Analysis of Steady State Behavior of Second Order Sliding Mode Algorithms

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***Abstract*—An analysis of a second order sliding mode algorithm known as the super-twisting algorithm is carried out in the frequency domain with the use of the describing function method. It is shown that in the presence of an actuator, the transient process converges to a periodic motion. Parameters of this periodic motion are analyzed. A comparison between the periodic solutions in the systems with higher order sliding mode controllers and the oscillations that occur in classical sliding mode systems with actuators is done.**

I. INTRODUCTION

HIGHER order sliding modes (SM) have received a lot of attention from the control research community over the last decade (see bibliography in [1-11]. The main reasons for the use of the higher order sliding mode algorithms are: a higher accuracy of resulting motions; the possibility of using a continuous control law (super twisting or twisting as a filter); the possibility of utilizing the Coulomb friction in the control algorithm [7]; the finite time convergence for the systems with arbitrary relative degree [1].

It is known that the first order SM in systems with actuators of relative degree *two* or more is realized as chattering [10,11]. For the same reason, it would be logical to expect a similar behavior from a real second order SM, as the second order SM algorithms contain the sign function or the infinite gain. The modes that occur in a relay feedback system with the plant being the order 1, 2, 3, etc. dynamics were studied in publications [12,13]. It has been proven in those works that for the plant of order 3 and higher the point of the origin cannot be a stable equilibrium point. A similar behavior, therefore, can be expected from a system with a second order SM algorithm. Thus, the objective of this paper is to analyze the motions that occur in a system with the super-twisting algorithm, to show the existence of the periodic motions, to assess the parameters of those motions to be able to generate requirements to the actuator dynamics, and to compare those parameters with the parameters of chattering in the corresponding first order SM [14] algorithms.

Given the objective of the outlined analysis and the facts that the introduction of an actuator increases the order of the system, and at least two nonlinearities are present in a second order SM algorithm, the analysis of corresponding Poincare maps becomes very complicated. In this case the describing function (DF) method [15] seems to be a good choice as a method of analysis, as it provides a relatively simple and efficient solution of the problem.

The paper is organized as follows. At first the model of the system involving the super-twisting algorithm suitable for the frequency domain analysis is obtained. Then the DF model of the algorithm is obtained. After that it is shown that a periodic motion occurs and the problem of finding the parameters of this periodic motion is considered. Finally, a number of examples are considered and a comparison is done page limits.

II. SUPER-TWISTING ALGORITHM AND ITS DF ANALYSIS

The super-twisting algorithm is one of the popular algorithms among the second order sliding mode algorithms. It is used for the plants with relative degree one. Let the plant (or plant plus actuator) be given by the following differential equations:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where A and B are matrices of respective dimensions, y can be treated as either the sliding variable or the output of the plant. We shall also use the plant description in the form of a transfer function *W(s)*, which can be obtained from the formulas (1) as follows:

$$W(s) = C(Is - A)^{-1} B$$

The control u for the super-twisting algorithm is given as

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a sum of two components [8,9]:

$$u(t) = u_1(t) + u_2(t)$$
$$\dot{u}_1 = -\gamma \;\; sign(y) \qquad (2)$$
$$u_2 = \begin{cases} -\lambda \, |s_0|^{\rho} \, sign(y) & if \; |y| > s_0 \\ -\lambda \, |y|^{\rho} \, sign(y) & if \; |y| \le s_0 \end{cases}$$

where γ, ρ and s_0 are design parameters. In the formula for u_2, ρ is suggested to be within the range from *0* to *1*. The typical values would be *0.5* and *1*. With ρ being *1*, the second component of the control: $u_2(t)$ becomes a linear function of the output y at small departures ($|y| \le s_0$): $u_2 = -\lambda |y| \, sign \, y = -\lambda y$, and the system can be analyzed as a conventional relay system.

The system under analysis can be represented in the form of the block diagram as follows:

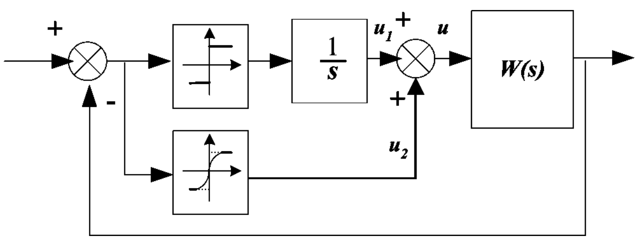

Fig.1. Block diagram of the system with the super-twisting algorithm

With ρ being *0.5* the function $u_2 = u_2\ (y)$ would be presented as shown in Fig. 2.

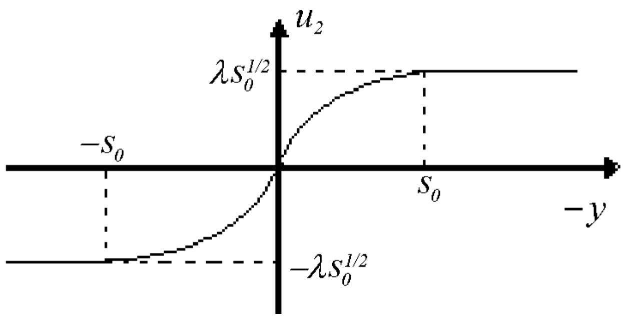

Fig. 2

Let us find the DF of the nonlinear function Fig. 2. By definition, a DF is given by the following formula:

$$N = \frac{1}{\pi A} \int_0^{2\pi} F(A \sin\psi) \sin\psi d\psi \qquad (3)$$

where $F(\psi)$ is the nonlinear function. With the square root nonlinearity ($\rho = 0.5$) the DF formula can be derived as:

$$N_2 = \frac{2}{\pi A} \int_0^{\pi} \sqrt{A sin\psi} \; sin\psi \, d\psi$$
$$= \frac{2\sqrt{A}}{A\sqrt{\pi}} \frac{\Gamma(1.25)}{\Gamma(1.75)} \approx \frac{1.1128}{\sqrt{A}} \qquad (4)$$

where A is the amplitude and $A \le s_0$ (that is considered the most important range of the amplitude values for the analysis of the steady state) and Γ is the gamma-function. With the nonlinearity given by Fig. 2 the DF formula can be derived as:

$$N_2 = 1.1128 \frac{\lambda \sqrt{s_0}}{\sqrt{A_y / s_0}} = 1.1128 \frac{\lambda}{\sqrt{A_y}} \qquad (5)$$
$$A_y \le s_0$$

where A_y is the amplitude of the variable y. For an arbitrary value of the power ρ in (2) and the amplitude $A_y \le s_0$, the formula of the DF of such nonlinear function can be given as follows:

$$N_2 = \frac{2\lambda A_y^{\;\rho-1}}{\pi} \int_0^{\pi} (sin\psi)^{\rho+1} \;\; d\psi, \quad 0 < \rho < 1$$

where the integral can be computed numerically.
The DF of the first component of the super-twisting algorithm can be written as follows:

$$N_1 = \frac{4\gamma}{\pi A_y} \frac{1}{s}$$

which is a result of the cascade connection of the ideal relay with the DF equal to $4\gamma/(\pi A_y)$ [15] and the integrator with the transfer function $1/s$. Taking into account both control components, the DF of the super-twisting algorithm can be written as:

$$N = N_1 + N_2 = \frac{4\gamma}{\pi A_y} \frac{1}{s} + 1.1128 \frac{\lambda}{\sqrt{A_y}} \qquad (6)$$

Let us note that the DF of the super-twisting algorithm depends on both: the amplitude and the frequency values. The parameters of the limit cycle can be found via the solution of the following complex equation [15]:

$$-\frac{1}{N(\Omega, A_y)} = W(j\Omega) \qquad (7)$$

where the DF N is given by (6). For solution of equation (7) the Laplace variable s in (6) can be replaced with $j\omega$. The function at the left-hand side of (7) can be represented by the following formula:

$$-\frac{1}{N} = -\frac{1}{1.1128\frac{\lambda}{\sqrt{A_y}} + \frac{4\gamma}{\pi A_y}\frac{1}{j\omega}} = -\frac{0.8986\frac{\sqrt{A_y}}{\lambda} + j1.1329\frac{\gamma}{\lambda^2}\frac{1}{\omega}}{1+1.3092\frac{\gamma^2}{\lambda^2}\frac{1}{A_y\omega^2}} \qquad (8)$$

The function $-1/N$ is a function of two variables: the amplitude and the frequency. It can be depicted as a number of plots representing the amplitude dependence, with each of those plots corresponding to a certain frequency. The frequency range that is of interest lies below the frequency corresponding to the intersection of the Nyquist plot and the real axis. The plots of function $-1/N$ are depicted in Fig. 3. The plots 1,2,3,4 correspond to four different frequencies, with the following relationship: $\omega_1>\omega_2>\omega_3>\omega_4$. Each of those plots represents the dependence of the DF on the amplitude value.

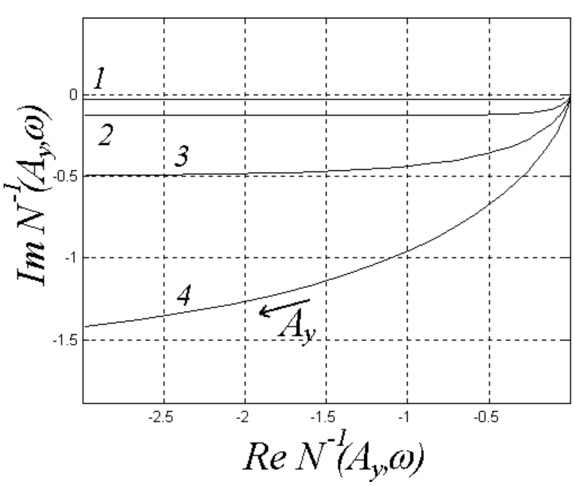

Fig.3. Plots of function $-1/N$

Function $-N^{-1}(A_y)$ (where ω=*const*) has two asymptotes. One of them is the imaginary axis, and the other one is the horizontal line $-j1.1329\,\gamma/(\lambda^2\omega)$. Considering the location of the plots of function $-N^{-1}(A_y)$ (including the asymptotic behavior) the following conclusion regarding the possibility of the periodic solution to take place can be made. The Nyquist plot of any system with relative degree higher than *one* will have a point of the intersection with the plot $-N^{-1}(A_y)$ because the former is located in the third quadrant of the complex plane and has a high-frequency asymptote coinciding with the real axis (for relative degree *two*) or goes through the third quadrant (for relative degrees higher than *two*). It is important that the point of the intersection is located in the third quadrant of the complex plane. Therefore, *if the transfer function of the plant (or plant plus actuator) has relative degree higher than one a periodic motion may occur in such a system*. For that reason, *if an actuator of first or higher order is added to the plant with relative degree one driven by the super-twisting controller a periodic motion may occur in the system*. From Fig. 3, it also follows that *the frequency of the periodic solution for the super-twisting algorithm is always lower than the frequency of the periodic motion in the system with the classical first order SM relay controller*, because the latter is determined by the point of the intersection of the Nyquist plot and the real axis.

The solution of equation (7) can be iterative with possible application of various techniques. However, complex equation (7) with two unknown variables: A_y and Ω can be reduced to one real equation having only one unknown variable Ω:

$$\Psi(\Omega) = \frac{4\gamma}{\pi\Omega}\frac{1}{Im W^{-1}(j\Omega)} - \left(\frac{1.1128\,\lambda}{Re W^{-1}(j\Omega)}\right)^2 = 0 \qquad (9)$$

Once equation (9) has been solved the amplitude A_y can be computed as follows:

$$A_y = \frac{4\gamma}{\pi\Omega}\frac{1}{Im W^{-1}(j\Omega)} \qquad (10)$$

Therefore, if a periodic motion occurs its parameters can be found from (9) and (10).

III. Examples of Analysis and Comparison of Results

Example 1. Let the plant be given by the following equations:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 + u_a$$
$$y = x_1 + x_2$$

and the actuator by: $0.01\dot{u}_a + u_a = u$. Carry out analysis of periodic motions in the systems with the super-twisting controller if the parameters of the algorithm are given as: $\rho=0.5$, $\gamma=0.8$, $\lambda=0.6$. The transfer function $W(s)$ of the actuator-plant can be derived from the original equations as:

$$W(s) = \frac{1}{0.01\,s+1}\;\frac{s+1}{s^2+s+1} \tag{11}$$

Equation (9) has a solution: $\Omega=66.16s^{-1}$. The amplitude A_y can be computed with the use of formula (10): $A_y=2.33\cdot10^{-4}$. The graphical illustration of the application of formula (7) to the analysis of the periodic motion is presented in Fig. 4. The plot $-N^{-1}(A_y)$ is drawn for the frequency of the periodic motion $\omega=66.16s^{-1}$ obtained above.

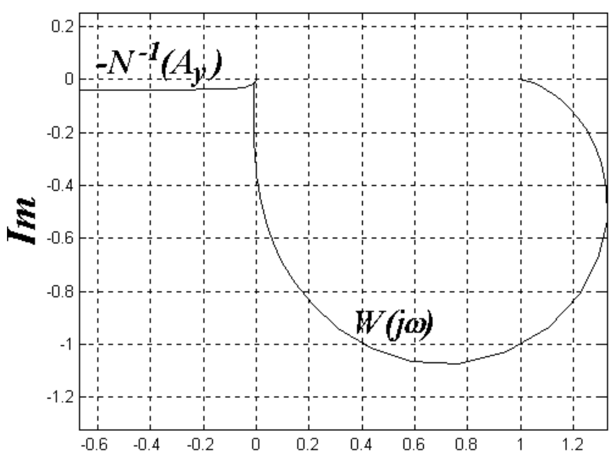

Fig.4. Negative reciprocal of DF $-N^{-1}(A_y)$ and the Nyquist plot $W(j\omega)$

Figure 4 provides a picture of the DF and the Nyquist plot location on the complex plane. The point of the intersection, however, cannot be seen from this picture, which is the result of the small value of the actuator time constant. A zoomed picture of the same plots in the vicinity of the intersection point is presented in Fig. 5.

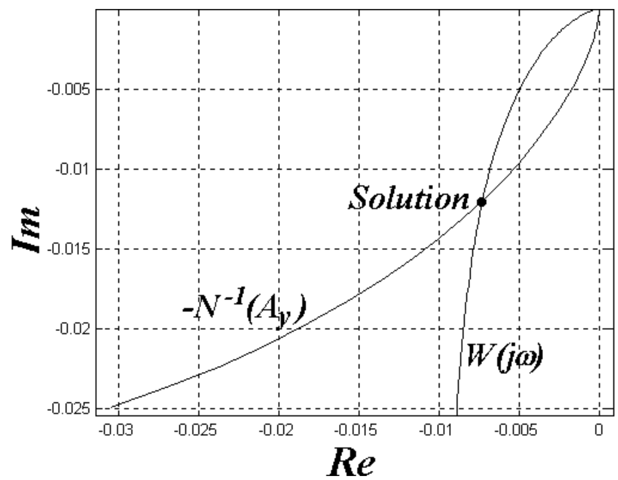

Fig.5. Negative reciprocal of DF $-N^{-1}(A_y)$ and the Nyquist plot $W(j\omega)$ (zoomed)

It is clearly seen in the Fig. 5 that the point of the intersection exists and the asymptotic behavior of the functions is in accordance with the above analysis.
The simulations of the original equations of the system with the super-twisting algorithm produce the following trajectory (Fig.6).

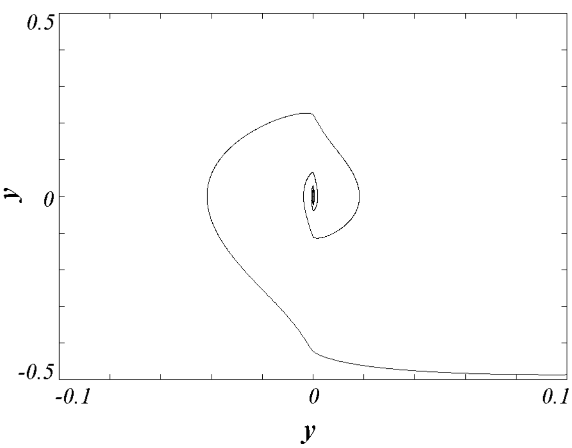

Fig. 6. Super-twisting trajectory

Figure 6 displays the trajectory typical of the super-twisting algorithm. The periodicity of the steady state motion is clearly seen in Fig. 7 where the control is presented as a function of time.

The frequency of the periodic motion obtained as a result of the simulation is $\Omega_{sim}=64.96s^{-1}$. One can see that the simulation result matches very well to the result of the DF analysis.

Some other examples of analysis are presented in Table 1. The actuator transfer function is denoted as $W_a(s)$, the plant transfer function as $W_p(s)$, and the transfer function from the plant input to the sliding variable is denoted as

$W_\sigma(s)$. As a result, the transfer function of the linear part $W(s)$ is the product of $W_a(s)$ and $W_\sigma(s)$.

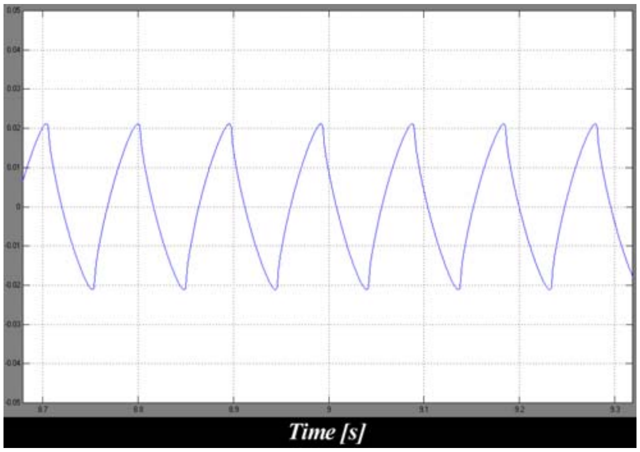

Fig. 7. Control *u(t)* at $t \rightarrow \infty$

One can see that the results of the DF analysis very well match to the results of the simulations. Also, the following properties are observed. A periodic motion occurs if the combined relative degree of the actuator and of the plant is higher than *one*. The frequency of the periodic motion in a system driven by the super-twisting controller is lower than the frequency of the periodic motion in the classical first order relay control – the fact that was predicted by the above analysis. The amplitudes of the chattering reflect the relationship between the frequency of the periodic motion and the decreasing character of the amplitude frequency response $|W(j\omega)|$ of the actuator-plant.

IV. CONCLUSIONS

A second order SM algorithm known as super-twisting is analyzed with the use of the describing function method. It is shown that if the combined relative degree of the actuator and the plant is higher than *one* a periodic motion may occur in the system with the super-twisting algorithm. An algorithm of finding the parameters of this periodic motion is presented. The performed analysis shows that the frequency of the periodic motion in the system with the super-twisting algorithm is always lower and the amplitude is higher than the respective parameters of the periodic motion in the first order SM system having the same plant. A few examples of analysis are provided.

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TABLE I
EXAMPLES OF ANALYSIS AND SIMULATIONS

	Super-twisting controller (ρ=0.5)	Super-twisting controller (ρ=0.5)	First order SM controller	First order SM controller
Plant $W_p(s)$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$	$W_p(s)=\frac{1}{s^2+s+1}$
Actuator $W_a(s)$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$	$W_a(s)=\frac{1}{0.01s+1}$	$W_a(s)=\frac{1}{0.0001s^2+0.01s+1}$
$W_\sigma(s)$	$W_\sigma(s)=\frac{s+1}{s^2+s+1}$	$W_\sigma(s)=\frac{s+1}{s^2+s+1}$	$W_\sigma(s)=\frac{s+1}{s^2+s+1}$	$W_\sigma(s)=\frac{s+1}{s^2+s+1}$
$W(s)$	$W=W_\sigma W_a$	$W=W_\sigma W_a$	$W=W_\sigma W_a$	$W=W_\sigma W_a$
Ω (DF analysis)	66.16	55.18	Infinite	100.00
Ω (simulations)	64.96	54.14	Converging to infinity	99.26
Amplitudes of chattering of plant output	2.33e-4	4.81e-4	0	1.30e-4