Research Article

Analysis of the Gyroscopic Stability of the Wheelset

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The wheelset of the railway vehicle is a rotor which itself has gyroscopic effect. Nowadays, the rolling stock has entered the era of high speed, and the wheel rotates faster than in the past. The influence of gyroscopic effect on stability is little understood. Metelitsyn's inequality theorem for asymptotic stability has some advantages to analyze this problem although this method is sufficient but not necessary condition. Based on its deduction, the extremal eigenvalues criterion and compared with Routh-Hurwitz criterion, both are applied to solve the critical value of speed. Further, according to the instability criterion, gyroscopic contributory ratio is derived to study how the role the gyroscopic effect plays in stability. Moreover, the effect of gyroscopic matrix or gyroscopic terms pitch rotor inertia Iy on stability coefficient is investigated. The results show that Iy is a key factor to wheelset gyroscopic stability. The gyroscopic effect becomes significant, and the stability increases with increasing Iy. The results also indicate that the critical value of speed solved by Metelitsyn theorem is more conservative than the one it solved by Hurwitz criterion, which proves that Metelitsyn inequality theorem for asymptotic stability is a sufficient but not necessary condition in the way of attaining the numerical simulation result. Finally, the test for the influence of gyroscopic effect on stability needs to be further studied.

1. Introduction

Dynamic stability problem of railway vehicle has been studied since hunting motion phenomenon was discovered by Stephenson from the United Kingdom 100 years ago [1]. For security operation of railway vehicle, stability problem is of primary importance. Instability may increase the derailment risk and lead to disastrous consequences. Guan investigated derailment caused by instability [2]. Goodall and Iwnicki proposed a new method to assess the vehicle stability which mainly refers to the equivalent conicity between the wheel and rail contact [3]. Chen et al. investigated the stability of high speed trains passing curved rail in crosswind [4]. Zeng et al. presented the random energy theorem to investigate the phenomenon of train derail, which gives an effect way to reveal the principle of the world's problem "derailment" [5–7].

Hunting phenomenon mentioned before always exists in the railway vehicle running process because the lateral and yaw motions of the vehicle are coupled. Specifically, the unconverged hunting means instability. As the vehicle system is usually considered a nonlinear system, the critical speed corresponds to the point of limit-cycle oscillation disappearing and the linear critical speed corresponds to the Hopf bifurcation point (the more detailed analysis is given by [8]). Furthermore, the former is usually lower than the latter. In this case, it means that if the running speed is lower than the former one, the stationary motion is global stability. Otherwise, sometimes the former equals the latter and it means that the stationary motion will lose its stability if the operation speed is higher than the critical speed. True gives the whole process to determine the critical speed in [9].

Moreover, the wheelset is the key component of the whole vehicle. Many researchers have focused on the wheelset stability to clarify the mechanism of instability. Wickens investigated the nonlinear stability of railway wheelset and bogie with worn wheel treads [10]. He also deduced the analytic solution of the linear critical speed for a simple wheelset with primary suspension [11]. Knudsen et al. applied Poincare sections and Poincare maps to analyze the bifurcation and chaos phenomenon of a single wheelset [12]. Ahmadian and Yang used asymptotic method to investigate the hunting stability of a wheelset with primary suspension [13, 14]. Liu et al. studied the random stability of single wheelset with Zhu's stochastic bifurcation theorem [15]. Compared with previous

study in deterministic systems, they yielded the critical speed range of wheelset in stochastic system and firstly found the first passage failure phenomenon of wheelset [16].

Many researchers solved the linear critical speed with using traditional Hurwitz criterion before. For example, Sedighi and Shirazi proposed equivalent functions method for complex nonlinearities to analyze the limit cycle of wheelset based on Yang and Ahmadian's asymptotic method, where the Hopf bifurcation point is solved also by Hurwitz criterion [17]. They also investigated the influence of parameters on linear critical speed [18]. He et al. used Hurwitz criterion and SQP algorithm to solve the nonlinear optimization stability problem of car [19]; see also the study by Law and Brand [20]. As for the nonlinear critical speed, it refers to the nonlinear bifurcation theory. Nevertheless, neither the nonlinear stability nor the traditional linear stability theorem is the concern of this paper. What we are concerned with is the gyroscopic effect analysis, and somehow the traditional linear stability theory is utilized for comparison.

Nowadays, we step into the age of high speed railway. China's high speed electric motor units (EMU) CRH2 and CRH3 run as fast as 300 km/h. Furthermore, in the near future, the operation speed of China's EMU CRH500 is possibly to get up to 500 km/h. With the operation speed increasing, the wheelset is a rotor, whose gyroscopic effect on stability must be further studied. In the study of rotor stability, Kirillov investigated the asymptotic stability domain of a linear autonomous nonconservative system with Lineard and Chipart criterion [21]. He found that the stability boundary was governed by Whitney umbrella singularity. Junfeng and Zhaolin investigated the stability of linear nonconservative systems subjected to potential, gyroscopic, and circulatory forces and Rayleigh damping with three stability theorems. They proved that the three theorems are convenient and useful [22]. Stephen C. Spry et al. discussed the stabilization of unstable vehicle, which is characterized by single wheel. Differing from railway vehicle of wheelset, the single wheel vehicle is stabilized only by gyroscopic effect [23]. In view of further development of the nonconservative stability theory of gyroscopic systems, Pommer and Kliem offer an overview of some of the recent researches on the stability and response bounds of linear system [24]. Kirillov analyzed the stability of a linear autonomous nonconservative system with an even number of DOF. In his study, an approximation of the boundary of the asymptotic stability domain near the singularity is explicitly found, and an analytical estimate of the critical gyroscopic parameter is obtained [25]. He also summarized a complete overview on the subject of nonconservative stability from the modern point of view [26].

Metelitsyn built his inequality theorem for asymptotic stability criterion [27, 28]. Compared with Hurwitz criterion, his criterion is only sufficient but not necessary. Not only did Kliem and Seyranian [29] give a counter example to prove the defect of Metelitsyn's criterion, but also they developed Metelitsyn's criterion. Therefore, Metelitsyn's theorem has some advantages for clear physical meanings and mechanism in derivation process. What is more is that the basic principles of Metelitsyn theorem and Hurwitz criterion are somehow the same. Specifically, they are both inequality criteria and

based on stationary motion (the details are presented in Section 2). As we mentioned above, Metelitsyn theorem is only sufficient but not necessary for inequality. This leads to the derived results differing from Hurwitz criterion as shown in Section 3. It is not the traditional linear critical speed that we focus on, but it is the value solved by Metelitsyn's criterion and the derived gyroscopic effect analysis that we investigate in this paper.

2. Metelitsyn's Inequality Theorem for Asymptotic Stability

Consider the following linear differential equation of the second order [22]:

$$[\mathbf{M}]\,\ddot{x} + [\mathbf{D}+\mathbf{G}]\,\dot{x} + [\mathbf{P}+\mathbf{E}]\,x = 0, \tag{1}$$

where the dot denotes time differentiation, $x \in R^m$, and $[\mathbf{P}]$ and $[\mathbf{D}]$ are symmetric matrices. $[\mathbf{G}]$ and $[\mathbf{E}]$ are skew-symmetric matrices. These matrices $[\mathbf{P}] = [\mathbf{P}]^T$, $[\mathbf{D}] = [\mathbf{D}]^T$, $[\mathbf{G}] = -[\mathbf{G}]^T$, $[\mathbf{E}] = -[\mathbf{E}]^T$ are related to potential forces matrix, damping matrix, gyroscopic matrix, and circulant matrix [30].

The eigenvalue problem of (1) is

$$\left(\lambda^2\,[\mathbf{M}] + \lambda\,[\mathbf{D} + \mathbf{G}] + [\mathbf{P} + \mathbf{E}]\right)u = 0, \quad u \neq 0. \tag{2}$$

With normalized eigenvectors $u^T u = 1$, we obtain the relation

$$M\lambda^2 + (D + iG)\,\lambda + (P + iE) = 0, \tag{3}$$

where M, D, P, G, and E are real quantities (*known as Rayleigh quotients*):

$$M = u^T\,[M]\,u, \qquad D = u^T\,[D]\,u,$$
$$P = u^T\,[P]\,u, \qquad iG = u^T\,[G]\,u, \tag{4}$$
$$iE = u^T\,[E]\,u.$$

Theorem 1. *If linearized system* (1) *is asymptotic stable, all the eigenvalues of* (2) *have negative parts* Re $\lambda < 0$ *[28].*

Besides, one of the sufficient and necessary conditions of all the eigenvalues Re $\lambda < 0$ is Hurwitz criterion. Meanwhile, Metelitsyn was the first to derive an inequality theorem for all the eigenvalues Re $\lambda < 0$. His theorem is depicted as follows [25, 26]:

$$D > 0,$$
$$ME^2 - GDE < D^2P. \tag{5}$$

Compared with Hurwitz criterion, Metelitsyn theorem is a sufficient but not necessary condition. But its matrices terms have physical meanings. This theorem provides a way to study the influence of gyroscopic effect on stability.
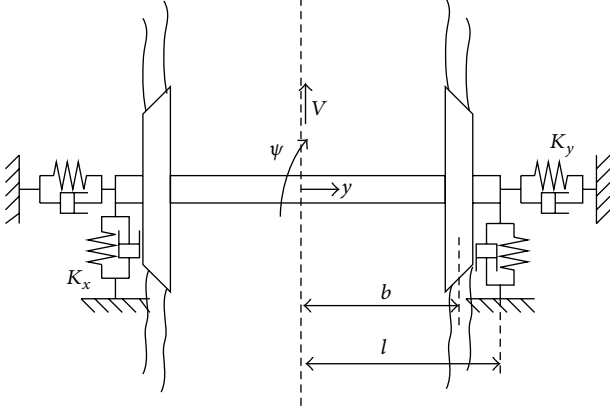
FIGURE 1: Dynamic model of wheelset with primary suspension.

Moreover, based on Lancaster and Tismenetsky's theorem [31], the corresponding quantities M, D, P (Rayleigh quotients) are therefore limited by their eigenvalues:

$$M_{\min} = \lambda_{\min} ([\mathbf{M}]) \leq M \leq \lambda_{\max} ([\mathbf{M}]) = M_{\max},$$

$$D_{\min} = \lambda_{\min} ([\mathbf{D}]) \leq D \leq \lambda_{\max} ([\mathbf{D}]) = D_{\max}, \qquad (6)$$

$$P_{\min} = \lambda_{\min} ([\mathbf{P}]) \leq P \leq \lambda_{\max} ([\mathbf{P}]) = P_{\max},$$

with

$$G_{\max} = \lambda_{\max} (i\,[\mathbf{G}]), \qquad E_{\max} = \lambda_{\max} (i\,[\mathbf{E}]). \qquad (7)$$

So we have

$$-G_{\max} \leq G \leq G_{\max}, \qquad -E_{\max} \leq E \leq E_{\max}, \qquad (8)$$

$$G_{\min} = -G_{\max}, \qquad E_{\min} = -E_{\max}. \qquad (9)$$

Therefore, inequality (5) yields a more sufficient stability condition:

$$D_{\min} (D_{\min} P_{\min} + G_{\min} E_{\min}) - M_{\max} E_{\max}^2 > 0. \qquad (10)$$

Compared with inequality (5), inequality (10) no longer depends on the eigenvectors u. This makes it suitable for engineering applications.

3. Wheelset Lateral Equation

Consider a wheelset model which contains the primary suspension. Its dynamic model is depicted in Figure 1. And (9) presents its lateral motion [13, 32]. Consider

$$\overbrace{\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}}^{M} \begin{bmatrix} \ddot{y} \\ \ddot{\varphi} \end{bmatrix} + \overbrace{\begin{bmatrix} \dfrac{2f_{22}}{V} & \dfrac{2f_{23}}{V} - \dfrac{I_y V \lambda}{r_0 b} \\ \dfrac{I_y V \lambda}{r_0 b} - \dfrac{2f_{23}}{V} & \dfrac{2f_{11} l^2}{V} \end{bmatrix}}^{C} \begin{bmatrix} \dot{y} \\ \dot{\varphi} \end{bmatrix}$$

$$+ \overbrace{\begin{bmatrix} 2k_y + \dfrac{W\lambda}{b} & -2f_{22} \\ \dfrac{2f_{11} b \lambda}{r_0} & 2l_0^2 k_x - W\lambda b + 2f_{23} \end{bmatrix}}^{K} \begin{bmatrix} y \\ \varphi \end{bmatrix} = 0. \qquad (11)$$

Therefore,

$$[\mathbf{D}] = \frac{[\mathbf{C} + \mathbf{C^T}]}{2} = \begin{bmatrix} \dfrac{2f_{22}}{V} & 0 \\ 0 & \dfrac{2f_{11} l^2}{V} \end{bmatrix},$$

$$[\mathbf{G}] = \frac{[\mathbf{C} - \mathbf{C^T}]}{2} = \begin{bmatrix} 0 & \dfrac{2f_{23}}{V} - \dfrac{I_y V \lambda}{r_0 b} \\ \dfrac{I_y V \lambda}{r_0 b} - \dfrac{2f_{23}}{V} & 0 \end{bmatrix}, \qquad (12)$$

$$[\mathbf{P}] = \frac{[\mathbf{K} + \mathbf{K^T}]}{2} = \begin{bmatrix} 2k_y + \dfrac{W\lambda}{b} & \dfrac{f_{11}\lambda b}{r_0} - f_{22} \\ \dfrac{f_{11}\lambda b}{r_0} - f_{22} & 2l_0^2 k_x - W\lambda b + 2f_{23} \end{bmatrix},$$

$$[\mathbf{E}] = \frac{[\mathbf{K} - \mathbf{K^T}]}{2} = \begin{bmatrix} 0 & -f_{22} - \dfrac{f_{11}\lambda b}{r_0} \\ f_{22} + \dfrac{f_{11}\lambda b}{r_0} & 0 \end{bmatrix}. \qquad (13)$$

4. Asymptotic Domain of Railway Wheelset by Metelitsyn's Theorem

Firstly, we yield the limited eigenvalues of the matrices M, D, P. With parameters in Table 1, we have

$$M_{\min} = \lambda_{\min} ([\mathbf{M}]) = J, \qquad M_{\max} = \lambda_{\max} ([\mathbf{M}]) = m,$$

$$D_{\min} = \lambda_{\min} ([\mathbf{D}]) = D_{\max} = \lambda_{\max} ([\mathbf{D}]) = \frac{2f_{22}}{V} = \frac{2f_{11} l^2}{V},$$

$$G_{\max} = \lambda_{\max} (i\,[\mathbf{G}]) = \left| \frac{2f_{23}}{V} - \frac{I_y V \lambda}{r_0 b} \right|,$$

$$E_{\max} = \lambda_{\max} (i\,[\mathbf{E}]) = f_{22} + \frac{f_{11}\lambda b}{r_0},$$

$$P_{\min} = \lambda_{\min} ([\mathbf{P}]) = \frac{A + \sqrt{A^2 + 4br_0^2 B}}{-2br_0^2},$$

$$P_{\max} = \lambda_{\max} ([\mathbf{P}]) = \frac{A - \sqrt{A^2 + 4br_0^2 B}}{-2br_0^2}, \qquad (14)$$

where

$$A = -2bf_{23}r_0^2 - 2bk_y r_0^2 - 2bk_x l^2 r_0^2 - \lambda r_0^2 W + b^2 \lambda r_0^2 W,$$

$$B = b^2 f_{11}^2 \lambda^2 - 2b^2 f_{11} f_{22} \lambda r_0 + bf_{22}^2 r_0^2 - 4bf_{23} k_y r_0^2 \\ - 4bk_x k_y l^2 r_0^2 - 2f_{23} \lambda r_0^2 W + 2b^2 k_y \lambda r_0^2 W \\ - 2k_x l^2 \lambda r_0^2 W + b\lambda^2 r_0^2 W^2. \qquad (15)$$

Substituting (14) into inequality (10) yields

$$\frac{2f_{22}}{V}\left(\frac{2f_{22}}{V}\times\frac{A+\sqrt{A^2+4br_0^2B}}{-2br_0^2}\right.$$

$$\left.+\left|\frac{2f_{23}}{V}-\frac{I_yV\lambda}{r_0b}\right|\left(f_{22}+\frac{f_{11}\lambda b}{r_0}\right)\right)\quad(16)$$

$$-m\left(f_{22}+\frac{f_{11}\lambda b}{r_0}\right)^2>0$$

$$\Longrightarrow V<\left(\left(-2Af_{22}^2-4b^2f_{11}f_{22}f_{23}\lambda r_0-4bf_{22}^2f_{23}r_0^2\right.\right.$$

$$\left.-2f_{22}^2\sqrt{A^2+4bBr^2}\right)$$

$$\times\left(-2bf_{11}f_{22}I_y\lambda^2+b^3f_{11}^2\lambda^2m\right.$$

$$-2f_{22}^2I_y\lambda r_0+2b^2f_{11}f_{22}\lambda mr_0$$

$$\left.\left.+bf_{22}^2mr^2\right)^{-1}\right)^{1/2}$$

$$(17)$$

$$\Longrightarrow V_{CM}=\left(\left(-2Af_{22}^2-4b^2f_{11}f_{22}f_{23}\lambda r_0-4bf_{22}^2f_{23}r_0^2\right.\right.$$

$$\left.-2f_{22}^2\sqrt{A^2+4bBr^2}\right)$$

$$\times\left(-2bf_{11}f_{22}I_y\lambda^2+b^3f_{11}^2\lambda^2m\right.$$

$$-2f_{22}^2I_y\lambda r_0+2b^2f_{11}f_{22}\lambda mr_0$$

$$\left.\left.+bf_{22}^2mr^2\right)^{-1}\right)^{1/2},$$

$$(18)$$

where V_{CM} represents the critical value of V solved by Metelitsyn theorem.

We can solve the linear critical speed according to (16) and Hurwitz criterion [17–20, 33], hereinafter referred to as V_{CM} and V_{cr}. With parameters in Table 1, it yields

$$V_{CM}=88.36\,\text{m/s},\qquad V_{cr}=126.38\,\text{m/s}.\quad(19)$$

As we know, Hurwitz criterion is the sufficient and necessary condition for all eigenvalues having negative part and the result $V_{CM}<V_{cr}$ proves that Metelitsyn theorem is the sufficient but not necessary condition for asymptotic stability. Besides, based on equality (16) and Hurwitz criterion, we can solve the influence of parameters on V_{CM} and V_{cr} as shown in Figures 2–4.

From Figures 2 and 3, we can also find $V_{CM}<V_{cr}$. It has been again proved that Metelitsyn theorem is the sufficient but not necessary condition for asymptotic stability. Therefore, when this theorem is used for analysis, the corresponding V_{CM} is safe but conservative.

TABLE 1: Parameters of wheelset.

| Parameters	Value	Commet
m	1500 kg	Wheelset mass
J	825.7 kg·m^2	Yaw inertia
f_{11}	6.728 MN	Longitudinal creep coefficient
f_{22}	6.728 MN	Lateral creep coefficient
f_{23}	1500 N·m	Spin creep coefficient
I_y	70 kg·m^2	Pitch rotor inertia
λ	0.28	Equivalent conicity
r_0	0.533 m	Wheel radius
b	0.7175 m	Half of rolling cycle gauge
l_0	1 m	Half of primary suspension gauge
k_x	5 MN/m	Primary longitudinal stiffness
k_y	5 MN/m	Primary lateral stiffness
W	$m*g$	Axle load
V	—	Operation speed
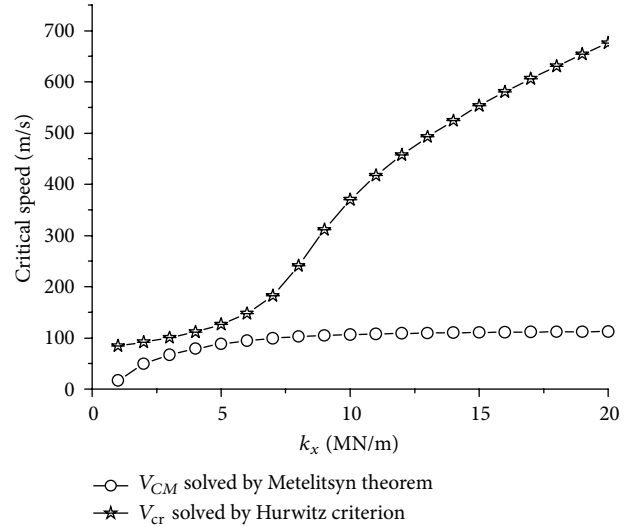
FIGURE 2: Influence of k_x on critical value V.

5. Gyroscopic Effect on Wheelset Stability

5.1. Gyroscopic Contributory Ratio. As mentioned in Section 1, the single wheel vehicle is stabilized only by gyroscopic effect [12]. But it is the railway wheelset that we discuss in this paper, with primary suspension, whose main factors for stabilization are the primary suspension. However, how is the gyroscopic effect for stabilization of railway wheelset represented?

To represent the gyroscopic effect, from inequality (8), we can define stability coefficient a_t with gyroscopic term and a_{nt} without gyroscopic term:

$$a_t=\frac{D_{\min}\left(D_{\min}P_{\min}+G_{\min}E_{\min}\right)}{M_{\max}E_{\max}^2},\qquad a_{nt}=\frac{D_{\min}^2P_{\min}}{M_{\max}E_{\max}^2}.\quad(20)$$

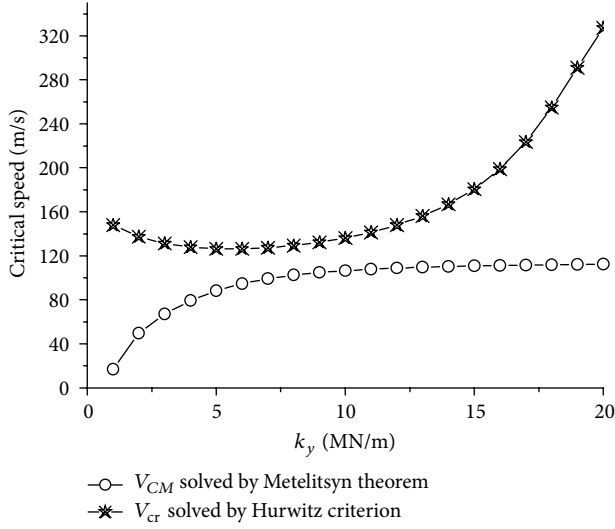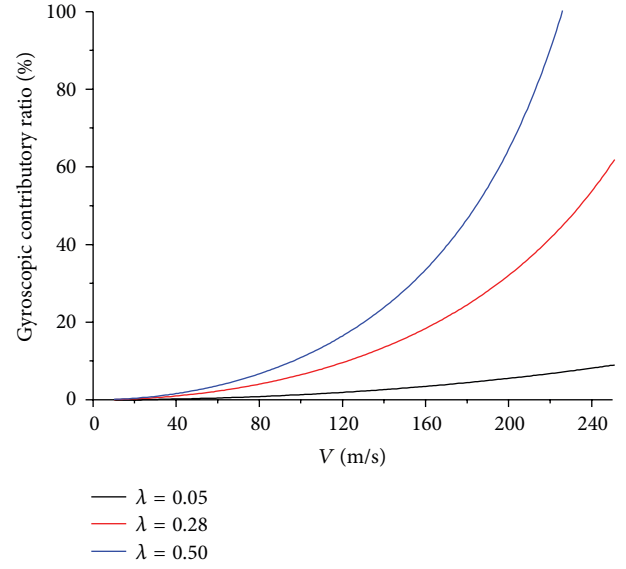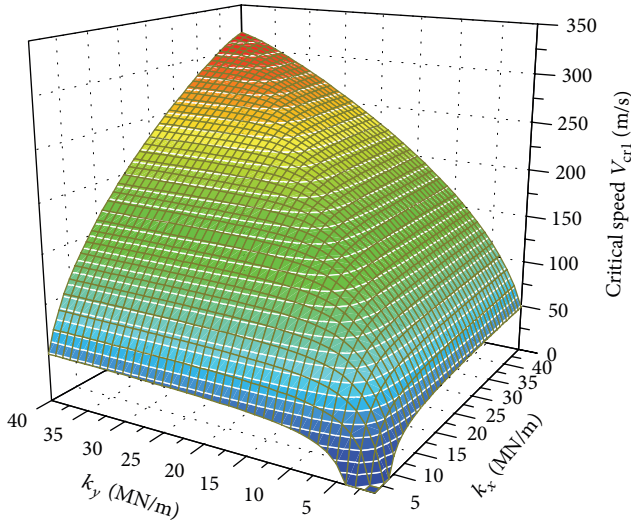Thus, the sufficient condition for asymptotical stability is $a_t>1$ or $a_{nt}>1$.

FIGURE 3: Influence of k_y on critical value V.

FIGURE 5: Gyroscopic contributory ratio.

FIGURE 4: Influence of k_x and k_y on V_{CM}.

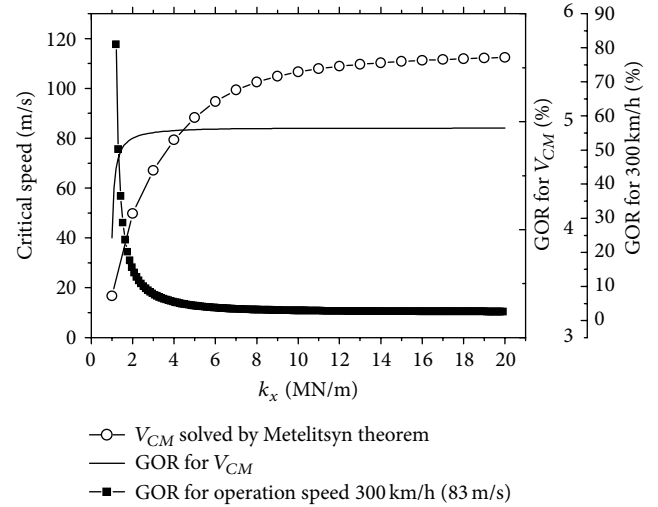FIGURE 6: Influence of k_x on V_{CM} and GOR.

Furthermore, gyroscopic contributory ratio (GCR) can be defined as

$$\text{GOR} = \frac{a_t - a_{nt}}{a_t} = \frac{G_{\max} E_{\max}}{D_{\min} K_{\min} + G_{\min} E_{\min}}. \tag{21}$$

With wheelset parameters, we can obtain the following.

It can be seen in Figure 5 that, with the equivalent conicity increasing, the gyroscopic contributory ratio for stabilization increases. As we know, the larger equivalent conicity leads to the worse stability. Meanwhile, the gyroscopic effect on stability becomes more obvious.

Next, we continue to discuss the gyroscopic effect on stability under $V = 300$ km/h (83 m/s) of high speed railway operation speed and V_{CM} condition. The influences of positioning stiffness k_x or k_y on V_{CM} and on GOR for both V_{CM} and high speed railway operation speed $V = 300$ km/h are presented in Figures 6-7.

As shown in Figures 6 and 7, the results show that GOR for V_{CM} increases slightly along with increasing k_x, and, further, with k_x increasing, both GOR for V_{CM} and V_{CM} itself tend to be levelled off. Hence, overall, the change of GOR for V_{CM} is small, and the value of GOR for V_{CM} is as low as less than 5%. If, however, GOR for high operation speed 300 km/h firstly decreases a lot along with increasing k_x and further with k_x increasing, it also tends to be levelled off. That is, GOR for high operation speed 300 km/h drops significantly from 80% to less than 5%. This phenomenon indicates that GOR is insignificant when the operation speed is low and even if it gets to V_{CM}. But when V_{CM} is low and the operation speed gets much higher than V_{CM}, GOR will become very significant.
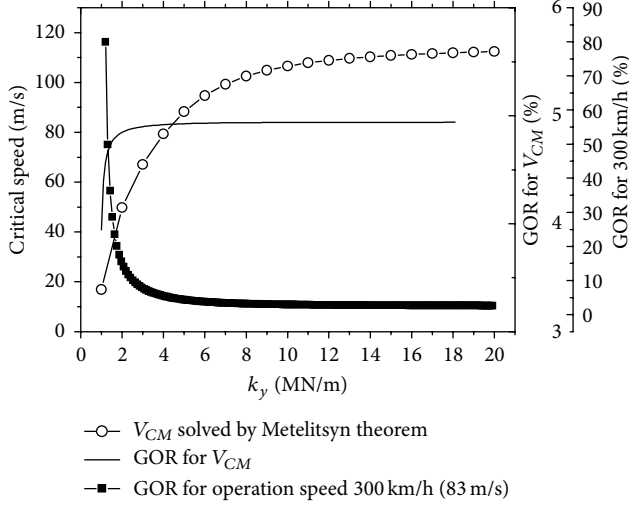
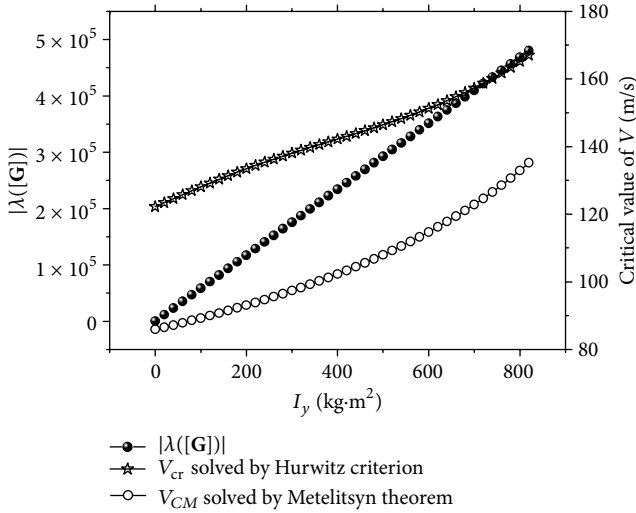FIGURE 7: Influence of k_y on V_{CM} and GOR.

FIGURE 8: Influence of I_y on $|\lambda([\mathbf{G}])|$ and critical value of V.

FIGURE 9: Influence of I_y on V_{CM} and GOR.

5.2. The Influence of Gyroscopic Term on Stability. We reconsider the gyroscopic matrix

$$[\mathbf{G}] = \begin{bmatrix} 0 & \dfrac{2f_{23}}{V} - \dfrac{I_y V \lambda}{r_0 b} \\ \dfrac{I_y V \lambda}{r_0 b} - \dfrac{2f_{23}}{V} & 0 \end{bmatrix}. \tag{22}$$

It contains the terms $f_{23}, I_y, r_0, b, \lambda$. Furthermore, among the above five terms, we can find that the pitch rotor inertia I_y is the only factor which just affects $[\mathbf{G}]$ but does not affect the other matrices. For this reason, the influence of univariate I_y, which means the gyroscopic effect, on stability needs to be studied. The influence of I_y on $\lambda([\mathbf{G}])$ and critical value of V are shown in Figure 9.

In Figures 8 and 9, the results indicate that, with increasing I_y, the critical values of V_{CM} and V_{cr} both increase but V_{CM} is lower than $V_{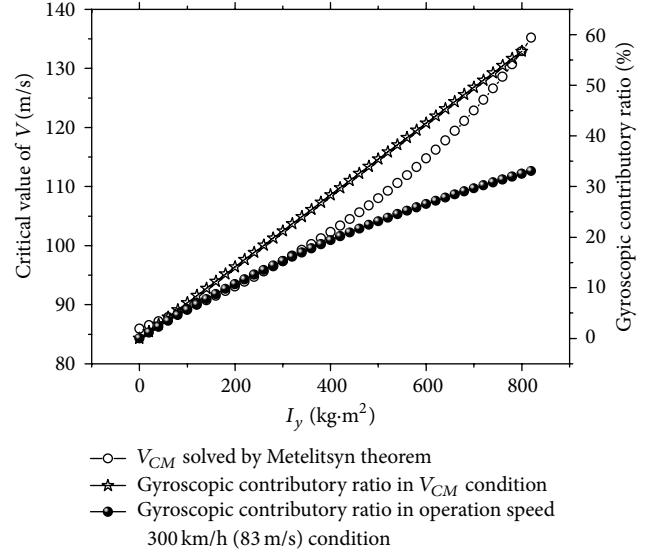cr}$. This again proves that the result attained by Metelitsyn theorem is more conservative than the one attained by Hurwitz criterion. And further, with increasing I_y, $|\lambda([\mathbf{G}])|$ and GOR also increase. This illustrates that I_y is a key factor in wheelset gyroscopic stability. If it is greater, the gyroscopic effect becomes more significant and V_{CM} gets higher.

6. Conclusion and Outlook

We derive the critical value of V and gyroscopic contributory ratio of wheelset equation with Metelitsyn inequality theorem. Analytical and numerical result shows that pitch rotor inertia I_y plays a key role in gyroscopic matrix. The gyroscopic effect becomes significant, and the V_{CM} gets higher with increasing I_y. However, when V_{CM} is low, but with high operation speed, gyroscopic contributory ratio is large. Therefore, to ensure high V_{CM} of vehicle as well, increasing pitch rotor inertia can further increase the stability. In practice, to get larger value of pitch rotor inertia, one way is to increase the mass of the wheelset, and another way is to increase the radius of the wheelset, both of which can get higher linear critical speed as generally realized.

What is mentioned above just refers to the theoretical analysis. As we know, the pitch rotor inertia I_y is the main factor in the gyroscopic system, and it would provide a method for testing verification. In the past EMU test on roller rig [34], we mainly focused on the influence of suspension parameters such as k_x and k_y on stability but ignored I_y. Nowadays, in the era of high speed EMU development, therefore, the new test item "gyroscopic effect on stability" is further required. Since China's roller rig has the condition of software and hardware, the following work of how to test the influence of gyroscopic effect on stability needs to be conducted.
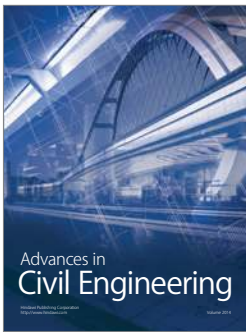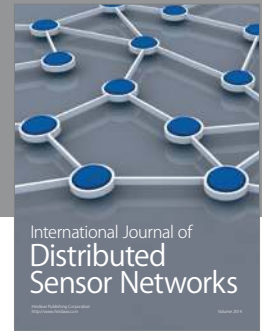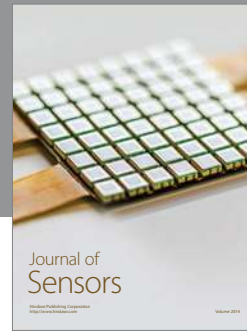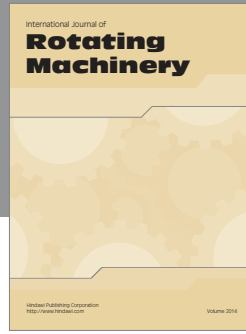
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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