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**ANALYSIS OF THE SPECTRAL VANISHING VISCOSITY
METHOD FOR PERIODIC CONSERVATION LAWS***

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ABSTRACT

We analyze the convergence of the spectral vanishing method for both the spectral and pseudospectral discretizations of the inviscid Burgers' equation. We prove that this kind of vanishing viscosity is responsible for a spectral decay of those Fourier coefficients located toward the end of the computed spectrum; consequently, the discretization error is shown to be spectrally small independent of whether the underlying solution is smooth or not. This in turn implies that the numerical solution remains uniformly bounded and convergence follows by compensated compactness arguments.

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INTRODUCTION

In this paper, we extend the analysis of the spectral vanishing viscosity method for stabilizing spectral approximations of nonlinear conservation laws. The spectral vanishing viscosity has been first introduced in [3], where it was shown that L^∞ -bounded spectral-Galerkin approximations converge strongly in $L^2_{loc}(x,t)$ to the exact entropy solutions of such conservation laws.

The analysis is performed on the 2π -periodic inviscid Burgers' equation

$$(1.1) \quad \frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} \left(\frac{u^2(x,t)}{2} \right) = 0,$$

submitted to the additional entropy condition

$$(1.2) \quad \frac{\partial}{\partial t} \left(\frac{u^2(x,t)}{2} \right) + \frac{\partial}{\partial x} \left(\frac{u^3(x,t)}{3} \right) \leq 0,$$

which singles out the unique "physically relevant" weak solution of (1.1). Both the spectral-Galerkin and pseudospectral-collocation methods for (1.1), (1.2) are treated, and to this end we proceed as follows.

Denote by $S_N u(x,t)$ the spectral-Fourier projection of $u(x,t)$,

$$(1.3) \quad S_N u(x,t) = \sum_{|k| \leq N} \hat{u}(k,t) e^{ikx}, \quad \hat{u}(k,t) = \frac{1}{2\pi} \cdot \int_0^{2\pi} u(x,t) e^{-ikx} dx,$$

and let $I_N u(x,t)$ denote the pseudospectral-Fourier projection of $u(x,t)$, which interpolate $u(x,t)$ at the $2N+1$ equidistant collocation points

$$x_v = vh, \quad h = \frac{2\pi}{2N+1}, \quad v = 0, \dots, 2N,$$

$$(1.4) \quad I_N u(x, t) = \sum_{|k| \leq N} \tilde{u}(k, t) e^{ikx}, \quad \tilde{u}(k, t) = \frac{h}{2\pi} \cdot \sum_{v=0}^{2N} u(x_v, t) e^{-ikx_v}.$$

These two projection operators differ by aliasing error--that is, we have

$$(1.5) \quad I_N = S_N + A_N$$

where the aliasing projection A_N is given by [2]

$$(1.6) \quad A_N u(x, t) = \sum_{|k| \leq N} \left[\sum_{j \neq 0} \hat{u}(k + j(2N + 1), t) \right] e^{ikx}.$$

Throughout this paper, we use

$$(1.7) \quad P_N = S_N + a \cdot A_N$$

as a concise notation for the two kinds of Fourier projections: either with $a = 0$, corresponding to the spectral projection, or with $a = 1$ which corresponds to the pseudospectral interpolation.

We approximate the Fourier projection of the exact solution $P_N u(x, t)$, by an N -trigonometric polynomial, $u_N(x, t)$

$$(1.8) \quad u_N(x, t) = \sum_{|k| \leq N} \hat{u}_k(t) e^{ikx},$$

which is determined by the approximate evolution equation

$$(1.9) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} \left(P_N \frac{1}{2} u_N^2(x, t) \right) = \epsilon \frac{\partial}{\partial x} \left(Q_N \frac{\partial}{\partial x} u_N(x, t) \right).$$

The expression on the right-hand side of (1.9) represents the spectral vanishing viscosity term. Here Q_N is the spectral viscosity operator which is defined as a convolution with a symmetric viscosity kernel $Q_N(x)$,

$$(1.10) \quad Q_N \frac{\partial}{\partial x} u_N(x, t) = Q_N(x) * \frac{\partial}{\partial x} u_N(x, t), \quad Q_N(x) = \sum_{|k| \leq N} \hat{Q}(k) e^{ikx}.$$

In the spectral case where $a = 0$, (1.9) amounts to

$$\frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} [S_N \frac{1}{2} u_N^2(x, t)] = \varepsilon \frac{\partial}{\partial x} [Q_N(x) * \frac{\partial}{\partial x} u_N(x, t)],$$

consisting of a nonlinear system of ordinary differential equations for the Fourier coefficients, $\hat{u}_k(t)$, which are coupled through the standard spectral convolution treatment of the nonlinear term. The interpretation of the scheme (1.9) in the pseudospectral case where $a = 1$ leads us to

$$\frac{\partial}{\partial t} u_N(x_v, t) + \frac{\partial}{\partial x} [I_N \frac{1}{2} u_N^2(x, t)] \Big|_{x=x_v} = \varepsilon \frac{\partial}{\partial x} [Q_N(x) * \frac{\partial}{\partial x} u_N(x, t)] \Big|_{x=x_v}, \quad 0 \leq v \leq 2N,$$

and consists in a complete statement of a standard collocation method with a pseudospectral treatment of the nonlinear term.

In both the spectral and pseudospectral cases, the spectral viscosity operator can be efficiently implemented in the Fourier rather than the physical space, i.e.,

$$\varepsilon \frac{\partial}{\partial x} (Q_N \frac{\partial}{\partial x} u_N(x, t)) \equiv \varepsilon \frac{\partial}{\partial x} [Q_N(x) * \frac{\partial}{\partial x} u_N(x, t)] = -\varepsilon \cdot \sum_{|k| \leq N} k^2 \hat{Q}(k) \hat{u}_k(t) e^{ikx}.$$

An essential ingredient of our spectral viscosity operator, Q_N , is that it

should operate only on the high portion of the spectrum, in order to retain the formal spectral accuracy of the method. Hence we make

ASSUMPTION I: There exists a constant $m \equiv m(N) < \frac{1}{4} N$, such that

$$\begin{cases} \hat{Q}(k) \equiv 0, & |k| < m, \\ 0 \leq \hat{Q}(k) \leq 1, & m \leq |k| \leq 2m, \\ \hat{Q}(k) \equiv 1, & 2m < |k| \leq N. \end{cases}$$

Then, with $Q_N = I - R_m$, we can rewrite (1.9) as

$$(1.11) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} (P_N \frac{1}{2} u_N^2(x, t)) = \epsilon \frac{\partial}{\partial x} [(I - R_m) \frac{\partial}{\partial x} u_N(x, t)],$$

where the corresponding kernel $R_m(x)$,

$$(1.12) \quad R_m(x) = \sum_{|k| \leq 2m} \hat{R}(k) e^{ikx},$$

is a trigonometric polynomial of degree $\leq 2m$, with Fourier coefficients

$$(1.13) \quad \begin{cases} \hat{R}(k) \equiv 1, & |k| < m, \\ 0 \leq \hat{R}(k) \leq 1, & m \leq |k| \leq 2m. \end{cases}$$

In order to guarantee the uniform boundedness of our approximation, $u_N(x, t)$, we shall need to control the size of this kernel; we therefore make

ASSUMPTION II: There exists a constant such that

$$(1.14) \quad \|R_m(\cdot)\|_{L^1(x)} \leq \text{Const.} \log m.$$

We remark that the assumption of a logarithmic upper bound for the size of $R_m(x)$ is plausible, since typical applications involve $\hat{R}(k)$ which decrease monotonically to zero and (1.14) is automatically fulfilled in such cases; consult Appendix A. To obtain, with the help of Assumption II, the promised uniform bound on $u_N(x,t)$, necessitates L^∞ -bounded initial data, $u_N(x,0)$. For technical reasons we shall need the slightly stronger

ASSUMPTION III: There exists a constant such that

$$\|u_N(x,t=0)\|_{L^\infty(x)} \leq \sum_{|k| \leq N} |\hat{u}_k(t=0)| \leq \text{Const}_0.$$

The spectral viscosity term on the right of (1.11) depends on two free parameters: the viscosity amplitude $\epsilon \equiv \epsilon(N)$ and the effective size of the inviscid spectrum $m \equiv m(N)$. These two parameters should be chosen to ensure the convergence of the method. In [3] it has been proved that in the absence of such viscosity term, $\epsilon = 0$, strong as well as weak convergence to the exact entropy solution fails.

The main result of this paper asserts

Theorem 1.1: Consider the Fourier approximation (1.11) of either spectral or pseudospectral type. Let the spectral viscosity in (1.12) - (1.14) be parameterized with (ϵ, m) as follows

$$(1.15) \quad \varepsilon \equiv \varepsilon(m) \sim \alpha \frac{1}{m^2 \cdot \|R_m(\cdot)\|_{L^1(x)}}, \quad m \equiv m(N) \sim \text{Const.} N^\beta, \quad 0 < \beta < \frac{1}{4}.$$

Then $u_N(x, t)$ converges boundedly a.e. to the unique entropy solution of the conservation law (1.1).

Let us examine for example the viscosity operator $Q_N = I - S_m$. Here $R_m(x)$ coincides with Dirichlet kernel $D_m(x)$, where [5, Chapter II]

$$D_m(x) = \sum_{|k| \leq m} e^{ikx} \equiv \frac{1}{2\pi} \frac{\sin(m + \frac{1}{2})x}{\sin \frac{1}{2}x}, \quad \|D_m(\cdot)\|_{L^1(x)} \sim \frac{4}{\pi} \log m,$$

so that Assumption II is fulfilled and Theorem 1.1 yields

Corollary 1.2: Consider the Fourier approximation

$$(1.16) \quad \frac{\partial}{\partial t} u_N(x, t) + \frac{\partial}{\partial x} (P_N \frac{1}{2} u_N^2(x, t)) = \varepsilon \frac{\partial}{\partial x} [(I - S_m) \frac{\partial}{\partial x} u_N(x, t)],$$

with

$$(1.17) \quad \varepsilon = \varepsilon(N) \sim \text{Const.} \frac{N^{-2\beta}}{\log N}, \quad m = m(N) \sim \text{Const.} N^\beta, \quad 0 < \beta < \frac{1}{4}.$$

Then $u_N(x, t)$ converges boundedly a.e. to the unique entropy solution of the conservation law (1.1).

The spectral portion of this result ($a = 0$), was derived in [3, Theorem 4.1] under the assumption that the numerical solution $u_N(x, t)$ remains uniformly bounded. The extension of Corollary 1.2 includes the pseudospectral

approximation ($a = 1$), and in addition, thanks to the slightly more stringent parametrization than that of [3, Theorem 4.1], contains a proof of the previously assumed L^∞ -bound.

In the last example the viscosity symbols $\hat{Q}(k)$ were discontinuous at $|k| = m$. It was suggested in [3] that the use of viscosity operators Q_N with smoothly varying symbols would be advantageous for the spectral viscosity method in (1.9). As our second and final example we consider the simplest viscosity operator of this type, namely

$$\begin{cases} \hat{Q}(k) \equiv 0, & |k| < m, \\ \hat{Q}(k) = \frac{|k|-m}{m}, & m \leq |k| \leq 2m, \\ \hat{Q}(k) \equiv 1, & 2m < |k| \leq N. \end{cases}$$

This kind of spectral viscosity is intimately related to the Fejer operator

$F_m = \frac{1}{m} \cdot \sum_{k=0}^{m-1} S_k$: if we let $K_m(x)$ denote the corresponding Fejer kernel [3, Chapter III]

$$K_m(x) = \sum_{|k| \leq m} (1 - \frac{|k|}{m}) e^{ikx} \equiv \frac{2}{\pi m} \left(\frac{\sin \frac{1}{2} mx}{\sin \frac{1}{2} x} \right)^2, \quad \|K_m(\cdot)\|_{L^1(x)} = 1,$$

then for $Q_N = I - R_m$ we have $R_m(x) = 2K_{2m}(x) - K_m(x)$. Hence the kernel associated with

$$R_m = 2F_{2m} - F_m \equiv \frac{1}{m} \cdot \sum_{k=m}^{2m-1} S_k$$

is L^1 -uniformly bounded,

$$\|R_m(\cdot)\|_{L^1(x)} \leq 2\|K_{2m}(\cdot)\|_{L^1(x)} + \|K_m(\cdot)\|_{L^1(x)} \leq 3,$$

so that Assumption II is fulfilled and Theorem 1.1 yields

Corollary 1.3: Consider the Fourier approximation

$$(1.18) \quad \frac{\partial}{\partial t} u_N(x,t) + \frac{\partial}{\partial x} (P_N \frac{1}{2} u_N^2(x,t)) = \varepsilon \frac{\partial}{\partial x} [(I - \frac{1}{m} \cdot \sum_{k=m}^{2m-1} S_k) \frac{\partial}{\partial x} u_N(x,t)],$$

with

$$(1.19) \quad \varepsilon = \varepsilon(N) \sim \text{Const.} \cdot N^{-2\beta}, \quad m = m(N) \sim \text{Const.} \cdot N^\beta, \quad 0 < \beta < \frac{1}{4}.$$

Then $u_N(x,t)$ converges boundedly a.e. to the unique entropy solution of the conservation law (1.1).

The paper is organized as follows. In Section 2 we derive a couple of basic L^2 -type a-priori energy estimates. In Section 3, these estimates are used in order to study the spectral decay rate of the Fourier coefficients. This in turn enables us, in Section 4, to obtain L^∞ a priori estimate on the numerical solution. Finally, based on the a priori estimates prepared in Sections 2, 3, and 4, Theorem 1.1 is proved in Section 5 along the lines of [3], using compensated compactness arguments.

2. L^2 -TYPE A PRIORI ESTIMATES

We consider the approximate Fourier method (1.9) which we rewrite as

$$(2.1) \quad \frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} (\frac{1}{2} u_N^2) = \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] + \varepsilon \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N] \equiv I + II.$$

In order to prove the convergence of this method we need a couple of a priori estimates on its solution. To this end, we multiply (2.1) by u_N

$$(2.2) \quad \begin{aligned} & \frac{\partial}{\partial t} \left(\frac{1}{2} u_N^2 \right) + \frac{\partial}{\partial x} \left(\frac{1}{3} u_N^3 \right) = \\ & = u_N \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] + \epsilon u_N \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N] \equiv \text{III} + \text{IV}, \end{aligned}$$

and integrate over a 2π period: the integral of the second term on the left vanishes by periodicity, and after integration by parts for the second term on the right we are left with

$$(2.3) \quad \frac{1}{2} \frac{d}{dt} \|u_N(\cdot, t)\|_{L^2(x)}^2 + \epsilon \int_0^{2\pi} \frac{\partial}{\partial x} u_N(x, t) Q_N \frac{\partial}{\partial x} u_N(x, t) dx = \int_0^{2\pi} u_N \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] dx.$$

Using (1.7) and the fact that $I - S_N$ is orthogonal to our N -space, we find that the right-hand side of (2.3) equals

$$\int_0^{2\pi} u_N \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] dx = -a \cdot \int_0^{2\pi} u_N \frac{\partial}{\partial x} [A_N \frac{1}{2} u_N^2] dx = -a \cdot \sum_{|p| \leq N} \hat{u}_{-p} \widehat{ip(A_N \frac{1}{2} u_N^2)}_p,$$

and by the aliasing relation (1.6), this does not exceed

$$\int_0^{2\pi} u_N \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] dx = \frac{a}{2} \sum_{|p+q+r|=2N+1} ip \hat{u}_p \hat{u}_q \hat{u}_r \leq \frac{|a|N}{2} \sum_{|p+q+r|=2N+1} |\hat{u}_p| |\hat{u}_q| |\hat{u}_r|.$$

In view of $|p + q + r| = 2N + 1$, at least two of the three indices $|p| \leq N$, $|q| \leq N$ and $|r| \leq N$ are greater in absolute value than $\frac{N}{2}$, and hence

$$\int_0^{2\pi} u_N \frac{\partial}{\partial x} [(I-P_N)^{\frac{1}{2}} u_N^2] dx \leq \frac{4|a|}{N} \sum_{\frac{N}{2} \leq |p| \leq N} \sum_{\frac{N}{2} \leq |q| \leq N} |p| |\hat{u}_p| |q| |\hat{u}_q| |\hat{u}_{\pm(2N+1)-(p+q)}|$$

$$\leq \frac{4|a|}{N} \left[\sum_{\frac{N}{2} \leq |p| \leq N} p^2 |\hat{u}_p|^2 \sum_{\frac{N}{2} \leq |q| \leq N} q^2 |\hat{u}_q|^2 \right]^{1/2} \cdot \left[\sum_{\frac{N}{2} \leq |p| \leq N} \sum_{\frac{N}{2} \leq |q| \leq N} |\hat{u}_{\pm(2N+1)-(p+q)}|^2 \right]^{1/2}.$$

Consequently, since $\hat{Q}(p) = \hat{Q}(q) \equiv 1$ for $|p|, |q| \geq \frac{N}{2}$, the expression on the right of (2.3) can be upper bounded by

$$(2.4a) \int_0^{2\pi} u_N \frac{\partial}{\partial x} [(I-P_N)^{\frac{1}{2}} u_N^2] dx \leq \frac{4|a|}{N^{1/2}} \cdot \|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)}^2 \cdot \|u_N(\cdot, t)\|_{L^2(x)}.$$

Moreover, since $0 \leq \hat{Q}(k) \leq 1$ we have for the second term on the left of (2.3)

$$(2.4b) \quad \varepsilon \int_0^{2\pi} \frac{\partial}{\partial x} u_N(x, t) Q_N \frac{\partial}{\partial x} u_N(x, t) dx =$$

$$= \varepsilon \sum_{|k| \leq N} k^2 \hat{Q}(k) |\hat{u}_k(t)|^2 \geq \varepsilon \|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)}^2.$$

Inserting this together with (2.4a) into (2.3) we end up with

$$(2.5) \quad \frac{1}{2} \frac{d}{dt} \|u_N(\cdot, t)\|_{L^2(x)}^2 + \left[\varepsilon - \frac{4|a|}{N^{1/2}} \|u_N(\cdot, t)\|_{L^2(x)} \right] \cdot \|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)}^2 \leq 0.$$

Thus, as long as

$$(2.6) \quad \varepsilon - \frac{4|a|}{N^{1/2}} \|u_N(\cdot, t)\|_{L^2(x)} > \frac{\varepsilon}{2},$$

we obtain

$$(2.7) \quad \frac{d}{dt} \|u_N(\cdot, t)\|_{L^2(x)}^2 + \varepsilon \|Q_N \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)}^2 \leq 0.$$

In particular, (2.7) implies that for $u_N(x, t) = \sum_{|k| \leq N} \hat{u}_k(t) e^{ikx}$ we have our first L^2 -type a priori estimate

$$(2.8) \quad \|u_N(\cdot, t)\|_{L^2(x)}^2 = \sum_{|k| \leq N} |\hat{u}_k(t)|^2 \leq E_0^2, \quad E_0 = \|u_N(\cdot, t=0)\|_{L^2(x)} \leq \text{Const}_0.$$

Hence (2.6), (2.7) and consequently (2.8) prevail for all time provided (2.6) is valid at $t = 0$, i.e., we require that in the pseudospectral case where $a = 1$ we shall have

$$(2.9) \quad \varepsilon(N) > 8E_0 \cdot N^{-1/2};$$

indeed, Assumption II tells us that this requirement is fulfilled, at least for sufficiently large N , for

$$(2.10) \quad \varepsilon > \text{Const.} \frac{N^{-2\beta}}{\log N} > 8 \cdot E_0 \cdot N^{-1/2}, \quad 2\beta < \frac{1}{2}.$$

Furthermore, temporal integration of (2.7) then gives us the second a priori estimate

$$(2.11) \quad \varepsilon \|Q_N \frac{\partial}{\partial x} u_N\|_{L_{loc}^2(x, t)}^2 \equiv \varepsilon \int_t \sum_{m < |k| \leq N} k^2 |\hat{Q}(k) \hat{u}_k(t)|^2 dt \leq J_0^2, \quad J_0 \leq \text{Const}_0.$$

3. THE DECAY RATE OF THE FOURIER COEFFICIENTS

Our Fourier approximation (2.1)

$$(3.1) \quad \frac{\partial}{\partial t} u_N + \frac{\partial}{\partial x} \left(\frac{1}{2} u_N^2 \right) = \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] + \epsilon \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N],$$

consists of two kinds of errors. The first term, $I \equiv \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2]$, represents the discretization error, which includes spectral truncation errors $\frac{\partial}{\partial x} [(I - S_N) \frac{1}{2} u_N^2]$ as well as additional aliasing errors $-a \frac{\partial}{\partial x} [A_N \frac{1}{2} u_N^2]$ in the pseudospectral case. In this section, we borrow from Kreiss [1], in order to show that due to the second error term of spectral vanishing viscosity, $II \equiv \epsilon \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N]$, there is spectral decay of the Fourier coefficients $|\hat{u}_k(t)|$, $|k| > \frac{1}{2} N$, and therefore, that the discretization error is spectrally small.

We begin by taking the $I - S_{2k}$ projection of (1.9): for $k > m$ we have by Assumption I, $(I - S_{2k})Q_N = I - S_{2k}$, and hence

$$(3.2) \quad \frac{\partial}{\partial t} [(I - S_{2k})u_N] + (I - S_{2k}) \frac{\partial}{\partial x} [P_N \frac{1}{2} u_N^2] = \epsilon \frac{\partial}{\partial x} [(I - S_{2k}) \frac{\partial}{\partial x} u_N], \quad m < k \leq N.$$

Multiplying by $(I - S_{2k})u_N$ and integrating by parts over a 2π -period, we find that

$$(3.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(I - S_{2k})u_N(\cdot, t)\|_{L^2(x)}^2 = \\ & = \frac{1}{2} \int_0^{2\pi} (I - S_{2k}) \frac{\partial}{\partial x} u_N \cdot (I - S_{2k}) P_N u_N^2 dx - \epsilon \|(I - S_{2k}) \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)}^2. \end{aligned}$$

The first integral on the right does not exceed

$$\begin{aligned}
 (3.4) \quad & \frac{1}{2} \int_0^{2\pi} (I-S_{2k}) \frac{\partial}{\partial x} u_N \cdot (I-S_{2k}) P_N u_N^2 dx \leq \\
 & \leq \frac{1}{2} \| (I-S_{2k}) \frac{\partial}{\partial x} u_N(\cdot, t) \|_{L^2(x)} \cdot \| (I-S_{2k}) P_N u_N^2(\cdot, t) \|_{L^2(x)}.
 \end{aligned}$$

In order to estimate the second term of the last product, we will make use of the following lemma whose proof is postponed to the end of this section.

Lemma 3.1: Let $f_N \equiv f_N(x)$ and $g_N \equiv g_N(x)$ be a couple of N -trigonometric polynomials. Then, for any $0 < 2k < N$ we have

$$\begin{aligned}
 (3.5) \quad & \| (I-S_{2k}) P_N (f_N g_N) \|_{L^2(x)} \leq \\
 & \leq \frac{2}{\sqrt{k}} [\| f_N \|_{L^2(x)} \cdot \| (I-S_k) \frac{\partial}{\partial x} g_N \|_{L^2(x)} + \| g_N \|_{L^2(x)} \cdot \| (I-S_k) \frac{\partial}{\partial x} f_N \|_{L^2(x)}].
 \end{aligned}$$

Lemma 3.1 with $f_N(\cdot) = g_N(\cdot) = u_N(\cdot, t)$ implies

$$(3.6) \quad \| (I-S_{2k}) P_N u_N^2(\cdot, t) \|_{L^2(x)} \leq \frac{4}{\sqrt{k}} \| u_N(\cdot, t) \|_{L^2(x)} \cdot \| (I-S_k) \frac{\partial}{\partial x} u_N(\cdot, t) \|_{L^2(x)}.$$

Equipped with (3.4), (3.6), and (2.8) we return to (3.3) to find that

$$\begin{aligned}
 (3.7) \quad & \frac{1}{2} \frac{d}{dt} \| (I-S_{2k}) u_N(\cdot, t) \|_{L^2(x)}^2 \leq -\varepsilon \| (I-S_{2k}) \frac{\partial}{\partial x} u_N(\cdot, t) \|_{L^2(x)}^2 + \\
 & + \frac{2}{\sqrt{k}} E_0 \cdot \| (I-S_{2k}) \frac{\partial}{\partial x} u_N(\cdot, t) \|_{L^2(x)} \cdot \| (I-S_k) \frac{\partial}{\partial x} u_N(\cdot, t) \|_{L^2(x)},
 \end{aligned}$$

which brings us to

Theorem 3.2: For any integer $s \geq 0$ there exists a constant $C_s = \text{Const}(s, E_0)$, such that for sufficiently large N , $N > 2^{s+4}m$, we have

$$(3.8) \quad \|(I - S_{\frac{N}{2}})u_N(\cdot, t)\|_{L^2(x)} \leq C_s \cdot \left[\left(\frac{1}{\varepsilon\sqrt{N}}\right)^s + \left(1 + \frac{1}{\varepsilon\sqrt{N}}\right)^{-(s+1)} \cdot e^{-4^s \cdot \varepsilon N^2 t} \right].$$

Proof: Let $E_k(t)$ abbreviates the quantity

$$(3.9) \quad E_k(t) \equiv \|(I - S_k)u_N(\cdot, t)\|_{L^2(x)}.$$

In view of the inverse inequalities

$$2kE_{2k}(t) \leq \|(I - S_{2k}) \frac{\partial}{\partial x} u_N(\cdot, t)\|_{L^2(x)} \leq NE_{2k}(t),$$

it follows from (3.7) that $E_k(t)$ satisfy

$$(3.10) \quad \frac{d}{dt} E_{2k}(t) \leq -4\varepsilon k^2 E_{2k}(t) + \frac{2E_0 \cdot N^2}{\sqrt{k}} E_k(t), \quad m < k \leq N.$$

Temporal integration yields that for any $0 < t_0 < t$ we have

$$(3.11) \quad E_{2k}(t) \leq \frac{2E_0 \cdot N^2}{\sqrt{k}} \int_{\tau=t_0}^t e^{-4\varepsilon k^2(t-\tau)} E_k(\tau) d\tau + e^{-4\varepsilon k^2(t-t_0)} \cdot E_{2k}(t_0),$$

and therefore

$$(3.12) \quad E_{2k}(t) \leq \frac{E_0 \cdot N^2}{2\varepsilon\sqrt{k} \cdot k^2} \cdot \max_{t_0 \leq \tau \leq t} E_k(\tau) + e^{-4\varepsilon k^2(t-t_0)} \cdot E_{2k}(t_0).$$

The a-priori estimate (2.8) implies that

$$\max_{0 \leq \tau \leq t} E_{2k}(\tau) < \max_{0 \leq \tau \leq t} E_0(\tau) \leq E_0,$$

and in view of (3.12) we have

$$(3.13)_{2k} \quad E_{2k}(t) \leq \frac{E_0 \cdot N^2}{2\epsilon\sqrt{k} \cdot k^2} \cdot E_0 + e^{-4\epsilon k^2 t} \cdot E_0, \quad k > m.$$

If we choose $t_0 = \frac{t}{2}$ in (3.12) we find that

$$(3.14) \quad E_{2k}(t) \leq \frac{E_0 \cdot N^2}{2\epsilon\sqrt{k} \cdot k^2} \cdot \max_{\frac{t}{2} \leq \tau \leq t} E_k(\tau) + e^{-2\epsilon k^2 t} \cdot E_{2k}\left(\frac{t}{2}\right), \quad k > m,$$

and following Kreiss [1], we can use this to improve our estimate (3.13).

Namely, for $k > 2m$ we can use $(3.13)_k$ to upper bound the terms $\max_{\tau} E_k(\tau)$ and $E_{2k}\left(\frac{t}{2}\right)$ on the right of (3.14), and obtain the improved bound

$$E_{2k}(t) < \left(\frac{8E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^2 \cdot E_0 + \left(1 + \frac{8E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right) e^{-\epsilon k^2 t} \cdot E_0, \quad k > 2m.$$

Now we can repeat this process, and by induction we obtain that for $k > 2^s \cdot m$ we have

$$(3.15) \quad E_{2k}(t) \leq \left(\frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^{s+1} \cdot E_0 + \left(1 + \frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^s e^{-4^{-s+1} \cdot \epsilon k^2 t} \cdot E_0.$$

Verification of the induction step is left to the Appendix. Finally, (3.15) implies that for sufficiently large $k = \frac{N}{4} > 2^s \cdot m$, we have

$$\begin{aligned} \|(I - S_N)u_N(\cdot, t)\|_{L^2(x)}^2 &\equiv E_N(t) \leq \\ &\leq \left(\frac{32 \cdot 8^s \cdot E_0}{\epsilon \sqrt{N}}\right)^{s+1} \cdot E_0 + \left(1 + \frac{32 \cdot 8^s \cdot E_0}{\epsilon \sqrt{N}}\right)^s e^{-4^{-(s+1)} \cdot \epsilon N^2 t \cdot E_0}, \end{aligned}$$

and 3.8 follows.

Our parametrization in (1.14), (1.15) implies that for sufficiently large N we have

$$4^{-(s+1)} \cdot \epsilon N^2 t \geq \text{Const.} \cdot \frac{N^{2-2\beta}}{4^{s+1} \cdot \log N} \cdot t \geq N^{3/2} \cdot t, \quad 0 < 2\beta < \frac{1}{2},$$

as well as

$$\frac{1}{\epsilon \sqrt{N}} \leq \text{Const.} \cdot \frac{\log N}{N^{\frac{1}{2} - 2\beta}} \leq \frac{1}{N^\gamma}, \quad 0 < \gamma < \frac{1}{2} - 2\beta,$$

and Theorem 3.2 tells us that

Corollary 3.3: For any integer $s \geq 0$ there exists a constant C_s such that

$$(3.16) \quad \|(I - S_N)u_N(\cdot, t)\|_{L^2(x)}^2 \leq C_s \cdot (N^{-s} + e^{-N^{3/2} \cdot t}).$$

Corollary 3.3 indicates the spectral decay of the Fourier coefficients $|\hat{u}(k)|$ with wavenumbers $|k| \geq \frac{1}{2} N$, which in turn implies a similar decay for the discretization error, I , on the right of (3.1). For the latter we

have

$$(3.17) \quad \|(I-P_N)^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2 \equiv \|(I-S_N)^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2 + a^2 \|A_N^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2.$$

The Fourier coefficients of the two expressions on the right are given respectively by

$$\begin{aligned} ((I - S_N)^{\frac{1}{2}} u_N^2(\cdot, t))_k &= \frac{1}{2} \sum_{p+q-k=0} u_p(t) \hat{u}_q(t), \quad |k| > N, \\ (A_N^{\frac{1}{2}} u_N^2(\cdot, t))_k &= \frac{1}{2} \sum_{|p+q-k|=2N+1} \hat{u}_p(t) \hat{u}_q(t), \quad |k| \leq N. \end{aligned}$$

In both cases, either $|p| > \frac{N}{2}$ or $|q| > \frac{N}{2}$; hence each one of these coefficients can be upper bounded in a standard fashion to yield

$$\begin{aligned} &\|(I-S_N)^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2 + a^2 \|A_N^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2 \leq \\ &\sum_{N \leq |k| \leq 2N} (1+a^2) \cdot \left[\sum_{|q| \leq N} |\hat{u}_q(t)|^2 \right] \cdot \left[\sum_{|p| > \frac{N}{2}} |\hat{u}_p(t)|^2 \right] \leq 4E_0^2 \cdot N \cdot \|(I-S_N)^{\frac{1}{2}} u_N(\cdot, t)\|_{L^2(x)}^2 \end{aligned}$$

and by (3.17) this is the same as

$$(3.18) \quad \|(I-P_N)^{\frac{1}{2}} u_N^2(\cdot, t)\|_{L^2(x)}^2 \leq 2E_0 \cdot \sqrt{N} \cdot \|(I - S_N)^{\frac{1}{2}} u_N(\cdot, t)\|_{L^2(x)}.$$

Corollary 3.3 together with 3.18 show that due to the presence of the spectral viscosity term II on the right of (2.1), the discretization error I decays to zero at a spectral rate independently whether the underlying solution is smooth or not. We state this as

Corollary 3.4: For any integer $s \geq 0$ there exists a constant C_s
such that for sufficiently large N we have

$$(3.19) \quad \|I \equiv [(I-P_N)^{\frac{1}{2}} u_N^2(\cdot, t)]\|_{L^2(x)} \leq C_s \cdot \sqrt{N} \cdot (N^{-s} + e^{-N^{3/2} \cdot t}), \quad s \geq 0.$$

We close this section with the promised

Proof (of Lemma 3.1): Starting with the identity

$$f_N g_N \equiv f_N (I - S_k) g_N + (I - S_k) f_N \cdot S_k g_N + S_k f_N \cdot S_k g_N$$

and subtracting from this $(I - S_{2k}) P_N [S_k f_N \cdot S_k g_N] \equiv (I - S_{2k}) [S_k f_N \cdot S_k g_N] \equiv 0$,
 we can write

$$(I - S_{2k}) P_N (f_N g_N) \equiv (I - S_{2k}) P_N [f_N (I - S_k) g_N + (I - S_k) f_N \cdot S_k g_N].$$

The quantity inside the right brackets is a trigonometric polynomial of degree $\leq 2N$ and hence, by Parseval relation, its $L^2(x)$ norm dominates the $L^2(x)$ norm of its P_N projection, i.e.,

$$(3.20) \quad \begin{aligned} \|(I - S_{2k}) P_N (f_N g_N)\|_{L^2(x)} &\leq \|P_N [\dots]\|_{L^2(x)} \leq \\ &\leq \|f_N (I - S_k) g_N + (I - S_k) f_N \cdot S_k g_N\|_{L^2(x)}. \end{aligned}$$

The norm on the right of (3.20) is upper bounded by

$$\|f_N(I-S_k)g_N + (I-S_k)f_N \cdot S_k g_N\|_{L^2(x)} \leq \quad (3.21)$$

$$\|f_N\|_{L^2(x)} \cdot \|(I-S_k)g_N\|_{L^\infty(x)} + \|g_N\|_{L^2(x)} \cdot \|(I-S_k)f_N\|_{L^\infty(x)}.$$

Finally, for h_N equals either f_N or g_N we have

$$\begin{aligned} \|(I-S_k)h_N\|_{L^\infty(x)} &\leq \sum_{|p|>k} |\hat{h}_p| \leq \\ (3.22) \quad &\leq \left[\sum_{|p|>k} \frac{1}{p^2} \right]^{1/2} \cdot \left[\sum_{|p|>k} p^2 |\hat{h}_p|^2 \right]^{1/2} < \frac{2}{\sqrt{k}} \|(I-S_k) \frac{\partial}{\partial x} h_N\|_{L^2(x)} \end{aligned}$$

and (3.5) follows from (3.20), (3.21), and (3.22).

4. L^∞ A PRIORI ESTIMATE

The classical energy method can be used to show that the solution of (2.1) remains uniformly bounded during a small finite time. The method reflects the fact that for sufficiently smooth initial data, say with $\frac{\partial^2}{\partial x^2} u_N(x, t=0)$ which are L^2 -bounded, the process of shock formation takes a finite time, during which $\frac{\partial}{\partial x} u_N(x, t)$ remains uniformly bounded and a couple of Sobolev norms could be a priori estimated during that time.

For a brief initial time intervals, we can do better with regard to the smoothness of the initial data, as told by

Lemma 4.1: Consider the Fourier approximation (1.11) - (1.14) with initial data $u_N(x, t=0)$ such that Assumption III holds, i.e.,

$$(4.1) \quad \sum_{|k| \leq N} |\hat{u}_k(t=0)| \leq \text{Const}_0.$$

Then for $t \lesssim \frac{1}{N}$ we have

$$(4.2) \quad \|u_N(\cdot, t)\|_{L^\infty(x)} \leq 2 \cdot \sum_{|k| \leq N} |\hat{u}_k(t=0)|, \quad t \leq \frac{1}{8\text{Const}_0 \cdot N}.$$

Proof: The Fourier transform of (1.9) reads

$$(4.3) \quad \frac{d}{dt} \hat{u}_k(t) + ik \left[\sum_{p+q=k} \hat{u}_p(t) \hat{u}_q(t) + a \cdot \sum_{|p+q-k|=2N+1} \hat{u}_p(t) \hat{u}_q(t) \right] = -\epsilon k^2 \hat{Q}(k) \hat{u}_k(t).$$

Multiply the real (imaginary) part of this by $\text{sgn}(\text{Re} \hat{u}_k(t))$ (respectively, $\text{sgn}(\text{Im} \hat{u}_k(t))$) and sum over all k 's: since the right-hand side is negative we obtain after summing both parts

$$(4.4) \quad \begin{aligned} \frac{d}{dt} \sum_{|k| \leq N} |\hat{u}_k(t)| &\leq (1 + |a|) \cdot 2N \cdot \sum_{|p| \leq N} |\hat{u}_p(t)| \cdot \sum_k |\hat{u}_{k-p}(t)| \leq \\ &\leq 4N \cdot \left(\sum_{|k| \leq N} |\hat{u}_k(t)| \right)^2. \end{aligned}$$

Integration in time yields

$$\|u_N(\cdot, t)\|_{L^\infty(x)} \leq \sum_{|k| \leq N} |\hat{u}_k(t)| \leq \frac{1}{1-4Nt \cdot \sum_{|k| \leq N} |\hat{u}_k(t=0)|} \cdot \sum_{|k| \leq N} |\hat{u}_k(t=0)|,$$

and (4.2) follows.

To obtain L^∞ bound for later time, we shall carefully iterate on the $L^p(x)$ norms of $u_N(x,t)$. To this end, we multiply (2.1) by pu_N^{p-1} and integrate over the 2π -period, obtaining

$$(4.5) \quad \begin{aligned} & \frac{d}{dt} \|u_N(\cdot, t)\|_{L^p(x)}^p + \frac{p}{p+1} u_N^{p+1}(x, t) \Big|_{x=0}^{x=2\pi} = \\ & = p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2 \right] dx + p \cdot \int_0^{2\pi} u_N^{p-1} \epsilon \frac{\partial}{\partial x} \left[Q_N \frac{\partial}{\partial x} u_N \right] dx. \end{aligned}$$

By Corollary 3.4, the discretization error is negligibly small: using (3.19) and the fact that $(I - P_N) \frac{1}{2} u_N^2$ is a trigonometric polynomial of degree $\leq 2N$, we have for any $s \geq 0$

$$(4.6) \quad \begin{aligned} & \left\| \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2(\cdot, t) \right] \right\|_{L^p(x)} \leq \\ & \sqrt{2N} \cdot \left\| \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2(\cdot, t) \right] \right\|_{L^2(x)} \leq C_s N^2 \cdot (N^{-s} + e^{-N^{3/2} \cdot t}). \end{aligned}$$

Therefore, by Hölder inequality, the first integral on the right of (4.5) does not exceed

$$(4.7) \quad \begin{aligned} & p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2 \right] dx \leq p \cdot \|u_N^{p-1}(\cdot, t)\|_{L^q(x)} \cdot \left\| \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2(\cdot, t) \right] \right\|_{L^p(x)} \leq \\ & \leq p \cdot \|u_N(\cdot, t)\|_{L^p(x)}^{p-1} \cdot C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2} \cdot t}), \quad \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

The second integral on the right of (4.5), with $Q_N = I - R_m$, equals

$$p \cdot \int_0^{2\pi} u_N^{p-1} \epsilon \frac{\partial}{\partial x} \left[(I - R_m) \frac{\partial}{\partial x} u_N \right] dx = -\epsilon p(p-1) \cdot \int_0^{2\pi} u_N^{p-2} \left(\frac{\partial u_N}{\partial x} \right)^2 dx - \epsilon p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial^2}{\partial x^2} (R_m u_N) dx.$$

The first term on the right hand side is negative for any even integer $p \geq 2$; for the second term we use Hölder inequality as before, obtaining

$$(4.8a) \quad \varepsilon p \cdot \int_0^{2\pi} u_N^{p-1} \frac{\partial^2}{\partial x^2} (R_m u_N) dx \leq \varepsilon p \|u_N(\cdot, t)\|_{L^p(x)}^{p-1} \cdot \left\| \frac{\partial^2}{\partial x^2} (R_m u_N) \right\|_{L^p(x)}.$$

Now, since $R_m u_N \equiv R_m(x) * u_N(x, t)$ is a trigonometric polynomial of degree $\leq 2m$, (consult (1.12)), we can estimate the $L^p(x)$ norm of its derivatives as follows [5, Chapter X]

$$(4.8b) \quad \begin{aligned} \left\| \frac{\partial^2}{\partial x^2} (R_m u_N) \right\|_{L^p(x)} &\leq (2m)^2 \cdot \|R_m(\cdot) * u_N(\cdot, t)\|_{L^p(x)} \leq \\ &\leq 4m^2 \|R_m(\cdot)\|_{L^1(x)} \cdot \|u_N(\cdot, t)\|_{L^p(x)}. \end{aligned}$$

Using (4.8a) and (4.8b) we conclude that the second integral on the right of (4.5) is upper bounded by

$$(4.9) \quad p \cdot \int_0^{2\pi} u_N^{p-1} \varepsilon \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N] dx \leq 4p \cdot \varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \cdot \|u_N(\cdot, t)\|_{L^p(x)}^p.$$

We recall that according to our parametrization (1.15), $\varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \leq \alpha$. Hence, equipped with (4.7) and (4.9) we return to (4.5) to find that

$$\begin{aligned} p \|u_N(\cdot, t)\|_{L^p(x)}^{p-1} \cdot \frac{d}{dt} \|u_N(\cdot, t)\|_{L^p(x)} &\leq \\ p \|u_N(\cdot, t)\|_{L^p(x)}^{p-1} [4\alpha \cdot \|u_N(\cdot, t)\|_{L^p(x)} + C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2} \cdot t})], \end{aligned}$$

or, after division by the common factor of $p \|u_N(\cdot, t)\|_{L^p(x)}^{p-1}$,

$$\frac{d}{dt} \|u_N(\cdot, t)\|_{L^p(x)} \leq 4\alpha \cdot \|u_N(\cdot, t)\|_{L^p(x)} + C_s \cdot (N^{2-s} + N^2 e^{-N^{3/2} \cdot t}).$$

Finally, we integrate in time obtaining by Gronwall's inequality that for any $0 \leq t_0 \leq t$,

$$(4.10) \quad \|u_N(\cdot, t)\|_{L^p(x)} \leq e^{4\alpha(t-t_0)} \cdot [\|u_N(\cdot, t_0)\|_{L^p(x)} + C_s \cdot (N^{2-s} \cdot (t-t_0) + \sqrt{N} \cdot e^{-N^{3/2} \cdot t_0})].$$

Letting p even tends to infinity, then (4.10) with $t_0 \equiv t_0(N) = \frac{1}{8\text{Const}_0 \cdot N}$ gives us

$$(4.11) \quad \|u_N(\cdot, t)\|_{L^\infty(x)} \leq e^{4\alpha t} \cdot [\|u_N(\cdot, t_0)\|_{L^\infty(x)} + C_s (N^{2-s} \cdot t + \sqrt{N} e^{-\text{Const} \cdot \sqrt{N}})]$$

and together with Lemma 4.1 we conclude with the desired L^∞ bound, namely,

Theorem 4.2: Consider the Fourier approximation (1.11)-(1.14). Then for any $s \geq 0$ there exist constants $\alpha > 0$ and C_s such that

$$(4.12) \quad \|u_N(\cdot, t)\|_{L^\infty(x)} \leq e^{4\alpha t} \cdot [4 \cdot \sum_{|k| \leq N} |\hat{u}_k(t=0)| + C_{s+2} \cdot (N^{-s} \cdot t + \sqrt{N} e^{-\text{Const} \cdot \sqrt{N}})].$$

Remarks: 1. We observe that the exponential time growth in (4.12) does not exceed 4α where $\alpha \sim \varepsilon m^2 \cdot \|R_m(\cdot)\|_{L^1(x)} \leq \text{Const}.$

2. The a priori $L^p(x)$ estimate derived in (4.10) is valid for arbitrary L^2 -bound initial data. We note, however, that the resulting L^∞ bound in such case is not uniform with respect to the initial time t_0 . That is, with arbitrary L^2 -bounded initial data, the solution $\|u_N(\cdot, t)\|_{L^\infty(x)}$ may still grow like $O(\sqrt{N})$. The point made in Lemma 4.1 was that with slightly strengthened regularity assumption on the initial data

$$\sum_{|k| \leq N} |\hat{u}_k(t=0)| \leq \text{Const}_0,$$

this growth is bounded for a brief time interval of length $\sim \frac{1}{N}$, after which the spectral viscosity becomes effective and guarantees the L^∞ bound later on.

5. CONVERGENCE TO THE ENTROPY SOLUTION

We follow [3], using compensated compactness arguments to conclude that $u_N(x, t)$ converges to the entropy solution of (1.1), (1.2).

Proof: (of Theorem 1.1). Consider the four terms on the right-hand side of (2.1) and (2.2). Using (3.18) together with (2.11) along the lines of [3, Lemma 3.1], we find that the term I satisfies

$$\begin{aligned} (5.1) \quad \|I\| &\equiv \frac{\partial}{\partial x} \left[(I - P_N) \frac{1}{2} u_N^2(\cdot, t) \right] \|_{H_{loc}^{-1}(x, t)} \leq \\ &\leq \| (I - P_N) \frac{1}{2} u_N^2(\cdot, t) \|_{L_{loc}^2(x, t)} \leq 4E_0 \cdot J_0 \cdot \frac{1}{\sqrt{\epsilon(N) \cdot N}} \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

Also, by the a priori estimate (2.11) we have

$$\begin{aligned}
 \text{III} &\equiv \varepsilon \frac{\partial}{\partial x} [Q_N \frac{\partial}{\partial x} u_N] \|_{H_{loc}^{-1}(x,t)} \leq \\
 (5.2) \quad &\leq \varepsilon \|Q_N \frac{\partial}{\partial x} u_N\|_{L_{loc}^2(x,t)} \leq \sqrt{\varepsilon(N)} \cdot J_0 \xrightarrow{N \rightarrow \infty} 0.
 \end{aligned}$$

Thus, in view of (5.1) and (5.2), the terms I and II on the right of (2.1) belong to the compact of $H_{loc}^{-1}(x,t)$.

Next we note that since $Q_N + R_m = I$ we have

$$\sqrt{\varepsilon} \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{loc}^2(x,t)} \leq \sqrt{\varepsilon} \left\| Q_N \frac{\partial}{\partial x} u_N \right\|_{L_{loc}^2(x,t)} + \sqrt{\varepsilon} \left\| \frac{\partial}{\partial x} (R_m u_N) \right\|_{L_{loc}^2(x,t)}.$$

The first term on the right is bounded by J_0 ; the second one--being the derivative of a trigonometric polynomial of degree $\leq 2m$, does not exceed $\sqrt{\varepsilon} \cdot 2m \cdot \|u_N\|_{L_{loc}^2(x,t)} \leq \text{Const.}$ Consequently,

$$(5.3) \quad \sqrt{\varepsilon} \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{loc}^2(x,t)} \leq \text{Const.}$$

Equipped with (5.3) we now turn to consider the right-hand side of (2.2). For the third term in (2.2),

$$\begin{aligned}
 \text{III} &= u_N \frac{\partial}{\partial x} [(I - P_N) \frac{1}{2} u_N^2] = \\
 (5.4a) \quad &= \frac{\partial}{\partial x} [u_N (I - P_N) \frac{1}{2} u_N^2] - \frac{\partial}{\partial x} u_N \cdot (I - P_N) \frac{1}{2} u_N^2 \equiv \text{III}_1 + \text{III}_2,
 \end{aligned}$$

we have by Theorem 4.2 and the estimate (5.1)

$$\|III_1\| \equiv \frac{\partial}{\partial x} [u_N(I - P_N)^{\frac{1}{2}} u_N^2] \|_{H_{loc}^{-1}(x,t)} \leq$$

(5.4b)

$$\leq \|u_N(I - P_N)^{\frac{1}{2}} u_N^2\|_{L_{loc}^2(x,t)} \leq \|u_N(\cdot, t)\|_{L^\infty(x)} \cdot \|(I - P_N)^{\frac{1}{2}} u_N\|_{L_{loc}^2(x,t)}^2 \xrightarrow{N \rightarrow \infty} 0,$$

and together with (5.3) we also have

$$\|III_2\| \equiv \frac{\partial}{\partial x} u_N \cdot (I - P_N)^{\frac{1}{2}} u_N^2 \|_{L_{loc}^1(x,t)} \leq$$

(5.4c)

$$\leq \sqrt{\varepsilon} \left\| \frac{\partial}{\partial x} u_N \right\|_{L_{loc}^2(x,t)} \cdot \frac{1}{\sqrt{\varepsilon}} \|(I - P_N)^{\frac{1}{2}} u_N^2\|_{L_{loc}^2(x,t)} \leq \text{Const.} \cdot 4E_0 J_0 \cdot \frac{1}{\varepsilon(N) \cdot \sqrt{N}} \xrightarrow{N \rightarrow \infty} 0.$$

Finally, for the fourth term in (2.2),

$$IV \equiv \varepsilon u_N \frac{\partial}{\partial x} (Q_N \frac{\partial}{\partial x} u_N) =$$

(5.5a)

$$= \varepsilon \frac{\partial}{\partial x} [u_N Q_N \frac{\partial}{\partial x} u_N] - \varepsilon \frac{\partial}{\partial x} u_N \cdot Q_N \frac{\partial}{\partial x} u_N \equiv IV_1 + IV_2,$$

we have by (2.8), (2.11) and the uniform bound in Theorem 4.2,

$$(5.5b) \quad \|IV_1\| \equiv \varepsilon \frac{\partial}{\partial x} [u_N Q_N \frac{\partial}{\partial x} u_N] \|_{H_{loc}^{-1}(x,t)} \leq \|u_N(\cdot, t)\|_{L^\infty(x)} \cdot \sqrt{\varepsilon(N)} \cdot J_0 \xrightarrow{N \rightarrow \infty} 0,$$

while $IV_2 \equiv -\varepsilon [Q_N \frac{\partial}{\partial x} u_N]^2 + \varepsilon (I - Q_N) \frac{\partial}{\partial x} u_N \cdot Q_N \frac{\partial}{\partial x} u_N$ satisfies

$$(5.5c) \quad \|IV_2\|_{L_{loc}^1(x,t)} \leq \varepsilon \|Q_N \frac{\partial}{\partial x} u_N\|_{L_{loc}^2(x,t)}^2 +$$

$$+ \sqrt{\varepsilon} \|R_m \frac{\partial}{\partial x} u_N\|_{L_{loc}^2(x,t)} \cdot \sqrt{\varepsilon} \|Q_N \frac{\partial}{\partial x} u_N\|_{L_{loc}^2(x,t)}$$

$$\leq J_0^2 + \sqrt{\varepsilon} \cdot 2m \cdot E_0 \cdot J_0 \leq \text{Const.}$$

Therefore, by Murat's lemma, [4], the inequalities (5.4), (5.5) imply that the terms III and IV are also in the compact of $H_{loc}^{-1}(x,t)$. In summary, we have shown that the right hand sides of (2.1), (2.2) lie in the compact of $H_{loc}^{-1}(x,t)$, and, according to Theorem 4.2, that $\|u_N(\cdot, t)\|_{L^\infty(x)}$ is bounded (in fact, $\|u_N\|_{L^p(x,t)}$ with $p > 6$ will do for our purpose). Hence we can apply the div-curl lemma [4] to the left-hand sides of (2.1), (2.2) and obtain that (a subsequence of) $u_N(x,t)$ converges strongly in $L_{loc}^2(x,t)$ to a weak limit solution $\bar{u}(x,t)$.

Moreover, we claim that this limit is the entropy solution of (1.1). To verify this claim we show that the right-hand side of (2.2), III + IV, tends weakly to a negative measure. Indeed, by (5.4) and (5.5b) the terms III and IV_1 tend weakly to zero, and hence it is left to show that the term IV_2 ,

$$IV_2 \equiv -\varepsilon \frac{\partial}{\partial x} u_N \cdot Q_N \frac{\partial}{\partial x} u_N = -\varepsilon (Q_N + R_m) \frac{\partial}{\partial x} u_N \cdot Q_N \frac{\partial}{\partial x} u_N, \quad Q_N + R_m = I,$$

tends weakly to a negative measure. To this end we proceed as in [3, Section 4] and rewrite IV_2 in the form

$$(5.6) \quad IV_2 = -\varepsilon [Q_N \frac{\partial}{\partial x} u_N]^2 - \varepsilon \frac{\partial}{\partial x} [\frac{\partial}{\partial x} (R_m u_N) \cdot Q_N u_N] + \varepsilon \frac{\partial^2}{\partial x^2} (R_m u_N) \cdot Q_N u_N.$$

Denote the three terms on the right of (5.6) by IV_{21} , IV_{22} and IV_{23} , respectively. By (2.11), IV_{21} tends weakly to a negative measure

$$(5.7a) \quad \lim_{N \rightarrow \infty} [IV_{21} \equiv -\varepsilon [Q_N \frac{\partial}{\partial x} u_N]^2] \leq 0.$$

If we integrate the second term, IV_{22} , against any C_0^∞ test function $\psi(x,t)$, we find

$$\begin{aligned} \int \int_{x,t} \psi \cdot IV_{22} \, dxdt &\leq \varepsilon \int \int_{x,t} \left| \frac{\partial \psi}{\partial x} \cdot \frac{\partial}{\partial x} (R_m u_N) \cdot Q_N u_N \right| dxdt \leq \\ &\leq \varepsilon \left\| \frac{\partial \psi}{\partial x} \right\|_{L^\infty_{loc}(x,t)} \cdot \left\| \frac{\partial}{\partial x} (R_m u_N) \right\|_{L^2_{loc}(x,t)} \cdot \|Q_N u_N\|_{L^2_{loc}(x,t)}, \end{aligned}$$

and since $R_m u_N$ is a trigonometric polynomial of degree $\leq 2m$, this is less than

$$(5.7b) \quad \int \int_{x,t} \psi \cdot IV_{22} \, dxdt \leq \varepsilon \cdot \left\| \frac{\partial \psi}{\partial x} \right\|_{L^\infty_{loc}(x,t)} \cdot 2m \cdot \|u_N\|_{L^2_{loc}(x,t)}^2 \leq \text{Const.} \cdot \frac{1}{m(N)} \xrightarrow{N \rightarrow \infty} 0.$$

Finally, for the third term

$$IV_{23} = \varepsilon \frac{\partial^2}{\partial x^2} (R_m u_N) \cdot Q_N (u_N - \bar{u}) + \varepsilon \frac{\partial^2}{\partial x^2} (R_m u_N) \cdot Q_N \bar{u} \equiv IV_{231} + IV_{232}$$

we have

$$\begin{aligned} (5.7c) \quad \|IV_{231}\| &\equiv \varepsilon \frac{\partial^2}{\partial x^2} (R_m u_N) \cdot Q_N (u_N - \bar{u}) \Big\|_{L^1_{loc}(x,t)} \leq \\ &\leq \varepsilon \cdot 4m^2 \cdot \|R_m u_N\|_{L^2_{loc}(x,t)} \cdot \|u_N - \bar{u}\|_{L^2_{loc}(x,t)} \leq \text{Const.} \cdot \|u_N - \bar{u}\|_{L^2_{loc}(x,t)} \rightarrow 0, \end{aligned}$$

and since $\varepsilon R_m u_N \cdot Q_N \bar{u}$ tends weakly to zero, so does the term IV_{232} ,

$$(5.7d) \quad \text{wlim}_{N \rightarrow \infty} [IV_{232}] \equiv \varepsilon \frac{\partial^2}{\partial x^2} (R_m u_N) \cdot Q_N \bar{u} = 0.$$

From (5.7a) - (5.7d) we conclude that the term IV_2 in (5.6)--and therefore that the right-hand side of (2.2), tends weakly to a negative measure. Thus, by taking the weak limit of (2.2) we recover (1.2) for our limit solution $\bar{u}(x,t)$. Consequently, the strong L^2_{loc} limit of $u_N(x,t) = \bar{u}(x,t)$ is the unique entropy solution of (1.1) as asserted.

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APPENDIX

A. THE L^1 -LOGARITHMIC BOUND OF MONOTONE VISCOSITY KERNELS

We consider symmetric viscosity kernels of the form

$$Q_N(x) = \sum_{|k| \leq 2m} \hat{Q}(k) e^{ikx} + \sum_{2m < |k| \leq N} e^{ikx},$$

with monotonically increasing Fourier coefficients. Then, the kernels which correspond to $R_m = I - Q_N$, are symmetric polynomials of degree $\leq 2m$

$$(a.1) \quad R_m(x) = 2 \cdot \sum_{|k| \leq 2m} \hat{R}(k) \cos kx$$

whose Fourier coefficients are monotonically decreasing, compare (1.13),

$$(a.2) \quad 1 \geq \hat{R}(k) \geq 0.$$

Such kernels satisfy Assumption II above, as told by

Lemma A.1: There exists a constant such that

$$(a.3) \quad \|R_m(\cdot)\|_{L^1(x)} \leq \text{Const.} \log m.$$

Proof: The result follows if we can show that $R_m(x)$ is majorized by $\text{Const.} \cdot \frac{1}{|x|}$, for then we have

$$\begin{aligned}
 (a.4) \quad \|R_m(\cdot)\|_{L^1(x)} &\leq \int_{|x| \leq \frac{1}{m}} \text{Const.} \cdot m \cdot dx + \int_{\frac{1}{m} \leq |x| \leq \pi} \text{Const.} \cdot \frac{1}{|x|} dx \leq \\
 &\leq \frac{2}{m} \text{Const.} \cdot m + 2 \cdot \text{Const.} \cdot \log |x| \Big|_{x=\frac{1}{m}}^{x=\pi} \leq \text{Const.} \cdot \log m.
 \end{aligned}$$

Since $0 \leq \hat{R}(k) \leq 1$ we have

$$|R_m(x)| \leq 2 \cdot \sum_{|k| \leq 2m} |\hat{R}(k)| \leq 4m;$$

furthermore, summation by parts yields

$$\begin{aligned}
 \left| \sin\left(\frac{x}{2}\right) \cdot R_m(x) \right| &= \left| \sum_{|k| \leq 2m} \hat{R}(k) \cdot \left[\sin\left(k + \frac{1}{2}\right)x - \sin\left(k - \frac{1}{2}\right)x \right] \right| \leq \\
 &\leq 4 + \sum_{1 \leq |k| \leq 2m-1} |\hat{R}(k+1) - \hat{R}(k)| \cdot \left| \sin\left(k + \frac{1}{2}\right)x \right|,
 \end{aligned}$$

and since $\hat{R}(k)$ are assumed to decrease monotonically

$$|R_m(x)| \leq \frac{6}{\left| \sin\left(\frac{x}{2}\right) \right|} \leq \text{Const.} \cdot \frac{1}{|x|},$$

which completes the proof.

B. THE DECAY RATE OF THE FOURIER COEFFICIENTS-REVISITED

In Section 3, we concluded that the quantities

$$E_k(t) \equiv \|(I-S_k)u_N(\cdot, t)\|_{L^2(x)}, \quad \text{satisfy for } k > m, \text{ the recursive inequality} \quad (3.14)$$

$$(b.1) \quad E_{2k}(t) \leq \frac{E_0 \cdot N^2}{2\epsilon\sqrt{k} \cdot k^2} \max_{\frac{t}{2} \leq \tau \leq t} E_k(\tau) + e^{-2\epsilon k^2 t} \cdot E_{2k}\left(\frac{t}{2}\right).$$

In this section we complete the details for the solution of these recurrence relations, and obtain that for $k > 2^s \cdot m$ we have

$$(b.2) \quad E_{2k}(t) \leq \left(\frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^{s+1} \cdot E_0 + \left(1 + \frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^s e^{-4^{-s+1} \cdot \epsilon k^2 t} \cdot E_0,$$

i.e., (3.15) holds. For $s = 0$, (b.2) is reduced to (3.13); now assume that (b.2) is valid for any $k > 2^s \cdot m$: in particular, for $k > 2^{s+1} \cdot m$ we can use (b.2) with k replaced by $\frac{k}{2}$, $\frac{k}{2} > 2^s \cdot m$, and obtain that

$$(b.3) \quad \max_{\frac{t}{2} \leq \tau \leq t} E_k(\tau) \leq \left(\frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^{s+1} \cdot E_0 + \left(1 + \frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^s \cdot e^{-4^{-s} \epsilon k^2 t} \cdot E_0.$$

Furthermore, we have

$$(b.4) \quad E_{2k}\left(\frac{t}{2}\right) \leq \left(\frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^{s+1} \cdot E_0 + \left(1 + \frac{8^s E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2}\right)^s \cdot e^{-2 \cdot 4^{-s} \epsilon k^2 t} \cdot E_0.$$

Using (b.3) and (b.4) to upper bound the right hand side of (b.1) we find

$$\begin{aligned}
 E_{2k}(t) \leq & \frac{E_0 \cdot N^2}{2\epsilon\sqrt{k} \cdot k^2} \cdot \left(\frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+1} \cdot E_0 + \\
 & + \frac{E_0 N^2}{2\epsilon\sqrt{k} \cdot k^2} \cdot \left(1 + \frac{8^{s+1} E_0 N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^s \cdot e^{-4^{-s} \epsilon k^2 t} \cdot E_0 \\
 & + e^{-2\epsilon k^2 t} \cdot \left(\frac{8^s E_0 N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+1} \cdot E_0 \\
 & + e^{-2\epsilon k^2 t} \cdot \left(1 + \frac{8^s E_0 N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^s \cdot e^{-2 \cdot 4^{-s} \epsilon k^2 t} \cdot E_0.
 \end{aligned}$$

The first of the four terms on the right is less than $\left(\frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+2} \cdot E_0$;
the sum of the remaining three terms does not exceed

$$\left(1 + \frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+1} \cdot e^{-4^{-s} \epsilon k^2 t} \cdot E_0,$$

and hence for $k > 2^{s+1} \cdot m$ we have

$$E_{2k}(t) \leq \left(\frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+2} \cdot E_0 + \left(1 + \frac{8^{s+1} E_0 \cdot N^2}{\epsilon\sqrt{k} \cdot k^2} \right)^{s+1} e^{-4^{-s} \epsilon k^2 t} \cdot E_0$$

which completes the induction proof of (b.4).



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