

ANALYSIS OF THE FLOWS OF INCOMPRESSIBLE FLUIDS WITH PRESSURE DEPENDENT VISCOSITY FULFILLING

$$\nu(p, \cdot) \rightarrow +\infty \text{ AS } p \rightarrow +\infty.$$

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ABSTRACT. Over a large range for the pressure, one cannot ignore the fact that the viscosity grows significantly (even exponentially) with increasing pressure. This paper concerns long-time and large-data existence results for a generalization of the Navier-Stokes fluid whose viscosity depends on the shear rate and the pressure. The novelty of this result stems from the fact that we allow the viscosity to be an unbounded function of pressure at infinity. In order to include a large class of viscosities and in order to explain the main idea in as a simple manner as possible, we restrict ourselves to a discussion of the spatially periodic problem.

1. INTRODUCTION

It is well documented in the case of several liquids that, over a sufficiently large range for the pressure, the viscosity grows significantly with varying values of the pressure. Since the variation in the density, in comparison to the variations in the viscosity are negligible, we can model these liquids as being incompressible, their viscosity however depending on the pressure (which in this case becomes the mean normal stress for those fluids that are a generalization of the Navier-Stokes model). Contrary to the mathematical theory for the incompressible Navier-Stokes equations (NSEs), for which results for large data are well known since the work of Leray [10], there is no mathematical result concerning large data or long-time existence of solution to a model that generalizes the NSEs allowing the viscosity to depend on the pressure. We refer the interested reader to the survey article by Málek and Rajagopal [13] for more details.

Recently, there have been several mathematical studies (see [11] and [9] for three- and two-dimensional evolutionary spatially periodic problem respectively, [8] for three-dimensional stationary Dirichlet problem, [6] for three-dimensional unsteady flows fulfilling Navier's slip on the boundary, and [7] for three-dimensional unsteady flows that take into consideration thermal effects and that are subject to

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the Navier's slip boundary conditions) establishing long-time and large-data existence results for a class of incompressible fluids with the viscosities depending specifically on the pressure, (the temperature) and the shear rate. It is worth of emphasizing that a suitable dependence of the viscosity on the shear rate is crucial for all these investigations, and in fact in all of them the viscosity decreases with increasing shear-rate (i.e., the fluid shear thins). While such fluid models capture the pressure thickening phenomenon observed for most liquids (the viscosity increasing with increasing pressure), the fact that the viscosity remains bounded as the pressure tends to infinity can be considered as a drawback of the above mentioned studies as in several liquids the viscosity grows with increasing pressure, even exponentially (see the Barus' [4] or Andrade's [1] formulas in the book by Bridgman [5] or in the survey paper by Málek and Rajagopal [13]). In fact, recent articles by Bair and Kottke [3] and by Bair [2] report on even drastically faster dependence of the viscosity on the pressure.

Of course, in reality we cannot apply infinite pressure. However, the experiments suggest an exponential increase of the viscosity with pressure for the range of pressure for which the experiments were carried out, which in its mathematical idealization would lead to viscosities which become unbounded with pressure. Idealized mathematical problems in unbounded domains wherein the pressure or velocity can become unbounded serve a very useful purpose. For instance, the classical solution for the Poiseuille flow problem is obtained under the assumption that the pressure at both ends is infinite (this follows from the fact that the pipe is assumed to be infinitely long and the pressure gradient along the axis of the pipe is a constant). However, the result obtained under such an assumption agrees remarkably well with experimental results for laminar flow in a sufficiently long pipe. Similarly, the solution obtained using Karman's assumption due to an infinite rotating plate implies infinite velocities and the results stemming from Karman's assumptions lead to excellent estimates for the development of the boundary layer due to a rotating plate of sufficient diameter. Thus, studies such as the one carried here have to be viewed within the context of whether the idealized mathematical problem serves some useful purpose. It is definitely worth knowing if flows of fluids with pressure dependent viscosity that grow unboundedly with the pressure admit solutions, as they could, and possibly do, have relevance to viscosities that do increase in such a manner within the realm of experimentation.

The aim of this paper is to relax the assumption on the boundedness of the viscosity, and to identify the class of fluid models (as large as possible) with the viscosity converging to infinity as pressure tends to infinity. Of course, the question of the rate of such convergence is important. In this paper, we are able to incorporate almost linear growth in the pressure. The task of how to include higher order, i.e., polynomial or exponential growths, is left completely open. In addition, we are able to treat fluids that can shear thicken (the viscosity increases with increasing shear rate). In order to look for a sufficiently robust class of models we start with very general, nevertheless natural conditions on the structure of the admissible viscosities, and look for restrictions required within the course of the existence proof. In order to avoid the additional restrictions caused by the choice of boundary conditions and in order to keep the presentation as simple as possible we deal with the spatially periodic problem. This permits us to have a relatively simple equation for the pressure, and to incorporate with ease the regularity method to obtain the

compactness for the velocity gradient and consequently for the pressure. Of course, one could be interested in knowing whether the same (or slightly modified) assumptions on the viscosity are applicable to other boundary value problems as well. We feel that the answer will be affirmative. From this perspective, we wish to mention that we do not intend to flood the mathematical community with, according to us, minor modifications based on a combination of the results in this paper with already established results. We feel that it is sufficient to address the main point just once; the establishment of long-time and large data mathematical theory for a class of fluids with the viscosity being a suitable function of the pressure and the shear rate, wherein the viscosity can grow almost linearly with the pressure and can be either an increasing or decreasing function of the shear rate. A more precise formulation of this statement and its proof form the content of the remaining sections of the paper.

We complete this introductory section by formulating the problem.

Let $\Omega := (0, L_1) \times (0, L_2) \times (0, L_3)$ denote a cube in \mathbb{R}^3 , and let $T \in (0, \infty)$ be the length of the time interval of interest. We say that a function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ is Ω -periodic if u is L_i -periodic in the variable x_i . Any vectorial or tensorial function is Ω -periodic if each of its components is Ω -periodic.

For a given $\mathbf{v}_0 = (v_{01}, v_{02}, v_{03})$ being Ω -periodic and satisfying $\operatorname{div} \mathbf{v}_0 = 0$, we look for a couple of functions $(\mathbf{v}, p) = (v_1, v_2, v_3, p) : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^4$ being Ω -periodic, with zero mean value, and solving the system

$$\begin{aligned} & \operatorname{div} \mathbf{v} = 0, \\ (\mathcal{U}) \quad & \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}(\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v})) = -\nabla p. \end{aligned}$$

The first equation expresses the fact that the fluid is incompressible and homogeneous (leading to the conclusion that the density is everywhere the same, equal to a positive constant ϱ^*), the other set of equations is a consequence of the balance of linear momentum

$$(1) \quad \varrho^* (\mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v})) = \operatorname{div} \mathbf{T} + \varrho^* \mathbf{f},$$

by neglecting the specific body force \mathbf{f} , dividing by ϱ^* and incorporating the constitutive equation for the Cauchy stress \mathbf{T} of the form (ν is the generalized (kinematic) viscosity):

$$(2) \quad \mathbf{T} = \varrho^* \left(-p \mathbf{I} + \nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}) \right),$$

$\mathbf{D}(\mathbf{v})$ being the symmetric part of the velocity gradient.

We also formulate the problem governing steady flows. For a given $\mathbf{f} = (f_1, f_2, f_3)$ being Ω -periodic we look for a couple of functions $(\mathbf{v}, p) = (v_1, v_2, v_3, p) : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ being Ω -periodic with zero mean value solving the system

$$\begin{aligned} & \operatorname{div} \mathbf{v} = 0, \\ (\mathcal{S}) \quad & \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}(\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v})) = -\nabla p + \mathbf{f}. \end{aligned}$$

Most, if not all, textbooks on continuum fluid mechanics do not consider the possibility that the viscosity can depend on the pressure. This stems from the fact that in these textbooks an a priori assumption is made concerning internal constraints, namely they require that the internal constraints do no work. This naturally leads one to ask the question whether the constitutive equation (2) is consistent with basic principles of continuum physics. The answer is an unequivocal yes. Currently,

there are three ways in which we can go about developing such models within a thermomechanically consistent framework: (i) to reject the assertion (principle) that internal constraints do no work (this is addressed in the work by Rajagopal and Srinivasa [16]), (ii) the model can be easily incorporated into the thermomechanical framework developed by Rajagopal and his co-workers (here in particular) based on the maximization of the rate of dissipation with respect to the symmetric part of the velocity gradient taking into account the incompressibility conditions and the reduced thermomechanical identity as the constraints (see Málek and Rajagopal [13] for details), and (iii) the model can be easily implemented into the framework of implicit constitutive equations developed by Rajagopal, see [14, 15].

The structure of the paper is as follows. In Section 2 we start delineating the assumptions concerning the structure of the viscosities, followed by two examples that satisfy them for some range of parameters. Then we introduce the function spaces needed to define what we mean by a solution to problems (\mathcal{S}) and (\mathcal{U}) . Finally, we formulate two main existence theorems. Section 3 is devoted to their proofs. In the last section, we show that the viscosities given in Section 2 satisfy the assumptions needed for establishing the proof.

2. ASSUMPTIONS ON VISCOSITY. EXAMPLES. FUNCTION SPACES. MAIN THEOREMS.

Our aim in this section is to characterize admissible viscosities and to show that this class contains several interesting examples. Calculations illustrating under what range of model parameters the examples fulfil the required assumptions are postponed to Section 4. Here, after introducing various function spaces, we define what we mean by a weak solution to Problem (\mathcal{S}) and Problem (\mathcal{U}) and formulate main theorems.

2.1. Assumptions on the viscosity. We assume that there are Lipschitz continuous functions $\gamma_1 := \gamma_1(p, |\mathbf{D}|^2)$, $\gamma_2 := \gamma_2(p, |\mathbf{D}|^2)$ and $\gamma_3 := \gamma_3(p, |\mathbf{D}|^2)$ such that the viscosity $\nu(p, |\mathbf{D}|^2)$ satisfies for all $\mathbf{B}, \mathbf{D} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$

$$(3) \quad \gamma_1(p, |\mathbf{D}|^2) |\mathbf{B}|^2 \leq \frac{\partial(\nu(p, |\mathbf{D}|^2) \mathbf{D}_{ij})}{\partial \mathbf{D}_{kl}} \mathbf{B}_{ij} \mathbf{B}_{kl} \leq \gamma_2(p, |\mathbf{D}|^2) |\mathbf{B}|^2,$$

$$(4) \quad |\partial_p \nu(p, |\mathbf{D}|^2) \mathbf{D}| \leq \gamma_3(p, |\mathbf{D}|^2) < \frac{1}{2},$$

and in addition there are real parameters $r > 1$, $q \in (1, r - 1)$ and $\sigma \in (0, 1)$ such that for arbitrarily small $a > 0$

$$(5) \quad c(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} \leq \gamma_1(p, |\mathbf{D}|^2) \leq \gamma_2(p, |\mathbf{D}|^2) \leq \sigma \frac{|p|}{a + |\mathbf{D}|} + C(1 + |\mathbf{D}|^q).$$

Next, we define two auxiliary quantities:

$$(6) \quad \gamma_p(p, |\mathbf{D}|^2) := p \partial_p \left(\frac{\gamma_3^2(p, |\mathbf{D}|^2)}{\gamma_1(p, |\mathbf{D}|^2)} \right),$$

$$(7) \quad \gamma_D(p, |\mathbf{D}|^2) := p \partial_{|\mathbf{D}|^2} \left(\frac{\gamma_3^2(p, |\mathbf{D}|^2)}{\gamma_1(p, |\mathbf{D}|^2)} \right),$$

and introduce several auxiliary numbers ($p \in \mathbb{R}$, $d \in \mathbb{R}^+$):

$$(8) \quad \alpha_0 := \sup_{p,d} \gamma_3(p, d^2),$$

$$(9) \quad \alpha_1 := \sup_{p,d} \frac{\gamma_3(p, d^2)\gamma_2(p, d^2)}{\gamma_1(p, d^2)},$$

$$(10) \quad \alpha_2 := \sup_{p,d} \frac{|p|\gamma_3^2(p, d^2)}{\gamma_1(p, d^2)(1 + d^2)^{\frac{1}{2}}},$$

$$(11) \quad \alpha_2^* := \sup_{p,d} \frac{\gamma_3^2(p, d^2)}{\gamma_1(p, d^2)},$$

$$(12) \quad \alpha_3 := \sup_{p,d} \frac{|\gamma_p(p, d^2)|\gamma_1(p, d^2)}{\gamma_3(p, d^2)},$$

$$(13) \quad \alpha_4 := \sup_{p,d} \frac{|\gamma_D(p, d^2)|\gamma_2(p, d^2)d}{\gamma_1(p, d^2)},$$

$$(14) \quad \alpha_5 := \sup_{p,d} \frac{|\gamma_p(p, d^2)|\gamma_2(p, d^2)}{\gamma_3(p, d^2)},$$

$$(15) \quad \alpha_6 := \sup_{p,d} |\gamma_D(p, d^2)|d,$$

$$(16) \quad \alpha_7 := \inf_{p,d} \frac{\gamma_p(p, d^2)\gamma_1(p, d^2)}{\gamma_3^2(p, d^2)},$$

$$(17) \quad \alpha_8 := \sup_{p,d} \frac{|\gamma_D(p, d^2)|d}{\gamma_3(p, d^2)}.$$

2.2. Examples. In this subsection we give two classes of viscosities that satisfy the assumptions (3)-(5) wherein auxiliary quantities $\alpha_0, \dots, \alpha_8$ are finite. The proofs of these properties for and some further related inequalities are given in Section 4.

Example 2.1. For $\alpha \in [0, 1]$, $\beta \geq 0$ and $r \in (1 + \alpha\beta, 1 + \beta]$, viscosities of the form

$$(18) \quad \nu(p, |\mathbf{D}|^2) := (1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}}\right)^\alpha,$$

where $f(p)$ has the following structure:

$$(19) \quad f(p) := \begin{cases} a(1 + bp^2)^c & \text{if } p \geq 0 \\ a & \text{if } p \leq 0 \end{cases} \quad (a, b, c \geq 0)$$

with $c \in [\frac{1}{2}, \frac{\beta}{2(r-1)}]$, satisfy our requirements.

Example 2.2. For $r \in (1, \infty)$, viscosities of the form

$$(20) \quad \nu(p, |\mathbf{D}|^2) := \nu_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} + \frac{f(p)}{\sqrt{\varepsilon + |\mathbf{D}|^2}},$$

where the function $f(p)$ is given either by (19) with $c \leq \frac{1}{2} \frac{r-1}{r+1} < \frac{1}{2}$, or by

$$(21) \quad f(p) := \begin{cases} a \ln^2(1 + bp) + c & \text{if } p \geq 0 \\ c & \text{if } p \leq 0 \end{cases} \quad (a, b, c \geq 0),$$

satisfy our requirements.

The structure of (20) has relevance to the viscosity used by Schaeffer [17] in modeling and analyzing flows of sand in silos.

2.3. Definition of a weak solution to Problem (S) and Problem (U). We start by introducing function spaces necessary to provide a proper definition of what we mean by a solution. The standard Lebesgue and Sobolev spaces (of Ω -periodic functions) are denoted by L^r and $W^{1,r}$, respectively. We do not distinguish between function spaces whose elements are scalars, vectors or tensors. A subspace of any space X consisting of functions belonging to X and having mean value zero is denoted by $\overset{\circ}{X}$. A subspace of X consisting of vector-valued functions belonging to X and being divergenceless is denoted by X_{div} . Moreover, in order to simplify the notation, we introduce for a, b being scalar-, vector- or tensor-valued function and fulfilling $a \cdot b \in L^1$ the notation $(a, b)_2 := \int_{\Omega} a \cdot b \, dx$. Similarly, for $a \in X^*$, $b \in X$, X^* being the dual space to X , we simply write $\langle a, b \rangle := \langle a, b \rangle_{X, X^*}$.

We also recall the standard result that is a consequence of the theory of multipliers.

Lemma 2.1. *Let $q_1 \in (1, \infty)$ and $q_2 \in (1, \infty)$.*

There exists a constant $C_{q_1} > 0$ such that for any $\mathbf{G} \in L^{q_1}$ there exists a unique $\pi \in \overset{\circ}{L}^{q_1}$ solving in the sense of distributions

$$-\Delta \pi = \operatorname{div} \operatorname{div} \mathbf{G}$$

and satisfying

$$(22) \quad \|\pi\|_{q_1} \leq C_{q_1} \|\mathbf{G}\|_{q_1}.$$

In addition, there exists a constant $C_{q_2, reg}$ such that for any $\mathbf{G} \in W^{1, q_2}$ the following inequality holds:

$$(23) \quad \|\nabla \pi\|_{q_2} \leq C_{q_2, reg} \|\nabla \mathbf{G}\|_{q_1}.$$

Next, we define what we mean by a solution to (S) and (U).

Definition 2.1. *Let $\mathbf{f} \in (W^{1, r})^*$. Let the viscosity satisfy (3)-(5) with $r > \frac{6}{5}$. We say that a couple (\mathbf{v}, p) is a weak solution to (S) if*

$$(24) \quad \mathbf{v} \in W_{div}^{1, r}, \quad p \in \overset{\circ}{L}^s \quad \text{with } s := \begin{cases} \min\left(\frac{3r}{2(3-r)}, \frac{r}{q+1}\right) & \text{if } r \leq 3, \\ \frac{r}{q+1} & \text{if } r \geq 3, \end{cases}$$

and

$$(25) \quad -(\mathbf{v} \otimes \mathbf{v}, \nabla \varphi)_2 + (\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}), \mathbf{D}(\varphi))_2 = (p, \operatorname{div} \varphi)_2 + (\mathbf{f}, \varphi)_2$$

holds for all $\varphi \in W^{1, \infty}$.

Definition 2.2. *Let $\mathbf{v}_0 \in \overset{\circ}{L}^2_{div}$. Let the viscosity satisfy (3)-(5) with $r > \frac{6}{5}$. We say that a couple (\mathbf{v}, p) is a weak solution to (U) if*

$$(26) \quad p \in \overset{\circ}{L}^s \quad \text{with } s := \min\left(\frac{5r}{6}, \frac{r}{q+1}\right),$$

$$(27) \quad \mathbf{v} \in L^\infty(0, T; L^2) \cap L^r(0, T; W_{div}^{1, r}),$$

$$(28) \quad \mathbf{v}_{,t} \in L^s(0, T; (W^{1, s'})^*),$$

$$\begin{aligned}
 (29) \quad & \int_0^T \langle \mathbf{v}_t, \boldsymbol{\varphi} \rangle - (\mathbf{v} \otimes \mathbf{v}, \nabla \boldsymbol{\varphi})_2 + (\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi}))_2 \\
 & = \int_0^T (p, \operatorname{div} \boldsymbol{\varphi})_2 + \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \quad \text{holds for all } \boldsymbol{\varphi} \in L^\infty(0, T; W^{1, \infty}),
 \end{aligned}$$

and the velocity \mathbf{v} attains the initial condition \mathbf{v}_0 in the following sense

$$(30) \quad \lim_{t \rightarrow 0} \|\mathbf{v}(t) - \mathbf{v}_0\|_2 = 0.$$

Note that the assumptions (3)-(5) imply that

$$\begin{aligned}
 (31) \quad & \left| \int_0^T (\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}), \mathbf{D}(\boldsymbol{\varphi})) \right| \stackrel{(5)}{\leq} C \int_0^T \int_\Omega |\nabla \boldsymbol{\varphi}| (1 + |p| + |\mathbf{D}(\mathbf{v})|^{q+1}) \\
 & \stackrel{(q+1 < r)}{\leq} C \|\nabla \boldsymbol{\varphi}\|_\infty \int_0^T (1 + \|p\|_1 + \|\mathbf{D}(\mathbf{v})\|_r^r)
 \end{aligned}$$

and using (24) or (27) we find that the integral estimated in (31) is finite.

Theorem 2.1 (Existence result for Problem (S)). *Let $\mathbf{f} \in L^{r'} \cap L^2$. For s defined in (26), let C_s denote the constant that appears in (22). Assume that the viscosity ν satisfies (3)-(5) with $\sigma < \frac{1}{C_s}$ and $r > \frac{9}{5}$. Moreover, let $\alpha_0, \alpha_1, \alpha_2^*$ defined in (8)-(9) and (11) be finite and fulfil*

$$(32) \quad \alpha_0 C_{reg,2} < 1,$$

$$(33) \quad \frac{\alpha_1}{1 - \alpha_0} < 1.$$

Then there exists a weak solution to (S).

Theorem 2.2 (Existence result for Problem (U)). *Let $\mathbf{v}_0 \in \overset{\circ}{L}{}^2_{\operatorname{div}}$. For s defined in (26), let C_s denote the constant that appears in (22). Assume that the viscosity ν satisfy (3)-(5) with $\sigma < \frac{1}{C_s}$ and $r > \frac{9}{5}$. Moreover, let $\alpha_0, \dots, \alpha_8$ defined in (8)-(17) be finite and fulfil (32) and*

$$(34) \quad 0 \leq \alpha_1 + \sqrt{2\alpha_4} + \alpha_5 + 2\alpha_6 + 2\alpha_8 < 1 - \alpha_0 - \alpha_3 + \alpha_7.$$

Then there exists a weak solution to (U).

In Section 4 we will show, among others, that Example 2.1 fulfills the assumptions (32) and (34) for certain range of parameters that appear in the model.

3. PROOF OF THE THEOREM 2.2

We prove only Theorem 2.2, since the proof of Theorem 2.1 can be established in a similar manner we do not provide it.

We split the proof of Theorem 2.2 into three parts. First, we summarize those properties of p that follows from certain properties of \mathbf{v} . Then we introduce suitable ε -approximations to Problem (U) and establish existence and regularity results for them. Finally, we develop the uniform estimates that are sufficient to prove the compactness of the velocity gradient, and to complete the proof.

3.1. Properties of pressure. Before we start with the proof of the main theorem, we prove an important lemma about the behavior of the pressure.

Lemma 3.1. *Let the viscosity satisfy (3)-(5) with $\sigma < \frac{1}{C_s}$ for $s := \min(\frac{5r}{6}, \frac{r}{q+1})$ and $\alpha_0 < \frac{1}{C_\beta}$ for some $\beta \in (1, s)$, C_s and C_β being introduced in (22). Assume that $\mathbf{v} \in L^\infty(0, T; L^2) \cap L^r(0, T; W_{\text{div}}^{1,r})$. Then there exists a pressure $\mathcal{P}(\mathbf{v}) =: p \in L^s(0, T; \mathring{L}^s)$ solving in the sense of distributions*

$$(35) \quad \Delta p = \text{div div} (\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}) - \mathbf{v} \otimes \mathbf{v})$$

and fulfilling the following estimate

$$(36) \quad \int_0^T \|p\|_s^s \leq C(\|\mathbf{v}\|). \quad (C(\|\mathbf{v}\|) \text{ defined in (39)})$$

Moreover, let $\alpha_0 < 1$. Then for almost all time $t \in (0, T)$ we have

$$(37) \quad \|\nabla p\|_2 \leq C (\|\gamma_2(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\nabla \mathbf{v})\|_2 + \|\mathbf{v}\| \|\nabla \mathbf{v}\|_2),$$

provided that the right hand side is finite.

Moreover, if p_1, p_2 are two solutions to (35) corresponding to $\mathbf{v}_1, \mathbf{v}_2$, then

$$(38) \quad \begin{aligned} \int_0^T \|p_1 - p_2\|_\beta^\beta &\leq C \int_0^T \|\mathbf{v}_1 - \mathbf{v}_2\|_{2\beta}^\beta \|\mathbf{v}_1 + \mathbf{v}_2\|_{2\beta}^\beta \\ &+ C \int_0^T \left\| \int_0^1 \gamma_2(p_2 - s(p_2 - p_1), |\mathbf{D}(\mathbf{v}_2 - s(\mathbf{v}_2 - \mathbf{v}_1))|^2) |\mathbf{D}(\mathbf{v}_1 - \mathbf{v}_2)| ds \right\|_\beta^\beta \end{aligned}$$

provided that right hand side is finite.

Proof. We prove Lemma 3.1 only for smooth \mathbf{v} . For general \mathbf{v} one can argue by using density arguments and the estimate (38). First, for a smooth velocity \mathbf{v} it is proved¹ in [11] that there exist a pressure p solving (35). Thus, it remains to show that the relations (36)-(38) hold. First, using Lemma 2.1 we see that for a.a. $t \in (0, T)$ we have²

$$\begin{aligned} \|p\|_s &\leq C_s \|\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v})\|_s + C_s \|\mathbf{v} \otimes \mathbf{v}\|_s \\ &\stackrel{(5)}{\leq} C_s \sigma \|p\|_s + C(1 + \|\mathbf{v}\|_{2s}^2 + \|\mathbf{D}(\mathbf{v})\|_{s(q+1)}^{q+1}). \end{aligned}$$

Since $C_s \sigma < 1$ and $1 < s = \min(\frac{5r}{6}, \frac{r}{q+1})$ we finally conclude (by virtue of assumptions concerning \mathbf{v} and standard interpolation argument) that

$$(39) \quad \int_0^T \|p\|_s^s \leq C + C \int_0^T \|\mathbf{v}\|_{2s}^{2s} + \|\mathbf{D}(\mathbf{v})\|_r^r =: C(\|\mathbf{v}\|),$$

which is exactly (36).

¹To prove the existence of p it is enough to use Lemma 2.1 and Banach fixed point theorem.

²We use the fact that $\nu(p, |\mathbf{D}|^2) \leq \gamma_2(p, |\mathbf{D}|^2)$.

To prove (37) it is enough to multiply (35) by p and after integration by parts we obtain

$$\begin{aligned} \|\nabla p\|_2^2 &\leq \|\nabla p\|_2 \|\operatorname{div} \nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}) + \operatorname{div}(\mathbf{v} \otimes \mathbf{v})\|_2 \\ &\leq \|\nabla p\|_2 \|\partial_p \nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}) \nabla p + \partial_{\mathbf{D}} \left(\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v}) \right) \cdot \mathbf{D}(\mathbf{v}) \otimes \mathbf{D}(\nabla \mathbf{v})\|_2 \\ &\quad + \|\nabla p\|_2 \|\operatorname{div}(\mathbf{v} \otimes \mathbf{v})\|_2 \\ &\stackrel{(3),(4)}{\leq} \alpha_0 \|\nabla p\|_2^2 + \|\nabla p\|_2 (\|\gamma_2(p, |\mathbf{D}|^2) \mathbf{D}(\nabla \mathbf{v})\|_2 + \|\mathbf{v}\| \|\nabla \mathbf{v}\|_2). \end{aligned}$$

Using the fact that $\alpha_0 < 1$ and Young inequality we establish (37).

The last step is to prove (38). Since p_1, p_2 solves (35), it follows that

$$\begin{aligned} -\Delta(p_1 - p_2) &= \operatorname{div} \operatorname{div}(\mathbf{v}_1 \otimes \mathbf{v}_1 - \mathbf{v}_2 \otimes \mathbf{v}_2) \\ &\quad - \operatorname{div} \left(\nu(p_1, |\mathbf{D}(\mathbf{v}_1)|^2) \mathbf{D}(\mathbf{v}_1) - \nu(p_2, |\mathbf{D}(\mathbf{v}_2)|^2) \mathbf{D}(\mathbf{v}_2) \right) \end{aligned}$$

and due to Lemma 2.1 we derive

$$\begin{aligned} \|p_1 - p_2\|_\beta &\leq C_\beta \|\mathbf{v}_1 \otimes \mathbf{v}_1 - \mathbf{v}_2 \otimes \mathbf{v}_2\|_\beta \\ &\quad + C_\beta \left\| \nu(p_1, |\mathbf{D}(\mathbf{v}_1)|^2) \mathbf{D}(\mathbf{v}_1) - \nu(p_2, |\mathbf{D}(\mathbf{v}_2)|^2) \mathbf{D}(\mathbf{v}_2) \right\|_\beta =: I_1 + I_2. \end{aligned}$$

The integral I_1 can be simply estimated as

$$I_1 \leq C_\beta \|\mathbf{v}_1 - \mathbf{v}_2\|_{2\beta} \|\mathbf{v}_1 + \mathbf{v}_2\|_{2\beta}.$$

To estimate I_2 , we use the notation $p_s := p_2 - s(p_2 - p_1)$, $\mathbf{v}_s := \mathbf{v}_2 - s(\mathbf{v}_2 - \mathbf{v}_1)$, and obtain that

$$\begin{aligned} \nu(p_1, |\mathbf{D}(\mathbf{v}_1)|^2) \mathbf{D}(\mathbf{v}_1) - \nu(p_2, |\mathbf{D}(\mathbf{v}_2)|^2) \mathbf{D}(\mathbf{v}_2) &= \int_0^1 \frac{d}{ds} \nu(p_s, |\mathbf{D}(\mathbf{v}_s)|^2) \mathbf{D}(\mathbf{v}_s) ds \\ &= \int_0^1 \partial_p \nu(p_s, |\mathbf{D}(\mathbf{v}_s)|^2) \mathbf{D}(\mathbf{v}_s) (p_1 - p_2) + \partial_{\mathbf{D}} \nu(p_s, |\mathbf{D}(\mathbf{v}_s)|^2) \mathbf{D}(\mathbf{v}_s) \mathbf{D}(\mathbf{v}_1 - \mathbf{v}_2) ds. \end{aligned}$$

Hence, using the assumptions (3)-(5) we find that

$$(40) \quad I_2 \leq C_\beta \alpha_0 \|p_1 - p_2\|_\beta + C_\beta \left\| \int_0^1 \gamma_2(p_s, |\mathbf{D}(\mathbf{v}_s)|^2) |\mathbf{D}(\mathbf{v}_1 - \mathbf{v}_2)| ds \right\|_\beta.$$

Finally, using the fact that $\alpha_0 C_\beta < 1$ and combining all the estimates given above we deduce (38). \square

Corollary 3.1. *Let all the assumptions of Lemma 2.1 be fulfilled. Let the sequence $\{\mathbf{v}^n\}$ be bounded in the space $L^\infty(0, T; L^2) \cap L^r(0, T; W_{\operatorname{div}}^{1,r})$ and let $\{\mathbf{v}^n\}$ converge to some \mathbf{v} strongly in $L^1(0, T; W^{1,1})$. Then there exists a corresponding sequence $\{p^n\}$ solving (35) and satisfying (36) and there is a p such that*

$$(41) \quad p^n \rightharpoonup p \text{ weakly in } L^s \text{ with } s := \min\left(\frac{5r}{6}, \frac{r}{q+1}\right),$$

$$(42) \quad p^n \rightarrow p \text{ strongly in } L^1,$$

and the couple (\mathbf{v}, p) again solves (35).

Proof. The existence of a sequence p^n is a direct consequence of Lemma 2.1. Also the property (41) directly follows from (36) (modulo a subsequence). It remains to prove the validity of (35) for the limit couple (\mathbf{v}, p) and also the validity of (42). Assuming for a moment that (42) is valid, then there are subsequences that we do not relabel such that $\nabla \mathbf{v}^n$ and p^n converge to $\nabla \mathbf{v}$ and p almost everywhere. With

the help of Vitali's theorem, we can easily prove the relation (35). Thus, it remains to show (42). To prove it, we show that the sequence p^n is Cauchy in L^1 . Let p^n, p^m be the pressures corresponding to $\mathbf{v}^n, \mathbf{v}^m$. Using (38) we find that

$$(43) \quad \int_0^T \|p^n - p^m\|_\beta^\beta \leq C(I_1 + I_2),$$

where

$$I_1 = \int_0^T \|\mathbf{v}^n - \mathbf{v}^m\|_{2\beta}^\beta \|\mathbf{v}^n + \mathbf{v}^m\|_{2\beta}^\beta,$$

$$I_2 = \int_0^T \left\| \int_0^1 \gamma_2(p^n - s(p^n - p^m), |\mathbf{D}(\mathbf{v}^n - s(\mathbf{v}^n - \mathbf{v}^m))|^2) |\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)| ds \right\|_\beta^\beta.$$

Due to the choice of β we see (after using standard interpolation inequalities) that

$$I_1 \leq f(n, m),$$

where $f(n, m)$ has *Cauchy* property³. In order to estimate I_2 we use (5) to obtain

$$I_2 \leq C \int_0^T \|(1 + |\mathbf{D}(\mathbf{v}^n)|^q + |\mathbf{D}(\mathbf{v}^m)|^q) |\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)|\|_\beta^\beta$$

$$+ \int_0^T \left\| C(|p^n| + |p^m| + 1) \int_0^1 \frac{|\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)|}{a + |\mathbf{D}(\mathbf{v}^n - s(\mathbf{v}^n - \mathbf{v}^m))|} ds \right\|_\beta^\beta =: I_3 + I_4$$

First, by applying Hölder's inequality, we obtain

$$I_3 \leq C \left(\int_0^T \|\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)\|_{\frac{r\beta}{r-q\beta}}^{\frac{r\beta}{r-q\beta}} \right)^{\frac{r-q\beta}{r}} \leq f(n, m),$$

where the last inequality follows from the fact that $\frac{r\beta}{r-q\beta} < r$ (which is a consequence of $\beta < s < \frac{r}{q+1}$). To estimate I_4 , we observe first that

$$(44) \quad \int_0^1 \frac{ds}{a + |\mathbf{D}(\mathbf{v}^n - s(\mathbf{v}^n - \mathbf{v}^m))|} \leq \int_0^1 \frac{ds}{a + \|\mathbf{D}(\mathbf{v}^n) - s\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)\|}$$

$$\leq C(a) \frac{\ln(1 + |\mathbf{D}(\mathbf{v}^n)| + |\mathbf{D}(\mathbf{v}^m)|)}{|\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)|}.$$

Using this estimate and the assumption on \mathbf{v}^n we easily show that

$$(45) \quad \int_0^T \left\| \int_0^1 \frac{|\mathbf{D}(\mathbf{v}^n - \mathbf{v}^m)|}{a + |\mathbf{D}(\mathbf{v}^n - s(\mathbf{v}^n - \mathbf{v}^m))|} ds \right\|_b^b dt \leq f(n, m)$$

for all $b \in (1, \infty)$. Finally since $\beta < s$ and the pressures p^n are uniformly bounded in $L^s(0, T; L^s)$ we can apply Hölder's inequality to I_4 and after using (45) we conclude the proof with the observation $I_4 \leq f(n, m)$. Taking all the above estimates into account, it follows from (43) that $p^n \rightarrow p$ strongly in L^β . Thus, (42) is proved. \square

³It means that for all ϵ there is an n_0 such that for all $n, m \geq n_0$: $f(n, m) \leq \epsilon$.

3.2. Existence of an ε -approximation. For the sake of completeness we introduce an ε -approximation to Problem (U) and discuss its solvability, and regularity.

First we mollify the initial condition through $\mathbf{v}_0^\varepsilon := \mathbf{v}_0 * \eta_\varepsilon$, where η_ε is a standard mollification kernel. Then we look for a solution of the following system

$$(46) \quad \begin{aligned} \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{v}_{,t} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) + \varepsilon \Delta^{(2z)} \mathbf{v} - \operatorname{div}(\nu(p, |\mathbf{D}(\mathbf{v})|^2) \mathbf{D}(\mathbf{v})) &= -\nabla p, \end{aligned}$$

where $z \in \mathbb{N}$, $z \geq 4(r+1)$. We construct the solution by using the standard Galerkin approximation (for details see [11]). Let $\{\mathbf{w}_k\}_{k=1}^\infty$ be a basis of $W_{\operatorname{div}}^{1,r}$ consisting of eigenfunctions of the Stokes operator. Note that such functions are smooth. Next, we look for \mathbf{v}^N of the form $\mathbf{v}^N(t, x) := \sum_1^N c_k^N(t) \mathbf{w}_k(x)$ where the coefficients $c^N = (c_1^N(t), \dots, c_N^N(t))$ solve the system of ordinary differential equations and

$$(47) \quad \begin{aligned} (\mathbf{v}_{,t}^N, \mathbf{w}_i)_2 - (\mathbf{v}^N \otimes \mathbf{v}^N, \nabla \mathbf{w}_i)_2 + \varepsilon (\nabla^{(2z)} \mathbf{v}^N, \nabla^{(2z)} \mathbf{w}_i)_2 \\ + (\nu(\mathcal{P}(\mathbf{v}^N), |\mathbf{D}(\mathbf{v}^N)|^2) \mathbf{D}(\mathbf{v}^N), \mathbf{D}(\mathbf{w}_i))_2 = 0 \quad \text{for } i = 1, \dots, N. \end{aligned}$$

By using Caratheodory theory, we find that for arbitrary N the solution to (47) exists at least for some small time interval. However, it can be extended onto the whole time interval $(0, T)$ by using the uniform estimates proved below.

3.2.1. First uniform estimate. Multiplying the i -th equation in (47) by $c_i^N(t)$ and taking the sum over $i = 1, \dots, N$ we obtain with help of (3) the following estimate

$$(48) \quad \sup_t \|\mathbf{v}^N(t)\|_2^2 + \int_0^T \varepsilon \|\nabla^{(2z)} \mathbf{v}^N\|_2^2 + \int_\Omega \gamma_1(p^N, |\mathbf{D}(\mathbf{v}^N)|^2) |\mathbf{D}(\mathbf{v}^N)|^2 dx dt \leq C,$$

that directly implies (after using (5)) that

$$\int_0^T \|\mathbf{D}(\mathbf{v}^N)\|_r^r \leq C.$$

Next, using Lemma 3.1, (36), we conclude that

$$(49) \quad \int_0^T \|p^N\|_s^s dt \leq C.$$

Consequently, having (48) and (49) we can deduce that

$$(50) \quad \int_0^T \|\mathbf{v}_{,t}\|_{(W_{\operatorname{div}}^{2z,2})^*}^{\frac{r}{r-1}} dt \leq C.$$

Finally, using Aubin-Lions lemma and Lemma 3.1 we see that we can easily let $N \rightarrow \infty$ in (47) and to find that (\mathbf{v}, p) is a weak solution of (46).

3.2.2. Further estimates. Because in the next subsection we need to test the equation by $\Delta \mathbf{v}$ we need to have some regularity result for the couple (\mathbf{v}, p) . This subsection is devoted to deriving the relevant estimates. To get them, we first test (46) by $\mathbf{v}_{,t}^N$ (i.e., we multiply (47) for i by $c_i'(t)$ and sum over $i = 1, \dots, N$) to get (after using (3)-(5))

$$(51) \quad \begin{aligned} \|\mathbf{v}_{,t}^N\|_2^2 + \varepsilon \frac{d}{dt} \|\nabla^{(2z)} \mathbf{v}^N\|_2^2 \leq C (\|\nabla \mathbf{v}^N\|_3^4 + \|\nabla p^N\|_2^2 + \|p^N\| \|\mathbf{D}(\nabla \mathbf{v}^N)\|_2^2 \\ + \| |\mathbf{D}(\mathbf{v}^N)|^q \mathbf{D}(\nabla \mathbf{v}^N) \|_2^2). \end{aligned}$$

Next, it is a simple consequence of standard interpolation inequalities that:

$$\begin{aligned}
\|\nabla \mathbf{v}^N\|_3^4 &\leq C \|\nabla^2 \mathbf{v}^N\|_2^4 \leq C \|\mathbf{v}\|_2^{4-\frac{4}{z}} \|\nabla^{(2z)} \mathbf{v}^N\|_2^{\frac{4}{z}} \stackrel{z \geq 4}{\leq} C + \|\nabla^{(2z)} \mathbf{v}^N\|_2^2, \\
\| |\mathbf{D}(\mathbf{v}^N)|^q |\mathbf{D}(\nabla \mathbf{v}^N)| \|_2^2 &\leq C \|\nabla^3 \mathbf{v}^N\|_2^{2q+2} \leq C \|\mathbf{v}^N\|_2^{2q+2-\frac{3(q+1)}{z}} \|\nabla^{(2z)} \mathbf{v}^N\|_2^{\frac{3(q+1)}{z}} \\
&\stackrel{z \geq 3(q+1)}{\leq} C + \|\nabla^{(2z)} \mathbf{v}^N\|_2^2, \\
\| |p^N| |\mathbf{D}(\nabla \mathbf{v}^N)| \|_2^2 &\leq C \|p^N\|_2^2 \|\nabla^4 \mathbf{v}^N\|_2^2 \\
&\stackrel{\text{Lemma 3.1}}{\leq} C \|\nabla^4 \mathbf{v}^N\|_2^{2q+4} \leq \|\mathbf{v}^N\|_2^{2q+4-\frac{4q+8}{z}} \|\nabla^{(2z)} \mathbf{v}^N\|_2^{\frac{4q+8}{z}} \\
&\stackrel{z \geq 4(r+1)}{\leq} C + \|\nabla^{(2z)} \mathbf{v}^N\|_2^2,
\end{aligned}$$

Combining all these estimates and using Lemma 3.1 we also get that

$$\|\nabla p^N\|_2^2 \leq C + \|\nabla^{(2z)} \mathbf{v}^N\|_2^2.$$

Thus, inserting these estimates into (51) and using Gronwall lemma we obtain that

$$(52) \quad \sup_t \|\nabla^{(2z)} \mathbf{v}^N\|_2^2 \leq C(\varepsilon).$$

Consequently, we also easily establish that

$$(53) \quad \int_0^T \| |\operatorname{div} \nu(p^N, |\mathbf{D}(\mathbf{v}^N)|^2) \mathbf{D}(\mathbf{v}^N)| + |\operatorname{div} \mathbf{v}^N \otimes \mathbf{v}^N| + |\nabla p^N| + |\mathbf{v}_{,t}^N| \|_2^2 \leq C(\varepsilon).$$

Using weak lower semicontinuity of norms and pointwise convergence of pressure and velocity gradients we find that (53) also remains valid when we replace (\mathbf{v}^N, p^N) by (\mathbf{v}, p) .

3.3. Limit $\varepsilon \rightarrow 0$. In this subsection, we denote by $(\mathbf{v}^\varepsilon, p^\varepsilon)$ the solution of (46). To obtain uniform estimates we can multiply (46)₂ by \mathbf{v}^ε and after using assumptions (3)-(5) we get

$$(54) \quad \sup_t \|\mathbf{v}^\varepsilon(t)\|_2^2 + \int_0^T \|\mathbf{D}(\mathbf{v}^\varepsilon)\|_r^r + \int_\Omega \gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) |\mathbf{D}(\mathbf{v}^\varepsilon)|^2 dx dt \leq C.$$

Next, having an estimate for \mathbf{v}^ε we can use Lemma 3.1 to deduce that

$$(55) \quad \int_0^T \|p^\varepsilon\|_s^s \leq C,$$

and consequently we can again derive the following uniform estimate

$$(56) \quad \int_0^T \|\mathbf{v}_{,t}^\varepsilon\|_{(W_{\operatorname{div}}^{2z,2})^*}^{\frac{r}{q+1}} dt \leq C$$

Thus, using Aubin-Lions lemma again and Lemma 3.1 we can easily pass to the limit with respect to ε to get the weak solution to (\mathcal{U}) provided that $\mathbf{D}(\mathbf{v}^\varepsilon)$ converges almost everywhere in $(0, T) \times \Omega$. The rest of this section is devoted to the proof of this property.

In order to simplify the notation we denote

$$\begin{aligned}
\mathcal{I}_1 &:= \int_\Omega \gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2 dx, \\
\mathcal{I}_2 &:= \int_\Omega \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} |\nabla p^\varepsilon|^2 dx.
\end{aligned}$$

Next, we multiply (46)₂ by⁴ $-\Delta \mathbf{v}^\varepsilon$, integrate over Ω , integrate by parts and obtain

$$(57) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{v}^\varepsilon\|_2^2 + \varepsilon \|\nabla^{(2z+1)} \mathbf{v}^\varepsilon\|_2^2 + \mathcal{I}_1 \\ & \leq C + \|\nabla \mathbf{v}^\varepsilon\|_3^3 + \int_{\Omega} \gamma_3(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) |\nabla p^\varepsilon| |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)| \, dx =: C + I_1 + I_2. \end{aligned}$$

Thus, by using Hölder inequality we get that

$$(58) \quad I_2 \leq (\mathcal{I}_1)^{\frac{1}{2}} (\mathcal{I}_2)^{\frac{1}{2}}.$$

To estimate \mathcal{I}_2 we take the L^2 -scalar product of (46)₂ with $\boldsymbol{\psi} := -\frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \nabla p^\varepsilon$ and observe that

$$(59) \quad \begin{aligned} \mathcal{I}_2 &= \left(\mathbf{v}_{,t}^\varepsilon + \varepsilon \Delta^{(2z)} \mathbf{v}^\varepsilon + \operatorname{div}(\mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon), \boldsymbol{\psi} \right)_2 - \left(\operatorname{div} \nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) \mathbf{D}(\mathbf{v}^\varepsilon), \boldsymbol{\psi} \right)_2 \\ &=: Y_1 + Y_2. \end{aligned}$$

Hence, using the assumptions (3)-(5) and (8) we get that

$$\begin{aligned} Y_2 &\leq \alpha_0 \mathcal{I}_2 + \int_{\Omega} \frac{\gamma_2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) \gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)| |\nabla p^\varepsilon| \, dx \\ &\stackrel{(9)}{\leq} \alpha_0 \mathcal{I}_2 + \alpha_1 \mathcal{I}_2^{\frac{1}{2}} \mathcal{I}_1^{\frac{1}{2}}. \end{aligned}$$

Next, we estimate Y_1 . To do this, we first integrate by parts and then substitute the resulting relation into (46)₂. Denoting $\mathbf{b}^\varepsilon := p^\varepsilon \nabla \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}$, we arrive at

$$\begin{aligned} Y_1 &= - \left(\mathbf{v}_{,t}^\varepsilon + \varepsilon \Delta^{(2z)} \mathbf{v}^\varepsilon + \operatorname{div}(\mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon), \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \nabla p^\varepsilon \right)_2 \\ &\stackrel{\operatorname{div} \mathbf{v}^\varepsilon = 0}{=} \left(\mathbf{v}_{,t}^\varepsilon + \varepsilon \Delta^{2z} \mathbf{v}^\varepsilon + \operatorname{div}(\mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon), p^\varepsilon \nabla \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \right)_2 \\ &\quad + \left(\operatorname{div} \operatorname{div}(\mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon), p^\varepsilon \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \right)_2 \\ &= \left(\operatorname{div} \operatorname{div}(\mathbf{v}^\varepsilon \otimes \mathbf{v}^\varepsilon), p^\varepsilon \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \right)_2 + \left(\operatorname{div} \nu(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) \mathbf{D}(\mathbf{v}^\varepsilon), \mathbf{b}^\varepsilon \right)_2 \\ &\quad - \left(\nabla p^\varepsilon, \mathbf{b}^\varepsilon \right)_2 =: Z_1 + Z_2 + Z_3 \end{aligned}$$

Thus, we can obtain the estimates (after using (3)-(4))

$$\begin{aligned} Z_1 &\stackrel{\operatorname{div} \mathbf{v}^\varepsilon = 0}{\leq} \int_{\Omega} |\nabla \mathbf{v}^\varepsilon|^2 |p^\varepsilon| \frac{\gamma_3^2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)}{\gamma_1(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)} \, dx \leq \alpha_2 (C + \|\nabla \mathbf{v}^\varepsilon\|_3^3), \\ Z_2 &\leq \int_{\Omega} \left(\gamma_3(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) |\nabla p^\varepsilon| + \gamma_2(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)| \right) \\ &\quad \cdot \left(|\gamma_p(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)| |\nabla p^\varepsilon| + 2|\gamma_D(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)| |\mathbf{D}(\mathbf{v}^\varepsilon)| |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)| \right) \, dx \\ &\stackrel{(12)-(15)}{\leq} \alpha_3 \mathcal{I}_2 + 2\alpha_4 \mathcal{I}_1 + (\alpha_5 + 2\alpha_6) \mathcal{I}_1^{\frac{1}{2}} \mathcal{I}_2^{\frac{1}{2}}, \\ Z_3 &\leq \int_{\Omega} -|\nabla p^\varepsilon|^2 \gamma_p(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2) + 2|\nabla p^\varepsilon| |\gamma_D(p^\varepsilon, |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)| |\mathbf{D}(\mathbf{v}^\varepsilon)| |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)| \, dx \\ &\stackrel{(16)-(17)}{\leq} -\alpha_7 \mathcal{I}_2 + 2\alpha_8 \mathcal{I}_1^{\frac{1}{2}} \mathcal{I}_2^{\frac{1}{2}}. \end{aligned}$$

⁴Note that thanks to the estimates (52) and (53), $-\Delta \mathbf{v}^\varepsilon$ is an admissible multiplier.

Next, putting all the inequalities together, we see that if $1 - \alpha_0 - \alpha_3 + \alpha_7 > 0$ and $\frac{\alpha_1 + \sqrt{2\alpha_4 + \alpha_5} + 2\alpha_6 + 2\alpha_8}{1 - \alpha_0 - \alpha_3 + \alpha_7} < 1$, i.e., (34) holds, then we can conclude that

$$(60) \quad \mathcal{I}_2 \leq \delta \mathcal{I}_1 + C(\delta)(1 + \|\nabla \mathbf{v}^\varepsilon\|_3^3)$$

for some δ , $0 \leq \delta < 1$. Finally, substituting this estimate into (57), and using (3) we come to the following essential estimate

$$(61) \quad \frac{d}{dt} \|\nabla \mathbf{v}^\varepsilon\|_2^2 + \int_\Omega (1 + |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)^{\frac{r-2}{2}} |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2 \leq C(1 + \|\nabla \mathbf{v}^\varepsilon\|_3^3).$$

Thus, for $r \geq 3$ we can easily finish the proof because the right hand side of (61) is uniformly integrable over time, therefore⁵ $\int_0^T \int_\Omega |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2 \leq C$ and Aubin-Lions lemma then completes the proof.

For $r \in (\frac{9}{5}, 3)$, we use approach described in [12, Section 5.3.2]. By just using interpolation and Young inequalities we can deduce from (61) that

$$(62) \quad \begin{aligned} \frac{d}{dt} (1 + \|\nabla \mathbf{v}^\varepsilon\|_2^2) + \int_\Omega (1 + |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)^{\frac{r-2}{2}} |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2 \\ \leq C(1 + \|\nabla \mathbf{v}^\varepsilon\|_r^r)(1 + \|\nabla \mathbf{v}^\varepsilon\|_2^2)^\lambda \end{aligned}$$

with $\lambda := \frac{2(3-r)}{3r-5}$. Then dividing (62) by $(1 + \|\nabla \mathbf{v}^\varepsilon\|_2^2)^\lambda$ and integrating the resulting equation over $t \in (0, T)$ we obtain that

$$\int_0^T (1 + \|\mathbf{v}^\varepsilon\|_2^2)^{-\lambda} \int_\Omega (1 + |\mathbf{D}(\mathbf{v}^\varepsilon)|^2)^{\frac{r-2}{2}} |\mathbf{D}(\nabla \mathbf{v}^\varepsilon)|^2 dx dt \leq C$$

and following the procedure explained in [12, pages 233-237], we can deduce that there exists $\eta > 0$ such that (because $r > \frac{9}{5}$)

$$\int_0^T \|\mathbf{v}^\varepsilon\|_{1+\eta, r}^{1+\eta} \leq C.$$

Aubin-Lions lemma then completes the proof of compactness of velocity gradients. To get compactness of pressure, we simply use Lemma 3.1 and the proof is finished.

To show that the initial condition is met we can appeal to standard procedures.

4. EXAMPLES OF ADMISSIBLE VISCOSITIES

Lemma 4.1. *For any $\beta \geq 0$, $r \in (1, 1 + \beta)$, $\alpha \in [0, (r - 1)/\beta]$ and $a, b, c \geq 0$, the numbers $\alpha_0, \dots, \alpha_8$ defined for the viscosity $\nu(p, |\mathbf{D}|^2)$ given by Example 2.1 are finite. Moreover, for any $\beta \geq 0$, $r \in (1, 1 + \beta)$ there are $\alpha, a, b, c \geq 0$ such that (32), (34), and the assumptions (3)–(5) are satisfied.*

Proof. Since,

$$\begin{aligned} \partial_{\mathbf{b}_{kl}} (\nu(p, |\mathbf{D}|^2) \mathbf{D}_{ij}) \mathbf{B}_{kl} \mathbf{B}_{ij} &= \nu(p, |\mathbf{D}|^2) |\mathbf{B}|^2 \\ &+ \left(r - 2 + (r - 2 - \alpha\beta) f(p) (1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right) (\mathbf{D} \cdot \mathbf{B})^2 \\ &\cdot (1 + f(p) (1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}})^{\alpha-1} (1 + |\mathbf{D}|^2)^{\frac{r-4}{2}} \end{aligned}$$

it is easy to observe that that we can define

$$(63) \quad \gamma_1(p, |\mathbf{D}|^2) := C_1 \nu(p, |\mathbf{D}|^2), \quad \gamma_2(p, |\mathbf{D}|^2) := C_2 \nu(p, |\mathbf{D}|^2),$$

⁵This is valid only if $\mathbf{v}_0 \in W^{1,2}$. However, one can use the same arguments as in [12, proof of Theorem 3.79] to extend whole theory also to the case when the initial velocity satisfies only $\mathbf{v}_0 \in L^2$.

where C_1, C_2 are given by the following relations

$$C_1 := \begin{cases} r - 1 - \alpha\beta \\ r - 1 - \alpha\beta \\ 1 \end{cases} \quad C_2 := \begin{cases} 1 & \text{if } r \leq 2, \\ r - 1 & \text{if } r \in (2, 2 + \alpha\beta), \\ r - 1 & \text{if } r \geq 2 + \alpha\beta. \end{cases}$$

Note, that by simple algebraic manipulation one can easily find that (5) is valid with arbitrary $\sigma > 0$. Similarly, we easily obtain the following relation for γ_3 :

$$(64) \quad \gamma_3(p, |\mathbf{D}|^2) := \alpha f'(p)(1 + |\mathbf{D}|^2)^{\frac{r-1-\beta}{2}}(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}})^{\alpha-1}.$$

Before we start deriving the explicit estimates for the numbers $\alpha_0, \dots, \alpha_8$ we give a simple algebraic inequality that is valid for all $c, b, \beta > 0$ and all continuous function $f(p) > 0$:

$$(65) \quad \sup_{x \in (1, \infty)} x^{-c}(1 + f(p)x^{-\frac{\beta}{2}})^{-b} \leq \begin{cases} (f(p))^{-b} & \text{if } c \geq \frac{\beta b}{2}, \\ C_h(\beta, c, b)(f(p))^{-\frac{2c}{\beta}} & \text{if } c < \frac{\beta b}{2}, \end{cases}$$

where

$$C_h(\beta, c, b) := \frac{c^{\frac{2c}{\beta}} \left(b\frac{\beta}{2} - c\right)^{b - \frac{2c}{\beta}}}{\left(b\frac{\beta}{2}\right)^b}.$$

Using the estimate (65) we find that for all r in the range of interest

$$\gamma_3(p, |\mathbf{D}|^2) \leq \alpha C_3 f'(p)(f(p))^{\frac{r-1-\beta}{\beta}},$$

where

$$C_3 := \left(\frac{\beta + 1 - r}{2}\right)^{\frac{\beta+1-r}{\beta}} \left(\frac{(1-\alpha)\beta}{2}\right)^{\alpha-1} \left(\frac{r - \alpha\beta - 1}{2}\right)^{\frac{r-\alpha\beta-1}{\beta}}$$

Since $f'(p) = 2abc p(1 + bp^2)^{c-1}$, we can finally find that (after using the assumption $c^{\frac{r-1}{\beta}} \leq \frac{1}{2}$)

$$\begin{aligned} \gamma_3 &\leq 2\alpha C_3 a^{\frac{r-1}{\beta}} b c p(1 + bp^2)^{-1+c\frac{r-1}{\beta}} \leq 2\alpha C_3 a^{\frac{r-1}{\beta}} \sqrt{bc}(1 + bp^2)^{-\frac{1}{2}+c\frac{r-1}{\beta}} \\ &\leq 2\alpha C_3 a^{\frac{r-1}{\beta}} \sqrt{bc} \end{aligned}$$

Thus, we can simply conclude with the following observation

$$\alpha_0 \leq 2\alpha C_3 a^{\frac{r-1}{\beta}} \sqrt{bc}.$$

Also by using the definition (9), we simply deduce with help of the previous estimate that

$$\alpha_1 = \alpha_0 \frac{C_2}{C_1} \leq 2\alpha \frac{C_2 C_3}{C_1} a^{\frac{r-1}{\beta}} \sqrt{bc}.$$

Next, for α_2 we can find that for positive p the following holds

$$\frac{p\gamma_3^2}{\gamma_1(1 + |\mathbf{D}|^2)^{\frac{1}{2}}} \leq \alpha^2 p C_1^{-1} (f'(p))^2 (1 + |\mathbf{D}|^2)^{\frac{r-1}{2}-\beta} (1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}})^{\alpha-2}.$$

Thus, using the estimate (65) once again, we are led to

$$\alpha_2 \leq 4C_4 \alpha^2 C_1^{-1} a^{\frac{r-1}{\beta}} c^2 \sqrt{b}$$

with the constant C_4 defined as

$$C_4 := \left(\frac{(2-\alpha)\beta}{2} \right)^{\alpha-2} \left(\beta - \frac{r-1}{2} \right)^{\frac{2\beta-r+1}{\beta}} \left(\frac{r-1-\alpha\beta}{2} \right)^{\frac{r-1-\alpha\beta}{\beta}}.$$

In order to estimate α_3, α_5 and α_7 we need the following relation for γ_p :

$$\begin{aligned} \gamma_p(p, |\mathbf{D}|^2) &= p \partial_p \frac{\gamma_3^2(p, |\mathbf{D}|^2)}{\gamma_1(p, |\mathbf{D}|^2)} \\ (66) \quad &= \alpha^2 C_1^{-1} (1 + |\mathbf{D}|^2)^{\frac{r}{2}-\beta} p \partial_p \left((f'(p))^2 (1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}})^{\alpha-2} \right) \\ &= \alpha^2 C_1^{-1} (1 + |\mathbf{D}|^2)^{\frac{r}{2}-\beta} p f'(p) \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha-3} \\ &\quad \cdot \left(2f''(p) + (2f''(p)f(p) + (f'(p))^2(\alpha-2)) (1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right). \end{aligned}$$

Once we have the above expression, we can continue by obtaining the following estimates

$$\begin{aligned} \frac{|\gamma_p| \gamma_1}{\gamma_3} &\leq \alpha (1 + |\mathbf{D}|^2)^{\frac{r-1-\beta}{2}} p \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha-2} \\ &\quad \cdot \left(2f''(p) + (2f''(p)f(p) + (f'(p))^2(\alpha-2)) (1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right) \\ &\leq 2\alpha (1 + |\mathbf{D}|^2)^{\frac{r-1-\beta}{2}} f''(p) p \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha-2} \\ &\quad \cdot \left(1 + \left(f(p) + \frac{(f'(p))^2}{f''(p)} \left(\frac{\alpha-2}{2} \right) \right) (1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right). \end{aligned}$$

Since $f''(p) = abc(1 + bp^2)^{c-2}(1 + bp^2(2c-1))$ we can easily estimate that

$$\leq \frac{2\alpha|\alpha c - 1|}{2c-1} (1 + |\mathbf{D}|^2)^{\frac{r-1-\beta}{2}} f''(p) p \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha-1},$$

using (65) we can continue as

$$\leq \frac{2\alpha|\alpha c - 1|}{2c-1} C_3 f''(p) p f^{\frac{r-1-\beta}{\beta}}(p) \leq 4\alpha a^{\frac{r-1}{\beta}} c \sqrt{b} C_3 \frac{2|\alpha c - 1|}{2c-1} \max\{1, 2c-1\}.$$

Consequently, we have

$$\alpha_3 \leq \bar{C}_3 a^{\frac{r-1}{\beta}} c \sqrt{b},$$

where $\bar{C}_3 := 4\alpha C_3 \frac{2|\alpha c - 1|}{2c-1} \max\{1, 2c-1\}$. By using the relation $\gamma_2 = \frac{C_2}{C_1} \gamma_1$ we obtain that

$$\alpha_5 \leq \frac{C_2}{C_1} \alpha_3 \leq \frac{C_2}{C_1} \bar{C}_3 a^{\frac{r-1}{\beta}} c \sqrt{b}.$$

Repeating the same scheme again, we can also deduce that

$$\frac{\gamma_p \gamma_1}{\gamma_3^2} \geq -\frac{2\alpha|\alpha c - 1|}{2c-1} \frac{f''(p)p}{f'(p)},$$

and then using the inequality $\frac{f''(p)p}{f'(p)} \leq \max\{1, 2c-1\}$ leads to the following conclusion

$$(67) \quad \alpha_7 \geq -\frac{2\alpha|\alpha c - 1|}{2c-1} \max\{1, 2c-1\}.$$

It remains to estimate all the α 's in which the function γ_D appears. Thus, we can derive

$$\begin{aligned} \gamma_D(p, |\mathbf{D}|^2) &= p \partial_{|\mathbf{D}|^2} \frac{\gamma_3^2(p, |\mathbf{D}|^2)}{\gamma_1(p, |\mathbf{D}|^2)} \\ &= \alpha^2 C_1^{-1} p (f'(p))^2 \partial_{|\mathbf{D}|^2} \left((1 + |\mathbf{D}|^2)^{\frac{r}{2} - \beta} (1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}})^{\alpha - 2} \right) \\ &= \alpha^2 C_1^{-1} p (f'(p))^2 (1 + |\mathbf{D}|^2)^{\frac{r-2}{2} - \beta} \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha - 3} \\ &\quad \cdot \left(\frac{r}{2} - \beta + \frac{r - \alpha\beta}{2} \cdot f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right) \end{aligned}$$

and after simple calculation we obtain that

$$|\gamma_D(p, |\mathbf{D}|^2)| \leq C_5 \alpha^2 C_1^{-1} p (f'(p))^2 (1 + |\mathbf{D}|^2)^{\frac{r-2}{2} - \beta} \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha - 2}$$

where

$$C_5 := \max \left\{ \frac{|r - \alpha\beta|}{2}, \frac{|r - 2\beta|}{2} \right\}.$$

Thus,

$$\gamma_D |\mathbf{D}| \leq C_5 \alpha^2 C_1^{-1} p (f'(p))^2 (1 + |\mathbf{D}|^2)^{\frac{r-1}{2} - \beta} \left(1 + f(p)(1 + |\mathbf{D}|^2)^{-\frac{\beta}{2}} \right)^{\alpha - 2}.$$

Next, using (65), the assumption on c and the relation for $f'(p)$ we find that

$$\alpha_6 \leq 4C_5 C_6 \alpha^2 C_1^{-1} a^{\frac{r-1}{\beta}} c^2 \sqrt{b},$$

where

$$C_6 := \left(\beta - \frac{r-1}{2} \right)^{2 - \frac{r-1}{\beta}} \left(\frac{r-1-\alpha\beta}{2} \right)^{\frac{r-1-\alpha\beta}{\beta}} \left((2-\alpha) \frac{\beta}{2} \right)^{\alpha - 2}.$$

Consequently

$$\alpha_4 \leq \frac{C_2}{C_1} \alpha_6 \leq 4C_5 C_6 \alpha^2 C_1^{-1} a^{\frac{r-1}{\beta}} c^2 \sqrt{b}$$

and by using similar arguments we finally find that

$$\alpha_8 \leq 2c C_5 \alpha C_1^{-1}.$$

Thus, it is easy to observe that for fixed $\beta \geq 0$, $r \in (1, 1 + \beta)$ we can always find constants α, a, b, c such that all numbers $\alpha_0, \dots, \alpha_8 \ll 1$ and consequently they satisfy assumption (34). \square

Remark 4.1. *In order to make the statements of Lemma 4.1 even more transparent, we take, in Example 2.2, $r \in [2, 3]$ and set $\alpha = \frac{r-2}{2}$ and $\beta = 2$. Consequently, the viscosity $\nu(p, |\mathbf{D}|^2)$ then has the form*

$$\nu(p, |\mathbf{D}|^2) = (1 + f(p) + |\mathbf{D}|^2)^{\frac{r-2}{2}}.$$

Note that then the parameter c fulfills $c \in [\frac{1}{2}, \frac{1}{r-1}]$. With this special choice of α and β it is then easy to estimate constants appearing in the proof of Lemma 4.1. Thus, we have

$$C_1 = 1, C_2 = r - 1, C_3 \leq e, C_4 \leq e, \bar{C}_3 \leq \frac{4e(r-2)}{2c-1}, C_5 \leq 1, C_6 \leq e.$$

Next denoting $U := a^{\frac{r-1}{2}} b^{\frac{1}{2}} c$ we get

$$\begin{aligned} \alpha_0 &\leq (r-2)eU, \quad \alpha_1 2(r-2)eU, \quad \alpha_3 \leq \frac{4e(r-2)}{2c-1}U, \quad \alpha_4 \leq 2(r-2)^2U, \\ \alpha_5 &\leq \frac{8e(r-2)}{2c-1}U, \quad \alpha_6 \leq (r-2)^2U, \quad \alpha_7 \geq -\frac{r-2}{2c-1}, \quad \alpha_8 \leq r-2. \end{aligned}$$

Therefore, we observe that the assumption (34) is satisfied provided that the following inequality holds:

$$(r-2) \left(\frac{12}{2c-1} + 2r-1 \right) eU + 2(r-2)\sqrt{U} < 5 - 2r - \frac{r-2}{2c-1}.$$

This is however the condition on smallness of α and β .

Lemma 4.2. *The viscosity given in Example 2.2 satisfies (3)–(4). Let the function f be given by (19) or by (21), then $\alpha_0, \dots, \alpha_2$ are finite. Moreover, if a, b, c are small enough then the assumptions (33)–(32) required by Theorems 2.1 are fulfilled.*

Moreover, if f is given by (21) and $r \geq 3$ then $\alpha_3, \dots, \alpha_8$ are finite and if a, b, c are sufficiently small then the assumption (34) required by 2.2 is also fulfilled.

Proof. First, we see that the function γ_3 can be simply defined as

$$\gamma_3 := f'(p).$$

Next, we can compute

$$\frac{\partial \nu(p, |\mathbf{D}|^2) \mathbf{D}_{ij} \mathbf{B}_{kl}}{\partial \mathbf{D}_{kl}} = \nu(p, |\mathbf{D}|^2) |\mathbf{B}|^2 + \left(\nu_1(r-2)(1 + |\mathbf{D}|^2)^{\frac{r-4}{2}} - \frac{f(p)}{(\varepsilon + |\mathbf{D}|^2)^{\frac{3}{2}}} \right) (\mathbf{D} \cdot \mathbf{B})^2,$$

and it easily follows that we can define

$$(68) \quad \gamma_2(p, |\mathbf{D}|^2) := B_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} + \frac{f(p)}{\sqrt{\varepsilon + |\mathbf{D}|^2}},$$

$$(69) \quad \gamma_1(p, |\mathbf{D}|^2) := B_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} + \frac{\varepsilon f(p)}{(\varepsilon + |\mathbf{D}|^2)^{\frac{3}{2}}},$$

where

$$\begin{aligned} B_2 &:= \nu_1, \quad B_1 := \nu_1(r-1), & \text{if } r < 2, \\ B_2 &:= \nu_1(r-1), \quad B_1 := \nu_1, & \text{if } r \geq 2. \end{aligned}$$

First, it is easy to show (because f is sublinear) that γ_1, γ_2 satisfy (5) with parameters $q := r-2$ and arbitrary $\sigma > 0$.

We also derive the following estimate for $\frac{\gamma_2}{\gamma_1}$:

$$(70) \quad \begin{aligned} \frac{\gamma_2}{\gamma_1} &\leq \frac{B_2}{B_1} + \frac{\frac{f(p)}{\sqrt{\varepsilon + |\mathbf{D}|^2}}}{B_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}} + \frac{\varepsilon f(p)}{(\varepsilon + |\mathbf{D}|^2)^{\frac{3}{2}}}} \\ &= \frac{B_2}{B_1} + \frac{f(p)(\varepsilon + |\mathbf{D}|^2)}{B_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}(\varepsilon + |\mathbf{D}|^2)^{\frac{3}{2}} + \varepsilon f(p)} \leq \frac{B_2}{B_1} + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} f(p)^{\frac{2}{r+1}} \end{aligned}$$

where $B_3 := \left(\frac{2}{r-1}\right)^{\frac{1-r}{r+1}}$ for $r \geq 2$ and $B_3 := \varepsilon^{\frac{r-2}{r+1}} \left(\frac{2}{r-1}\right)^{\frac{1-r}{r+1}}$ if $r < 2$. We see that the number α_1 can then be estimated as

$$\alpha_1 := \sup_p \left(f'(p) \left(\frac{B_2}{B_1} + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} f(p)^{\frac{2}{r+1}} \right) \right).$$

Thus, if f is defined by the relation (19) then we can derive the following estimates

$$\begin{aligned} \alpha_0 &\leq 2^c ac \sqrt{b} (1-2c)^{\frac{1}{2}-c}, \\ \alpha_1 &\leq \frac{B_2}{B_1} \alpha_0 + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} a^{\frac{3+r}{r+1}} \sqrt{bc} \end{aligned}$$

and we conclude the proof for such f . The rest of the proof is devoted to f given by (21).

First, note that $f'(p) = \frac{2ab \ln(1+bp^2)}{1+bp}$ and $f''(p) = \frac{2ab^2(1-\ln(1+bp))}{(1+bp)^2}$ for all $p > 0$. In what follows we also frequently use the inequality

$$(71) \quad \ln^b(y) \leq b^b y, \quad \text{valid for all } b, y \geq 1.$$

Having this we can simply estimate the numbers α_0, α_1 as (assuming $c \geq 1$)

$$\begin{aligned} \alpha_0 &\leq 2ab, \\ \alpha_1 &\leq 2ab \frac{B_2}{B_1} + B_3 B_1 \varepsilon^{\frac{1-r}{1+r}} \left(2abc^{\frac{2}{r+1}} + 2a^{\frac{3+r}{r+1}} b \left(\frac{5+r}{r+1} \right)^{\frac{5+r}{r+1}} \right). \end{aligned}$$

Moreover, we can obtain the estimate

$$\alpha_2 \leq \sup_{p, d^2} \frac{1}{B_1} |p| \gamma_3^2(p, d^2) \leq 16a^2 b B_1^{-1}.$$

Next, we derive an explicit formula for γ_p .

$$\gamma_p = p \gamma_3 \frac{2\gamma_3' \gamma_1 - \gamma_3 \gamma_1'}{\gamma_1^2} = \frac{2p f'(p) f''(p)}{\gamma_1} - \frac{\varepsilon p (f'(p))^3}{\gamma_1^2 (\varepsilon + |\mathbf{D}|^2)^{\frac{3}{2}}}.$$

Thus, we can deduce that

$$\begin{aligned} \alpha_3 &\leq \sup_p (2|p f''(p)| + B_1^{-1} \varepsilon^{-\frac{1}{2}} |p (f'(p))^2|) \leq 4ab + 4\varepsilon B_1^{-1} a^2 b, \\ \alpha_5 &\leq \sup_{p, d^2} \frac{\gamma_2(p, d^2)}{\gamma_1(p, d^2)} (2|p f''(p)| + \varepsilon^{-\frac{1}{2}} p (f'(p))^2 B_1^{-1}) \\ &\stackrel{(70)}{\leq} \sup_p \left(\frac{B_2}{B_1} + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} f(p)^{\frac{2}{r+1}} \right) (2|p f''(p)| + \varepsilon^{-\frac{1}{2}} p (f'(p))^2 B_1^{-1}) \\ &\leq \left(\frac{B_2}{B_1} + c^{\frac{2}{r+1}} B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{-2r}{1+r}} \right) \alpha_3 + 2B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} a^{\frac{3+r}{r+1}} \left(\frac{5+r}{r+1} \right)^{\frac{5+r}{r+1}}, \\ \alpha_7 &\geq \inf_p \left(p \frac{f''(p)}{f'(p)} \right) - \sup_{p, d^2} \frac{\varepsilon p f'(p)}{\gamma_1(p, d^2) (\varepsilon + d^2)^{\frac{3}{2}}} \geq -b - \sup_p \frac{p f'(p)}{B_1 + f(p)} \\ &\geq -b - \frac{2a}{\sqrt{B_1 + c}}. \end{aligned}$$

The last step is to derive the formula for γ_D and to prove the desired remaining estimates. Since

$$(72) \quad \gamma_D = -2 \frac{pf'(p)^2}{\gamma_1^2(p, |\mathbf{D}|^2)} \left(B_1(r-2)(1+|\mathbf{D}|^2)^{\frac{r-4}{2}} - 3\varepsilon f(p)(\varepsilon+|\mathbf{D}|^2)^{-\frac{5}{2}} \right),$$

we obtain the estimate

$$(73) \quad |\gamma_D|(\varepsilon+|\mathbf{D}|^2) \leq \max(|r-2|, 3) \frac{|p|(f'(p))^2}{\gamma_1} =: B_4 \frac{|p|(f'(p))^2}{\gamma_1}.$$

Using this estimate and taking into account (70) we get

$$\begin{aligned} \alpha_4 &\leq \sup_p 2B_4 \left(\frac{B_2}{B_1} + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} f(p)^{\frac{2}{r+1}} \right) B_1^{-1} \varepsilon^{-\frac{1}{2}} pf'(p)^2 \\ &\leq B_4 \left(\frac{B_2}{B_1} + B_3 B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} \right) 30 B_1^{-1} \varepsilon^{-\frac{1}{2}} b \left(a^{\frac{4+2r}{r+1}} + c^{\frac{2}{r+1}} \right), \\ \alpha_6 &\stackrel{(72)}{\leq} \sup_{p,d} B_4 \frac{pf'(p)^2}{\gamma_1(p, d^2)(\varepsilon+d^2)^{\frac{1}{2}}} \leq \sup_p B_4 B_1^{-1} \varepsilon^{-\frac{1}{2}} pf'(p)^2 \leq 16a^2 b B_4 B_1^{-1} \varepsilon^{-\frac{1}{2}}. \end{aligned}$$

Finally, we deduce the estimate on α_8 that is valid if $r \geq 3$

$$\begin{aligned} \alpha_8 &\leq \sup_{p,d} \frac{pf'(p)}{\gamma_1(p, d^2)(\varepsilon+d^2)^{\frac{1}{2}}} \\ &\leq \sup_p \varepsilon^{-\frac{1}{2}} 4a B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} (1 + \ln(1+bp))^{\frac{3-r}{r+1}} \stackrel{(r \geq 3)}{\leq} \varepsilon^{-\frac{1}{2}} 4a B_1^{-\frac{2}{r+1}} \varepsilon^{\frac{1-r}{1+r}} \end{aligned}$$

We see, that in all α 's there appear numbers a, b and we can chose them in such a way that all numbers $\alpha_0, \dots, \alpha_8$ are small enough to validate the assumptions (33)–(34). \square

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