

NATIONAL ADVISORY COMMITTEE  
FOR AERONAUTICS

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REPORT 1015

ANALYSIS OF TURBULENCE-FREE CONVECTION  
BOUNDARY LAYER ON FLAT PLATE



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## AERONAUTIC SYMBOLS FUNDAMENTAL AND DERIVED UNITS

	Symbol	Abbreviation	Unit	Abbreviation
Length	$l$	meter	foot (or mile)	ft (or mi)
	$T$	second	second (or hour)	sec (or hr)
Force	$F$	weight of 1 kilogram	weight of 1 pound	lb
	$P$	horsepower (metric)	horsepower	hp
Power	$P$	horsepower (metric)	hills per hour	mph
	$P$	horsepower (metric)	feet per second	fps

AERONAUTIC SYMBOLS  
FUNDAMENTAL AND DERIVED UNITS

Standard weight of standard air, 1.2255 kg/m<sup>3</sup> or 0.002378 lb/ft<sup>3</sup> (at sea level, 15° C)

Standard velocity of standard air, 1.2255 kg/m<sup>3</sup> or 0.002378 lb/ft<sup>3</sup> (at sea level, 15° C)

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$D_L$  induced drag absolute coefficient  $C_{D_i} = \frac{D_i}{qS}$

$D_P$  parasite drag absolute coefficient  $C_{D_p} = \frac{D_p}{qS}$

$D_T$  total drag absolute coefficient  $C_{D_T} = \frac{D_T}{qS}$

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**REPORT 1015**

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**ANALYSIS OF TURBULENT FREE-CONVECTION  
BOUNDARY LAYER ON FLAT PLATE**

By **E. R. G. ECKERT** and **THOMAS W. JACKSON**

Lewis Flight Propulsion Laboratory  
Cleveland, Ohio

TR 1015~

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# REPORT 1015

## ANALYSIS OF TURBULENT FREE-CONVECTION BOUNDARY LAYER ON FLAT PLATE<sup>1</sup>

By E. R. G. ECKERT AND THOMAS W. JACKSON

### SUMMARY

*With the use of Kármán's integrated momentum equation for the boundary layer and data on the wall-shearing stress and heat transfer in forced-convection flow, a calculation was carried out for the flow and heat transfer in the turbulent free-convection boundary layer on a vertical flat plate. The calculation is for a fluid with a Prandtl number that is close to 1.*

*A formula was derived for the heat-transfer coefficient that was in good agreement with experimental data in the range of Grashof numbers from  $10^0$  to  $10^{12}$ . Because of the good agreement between the theoretical formula and the experimental data, the formula may be used to obtain data for higher Grashof numbers. The calculation also yielded formulas for the maximum velocity in the boundary layer and for the boundary-layer thickness.*

### INTRODUCTION

Recent developments in the field of gas turbines have revealed the need for data on heat transfer in turbulent free-convection flow at very high Grashof numbers ( $10^{12}$  to  $10^{16}$ ). For example, in using the method of free-convection cooling for turbine blades, the centrifugal forces generate a free-convection flow of the cooling fluid in the blade passages that is within the preceding range of Grashof numbers. Free-convection flow is also superimposed on the flow of the cooling air in the hollow blades of air-cooled gas turbines and may influence considerably the heat transfer under certain conditions. The radial flow present in the cooled boundary layers on the outside of cooled turbine blades is also of a type similar to the free-convection flow in the blade coolant passages.

In order to understand and to evaluate such cooling processes, information on the heat transfer, the boundary-layer thickness, and the velocities connected with free-convection flow is necessary. The knowledge of turbulent free-convection flow, however, is limited. Experiments on plane vertical surfaces give heat-transfer data up to Grashof times Prandtl numbers of  $10^{12}$  (summarized in references 1 and 2). Griffiths and Davis (reference 3) determined, in addition, temperature and velocity profiles. Watzinger and Johnson (reference 4) measured heat transfer and temperature and velocity profiles in a superimposed forced- and free-convection flow in a vertical tube. Brown and Marko (reference 5) show by theoretical considerations that a gen-

eral relation exists between the Grashof number that characterizes free-convection flow and the Reynolds number that characterizes forced flow. Colburn and Hougen (reference 6) derive a formula for the heat transfer in turbulent free-convection flow on a vertical plate. However, only the laminar sublayer in the immediate neighborhood of the wall was investigated in reference 6 and the thickness of this sublayer, made dimensionless by the shearing stress velocity and the kinematic viscosity, was assumed to be the same as in forced flow. This analysis therefore gives no information on the whole boundary-layer thickness and on the velocities.

The problem of turbulent free-convection flow is approached herein using another method. Certain shapes are assumed for the temperature and velocity profiles in the free-convection boundary layer. In addition, an empirical relation for the shearing stress on the wall and a heat-transfer coefficient derived from forced-convection flow are used to estimate the boundary-layer thickness, the maximum velocity within the boundary layer, and a heat-transfer coefficient for free-convection flow. No experimental free-convection measurements are used in the calculations.

### SHEARING STRESS AND HEAT FLOW IN FORCED-CONVECTION BOUNDARY LAYER

The relations for forced-convection flow that are needed in the theoretical free-convection calculations are compiled in this section. It is known that for Reynolds numbers that are not too high, the velocity profile in a turbulent boundary layer on a flat plate can be represented by the equation

$$u = u_1 \left( \frac{y}{\delta} \right)^{1/7} \quad (1)$$

(All symbols are defined in the appendix.) The shearing stress on the plate surface in such a flow is given by the equation (reference 7),

$$\tau_w = 0.0225 \rho u^2 \left( \frac{\nu}{u_1 y} \right)^{1/4} \quad (2)$$

This equation is the relation for the shearing stress on the wall that is used in the derivation of the boundary-layer equations for the turbulent free-convection boundary layer. By introducing equation (1), the velocity profile, the shear-

<sup>1</sup> Supersedes NACA TN 2207, "Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate" by E. R. G. Eckert and Thomas W. Jackson, 1950.

ing stress can also be expressed by the velocity  $u_1$  at the outer edge of the boundary layer and by the boundary-layer thickness  $\delta$ :

$$\tau_w = 0.0225 \rho u_1^2 \left( \frac{\nu}{u_1 \delta} \right)^{1/4} \quad (3)$$

For fluids or gases with a Prandtl number equal to 1, the temperature profile is similar in shape to the velocity profile. When the temperatures within the boundary layer are measured with the temperature outside the boundary layer as reference, the expression for the temperature profile is

$$\theta = \theta_w \left[ 1 - \left( \frac{y}{\delta} \right)^{1/7} \right] \quad (4)$$

Equation (4) is valid for values of  $y$  less than  $\delta$ . For larger values of  $y$ ,  $\theta = 0$ .

Reynolds analogy between the turbulent exchange of momentum and heat gives the relation (reference 8)

$$\frac{q}{\tau} = g c_p \frac{dt}{du} \quad (5)$$

Because the temperature and velocity profiles are similar in shape for a Prandtl number equal to 1, the finite temperature and velocity differences at two arbitrary points in the flow can be introduced into the last equation instead of the differentials. Using the differences between the wall and the flow outside the boundary layer and specifying the heat flow and the shearing stress for the position at the wall give the following equation:

$$\frac{q_w}{\tau_w} = g c_p \frac{\theta_w}{u_1} \quad (6)$$

The introduction of the law (equation (3)) for the shearing stress into this expression yields the following equation:

$$q_w = 0.0225 g \rho c_p u_1 \theta_w \left( \frac{\nu}{u_1 \delta} \right)^{1/4} \quad (7)$$

Heat transfer is usually calculated with dimensionless moduli. Such a modulus is the Stanton number

$$St = \frac{q_w}{g \rho c_p u_1 \theta_w} = 0.0225 \left( \frac{\nu}{u_1 \delta} \right)^{1/4}$$

Experimental investigations show that for fluids with Prandtl numbers from approximately 0.5 to 50 (reference 1, p. 520) the same relation holds when it is multiplied by the factor  $P_r^{-2/3}$ .

$$St = \frac{q_w}{g \rho c_p u_1 \theta_w} = 0.0225 \left( \frac{\nu}{u_1 \delta} \right)^{1/4} (P_r)^{-2/3} \quad (8)$$

This relation for the heat flow  $q_w$  is used in the derivation of the boundary-layer equations for the turbulent free-convection boundary layer.

Replacing the boundary-layer thickness  $\delta$  by

$$\delta = 0.366 x \left( \frac{\nu}{u_1 x} \right)^{1/5}$$

which is the boundary-layer thickness on a flat plate in turbulent forced-convection flow (reference 1, p. 481), transforms equation (8) into the widely accepted formula for heat transfer on a flat plate in the turbulent range

$$Nu = \frac{q_w x}{k \theta_w} = 0.029 \left( \frac{u_1 x}{\nu} \right)^{0.8} (P_r)^{1/3} \quad (9)$$

## DETERMINATION OF TURBULENT FREE-CONVECTION BOUNDARY LAYER

### DERIVATION OF BOUNDARY-LAYER EQUATIONS

If a stationary plane vertical wall is heated to a temperature higher than the surroundings, the layer of fluid adjacent to the wall is heated by conduction from the wall. In this way, buoyant forces are generated that cause this layer to flow in an upward direction. This layer of fluid adjacent to the wall to which the vertical motion is confined is called the free-convection boundary layer. The boundary layer begins with zero thickness at the lower end of the vertical wall and increases in thickness in the upward direction. At a certain distance from the lower end of the wall the boundary layer becomes turbulent, depending on the critical Grashof number. The distance measured vertically from the lower end of the wall is called  $x$  and the distance normal to the wall  $y$ . In order to determine the boundary-layer thickness for steady state, a small stationary volume element in the turbulent region of the boundary layer is considered. Figure 1 shows this volume element. The dimensions of the element are  $dx$  along the wall and  $l$  normal to the wall. The length  $l$  should be larger than the boundary-layer thickness  $\delta$ . For two-dimensional flow, the dimension of the volume element normal to the plane of figure 1 may be considered to be 1. The upward velocity of the fluid in plane 1-1 at a distance  $y$  from the surface of the wall is  $u$ . Then the mass flow through a small area with a width  $dy$  is  $\rho u dy$  and the flow of momentum in the  $x$ -direction is  $\rho u^2 dy$ . The momentum flow in the  $x$ -direction entering the volume element through plane 1-1 is

$$\int_0^l \rho u^2 dy$$

In progressing to plane 2-2 the momentum flow changes by

$$\frac{d}{dx} \left( \int_0^l \rho u^2 dy \right) dx$$

The mass flow entering the volume element through plane 1-1 is generally different from the mass flow leaving the element through plane 2-2. Therefore fluid enters or leaves the volume element through the plane parallel to the wall at a distance  $l$ . Because it is assumed that the velocity in the  $x$ -direction outside the boundary layer is so small that it can be neglected, no momentum in the  $x$ -direction is carried through the plane.

The rate of change of momentum must be in equilibrium with the forces acting in the  $x$ -direction on the fluid within or on the surface of the volume element considered. A shearing stress  $\tau_w$  acts on the wall. The force connected with

this stress is  $\tau_w dx$ . No shearing stress occurs on the surface of the volume element that is parallel to the wall and at a distance  $l$  from the wall because outside the boundary layer the velocity in the  $x$ -direction is zero.

According to the boundary-layer theory, the pressure change can be neglected along any normal to the surface. A constant pressure difference  $dp$  therefore exists between planes 1-1 and 2-2. This pressure difference gives a force on the volume element of magnitude  $l dp$ . In addition to the forces on the surfaces, there is a force due to the weight of the fluid within the volume element

$$\left( \int_0^l \rho g dy \right) dx$$

The process of summing up all the forces and equating them to the change in momentum flow gives the momentum equation

$$\frac{d}{dx} \left( \int_0^l \rho u^2 dy \right) dx = l dp - \left( \int_0^l \rho g dy \right) dx - \tau_w dx$$

Because no flow exists outside the boundary layer, the pressure difference between planes 1-1 and 2-2 is balanced by the weight of the fluid layer between the planes

$$dp = \rho_s g dx$$

When both sides of this equation are multiplied by  $l$  and the right side of the equation is changed to the integral form, the following equation is obtained

$$l dp = \left( \int_0^l \rho_s g dy \right) dx$$

Introducing the preceding equation into the momentum equation gives

$$\frac{d}{dx} \left( \int_0^l \rho u^2 dy \right) dx = g \left[ \int_0^l (\rho_s - \rho) dy \right] dx - \tau_w dx$$

Introducing the expansion coefficient defined by the equation

$$\beta = \frac{v - v_s}{v_s} \left( \frac{1}{t - t_s} \right)$$

the first term on the right side of the preceding equation can be transformed into

$$g dx \int_0^l \beta \rho (t - t_s) dy$$

Designating the difference between the temperature  $t$  at the distance  $y$  and the temperature  $t_s$  outside the boundary layer by  $\theta$ , the following expression is obtained:

$$g dx \int_0^l \beta \rho \theta dy$$

In the applications considered, the essential influence of the density changes on the flow is taken into account by the introduction of the expansion coefficient  $\beta$ . The other influences of variable density on the flow and the variation of

the expansion coefficient  $\beta$  with small temperature differences are negligible. Therefore,  $\beta$  and  $\rho$  can be assumed constant in the preceding expressions. The momentum equation for the free-convection boundary layer therefore becomes

$$\frac{d}{dx} \int_0^l u^2 dy = g \beta \int_0^l \theta dy - \frac{\tau_w}{\rho} \quad (10)$$

A similar equation is set up for the heat flow through the volume element in figure 1. The heat carried with the fluid through plane 1-1 is

$$g \rho c_p \int_0^l u \theta dy$$

where the enthalpies  $c_p \theta$  are measured from the temperature outside the boundary layer. The specific heat and density are considered constant.

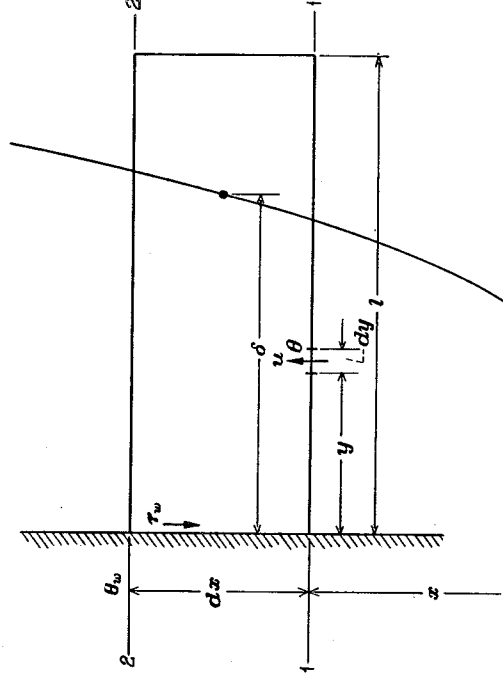


FIGURE 1.—Nomenclature in deriving equations for free-convection boundary layer.

The heat carried out of plane 2-2 differs from that carried into plane 1-1 by

$$g \rho c_p \frac{d}{dx} \left( \int_0^l u \theta dy \right) dx$$

The difference in the heat flow through planes 1-1 and 2-2 must come from the surface and the heat flow leaving the plate per unit time and area is therefore

$$q_w = g \rho c_p \frac{d}{dx} \int_0^l u \theta dy \quad (11)$$

Equation (11) is the heat-flow equation for the free-convection boundary layer.

Equations for the laminar free-convection boundary layer that are analogous to equations (10) and (11) were derived and solved by Squire. (See reference 9.)

#### SOLUTION OF BOUNDARY-LAYER EQUATIONS

Equations (10) and (11) are sufficient to calculate the boundary-layer thickness  $\delta$  and the velocity  $u$  when the shape of the velocity and temperature profiles within the boundary layer and the laws for the shearing stress and the

heat flow on the wall are known. It was suggested by von Kármán that an approximate solution may be obtained by introducing approximate shapes for both profiles (reference 7). The accuracy of the solution is better the more closely the assumed shapes correspond to the real profiles. The method, however, proved comparatively insensitive to changes in the profile shape.

As mentioned in the section "INTRODUCTION" some information on the shape of the velocity and temperature profiles in turbulent free-convection flow can be obtained from reference 3. In figure 2, the measured values of both profiles in the turbulent range are presented with the ratios  $\theta/\theta_w$  and  $u/u_{max}$  plotted against the distance from the wall in an arbitrary scale. The value  $u_{max}$  was determined by drawing curves through the measured velocity values given in table VI of reference 3.

The distance from the wall where the curves  $u/u_{max}$  have the value 0.5 was arbitrarily assigned the value 1. The lines in figure 2 are the curves by which the temperature and velocity profiles are approximated herein. These profiles are presented in figure 2 in such a way that the velocity profile has the value 0.2 at a distance 1 and the temperature profile the value 0.2 at a distance 0.5 from the wall. An indication of the shape of turbulent free-convection flow profiles can also be obtained from the measurements by Waizinger and Johnson (reference 4) on mixed forced- and free-convection flow in a vertical tube. Some of these measured profiles are presented in figure 3.

Equation (4), which describes the forced-convection temperature profile, is used for the temperature profile in turbulent free-convection flow. The profiles of figure 2 indicate that this assumption is reasonable. The velocity profile in free-convection flow, however, differs from the forced-flow profile by having a velocity of zero outside the boundary layer. Accordingly, equation (1) is multiplied by a function of  $y/\delta$ , which brings the velocities back to the value zero at the outer edge of the boundary layer where  $y=\delta$ . Two simple expressions that fulfill this condition and that agree quite well with the measured values are plotted in figure 2.

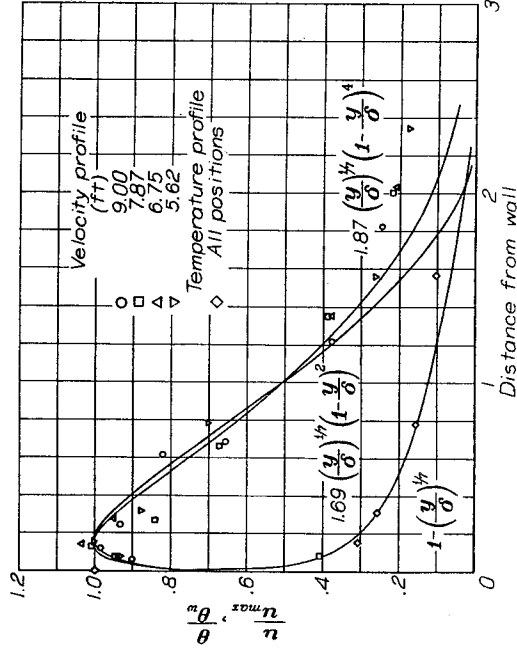
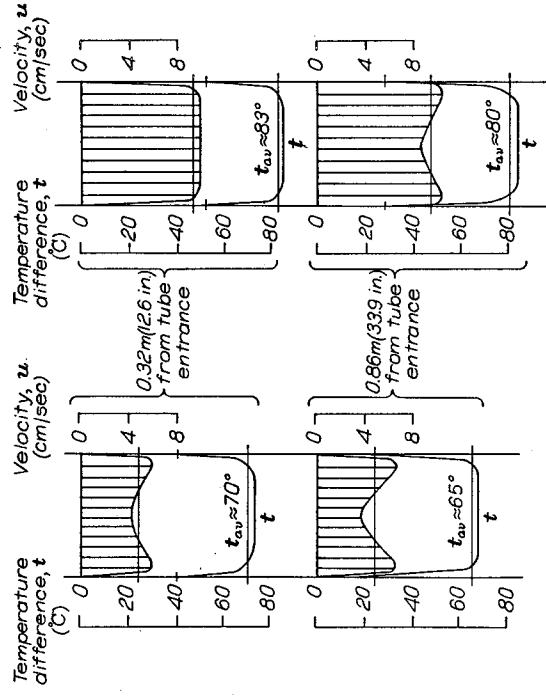


FIGURE 2.—Experimental and theoretical velocity and temperature profiles for turbulent free-convection flow on a flat plate. Experimental data from reference 3.



(a) Reynolds number of 6340 (corresponding to average velocity of 4.7 cm/sec). (b) Reynolds number of 14,200 (corresponding to average velocity of 9.4 cm/sec).

FIGURE 3.—Temperature and velocity profiles in mixed forced- and free-convection flow of water in vertical tube of 50.3 millimeter (1.98 in.) diameter. Data from figure 10 of reference 4.

The velocity profile that represents the measured points somewhat better is used in the following calculations:

$$u = u_1 \left( \frac{y}{\delta} \right)^{1/7} \left( 1 - \frac{y}{\delta} \right)^4 \quad (12)$$

Equation (12) is valid for values of  $y$  less than  $\delta$ . For larger values of  $y$ ,  $u=0$ . The maximum velocity  $u_{max}$  of this profile can be found by differentiating equation (12), giving

$$u_{max} = 0.537 u_1 \quad (13)$$

By use of equations (4) and (12) and the fact that  $u=0$  and  $\theta=0$  for distances greater than  $\delta$  from the wall, the integrals in the momentum and heat-flow equations (equations (10) and (11)) become

$$\int_0^t u^2 dy = 0.0523 \delta u_1^2$$

$$\int_0^t \theta dy = 0.125 \delta \theta_w$$

$$\int_0^t u \theta dy = 0.0366 \delta u_1 \theta_w$$

The same value of  $\delta$  was used for the velocity and temperature profiles. Calculations on forced-convection heat transfer indicate that the same value of  $\delta$  for the velocity and temperature profiles can be used for fluids that have a Prandtl number close to 1 (reference 8). For very large or small Prandtl numbers in forced flow, the thickness of the temperature boundary layer is considerably smaller or larger than the thickness of the velocity boundary layer. This condition is probably true for free-convection flow also.

In order to solve boundary-layer equations (10) and (11), the laws for the shearing stress on the wall and the heat flow must be introduced. It is assumed that, in the layers very



near the wall, the conditions are similar for free- and forced-convection flow and that the same laws for the shearing stress and the heat flow that are used in forced-convection flow can therefore be used in free-convection flow. A similar assumption for the shearing stress was used by von Kármán to calculate the radial flow on a rotating disk, with satisfactory results (reference 7).

As a consequence of the preceding assumption, equations (2) and (8) will be used for free-convection flow. Very near the wall the second term in equation (12) for the free-convection-velocity profile is 1 and, therefore, the same transformation that led in forced flow from equation (2) to equation (3) can be made for free-convection flow. By introducing the evaluated integrals and equations (3) and (8), the momentum and heat-flow equations become

$$0.0523 \frac{d}{dx} (u_1^2 \delta) = 0.125 g \beta \theta_w \delta - 0.0225 u_1^2 \left( \frac{\nu}{u_1 \delta} \right)^{1/4} \quad (14)$$

$$0.0366 \frac{d}{dx} (u_1 \delta) = 0.0225 u_1 \left( \frac{\nu}{u_1 \delta} \right)^{1/4} (Pr)^{-2/3} \quad (15)$$

Equations (14) and (15) are total differential equations from which the two unknown values  $\delta$  and  $u_1$  may be determined as functions of  $x$ .

Equations (14) and (15) can be solved by introducing

$$u_1 = C_u x^m \quad (16)$$

$$\delta = C_\delta x^n \quad (17)$$

The introduction of these values transforms equations (14) and (15) into

$$0.0523 (2m+n) C_u^2 C_\delta^2 x^{2m+n-1} = 0.125 g \beta \theta_w C_u x^n - \quad (18)$$

$$0.0366 (m+n) C_u C_\delta x^{m+n-1} = 0.0225 (Pr)^{-2/3} \nu^{1/4} C_u^{3/4} C_\delta^{-1/4} x^{(3m/4)-(n/4)} \quad (19)$$

Because equations (18) and (19) must be valid for any value of  $x$ , the exponents of  $x$  must be identical, that is,

$$2m+n-1 = n = \frac{7m}{4} - 4$$

$$m+n-1 = \frac{3m}{4} - 4$$

$$m = \frac{1}{2} \text{ and } n = \frac{7}{10} \quad (20)$$

With these values of  $m$  and  $n$ , equation (19) can be solved for the constant  $C_u$

$$C_u = 0.0689 \nu C_\delta^{-5} (Pr)^{-8/3} \quad (21)$$

Introducing this constant into equation (18) gives

$$C_\delta^{10} = 0.00338 \frac{\nu^2}{g \beta \theta_w} [1 + 0.494 (Pr)^{2/3}] (Pr)^{-16/3} \quad (22)$$

Introducing the Grashof number  $Gr = \frac{g \beta \theta_w x^3}{\nu^2}$  gives for the velocity

$$u_1 = 1.185 \frac{\nu}{x} (Gr)^{1/2} [1 + 0.494 (Pr)^{2/3}]^{-1/2} \quad (23)$$

and for the boundary-layer thickness

$$\delta = 0.565 x (Gr)^{-1/10} (Pr)^{-8/15} [1 + 0.494 (Pr)^{2/3}]^{1/10} \quad (24)$$

More significant than the value  $u_1$  is the maximum velocity  $u_{max}$  within the boundary layer. (See equation (13).) A Reynolds number is obtained to make this velocity dimensionless:

$$Re_{e,max} = \frac{u_{max} x}{\nu} = 0.636 (Gr)^{1/2} [1 + 0.494 (Pr)^{2/3}]^{-1/2} \quad (25)$$

A more significant value characterizing the thickness of the boundary layer than the value  $\delta$  used up to now is the displacement thickness  $\delta^*$ , which is defined for free-convection flow by the equation

$$\delta^* = \int_0^\infty \frac{u}{u_{max}} dy$$

With the use of equations (12) and (13) it is found that

$$\delta^* = 0.272 \delta$$

The displacement thickness made dimensionless by the distance  $x$  from the leading edge of the plate is therefore

$$\frac{\delta^*}{x} = 0.154 (Gr)^{-1/10} (Pr)^{-8/15} [1 + 0.494 (Pr)^{2/3}]^{1/10}$$

The heat-transfer coefficient  $h = q_w / \theta_w$  at the point  $x$  on the wall is found by introducing the boundary-layer thickness  $\delta$  and the velocity  $u_1$  into equation (8). Changing to the usually presented Nusselt number  $Nu = hx/k$  gives

$$Nu = 0.0295 (Gr)^{2/5} (Pr)^{7/15} [1 + 0.494 (Pr)^{2/3}]^{-2/5} \quad (26)$$

In order to compare this equation with experimental results, it is necessary to change from the local heat-transfer coefficient to the average value along the plate. By introducing the expression for the Grashof number into equation (26), it can be seen that the local heat-transfer coefficient is proportional to the power 0.2 of the distance  $x$  ( $h = C_h x^{0.2}$ ). With the assumption that the boundary layer is turbulent from the leading edge, the average heat-transfer coefficient becomes

$$h_{av} = \frac{1}{x} \int_0^x h dx = \frac{C_h}{x} \int_0^x x^{0.2} dx = \frac{C_h}{1.2} x^{0.2} = \frac{h}{1.2} \quad (27)$$

In reality the boundary layer is first laminar and only at a certain distance from the lower edge of the plate does it become turbulent. The preceding expression for the average heat-transfer coefficient can therefore be expected to represent the true value only at Grashof numbers so high that the extent of the laminar boundary layer is small compared

with the total length  $x$ . This limit for the Grashof number seems to be near  $10^{10}$ . For Grashof numbers higher than this value, the average Nusselt number can be calculated with equation (27); therefore,

$$Nu_{av} = 0.0246(Gr)^{2/5}(Pr)^{1/15}[1 + 0.494(Pr)^{2/3}]^{-2/5} \quad (28)$$

In order to determine to what extent the shape of the velocity profile influences the results, a calculation was made with the second velocity distribution shown in figure 2:

$$u = u_1 \left( \frac{y}{\delta} \right)^{1/7} \left( 1 - \frac{y}{\delta} \right)^2 \quad (29)$$

Only the numerical constants are influenced by the change in the profile shape. The constant preceding the value  $(Pr)^{2/3}$  changes from 0.494 to 0.342 in all equations. In addition, the value 0.636 changes to 0.487 in the equation for  $Re_{max}$ ; in the equation for  $\delta^*/x$  the value 0.154 changes to 0.200 and in the equation for  $Nu_{av}$  the constant 0.0246 changes to 0.0198.

#### COMPARISON WITH EXPERIMENTS

In figure 4 the results of experiments carried out in different investigations (references 1 and 2) are plotted as the average Nusselt number  $Nu_{av}$  against the product  $Gr Pr$ . For the

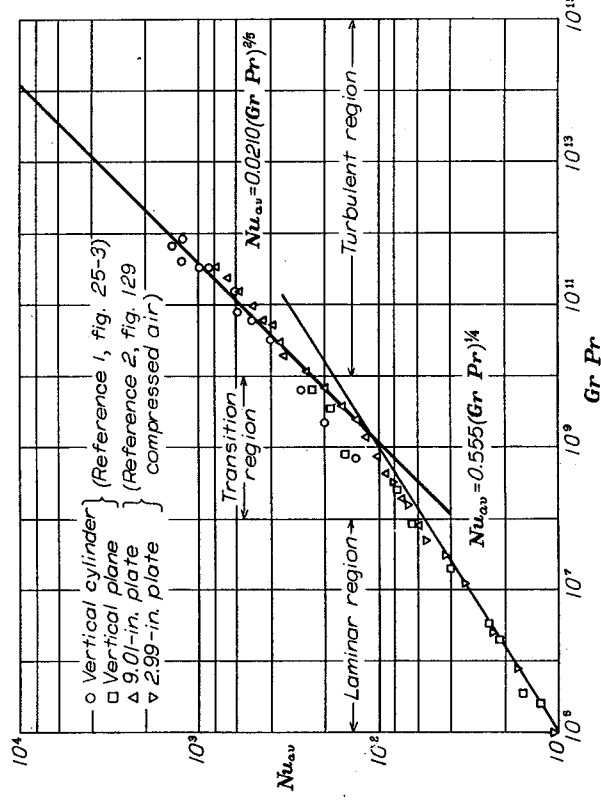


FIGURE 4.—Average Nusselt number for free-convection flows on a vertical plate. Experimental data from references 1 and 2.

lower values of  $Gr Pr$  the experimental results quite accurately fit the equation

$$Nu_{av} = 0.555(Gr Pr)^{1/4}$$

which was, with a slight adjustment of the constant, theoretically derived for laminar free-convection flow. The experimental results in the turbulent range ( $Gr Pr = 10^{10}$  to  $10^{12}$ ) may be represented by the equation

$$Nu_{av} = 0.0210(Gr Pr)^{2/5} \quad (30)$$

The exponent of the Grashof number in this equation is the same as that of equation (28) derived in the previous section. When equation (28) is transformed into the form of equation (30) in such a way that the values of both equations are the same for  $Pr = 0.72$ , the constant of equation (28) becomes 0.0210. The heat-transfer coefficient derived with equation (12) for the velocity profile is in perfect agreement with the experimental results. Such agreement is probably a coincidence. The profile given by equation (29), which does not fit the velocity distributions measured by Griffiths and Davis in reference 3 as well as does the first profile, gives heat-transfer coefficients that are 17 percent lower than the measured values.

The values for the maximum velocity within the boundary layer and the boundary-layer-displacement thickness agree poorly with the values measured in reference 3 in the turbulent range. Heat-transfer coefficients were not measured therein on the experimental apparatus on which velocity and temperature profiles were obtained.

Whereas the velocity and temperature profile shapes as measured in reference 3 are typical of turbulent free-convection flow, the order of magnitude of these profiles appears to be in error. It can be shown that there is disagreement within the measured values themselves. The heat given off by the plate to the air stream must be carried away within the boundary layer. The measured temperature and velocity profiles as well as the measured maximum velocity and the boundary-layer thickness vary little along the plate in the turbulent range, which means that only a small part of the heat given off by the wall is found in the boundary layer. The horizontal dimension of the plate may have been too small compared with the vertical length to make the flow in the center part two-dimensional and air may have flowed into the boundary layer from the sides.

#### SUMMARY OF RESULTS

With the use of Kármán's approximate method, a calculation was carried out for the flow and heat transfer in the turbulent free-convection boundary layer on a vertical flat plate. The calculation used relations for the heat flow and shearing stress on the wall developed for forced turbulent flow and velocity and temperature profiles that approximate well the shapes of profiles measured by Griffiths and Davis.

A formula was derived for the heat-transfer coefficient that was in good agreement with measured values in the range of Grashof numbers from  $10^{10}$  to  $10^{12}$  and that can be used to extrapolate the values into the range of higher Grashof numbers. The formula is valid for Prandtl numbers that are close to 1.

The calculation also yielded formulas for the maximum velocity in the boundary layer and for the boundary-layer thickness.

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## APPENDIX

## SYMBOLS

The following symbols are used in this report:

$a$	thermal diffusivity, sq ft/sec	$St$	Stanton number, $\frac{q_w}{g\rho c_p u_1 \theta_w}$
$C_h$	constant for variation of heat-transfer coefficient with $x^{0.2}$	$t$	temperature, °F
$C_u$	constant for variation of velocity $u_1$ in boundary layer with $x^m$	$u$	velocity component in $x$ -direction, ft/sec
$C_s$	constant for variation of boundary-layer thickness with $x^n$	$u_1$	velocity outside boundary layer of comparable forced-convection flow, ft/sec
$c_p$	specific heat at constant pressure, Btu/(lb) (°F)	$v$	specific volume, cu ft/lb
$Gr$	Grashof number, $\frac{g\beta\theta_w x^3}{\nu^2}$	$x$	coordinate (distance along plate from starting point of boundary layer), ft
$g$	acceleration due to gravity, ft/sec <sup>2</sup>	$y$	coordinate (distance from wall), ft
$h$	heat-transfer coefficient, Btu/(sq ft) (sec) (°F)	$\beta$	expansion coefficient, 1/°F
$k$	heat conductivity, Btu/(ft) (sec) (°F)	$\delta$	boundary-layer thickness, ft
$l$	length (fig. 1), ft	$\delta^*$	displacement thickness of boundary layer, ft
$m$	exponent	$\theta$	temperature difference, °F
$Nu$	Nusselt number, $\frac{hx}{k}$	$\theta_w$	temperature difference between wall and fluid outside of boundary layer, °F
$n$	exponent	$\mu$	absolute viscosity, $\rho\nu$ , lb-sec/sq ft
$Pr$	Prandtl number, $\frac{\nu}{a} = \frac{g c_p \mu}{k}$	$\nu$	kinematic viscosity, sq ft/sec
$p$	pressure, lb/sq ft	$\rho$	mass density, (lb) (sec <sup>2</sup> )/ft <sup>4</sup>
$q$	specific heat flow, Btu/(sec) (sq ft)	$\tau$	shearing stress, lb/sq ft
$Re_{max}$	Reynolds number based on maximum velocity $u_{max}$ , $\frac{u_{max} x}{\nu}$	Subscripts:	
		$av$	average value
		$max$	maximum value
		$w$	on wall
		$\delta$	at outer edge of boundary layer

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