Analysis of the Discrete-Time $G^{(G)}/Geom/c$ Queueing Model

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Abstract. This paper presents the steady-state analysis of a discrete-time infinite-capacity multiserver queue with c servers and independent geometrically distributed service times. The arrival process is a batch renewal process, characterized by general independent batch interarrival times and general independent batch sizes. The analysis has been carried out by means of an analytical technique based on generating functions, complex analysis and contour integration. Expressions for the generating functions of the system contents during an arrival slot as well as during an arbitrary slot have been obtained. Also, the delay in case of a first-come-first-served queueing discipline has been analyzed.

1 Introduction

The analysis of discrete-time queueing models has received considerable attention in the scientific literature over the past years in view of its applicability in the study of many computer and communication systems in which time is slotted, see e.g. [1,2,3] and the references therein. In most of the existing studies of discrete-time multiserver queueing models, however, the service times of customers are assumed to be constant, equal to one slot (see [4,5,6]) or multiple slots ([7]). On the other hand, very little seems to have been done on discrete-time multiserver queues with random service times. A multiserver queueing system with geometric service times and a general independent arrival process is analyzed in [8]. Geometric service times are also considered in [9,10,11], while [12] and [13] deal with general service times, but only for the single-server case.

In this paper, we present an analytical technique for the analysis of discrete-time multiserver queues with geometric service times and a batch renewal arrival process. This process is characterized by a sequence of independent and identically distributed (i.i.d.) batch interarrival times and a sequence of i.i.d. batch sizes, and can be used to model both first- and second-order correlation characteristics of a traffic stream ([14]). As far as know to the authors, an analysis of the considered queueing model has never been reported on in the literature.

The remainder of the paper is organized as follows. The assumptions of the queueing model under study and some basic terminology are given in Sect. 2.

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Next, the analysis of the queueing model is carried out and expressions are derived for the generating functions of the system contents during an arrival slot (Sect. 3), the system contents during an arbitrary slot (Sect. 4) and the waiting time and delay in case of a first-come-first-served queueing discipline (Sect. 5).

2 Model Description

In this paper, we consider a discrete-time buffer system with an infinite waiting room for customers and c servers. It is assumed the system has a clock such that time is divided in fixed-length *slots* s_j ($j = 1, 2, \ldots$), chronologically indexed, and separated by *slot boundaries* t_j, where $s_j \triangleq [t_j, t_{j+1})$.

Arrivals to the system occur solely on slot boundaries. Slot boundaries with arrivals are called *arrival instants*; the slot following an arrival instant is called an *arrival slot*. We denote the kth arrival instant by τ_k and the arrival slot following τ_k by I_k ($k = 1, 2, \ldots$). The time interval (expressed in slots) between two successive arrival instants is referred to as the *interarrival time* and is denoted by the symbol A. Specifically, A_k stands for the interarrival time starting at τ_k. The interarrival times are assumed to be independent and identically distributed (i.i.d.) random variables with common probability mass function (PMF) $a(n) \triangleq \text{Prob}[A_k = n]$ ($n = 1, 2, \ldots$), and probability generating function (PGF) $A(z)$.

At a given arrival instant, several customers may enter the system (batch arrivals). In particular, the random variable for the number of customers arriving at τ_k is denoted by B_k and is called the *batch size* for the arrival instant τ_k. The batch sizes are assumed to be i.i.d. random variables. Their common PGF is denoted by $B(z)$. All A_k's and B_k's are also mutually statistically independent.

Customers are queued for service according to a first-come-first-served discipline (FCFS), and receive service from any of the c servers. Hereby the order in which simultaneously arriving customers are queued for service is irrelevant for the analysis. Service can start solely at slot boundaries and always takes a positive integer number of full slots. A customer can be taken in service as soon as he arrives, provided, of course, there is a server available. After service completion customers leave the system immediately. Hence, the departures from the system also occur solely at slot boundaries. Note that the assumption of a FCFS queueing discipline has no influence on the distribution of the buffer contents.

The number of slots it takes to serve the lth customer is called the lth *service time*, and is denoted by D_l. The service times are assumed to be i.i.d. and geometrically distributed with parameter $1 - \sigma$ ($0 < \sigma \leq 1$), i.e., with PGF

$$D(z) = \frac{\sigma z}{1 - (1 - \sigma)z} \quad . \tag{1}$$

3 System Contents during an Arrival Slot

During any slot the number of customers in the system, referred to as the *system contents*, remains constant and therefore is well-defined. For any arrival slot I_k, let U_k be the system contents during I_k. Due to the memoryless nature of the service-time distribution, the random variables $\{U_k \mid k = 1, 2, \ldots\}$ form a Markov chain. Let $U_k(z)$ be the PGF of U_k. Under the assumption that the buffer system, on the average, receives less work than it can handle, i.e., $B'(1) < c\sigma A'(1)$, the system will, for large k, tend towards an equilibrium, where all U_k's have a common distribution with PGF $U(z)$.

Now, let us consider an arbitrary pair of two consecutive arrival slots I_k and I_{k+1}, and the time interval in between them. We define the random variable V_p $(p = 0, 1, \ldots, A_k)$ as the number of customers present in the system in slot I_k and still present in the system during the slot $s_p^* = [t_p^*, t_{p+1}^*)$, i.e., the pth slot after I_k (see Fig. 1). For $p < A_k$, V_p represents the actual system contents during s_p^*, while for $p = A_k$, we have $s_{A_k}^* = I_{k+1}$ and the system contents in this slot is

$$U_{k+1} = V_{A_k} + B_{k+1} \ . \tag{2}$$

The PMF of the random variable V_p is denoted by $v_p(n) \triangleq \mathrm{Prob}[V_p = n]$ $(n = 0, 1, \ldots)$, and its PGF by $V_p(z)$. It is clear that the distribution of V_p depends only on the distribution of $V_0 = U_k$ and the distribution of the total number of departures at the slot boundaries in the interval $]\tau_k, t_p^*]$. Between the random variables related to consecutive slots the following relation holds :

$$V_p = V_{p-1} - R_p \ , \quad p = 1, 2, \ldots, A_k \ , \tag{3}$$

where the random variable R_p indicates the number of departures at t_p^* (see Fig. 1). Since the service times are geometrically distributed with parameter $(1-\sigma)$, the number of departures at t_p^* $(p = 1, 2, \ldots, A_k)$ has a binomial distribution with parameters $r_p = \min(c, V_{p-1})$ and σ, r_p being the number of servers that are occupied during the preceding slot s_{p-1}^*. In terms of the conditional PGF

$$R(z \mid n) \triangleq \sum_{m=0}^{\min(c,n)} \mathrm{Prob}[R_p = m \mid V_{p-1} = n]\ z^m \ , \tag{4}$$

we have that

$$R(z \mid n) = \begin{cases} (1-\sigma+\sigma z)^n \triangleq [M(z)]^n \ , & 0 \le n \le c-1 \ ; \\ (1-\sigma+\sigma z)^c \triangleq [M(z)]^c \ , & c \le n \ . \end{cases} \tag{5}$$

Equations (3)–(5) then yield

$$\begin{aligned} V_p(z) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\min(c,n)} \mathrm{Prob}[V_{p-1} = n]\ \mathrm{Prob}[R_p = m \mid V_{p-1} = n]\ z^{n-m} \\ &= [M(1/z)]^c\, V_{p-1}(z) + \sum_{n=0}^{c-1} v_{p-1}(n)\, F_n(z) \ , \quad p = 1, 2, \ldots, A_k \ , \end{aligned} \tag{6}$$

where $\{F_n(z) \mid n = 0, 1, \ldots, c-1\}$ is a family of rational functions in z :

$$F_n(z) \triangleq z^n \left([M(1/z)]^n - [M(1/z)]^c\right) \ , \quad n = 0, 1, \ldots, c-1 \ . \tag{7}$$

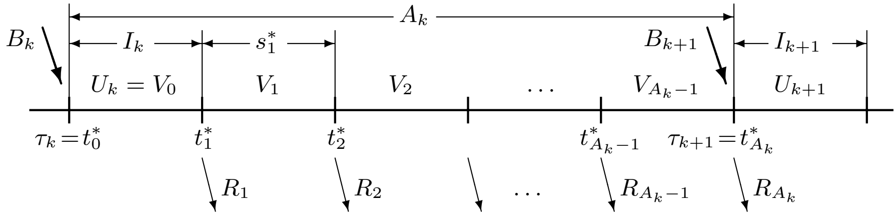

Fig. 1. System contents during an arrival slot

Next, applying (6) repeatedly, and taking into account that $V_0 = U_k$, or equivalently $V_0(z) = U_k(z)$, we get that

$$V_p(z) = [M(1/z)]^{p\,c}\, U_k(z) + \sum_{n=0}^{c-1} F_n(z) \sum_{m=0}^{p-1} v_m(n)\, [M(1/z)]^{(p-m-1)\,c} \ . \tag{8}$$

In view of (2), and since B_{k+1} is independent of V_{A_k}, we have

$$U_{k+1}(z\,|\,p) \triangleq E\big[z^{U_{k+1}}\,|\,A_k = p\big] = B(z)\, V_p(z) \ . \tag{9}$$

From (8)–(9), the unconditional PGF $U_{k+1}(z)$ of U_{k+1} is then derived as

$$\begin{aligned} U_{k+1}(z) &= B(z)\, A([M(1/z)]^c)\ U_k(z) \\ &\quad + B(z) \sum_{n=0}^{c-1} F_n(z) \sum_{p=1}^{\infty} a(p) \sum_{m=0}^{p-1} v_m(n)\, [M(1/z)]^{(p-m-1)\,c} \ . \end{aligned} \tag{10}$$

Under the assumption of equilibrium, both $U_k(z)$ and $U_{k+1}(z)$ can be substituted by $U(z)$. Solving for $U(z)$, we obtain

$$U(z) = \frac{B(z)}{1 - B(z)\, A([M(1/z)]^c)} \sum_{n=0}^{c-1} F_n(z)\, J_n(z) \ , \tag{11}$$

where the functions $J_n(z)$ are defined as

$$J_n(z) \triangleq \sum_{p=1}^{\infty} a(p)\, [M(1/z)]^{(p-1)\,c} \sum_{m=0}^{p-1} v_m(n)\, [M(1/z)]^{-m\,c} \ . \tag{12}$$

The right-hand side of (11) contains, through the functions $J_n(z)$, an infinite number of unknown coefficients $\{v_m(n)\,|\,m = 0, 1, \ldots ; n = 0, 1, \ldots , c{-}1\}$. Based on these probabilities, we define a family of c functions by their Taylor series expansions around $z = 0$:

$$H_n(z) \triangleq \sum_{m=0}^{\infty} v_m(n)\, z^m \ , \quad n = 0, 1, \ldots , c-1 \ . \tag{13}$$

These series converge at least for all z with $|z| < 1$, since the sum of all of the coefficients are bounded by 1. Since $\sigma > 0$, all customers present in the buffer during I_k will eventually leave the system, so that $\lim_{m\to\infty} v_m(n) = \delta(n)$, where

$$\delta(n) \triangleq \begin{cases} 0 \ , & \text{if } n \neq 0 \ ; \\ 1 \ , & \text{if } n = 0 \ . \end{cases} \tag{14}$$

Hence, we can expect the series $H_n(z)$, for all $n \in \{1, 2, \ldots, c-1\}$, to converge in a region that contains the unit disk, including its edge. On the other hand, $H_0(z)$ apparently has a pole in $z = 1$. The coefficient $v_m(n)$ can be written as the residue of the complex function $H_n(\zeta)\,\zeta^{-m-1}$ at $\zeta = 0$, so that

$$v_m(n) = \frac{1}{2\pi i} \oint_L \frac{H_n(\zeta)}{\zeta^{m+1}}\, d\zeta \ , \tag{15}$$

where i indicates the imaginary unit and L, for the time being, is an arbitrary closed contour around the origin $\zeta = 0$ in the complex ζ-plane, but not around any other singularity of $H_n(\zeta)$. After substitution in (12), under the assumption that a proper choice is made for the contour L and for proper z, the summations over p and m on the right-hand side of (12) can be brought behind the integration operator and summed. As a result, we get the following expression for $J_n(z)$:

$$J_n(z) = \frac{1}{2\pi i} \oint_L \frac{H_n(\zeta)}{\zeta\,[M(1/z)]^c - 1} \left[A([M(1/z)]^c) - A(1/\zeta)\right] d\zeta \ , \tag{16}$$

for all z such that $|M(1/z)|^c < \mathcal{R}_A$, where the notation $\mathcal{R}_X$ stands for the radius of convergence of the Taylor series expansion of the function $X(z)$ around the origin $z = 0$. This range for z contains at least the entire complex z-plane outside the unit circle and the unit circle itself. For ζ, on the other hand, a "proper" choice for the contour L means that

$$(\forall \zeta \in L)\,(1/\,|\zeta| < \mathcal{R}_A \text{ and } |\zeta| < \mathcal{R}_{H_n}) \ . \tag{17}$$

For all $n \in \{0, 1, \ldots, c-1\}$, we can choose for L an arbitrary circle with center in $\zeta = 0$ and radius a, where $1/\mathcal{R}_A < a < 1$, i.e., a circle smaller than the unit circle but still around all the singularities of $A(1/\zeta)$.

In view of Cauchy's residue theorem ([15]), $J_n(z)$ can be obtained as a sum of integrals over small contours around the singularities of the integrand inside the contour L. These singularities are given by the set $\mathcal{S}_A^{-1}$, i.e., the set of singular points of the function $A(1/\zeta)$. They all have a modulus smaller than $1/\mathcal{R}_A$ and therefore lie within the contour L, so that

$$J_n(z) = \sum_{\alpha \in \mathcal{S}_A^{-1}} \frac{1}{2\pi i} \oint_{L_\alpha} \frac{H_n(\zeta)}{1 - \zeta\,[M(1/z)]^c} A(1/\zeta)\, d\zeta \ . \tag{18}$$

Note that there is no contribution of the term with $A([M(1/z]^c)$ because it remains regular for all $\zeta \in \mathcal{S}_A^{-1}$. In (18), L_α is a "sufficiently" small contour around $\alpha \in \mathcal{S}_A^{-1}$, i.e., inside the circle $C(0, 1/\mathcal{R}_A)$ and not around any other singularity of $A(1/\zeta)$. Because in (16), z solely occurs under the form $[M(1/z)]^c$, it is expedient to make a change of variable towards

$$u = u(z) \triangleq [M(1/z)]^{-1} \equiv D^{-1}(z) \Leftrightarrow z = D(u) \ , \tag{19}$$

so that we get for $J_n(z)$:

$$J_n(z) \triangleq \tilde{J}_n(u^c) = u^c \sum_{\alpha \in \mathcal{S}_A^{-1}} \tilde{J}_{n,\alpha}(u^c) \ , \tag{20}$$

where

$$\tilde{J}_{n,\alpha}(x) \triangleq \frac{1}{2\pi i} \oint_{L_\alpha} \frac{H_n(\zeta)}{x - \zeta} A(1/\zeta)\, d\zeta \ . \tag{21}$$

For any x outside L_α, the contribution $\tilde{J}_{n,\alpha}(x)$ for any $\alpha \in \mathcal{S}_A^{-1}$, is the integral along a *finite* contour of an integrand that remains regular along that contour, and therefore is a well-defined complex number. Further, the integrand is an analytical function of x for all x outside of L_α, so that also $\tilde{J}_{n,\alpha}(x)$ is analytical outside of L_α. Finally, since L_α can be chosen arbitrarily small, $\tilde{J}_{n,\alpha}(x)$ is a regular analytical function for all $x \neq \alpha$. At $x = \alpha$, however, $\tilde{J}_{n,\alpha}(x)$ has a singularity, exactly of the same type as for $A(1/\zeta)$ at $\zeta = \alpha$. Also, $\lim_{x\to\infty} \tilde{J}_{n,\alpha}(x) = 0$, so that the Laurent series expansion around $x = \alpha$ for $\tilde{J}_{n,\alpha}(x)$ does not contain terms with positive powers of $(x - \alpha)$. For an essential singularity, little more can be said because the functions $H_n(\zeta)$ are unknown. However, if $\alpha \in \mathcal{S}_A^{-1}$ is a pole with multiplicity m, $\tilde{J}_{n,\alpha}(x)$ is of the following form :

$$\tilde{J}_{n,\alpha}(x) = \frac{\Lambda_{-m}}{(x-\alpha)^m} + \frac{\Lambda_{-m+1}}{(x-\alpha)^{m-1}} + \ldots + \frac{\Lambda_{-1}}{x-\alpha} \ , \tag{22}$$

where

$$\Lambda_{-k} = \frac{1}{(m-k)!} \lim_{\zeta\to\alpha} \frac{\mathrm{d}^{m-k}}{\mathrm{d}\zeta^{m-k}} \left[H_n(\zeta)\, A(1/\zeta)\, (\zeta-\alpha)^m\right] \ , \ \ k = 1, 2, \ldots, m \ . \tag{23}$$

The Λ_{-k}'s still depend on $H_n(\alpha), H_n'(\alpha), \ldots, H_n^{(m-k)}(\alpha)$ and therefore are to be considered as unknown quantities. However, further analysis will show that it is not necessary to determine these coefficients.

From this point on we assume that $A(z)$ is a rational function. When $A(z)$ is rational, all the singularities of $A(1/z)$ are poles and all of them give contributions of the form of (22). Let us write

$$A(1/z) \triangleq \frac{P_A(z)}{Q_A(z)} \ ; \tag{24}$$

$$Q_A(z) \triangleq \prod_{\alpha \in \mathcal{S}_A^{-1}} (z-\alpha)^{m_\alpha} \ , \tag{25}$$

where m_α indicates the multiplicity of $\alpha \in \mathcal{S}_A^{-1}$ and $P_A(z)$ is a polynomial function with $\deg P_A < \deg Q_A$, since $\lim_{z\to\infty} A(1/z) = 0$. Summing over the contributions of all poles, we can easily see that

$$\sum_{\alpha\in\mathcal{S}_A^{-1}} \tilde{J}_{n,\alpha}(x) = \frac{P_{U,n}(x)}{Q_A(x)} \ , \quad n = 0, 1, \ldots, c-1 \ , \tag{26}$$

with $P_{U,n}(x)$ a yet unknown polynomial function of degree $\deg P_{U,n} = \deg Q_A - 1$. Hence, (20) becomes

$$J_n(z) = \tilde{J}_n(u^c) = u^c \, \frac{P_{U,n}(u^c)}{Q_A(u^c)} \ . \tag{27}$$

Next, from (11) and (27), in view of

$$F_n(z) \triangleq \tilde{F}_n(u) = [D(u)]^n \left(u^{-n} - u^{-c}\right) \ , \quad n = 0, 1, \ldots, c-1 \ , \tag{28}$$

we readily get for $U(z)$:

$$U(z) = \tilde{U}(u) = \frac{B(D(u))\,\tilde{P}_U(u)}{[1-(1-\sigma)u]^{c-1}\,[Q_A(u^c) - B(D(u))\,P_A(u^c)]} \ , \tag{29}$$

where

$$\tilde{P}_U(u) \triangleq \sum_{n=0}^{c-1} \sigma^n \,(u^c - u^n)\,[1-(1-\sigma)u]^{c-n-1}\,P_{U,n}(u^c) \ , \tag{30}$$

which is the sole unknown function on the right-hand side of (29). $\tilde{P}_U(u)$ is a polynomial function in u of degree $(c\deg Q_A + c - 1)$, though, the coefficients have only $c\deg Q_A$ degrees of freedom, so that $c\deg Q_A$ linearly independent conditions on the coefficients suffice to determine the polynomial $\tilde{P}_U(u)$ completely. These conditions are obtained by invoking the analyticity of the PGF $U(z)$ in the unit disk, i.e., the analyticity of $\tilde{U}(u)$ in the image of the unit disk under the transformation $z \to u = D^{-1}(z)$ and more particularily in the zeros of the denominator of (29) in that area. An application of Rouché's theorem ([16]) shows that the factor

$$Q_U(z) \triangleq Q_A([M(1/z)]^{-c}) - B(z)\,P_A([M(1/z)]^{-c}) \tag{31}$$

has as many zeros within the unit disk as $Q_A([M(1/z)]^{-c})$, counting multiple zeros several times. This number of zeros is precisely $c\deg Q_A$. One of these zeros of $Q_U(z)$ is 1, all other zeros lie strictly within the unit circle. Therefore, also

$$\tilde{Q}_U(u) = Q_A(u^c) - B(D(u))\,P_A(u^c) \tag{32}$$

has $c\deg Q_A$ zeros within the image of the unit disk under the mapping $z \to D^{-1}(z)$. One of these zeros is 1, where $\tilde{P}_U(u)$ vanishes regardless the coefficients of the polynomials $P_{U,n}$. Let us denote the set of the $c\deg Q_A - 1$ other zeros by $Q_U = \{u : \tilde{Q}_U(u) = 0 \text{ and } |D(u)| < 1\}$. In these points, $\tilde{U}(u)$ must remain regular, and therefore they have to be zeros of $\tilde{P}_U(u)$ as well, with at least the same multiplicity, so that $c \deg Q_A - 1$ linear equations in the $c \deg Q_A$ unknown coefficients are obtained. An additional equation follows from the normalization condition $\tilde{U}(1) = 1$. Hence, we have a set of $c \deg Q_A$ linearly independent conditions on the $c \deg Q_A$ unknown coefficients, so that the polynomial $\tilde{P}_U(u)$ can be determined completely. Note that $\tilde{P}_U(u)$ will be of the form

$$\tilde{P}_U(u) = \Psi(u) \prod_{\beta \in Q_U} (u - \beta)^{m_\beta} \quad , \tag{33}$$

where $\Psi(u)$ is a polynomial of degree c, and m_β indicates the multiplicity of the zero β of $\tilde{Q}_U(u)$ (and $\tilde{P}_U(u)$). Thus finally, $U(z)$ is completely expressed in terms of known quantities only, i.e., the PGFs $A(z)$ and $B(z)$, the parameter σ, the $\deg Q_A$ poles of the function $A(1/z)$, and the $c \deg Q_A - 1$ solutions of $Q_U(z) = 0$, or, equivalently, of the characteristic equation $1 = B(z)\, A([M(1/z)]^c)$ strictly inside the unit disk.

4 System Contents during an Arbitrary Slot

Under the assumption that the buffer system has reached a stochastic equilibrium, we now consider an arbitrary slot s and we let N denote the system contents during slot s (see Fig. 2). The start of s is indicated by t. Also, let τ be the preceding arrival instant (if t is an arrival instant, let $\tau = t$), let I be the arrival slot starting at τ and let U be the system contents during I.

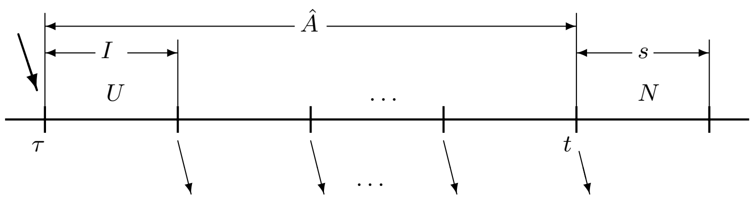

Fig. 2. System contents during an arbitrary slot

It is clear that the distribution of N depends only on the distribution of U and the distribution of the total number of departures at the slot boundaries in the interval $]\tau, t]$. Let the random variable $\hat{A}$, with PMF $\hat{a}(n)$ $(n = 0, 1, \dots)$ and PGF $\hat{A}(z)$, be the number of slots between τ and t, then $\hat{A}(z)$ is given by (see e.g. [2]) :

$$\hat{A}(z) = \frac{A(z) - 1}{A'(1)\,(z - 1)} \quad , \tag{34}$$

which implies that the singularities of $\hat{A}(z)$ and $A(z)$ coincide. By applying the method introduced in Sect. 3, we can derive the PGF $N(z)$ of N in terms of the PGF $U(z)$ of U as

$$N(z) = U(z)\,\hat{A}([M(1/z)]^c) + \sum_{n=0}^{c-1} F_n(z)\,K_n(z) \ , \tag{35}$$

where the functions $K_n(z)$ are defined as

$$K_n(z) \triangleq \sum_{p=0}^{\infty} \hat{a}(p) \sum_{m=0}^{p-1} v_m(n)\,[M(1/z)]^{(p-m-1)\,c} \ , \quad n = 0, 1, \ldots, c-1 \ . \tag{36}$$

Again, we now assume that $A(z)$ is rational. In this case, $\hat{A}(z)$ is a rational function as well. Also $A(1/z)$ and $\hat{A}(1/z)$ have exactly the same set of poles, i.e.,

$$\hat{A}(1/z) = \frac{P_{\hat{A}}(z)}{Q_A(z)} \ , \tag{37}$$

where $Q_A(z)$ is given by (25), and $P_{\hat{A}}(z)$ is a polynomial function with $\deg P_{\hat{A}} = \deg Q_A$, since $\lim_{z\to\infty} \hat{A}(1/z) = 1/A'(1)$. In a similar way as explained for the functions $J_n(z)$ in the previous section, it can be shown that the functions $K_n(z)$ are of the following form :

$$K_n(z) = \tilde{K}_n(u^c) = u^c\,\frac{P_{N,n}(u^c)}{Q_A(u^c)} \ , \tag{38}$$

where the $P_{N,n}(x)$ are unknown polynomials in x of degree $\deg P_{N,n} = \deg Q_A - 1$, and again we have made a change of variable according to (19). Finally, combining (28), (35) and (38), we get the PGF $N(z)$ as

$$N(z) = \tilde{N}(u) = \tilde{U}(u)\,\frac{P_{\hat{A}}(u^c)}{Q_A(u^c)} + \frac{\tilde{P}_N(u)}{[1-(1-\sigma)u]^{c-1}\,Q_A(u^c)} \ , \tag{39}$$

where

$$\tilde{P}_N(u) \triangleq \sum_{n=0}^{c-1} \sigma^n\,(u^c - u^n)\,[1-(1-\sigma)u]^{c-n-1}\,P_{N,n}(u^c) \tag{40}$$

is a polynomial in u of degree $(c\deg Q_A + c - 1)$, with $c\deg Q_A$ unknown coefficients. Again, the required conditions for the determination of $\tilde{P}_N(u)$ are obtained by imposing the analyticity of the PGF $N(z)$ everywhere in the unit disk, or equivalently, the analyticity of $\tilde{N}(u)$ in the image of the unit disk under the transformation $z \to u = D^{-1}(z)$. As before, note that $Q_A([M(1/z)]^{-c})$ has exactly $c\deg Q_A$ zeros within the unit disk of the complex z-plane, and hence, $Q_A(u^c)$ has $c\deg Q_A$ zeros within the image of the unit disk under the transformation $z \to D^{-1}(z)$. In order for $\tilde{N}(u)$ to remain regular in these points, they also have to be zeros of the numerator of $\tilde{N}(u)$, with at least the same multiplicity. This gives us a set of $c\deg Q_A$ linearly independent equations in the $c\deg Q_A$ unknown coefficients of $\tilde{P}_N(u)$, so that $\tilde{P}_N(u)$, and hence $\tilde{N}(u)$, is completely determined.

5 Waiting Time and Delay

We now derive the PGF of the waiting time W of an arbitrary customer, denoted by C, arriving in the system when equilibrium has established itself. Let us denote the arrival instant of C by τ_a. Owing to the FCFS queueing discipline, the waiting time W of customer C depends on the total number of customers present in the buffer system during the arrival slot of C which have priority over customer C to be taken into service. This number of customers is a random variable, denoted by T, which is the sum of the number of customers staying in the system at τ_a (i.e., the number of customers present in the system both during the slot before and the slot after τ_a), and the number of customers arriving at τ_a (simultaneously with C) and being queued for service ahead of C. In stochastic equilibrium, the first component has PGF

$$V(z) = \frac{U(z)}{B(z)} \ , \tag{41}$$

and the second component has PGF (see e.g. [2])

$$\hat{B}(z) = \frac{B(z) - 1}{B'(1)\,(z-1)} \ , \tag{42}$$

since C is an arbitrary customer, and hence in an arbitrary position within the batch of customers arriving at τ_a. Clearly, the two components of T are independent random variables, so that the PGF of T is given by

$$T(z) = V(z)\,\hat{B}(z) \ . \tag{43}$$

Let us consider the conditional probabilities

$$g(n\,|\,k) \triangleq \mathrm{Prob}[W = n\,|\,T = k] \ , \quad n \geq 0 \ , \quad k \geq 0 \ . \tag{44}$$

The way in which the distribution of W depends on the distribution of T is determined uniquely by the departure process. When $T < c$, customer C will be taken into service as soon as he arrives and his waiting time will be zero. Hence,

$$g(n\,|\,k) = \delta(n) \ , \quad 0 \leq k < c \ . \tag{45}$$

For $T \geq c$, on the other hand, the waiting time of C cannot be zero. In that case, however, the following recurrence relationship holds :

$$\begin{cases} g(0\,|\,k) = 0 \ , & k \geq c \ ; \\ g(n\,|\,k) = \sum\limits_{l=0}^{c} r(l)\, g(n-1\,|\,k-l) \ , & k \geq c \ , \quad n \geq 1 \ . \end{cases} \tag{46}$$

Here $r(l)$ ($l = 0, 1, \ldots, c$) is the PMF of the number of departures at a slot boundary when a customer is waiting and hence all the servers are occupied.

Due to the geometric service-time distribution, this number of departures has a binomial distribution with parameters c and σ, i.e., a PGF

$$R(z) = (1 - \sigma + \sigma z)^c \quad . \tag{47}$$

The recurrence relationship (46) can be transformed in an algebraic one for

$$G(x \,|\, y) \triangleq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n \,|\, k)\, x^n\, y^k \quad . \tag{48}$$

More specifically, we get

$$\begin{aligned} G(x \,|\, y) &= \sum_{k=0}^{c-1} y^k + \sum_{n=1}^{\infty} \sum_{k=c}^{\infty} \sum_{l=0}^{c} r(l)\, g(n-1 \,|\, k-l)\, x^n\, y^k \\ &= \frac{1-y^c}{1-y} + x \left[y^c \sum_{l=0}^{c} r(l)\, \frac{1-y^l}{1-y} + R(y) \left(G(x \,|\, y) - \frac{1-y^c}{1-y} \right) \right] \quad . \end{aligned} \tag{49}$$

Solving for $G(x \,|\, y)$, we find

$$G(x \,|\, y) = \frac{1}{1-y} \left(1 - y^c\, \frac{1-x}{1-x\, R(y)} \right) \quad , \tag{50}$$

which is a valid expression for $G(x \,|\, y)$ wherever $|x| < 1/\,|R(y)|$ in view of the factor $[1 - x\, R(y)]$ in the denominator of (50).

Together with the distribution of T (equation (43)), this result suffices to obtain the PGF $W(z)$ of the waiting time W of an arbitrary customer. Indeed, with $t(k) \triangleq \text{Prob}[T = k]$ $(k = 0, 1, \dots)$,

$$W(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} g(n \,|\, k)\, t(k)\, z^n = \frac{1}{2\pi i} \oint_L \frac{T(\zeta)}{\zeta}\, G\!\left(z \,\middle|\, \frac{1}{\zeta} \right) d\zeta \quad , \tag{51}$$

with L a contour around the origin $\zeta = 0$, but not around any singularity of $T(\zeta)$ and such that for all $\zeta \in L$, the sum over k and n converges, i.e., for all $\zeta \in L$, $|z| < 1/\,|R(1/\zeta)|$. The integrand is given by

$$\frac{T(\zeta)}{\zeta}\, G\!\left(z \,\middle|\, \frac{1}{\zeta} \right) = \frac{T(\zeta)}{\zeta - 1} \left[1 - \frac{1-z}{\zeta^c - z\, \tilde{R}(\zeta)} \right] \quad , \tag{52}$$

where $\tilde{R}(\zeta) \triangleq \zeta^c\, R(1/\zeta)$ is a polynomial in ζ of degree c. From the Theorem of Rouché, it follows that the numerator factor $\zeta^c - z\, \tilde{R}(\zeta)$ has c zeros inside the unit circle, counting multiple zeros several times. Let us denote these zeros by $\beta_j(z)$ $(j = 0, 1, \dots, c-1)$. They are given by

$$\beta_j(z) = -\frac{\sigma}{1 - \sigma - |z|^{-1/c} \exp[\frac{i}{c}\,(-\arg z + 2\pi j)]} \quad , \qquad j = 0, 1, \dots, c-1 \quad , \tag{53}$$

where $\beta_0(1) = 1$. Apparently, for any $z \neq 0$ all $\beta_j(z)$ are different so that the contour integral in (51) can be obtained as the sum over the residues of the integrand in the simple poles of the integrand. The result reads

$$W(z) = \frac{z-1}{z} \sum_{j=0}^{c-1} \frac{T(\beta_j(z))}{[\beta_j(z)-1]\beta_j(z)^{c-2}R'(1/\beta_j(z))} \ . \tag{54}$$

Finally, the PGF of the complete system time or delay S of C, in which also the service time is included, is then obtained as

$$S(z) = W(z)\,D(z) \ . \tag{55}$$

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