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Analysis of the Global Relation for the Nonlinear Schrödinger Equation on the Half-line

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Abstract. It has been shown recently that the unique, global solution of the Dirichlet problem of the nonlinear Schrödinger equation on the half-line can be expressed through the solution of a 2×2 matrix Riemann–Hilbert problem. This problem is specified by the spectral functions $\{a(k), b(k)\}$ which are defined in terms of the initial condition $q(x, 0) = q_0(x)$, and by the spectral functions $\{A(k), B(k)\}$ which are defined in terms of the specified boundary condition $q(0, t) = g_0(t)$ and the unknown boundary value $q_x(0, t) = g_1(t)$. Furthermore, it has been shown that given q_0 and g_0, the function g_1 can be characterized through the solution of a certain 'global relation' coupling q_0, g_0, g_1, and $\Phi(t, k)$, where Φ satisfies the t-part of the associated Lax pair evaluated at $x = 0$. We show here that, by using a Gelfand–Levitan–Marchenko triangular representation of Φ, the global relation can be explicitly solved for g_1.

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1. Introduction

Let the complex-valued function $q(x, t)$ satisfy the nonlinear Schrödinger equation

$$iq_t + q_{xx} - 2\rho|q|^2 q = 0, \qquad \rho = \pm 1, \quad 0 < x < \infty,\ 0 < t < T, \tag{1}$$

and the initial and boundary conditions

$$q(x, 0) = q_0(x) \in \mathcal{S}(\mathbb{R}^+), \quad 0 < x < \infty, \tag{2a}$$

$$q(0, t) = g_0(t), \quad 0 < t < T, \tag{2b}$$

where T is a given fixed constant, $\mathcal{S}$ denotes the space of Schwartz functions, and the function $g_0(t)$ has sufficient smoothness. The solution of this initial boundary value (IBV) problem can be constructed as follows ([3, 6]):

- *Given $q_0(x)$ construct the spectral functions* $\{a(k), b(k)\}$. These functions are defined by

$$a(k) = \phi_2(0,k), \qquad b(k) = \phi_1(0,k), \tag{3}$$

where the vector

$$\phi(x,k) = \begin{pmatrix} \phi_1(x,k) \\ \phi_2(x,k) \end{pmatrix}$$

is the following solution of the x-problem of the associated Lax pair evaluated at $t = 0$:

$$\phi_x + ik\sigma_3\phi = Q_0(x)\phi, \quad 0 < x < \infty, \ \operatorname{Im} k \geqslant 0, \tag{4a}$$

$$\phi(x,k) = \mathrm{e}^{ikx}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathrm{o}(1)\right) \quad \text{as } x \to \infty, \tag{4b}$$

with σ_3 and $Q_0(x)$ defined by

$$\sigma_3 = \operatorname{diag}\{1, -1\}, \qquad Q_0(x) = \begin{pmatrix} 0 & q_0(x) \\ \rho\bar{q}_0(x) & 0 \end{pmatrix}. \tag{5}$$

- *Given $q_0(x)$ and $g_0(t)$ characterize $g_1(t)$* by the requirement that the spectral functions $\{A(t,k), B(t,k)\}$ satisfy the *global relation*

$$a(k)B(t,k) - b(k)A(t,k) = \mathrm{e}^{4ik^2t}c(t,k), \quad t \in [0,T], \ k \in \bar{D}, \tag{6}$$

where $\bar{D}$ denotes the closure of the first quadrant

$$D = \{k \in \mathbb{C} \mid \operatorname{Re} k > 0, \ \operatorname{Im} k > 0\},$$

and $c(t,k)$ is analytic in $k \in D$ and is $\mathrm{O}(1/k)$ as $k \to \infty$. The spectral functions $A(t,k)$, $B(t,k)$ are defined by

$$A(t,k) = \mathrm{e}^{2ik^2t}\overline{\Phi_2(t,\bar{k})}, \qquad B(t,k) = -\mathrm{e}^{2ik^2t}\Phi_1(t,k), \tag{7}$$

where the vector

$$\Phi(t,k) = \begin{pmatrix} \Phi_1(t,k) \\ \Phi_2(t,k) \end{pmatrix}$$

is the following solution of the t-problem of the associated Lax pair evaluated at $x = 0$:

$$\Phi_t + 2ik^2\sigma_3\Phi = (2kQ(t) + R(t))\Phi, \quad 0 < t < T, \ k \in \mathbb{C}, \tag{8a}$$

$$\Phi(0,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{8b}$$

with $Q(t)$ and $R(t)$ defined by

$$Q(t) = \begin{pmatrix} 0 & g_0(t) \\ \rho\bar{g}_0(t) & 0 \end{pmatrix}, \qquad R(t) = i\rho\begin{pmatrix} -|g_0(t)|^2 & \rho g_1(t) \\ -\bar{g}_1(t) & |g_0(t)|^2 \end{pmatrix}. \tag{9}$$

- *Given* $a(k)$, $b(k)$, $A(k) = A(T,k)$, $B(k) = B(T,k)$, *define a* 2×2 *matrix Riemann–Hilbert* (*RH*) *problem*. This problem has the distinctive feature that its jump has explicit (x,t)-dependence in the exponential form $\exp\{ikx + 2ik^2 t\}$. Determine $q(x,t)$ in terms of the solution of this RH problem. The function $q(x,t)$ solves the NLS equation with initial-boundary conditions $q(x,0) = q_0(x)$, $q(0,t) = g_0(t)$.

The most complicated step in the above construction is the characterization of g_1. In the particular case $q_0(x) \equiv 0$, it can be shown that the global relation is equivalent to the integral relation

$$\int_{\partial D} k \int_0^t e^{4ik^2(t'-t)+i\rho\int_0^{t'}|g_0|^2\,d\tau}(2kg_0(t') + ig_1(t'))\overline{\Phi_2(t',\bar{k})}\,dt'\,dk = 0, \tag{10}$$

where $\partial D = (i\infty, 0) \cup (0, \infty)$ is the oriented boundary of D. Given $g_0(t)$, the vector equation (8a) and the scalar equation (10) are two equations for the unknown vector-valued and scalar-valued functions $\Phi(t,k)$ and $g_1(t)$, respectively. It is shown in [6] that these equations constitute a system of *coupled* nonlinear Volterra integral equations for Φ and g_1.

1.1. A FORMULATION IN TERMS OF $\{M_j(t,s), L_j(t,s)\}_{j=1,2}$

In this Letter we will show that the above system *decouples*. Indeed, we will show that the global relation (6) can be solved *explicitly* for $g_1(t)$ in terms of $g_0(t)$ and $\Phi(t,k)$. For this purpose we will use the Gelfand–Levitan–Marchenko integral representation for $\Phi(t,k)$; namely we will express $\Phi(t,k)$ in terms of four scalar functions $\{M_j(t,s), L_j(t,s)\}_{j=1,2}$, see Proposition 2. The function Φ satisfies Equation (8) if and only if these four functions satisfy Equations (23), (24). Furthermore using definitions (7), it follows that $A(t,k)$ and $B(t,k)$ can be expressed in terms of these four functions by

$$A(t,k) = 1 + \int_0^t e^{4ik^2\tau}[2\bar{L}_2(t,t-2\tau) - i\rho g_0(t)\bar{M}_1(t,t-2\tau) + 2k\bar{M}_2(t,t-2\tau)]\,d\tau, \tag{11a}$$

$$B(t,k) = -\int_0^t e^{4ik^2\tau}[2L_1(t,2\tau-t) - ig_0(t)M_2(t,2\tau-t) + 2kM_1(t,2\tau-t)]\,d\tau. \tag{11b}$$

It turns out that if we replace A and B in the global relation (6) by the above expressions, then the global relation can be solved in closed form for $g_1(t)$ in terms of $g_0(t)$ and $\{M_j, L_j\}_{j=1,2}$, see Proposition 3. In the particular case $q_0(x) \equiv 0$, the following proposition is valid.

PROPOSITION 1. *Let* $B(t,k)$ *satisfy the global relation corresponding to the case of zero initial condition,* $q_0(x) \equiv 0$*, i.e., Equation* (6) *with* $a \equiv 1$, $b \equiv 0$:

$$B(t,k) = e^{4ik^2 t}c(t,k), \quad k \in \bar{D}, \text{ i.e. } \operatorname{Re} k \geqslant 0, \operatorname{Im} k \geqslant 0. \tag{12}$$

Let the solution $\Phi(t,k)$ of Equation (8) *be expressed in terms of the functions $M_j(t,s)$, $L_j(t,s)$, $j=1,\,2$, see Proposition* 2. *Then Equation* (12) *can be solved explicitly for $g_1(t)$ in terms of $g_0(t)$, M_1, and M_2*:

$$g_1(t) = g_0(t)M_2(t,t) + \frac{4i}{\pi}\int_{\partial D}\left[2k^2\int_0^t e^{4ik^2(\tau-t)}M_1(t,2\tau-t)\,\mathrm{d}\tau - \frac{g_0(t)}{2\mathrm{i}}\right]\mathrm{d}k. \tag{13}$$

Proof. We multiply the global relation (12) by $k\exp[-4\mathrm{i}k^2t']$, $t'<t$, and integrate along ∂D. Using the fact that $kc(k,t)$ is $\mathrm{O}(1)$ as $k\to\infty$, and that $\exp[4\mathrm{i}k^2(t-t')]$ is bounded in D, Jordan's lemma implies that the right-hand side of the resulting equation vanishes. Using for $B(t,k)$ the expression (11b) we find

$$-\int_{\partial D}\left[\int_0^t e^{4ik^2(\tau-t')}(2kL_1(t,2\tau-t) - ig_0(t)kM_2(t,2\tau-t) + {} \right.$$
$$\left. + 2k^2M_1(t,2\tau-t))\mathrm{d}\tau\right]\mathrm{d}k = 0. \tag{14}$$

The term involving M_1 can be written as

$$-\int_{\partial D}\left[\int_0^{t'} e^{4ik^2(\tau-t')}2k^2M_1(t,2\tau-t)\,\mathrm{d}\tau - \frac{M_1(t,2t'-t)}{2i}\right]\mathrm{d}k$$
$$-\int_{\partial D}\left[\int_{t'}^{t} e^{4ik^2(\tau-t')}2k^2M_1(t,2\tau-t)\,\mathrm{d}\tau + \frac{M_1(t,2t'-t)}{2i}\right]\mathrm{d}k.$$

Integration by parts implies that the bracket appearing in the second integral is bounded in D; since $\tau>t'$, this integral vanishes.

In order to evaluate the term in (14) involving L_1, we note that the contour ∂D involves an integral from 0 to ∞ which can be mapped to an integral from 0 to $-\infty$ by replacing k with $-k$, thus ∂D, can be replaced by $\partial\tilde{D}$, where $\tilde{D}$ denotes the second quadrant of the complex k-plane. Hence, this term can be written as

$$-\int_{\partial\tilde{D}^0}\left[\int_0^{t'} e^{4ik^2(\tau-t')}2kL_1(t,2\tau-t)\mathrm{d}\tau - \frac{L_1(t,2t'-t)}{2ik}\right]\mathrm{d}k \quad -\frac{L_1(t,2t'-t)}{2i}\int_{\partial\tilde{D}^0}\frac{\mathrm{d}k}{k}$$
$$-\int_{\partial D^0}\left[\int_{t'}^{t} e^{4ik^2(\tau-t')}2kL_1(t,2\tau-t)\mathrm{d}\tau + \frac{L_1(t,2t'-t)}{2ik}\right]\mathrm{d}k \quad +\frac{L_1(t,2t'-t)}{2i}\int_{\partial D^0}\frac{\mathrm{d}k}{k},$$

where the superscript zero in $\partial\tilde{D}^0$ and ∂D^0 indicates that we have deformed the contours to avoid $k=0$. The first and the third integrals vanish, since each of the brackets is bounded and analytic in $\tilde{D}^0$ and D^0, respectively. The remaining two integrals equal

$$-\frac{L_1(t,2t'-t)}{2i}\int_0^{\pi} i\,\mathrm{d}\theta = -\frac{\pi}{2}L_1(t,2t'-t).$$

Using the same method for the term in (14) involving M_2, we finally find

$$-\frac{\pi}{2}L_1(t,2t'-t)+\frac{i\pi}{4}g_0(t)M_2(t,2t'-t)-$$
$$-\int_{\partial D}\left[\int_0^{t'} e^{4ik^2(\tau-t')}2k^2M_1(t,2\tau-t)\,d\tau-\frac{M_1(t,2t'-t)}{2i}\right]dk=0. \tag{15}$$

Letting $t'\to t$ and expressing $L_1(t,t)$ in terms of $g_1(t)$, see (33a), Equation (15) becomes (13). □

1.2. A FORMULATION IN TERMS OF $\{\mu_j(t,k),\lambda_j(t,k)\}_{j=1,2}$

Equation (13) expresses g_1 in terms of $M_1(t,s)$ and $M_2(t,t)$, where the functions $M_j(t,s)$, $L_j(t,s)$, $j=1,2$ satisfy Equations (25), (26). It is more convenient to express these latter functions in terms of the functions $\{\mu_j(t,k),\lambda_j(t,k)\}_{j=1,2}$ defined by the equations ($j=1,2$)

$$\mu_j(t,k)=\int_{-t}^{t} e^{2ik^2(s-t)}M_j(t,s)\,ds, \quad t62;\,0,\ k\in\mathbb{C}, \tag{16a}$$

$$\lambda_j(t,k)=\int_{-t}^{t} e^{2ik^2(s-t)}L_j(t,s)\,ds, \quad t62;\,0,\ k\in\mathbb{C}. \tag{16b}$$

Using these functions, the expressions for $A(t,k)$ and $B(t,k)$, given by Equations (11), become

$$A(t,k)=1+\left(\overline{\lambda_2(t,\bar{k})}-\frac{i\rho}{2}g_0(t)\overline{\mu_1(t,\bar{k})}+k\overline{\mu_2(t,\bar{k})}\right), \tag{17a}$$

$$B(t,k)=-e^{4ik^2t}\left(\lambda_1(t,k)-\frac{i}{2}g_0(t)\mu_2(t,k)+k\mu_1(t,k)\right). \tag{17b}$$

Furthermore, rewriting Equations (13) and (25), (26), in terms of the new variables $\mu_j(t,k)$, $\lambda_j(t,k)$, $j=1,2$ we arrive at the following result.

THEOREM 1. *Let $q(x,t)$ satisfy the NLS equation on the half-line $0<x<\infty$, $t>0$ with the initial and boundary conditions*

$$q(x,0)=0, \qquad 0<x<\infty,$$
$$q(0,t)=g_0(t), \qquad t>0,$$

where $g_0(t)$ is a smooth function satisfying $g_0(0)=0$.

Then $g_1(t):=q_x(0,t)$ can be expressed explicitly in terms of $\mu_1(t,k)$ and $\mu_2(t,k)$ by the equation

$$g_1(t)=\frac{2g_0(t)}{\pi}\int_{\partial D}k\mu_2(t,k)\,dk+\frac{4i}{\pi}\int_{\partial D}\left[k^2\mu_1(t,k)-\frac{g_o(t)}{2i}\right]dk, \tag{18}$$

where the functions $\{\mu_j(t,k), \lambda_j(t,k)\}_{j=1,2}$ *for* $t > 0$, $k \in \mathbb{C}$, *satisfy the system of equations*

$$\lambda_{1t} + 4ik^2\lambda_1 = ig_1(t)\lambda_2 + \alpha(t)\mu_1 + \beta(t)\mu_2 + ig_1(t), \tag{19a}$$

$$\lambda_{2t} = -i\rho\bar{g}_1(t)\lambda_1 - \alpha(t)\mu_2 + \rho\bar{\beta}(t)\mu_1, \tag{19b}$$

$$\mu_{1t} + 4ik^2\mu_1 = 2g_0(t)\lambda_2 + ig_1(t)\mu_2 + 2g_0(t), \tag{19c}$$

$$\mu_{2t} = 2\rho\bar{g}_0(t)\lambda_1 - i\rho\bar{g}_1(t)\mu_1, \tag{19d}$$

with

$$\alpha(t) = \frac{\rho}{2}(g_0\bar{g}_1 - \bar{g}_0 g_1), \qquad \beta(t) = \frac{1}{2}\left(\frac{\mathrm{d}g_0}{\mathrm{d}t} - \rho|g_0|^2 g_0\right), \tag{20}$$

and the initial conditions

$$\lambda_j(0,k) = \mu_j(0,k) = 0, \quad j = 1,2. \tag{21}$$

We note that, in the framework of the Dirichlet problem, $M_2(t,t)$ is an *unknown* function; that is why we prefer to use the function $\mu_2(t,k)$ in the formulation of Theorem 1, Equation (18), as well as in Theorem 2 below.

Remark 1. Replacing $g_1(t)$ in Equations (19) by the explicit expression (18) we obtain a system of nonlinear Volterra integral equations for $\{\mu_j, \lambda_j\}_{j=1,2}$ in terms of the given function $g_0(t)$. The rigorous analysis of this system remains open.

Remark 2. Suppose that $q(x,t)$ satisfies the linearized Schrödinger equation $iq_t + q_{xx} = 0$. In this case the global relation is ([4])

$$\hat{g}(k,t) = -\hat{q}_0(k) + \mathrm{e}^{ik^2 t}c(k,t), \quad t \in [0,T],\ k \in D, \tag{22}$$

where $\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$, and $\hat{g}(k,t)$ is defined by

$$\hat{g}(k,t) = \int_0^t \mathrm{e}^{ik^2\tau}(ig_1(\tau) - kg_0(\tau))\,\mathrm{d}\tau. \tag{23}$$

Actually it can be shown that for small q,

$$a \to 1, \quad A \to 1, \quad b \to -\hat{q}_0(k), \quad B \to \hat{g},$$

and the global relation (6) yields Equation (22). The analysis of the latter equation plays a key role in the solution of the linearized Schrödinger equation. This analysis makes crucial use of the particular explicit k-dependence of $\hat{g}(k,t)$. Using this dependence one can either

(i) utilise the transformation $k \mapsto -k$ to eliminate the term containing the unknown function $g_1(\tau)$,

or

(ii) obtain explicitly g_1 in terms of g_0 by multiplying Equation (22) with $\exp(ik^2 t)$ and integrating w.r.t. k over ∂D.

It is remarkable, that if one uses the Gelfand–Levitan–Marchenko representation for Φ, then the global relation (6) has precisely the *same* k dependence as Equation (20), see the expressions (11) for A, B. Thus again one can obtain explicitly g_1 using the same procedure as in (ii) above.

Remark 3. The first attempt to characterize the spectral functions was made in [5] and led to a formal nonlinear RH problem. A similar formulation was presented in [2], where a different formulation was also presented based on an attempt to express g_1 explicitly in terms of Φ using certain analyticity arguments. However, all these formal attempts yield a system of nonlinear *Fredholm* integral equations for the spectral functions. This is to be contrasted with the formulation of [6] as well as with the simplified formulation presented here, which yield a system of nonlinear *Volterra* integral equations. The insurmountable problem with the former formulations is that, since the spectral functions are characterised to within an equivalent class ($c(t, k)$ in Equation (6) is arbitrary), one *cannot* rigorously establish solvability for the associated Fredholm equations.

Remark 4. For economy of presentation we have concentrated on the Dirichlet boundary value problem. This analysis applies mutatis-mutandis to the Neumann boundary value problem as well.

Organization of the Letter. In Section 2 we present the Gelfand–Levitan–Marchenko representation for Φ [1], and derive Equations (19). In Section 3, we present the analogue of Theorem 1 when $q(x, 0) \not\equiv 0$. In Section 4 we discuss linearizable boundary conditions; namely, it has been shown in [3, 6] that for some particular boundary conditions it is possible to bypass the nonlinear Volterra equations satisfied by g_1 and Φ, and to define the spectral functions $\{A(k), B(k)\}$ using only algebraic manipulations. These particular cases can also be analysed using the present formulation.

2. The Gelfand–Levitan–Marchenko Representation of $\Phi(t, k)$

PROPOSITION 2 ([1]). *Let the* 2*-vector function* $\Phi(t, k)$ *satisfy* (8). *Then:*

(i) $\Phi(t, k)$ *can be represented in the form*

$$\Phi(t,k) = \begin{pmatrix} 0 \\ e^{2ik^2 t} \end{pmatrix} + \int_{-t}^{t} \begin{pmatrix} L_1(t,s) - \frac{i}{2} g_0(t) M_2(t,s) + k M_1(t,s) \\ L_2(t,s) + \frac{i\rho}{2} \bar{g}_0(t) M_1(t,s) + k M_2(t,s) \end{pmatrix} e^{2ik^2 s}\, ds, \tag{24}$$

where the four functions L_1, L_2, M_1, *and* M_2 *satisfy the differential equations with* $t > 0$, $-t < s < t$

$$L_{1t} - L_{1s} = i g_1(t) L_2 + \alpha(t) M_1 + \beta(t) M_2, \tag{25a}$$

$$L_{2t} + L_{2s} = -i\rho\bar{g}_1(t)L_1 - \alpha(t)M_2 + \rho\bar{\beta}(t)M_1, \tag{25b}$$

$$M_{1t} - M_{1s} = 2g_0(t)L_2 + ig_1(t)M_2, \tag{25c}$$

$$M_{2t} + M_{2s} = 2\rho\bar{g}_0(t)L_1 - i\rho\bar{g}_1(t)M_1, \tag{25d}$$

as well as the boundary conditions

$$L_1(t,t) = \frac{i}{2}g_1(t), \quad L_2(t,-t) = 0, \tag{26a}$$

$$M_1(t,t) = g_0(t), \quad M_2(t,-t) = 0, \tag{26b}$$

with α *and* β *defined by* (20).

(ii) *Let* $\{\mu_j(t,k), \lambda_j(t,k)\}_{j=1,2}$ *be defined in terms of* $\{M_j(t,s), L_j(t,s)\}_{j=1,2}$ *by Equations* (14). *Then Equations* (19) *are valid.*

Proof. (i) Let the 2×2 matrix-valued function $\Psi(t,k)$ satisfy Equation (8a) and the initial condition $\Psi(0,k) = I$. Using the representation

$$\Psi(t,k) = \mathrm{e}^{-2ik^2t\sigma_3} + \int_{-t}^{t}(N(t,s) + kM(t,s))\mathrm{e}^{-2ik^2s\sigma_3}\,\mathrm{d}s \tag{27}$$

into Equation (8a) and integrating by parts in order to eliminate integral terms containing $k^2\mathrm{e}^{-2ik^2s\sigma_3}$ and $k^3\mathrm{e}^{-2ik^2s\sigma_3}$, we find

$$[\ldots]k\mathrm{e}^{-2ik^2t\sigma_3} + [\ldots]\mathrm{e}^{-2ik^2t\sigma_3} + [\ldots]k\mathrm{e}^{2ik^2t\sigma_3} + [\ldots]\mathrm{e}^{2ik^2t\sigma_3} + \\ + \int_{-t}^{t}[\ldots]k\mathrm{e}^{-2ik^2s\sigma_3}\,\mathrm{d}s + \int_{-t}^{t}[\ldots]\mathrm{e}^{-2ik^2s\sigma_3}\,\mathrm{d}s = 0.$$

Hence, each of the above brackets $[\ldots]$ vanishes, which gives

$$M(t,t) - \sigma_3 M(t,t)\sigma_3 = 2Q(t), \tag{28a}$$

$$N(t,t) - \sigma_3 N(t,t)\sigma_3 = iQ(t)M(t,t)\sigma_3 + R(t), \tag{28b}$$

$$M(t,-t) + \sigma_3 M(t,-t)\sigma_3 = 0, \tag{28c}$$

$$N(t,-t) + \sigma_3 N(t,-t)\sigma_3 + iQ(t)M(t,-t)\sigma_3 = 0, \tag{28d}$$

$$M_t(t,s) + \sigma_3 M_s(t,s)\sigma_3 = 2Q(t)N(t,s) + R(t)M(t,s), \tag{28e}$$

$$N_t(t,s) + \sigma_3 N_s(t,s)\sigma_3 = -iQ(t)M_s(t,s)\sigma_3 + R(t)N(t,s). \tag{28f}$$

Denote by A_{diag} and A_{off} the diagonal and off-diagonal parts of a matrix A, respectively. Equation (28a) is consistent with Q being off-diagonal and it gives

$$M_{\mathrm{off}}(t,t) = Q(t). \tag{29}$$

The diagonal part of (28b) reads

$$R_{\text{diag}}(t) + iQ^2(t)\sigma_3 = 0,$$

which is consistent with the form of $R(t)$, see (9), whereas the off-diagonal part of (28b) gives

$$N_{\text{off}}(t,t) = \frac{1}{2}R_{\text{off}}(t) + \frac{i}{2}Q(t)\sigma_3 M_{\text{diag}}(t,t). \tag{30}$$

Equations (28c) and (28d) give

$$M_{\text{diag}}(t,-t) = 0 \tag{31}$$

and

$$N_{\text{diag}}(t,-t) = \frac{i}{2}Q(t)\sigma_3 M_{\text{off}}(t,-t), \tag{32}$$

respectively.

Equations (30), (32), and (28f) suggest the introduction of a new function L, to be used instead of N:

$$L(t,s) = N(t,s) - \frac{i}{2}Q(t)\sigma_3 M(t,s).$$

Then the boundary conditions (30) and (32) simplify to

$$L_{\text{off}}(t,t) = \frac{1}{2}R_{\text{off}}(t), \tag{33a}$$

$$L_{\text{diag}}(t,-t) = 0. \tag{33b}$$

Writing the differential equations (28e) and (28f) in terms of L and M we find

$$M_t(t,s) + \sigma_3 M_s(t,s)\sigma_3 = 2Q(t)L(t,s) + R_{\text{off}}(t)M(t,s), \tag{34a}$$

$$L_t(t,s) + \sigma_3 L_s(t,s)\sigma_3 = R_{\text{off}}(t)L(t,s) + W(t)M(t,s), \tag{34b}$$

where

$$W = \frac{i}{2}(R_{\text{off}}Q + QR_{\text{off}})\sigma_3 - \frac{i}{2}\frac{\mathrm{d}Q}{\mathrm{d}t}\sigma_3 - \frac{1}{2}Q^3. \tag{35}$$

Taking into account the particular form of Q and R (see (9)) and writing the matrices L and M as

$$L = \begin{pmatrix} \bar{L}_2 & L_1 \\ \rho\bar{L}_1 & L_2 \end{pmatrix}, \quad M = \begin{pmatrix} \bar{M}_2 & M_1 \\ \rho\bar{M}_1 & M_2 \end{pmatrix},$$

the matrix differential equations (34) reduce to the system of four scalar equations (23), whereas the boundary conditions (29), (30), and (32) reduce to (26). Equations (25) and (26) constitute a well-posed Goursat problem, the solution of which can be obtained via the solution of the associated system of linear Volterra integral equations.

(ii) Multiplying (25) by $e^{2ik^2 s}$, integrating with respect to s from $-t$ to t, and using the definitions (16) and the boundary conditions (26), we derive Equations (19). For this derivation we note that integration by parts yields

$$\int_{-t}^{t} e^{2ik^2 s} M_{j_s}(t,s)\,ds = e^{2ik^2 t} M_j(t,t) - e^{-2ik^2 t} M_j(t,-t) - 2ik^2 e^{2ik^2 t}\mu_j(t,k). \tag{36}$$

Also, differentiating μ_j with respect to t we find

$$\int_{-t}^{t} e^{2ik^2 s} M_{j_t}(t,s)\,ds = -e^{2ik^2 t} M_j(t,t) - e^{-2ik^2 t} M_j(t,-t) + (e^{2ik^2 t}\mu_j)_t(t,k). \tag{37}$$

The functions $L_j(t,s)$ satisfy similar equations. It is important to note that although the functions $M_1(t,-t)$, $M_2(t,t)$, $L_1(t,-t)$, and $L_2(t,t)$ are unknown, these functions cancel when we use Equations (36), (37), and the analogous equations involving L_j, in Equations (25):

$$\begin{aligned}
&\int_{-t}^{t} e^{2ik^2(s-t)}[M_{1t} - M_{1s}]\,ds = -2g_0(t) + \mu_{1t} + 4ik^2\mu_1,\\
&\int_{-t}^{t} e^{2ik^2(s-t)}[L_{1t} - L_{1s}]\,ds = -ig_1(t) + \lambda_{1t} + 4ik^2\lambda_1,\\
&\int_{-t}^{t} e^{2ik^2(s-t)}[M_{2t} + M_{2s}]\,ds = \mu_{2t},\\
&\int_{-t}^{t} e^{2ik^2(s-t)}[L_{2t} + L_{2s}]\,ds = \lambda_{2t}.
\end{aligned}$$

□

3. Solution of the Global Relation

PROPOSITION 3. *Let $A(t,k)$ and $B(t,k)$ satisfy the global relation* (6). *Let the solution $\Phi(t,k)$ of Equation* (8*a*) *be expressed in terms of the functions $\{M_j(t,s), L_j(t,s)\}_{j=1,2}$, see Proposition* 2. *Then Equation* (6) *can be solved explicitly for $g_1(t)$ in terms of $g_0(t)$, M_1, M_2, and L_2:*

$$\begin{aligned}
g_1(t) = {} & g_0(t)\left\{M_2(t,t) + \frac{4\rho}{\pi}\int_{\partial D} kR(k)\int_0^t e^{4ik^2(\tau-t)}\bar{M}_1(t,t-2\tau)\,d\tau\,dk\right\} + \\
& + \frac{4i}{\pi}\int_{\partial D} kR(k)e^{-4ik^2 t}\,dk + \frac{4i}{\pi}\int_{\partial D}\left[2k^2\int_0^t e^{4ik^2(\tau-t)}M_1(t,2\tau-t)\,d\tau - \frac{g_0(t)}{2i}\right]dk + \\
& + \frac{8i}{\pi}\int_{\partial D} kR(k)\int_0^t e^{4ik^2(\tau-t)}\left[\bar{L}_2(t,t-2\tau) + k\bar{M}_2(t,t-2\tau)\right]d\tau\,dk,
\end{aligned} \tag{38}$$

where $R(k) = b(k)/a(k)$ and, if $a(k)$ has zeros in D, the contour ∂D has to be deformed to pass 'above' all the zeros.

Proof. Write the global relation (6) in the form

$$B(t,k) = R(k)A(t,k) + e^{4ik^2t}\frac{c(k)}{a(k)}, \tag{39}$$

multiply it by $k\exp[-4ik^2t']$, $t' < t$, and integrate along ∂D. Since $kR(k)A(t,k)$ is O(1) as $k \to \infty$, the integral involving this term is well-defined. Then the proof of Proposition 3 follows the same lines as the proof of Proposition 1. □

In terms of the functions $\{\mu_j, \lambda_j\}_{j=1,2}$ the analogue of Proposition 3 is the following theorem.

THEOREM 2. *Let $q(x,t)$ satisfy the NLS equation on the half-line $0 < x < \infty$, $t > 0$ with the initial and boundary conditions*

$$q(x,0) = q_0(x) \in \mathscr{S}(\mathbb{R}^+), \qquad 0 < x < \infty,$$
$$q(0,t) = g_0(t), \quad t > 0,$$

where $g_0(t)$ is a smooth function satisfying $g_0(0) = q_0(0)$. Then $g_1(t) := q_x(0,t)$ can be expressed in terms of the functions $\mu_1(t,k)$, $\mu_2(t,k)$, and $\lambda_2(t,k)$ by the equation

$$\begin{aligned} g_1(t) = {}& \frac{4i}{\pi}\int_{\partial D} e^{-4ik^2t}kR(k)\,dk + \frac{2g_0(t)}{\pi}\int_{\partial D} k\Big\{\mu_2(t,k) + \rho e^{-4ik^2t}R(k)\overline{\mu_1(t,\bar{k})}\Big\}dk + \\ & + \frac{4i}{\pi}\int_{\partial D}\Big\{k^2\mu_1(t,k) - \frac{g_0(t)}{2i} + e^{-4ik^2t}kR(k)\Big[\overline{\lambda_2(t,\bar{k})} + k\overline{\mu_2(t,\bar{k})}\Big]\Big\}dk, \end{aligned} \tag{40}$$

where $R(k) = b(k)/a(k)$ and the functions $\{\mu_j(t,k), \lambda_j(t,k)\}_{j=1,2}$ satisfy Equations (19), (21). *The contour ∂D is the boundary of the first quadrant of the complex k-plane; if $a(k)$ has zeros in D, ∂D has to be deformed to pass 'above' all the zeros.*

Remark 5 (The linear limit). In the approximation of small q_0, g_0, g_1 (or small q_0 and t), from (25) and (26) we find

$$L_1(t,s) \approx \frac{i}{2}g_1\Big(\frac{t+s}{2}\Big), \qquad M_1(t,s) \approx g_0\Big(\frac{t+s}{2}\Big), \tag{41a}$$

$$L_2(t,s) \approx 0, \qquad M_2(t,s) \approx 0, \tag{41b}$$

as well as

$$a(k) \approx 1, \qquad R(k) \approx b(k) \approx -\int_0^\infty q_0(y)e^{2iky}\,dy. \tag{42}$$

Using (41) and (42) in (38), we retrieve the formula relating the initial and boundary values for the linearized NLS equation:

$$g_1(t) \approx -\frac{4i}{\pi}\int_{\partial D}\Big[ke^{-4ik^2t}\Big\{\int_0^\infty q_0(y)e^{2iky}\,dy - 2k\int_0^t g_0(\tau)e^{4ik^2\tau}d\tau\Big\} + \frac{g_o(t)}{2i}\Big]dk. \tag{43}$$

4. Linearizable Boundary Conditions

PROPOSITION 4 (Linearizable cases). *Let $q(x,t)$ satisfy the NLS equation, the initial condition $q(x,0) = q_0(x)$, and one of the following boundary conditions, either*

(i) $q(0,t) = 0$,

or

(ii) $q_x(0,t) - \chi q(0,t) = 0$, *where χ is a real constant.*

Let the solution $\Phi(t,k)$ of Equation (8a) *be expressed in terms of the functions $\mu_j(t,k)$, $\lambda_j(t,k)$, $j = 1, 2$, see Proposition* 2.

Then the spectral functions A, B can be represented as follows:

(i)

$$A(t,k) = 1 + \overline{\lambda_2(t,\bar{k})}, \tag{44a}$$

$$B(t,k) = -e^{4ik^2t}\lambda_1(t,k). \tag{44b}$$

(ii)

$$A(t,k) = 1 + \left(\overline{\lambda_2(t,\bar{k})} - \frac{i\rho}{2} g_0(t)\overline{\mu_1(t,\bar{k})}\right), \tag{45a}$$

$$B(t,k) = -\left(\frac{i\chi}{2} + k\right)e^{4ik^2t}\mu_1(t,k). \tag{45b}$$

Equations (44) *and* (45) *imply, respectively*

$$A(t,k) = A(t,-k), \qquad B(t,k) = B(t,-k), \tag{46}$$

$$A(t,k) = A(t,-k), \qquad \frac{B(t,k)}{i\chi + 2k} = \frac{B(t,-k)}{i\chi - 2k}. \tag{47}$$

Proof. Let us show that the linearizable boundary conditions correspond to particular reductions of (25), (26), such that a part of this system constitutes a closed system of homogeneous Volterra integral equations, whose only solution is the trivial solution.

First, note that in the linearizable cases, $\bar{g}_0 g_1 - \bar{g}_1 g_0 = 0$, which, in terms of (25), reads $\alpha(t) = 0$.

Case (i) $q(0,t) = 0$. In this case we also have $\beta(t) = 0$ and $M_1(t,t) = 0$, so that M_1 and M_2 satisfy the system of equations

$$M_{1t} - M_{1s} = ig_1 M_2, \tag{48a}$$

$$M_{2t} + M_{2s} = -i\rho\bar{g}_1 M_1, \tag{48b}$$

$$M_1(t,t) = 0, \tag{48c}$$

$$M_2(t,-t) = 0. \tag{48d}$$

The unique solution of (48) is trivial: $M_1(t,s) = M_2(t,s) \equiv 0$. Hence, (11a) and (11b) become (44a) and (44b), respectively.

Case (ii) $q_x(0,t) - \chi q(0,t) = 0$. Introduce $P(t,s)$ by

$$P(t,s) = L_1(t,s) - \frac{i\chi}{2} M_1(t,s). \tag{49}$$

Then, in terms of P and M_2, we again obtain from (25), (26) a homogeneous system of equations

$$P_t - P_s = \left(\beta + \frac{\chi^2}{2} g_0\right) M_2, \tag{50a}$$

$$M_{2t} + M_{2s} = 2\rho \bar{g}_0 P, \tag{50b}$$

$$P(t,t) = 0, \tag{50c}$$

$$M_2(t,-t) = 0, \tag{50d}$$

whose only solution is the trivial solution $P(t,s) = M_2(t,s) \equiv 0$. In view of (49), (11a) and (11b) become (45a) and (45b), respectively.

Using the fact that λ_j and μ_j are even functions of k (see (16)), Equations (44) and (45) imply Equations (46) and (47). □

Remark 6. Using the symmetry relations (46), (47), the global relation (6), and the fact that the solution of the NLS equation for $0 < t < t^*$ does not depend on $g_0(t)$, $g_1(t)$, for $t > t^*$, it can be shown [3] that the ratio $B(k)/A(k)$ (which is needed for the relevant RH problem) can be expressed in terms of $b(k)/a(k)$ and χ.

Remark 7. In the case of the linearizable boundary condition $g_0(t) \equiv 0$, we find the following system characterizing $g_1(t)$:

$$\lambda_{1t}(t,k) + 4ik^2 \lambda_1(t,k) = ig_1(t)\lambda_2(t,k) + ig_1(t),$$

$$\lambda_{2t}(t,k) = -i\rho \bar{g}_1(t)\lambda_1(t,k),$$

$$g_1(t) = \frac{4i}{\pi} \int_{\partial D} \mathrm{e}^{-4ik^2 t} k R(k)\left(1 + \overline{\lambda_2(t,\bar{k})}\right) \mathrm{d}k.$$

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