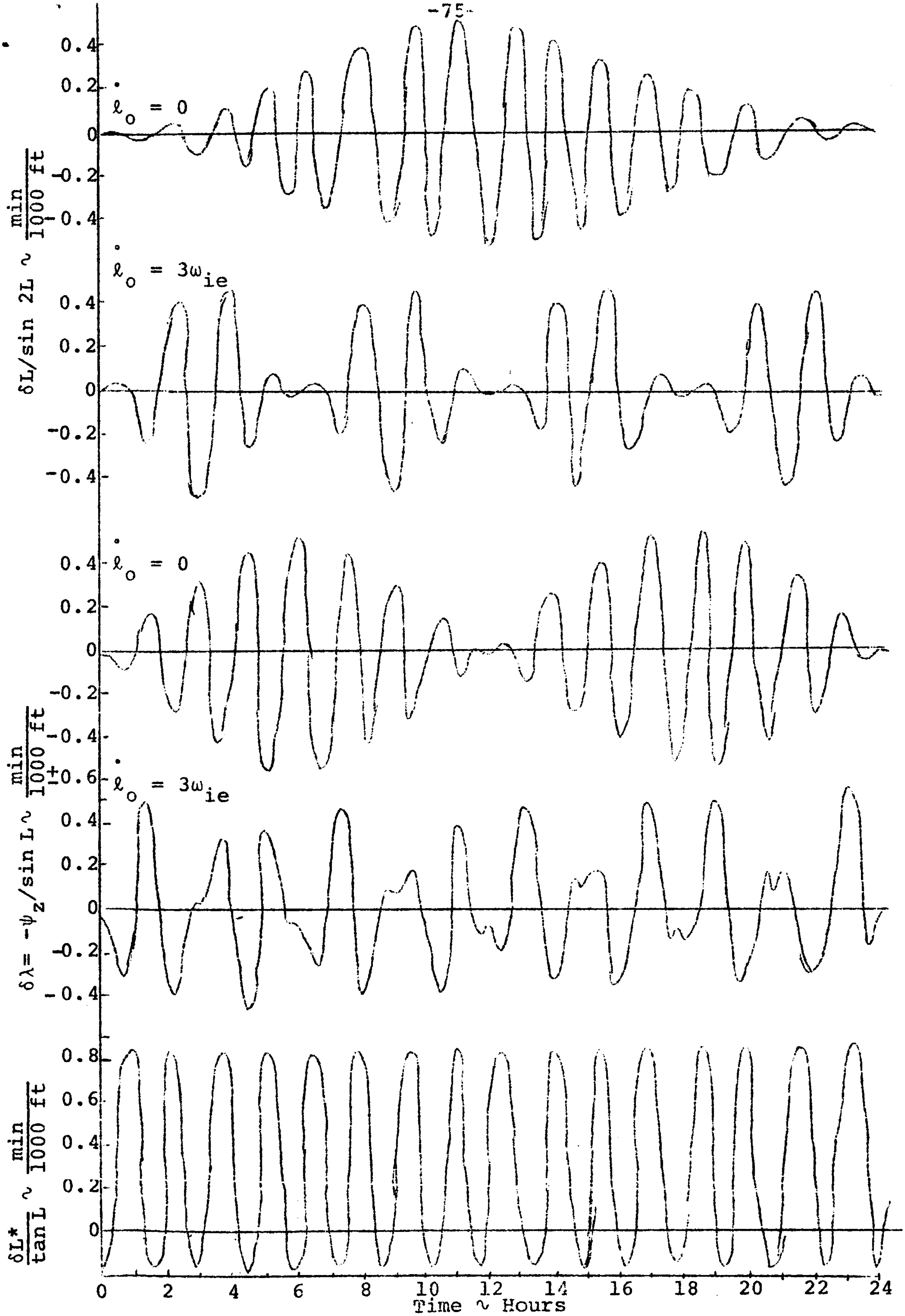


Figure 11 ∿ Altimeter-caused error - inertial computation

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