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ANALYSIS OF UNSTEADY TRANSONIC CHANNEL FLOW WITH SHOCK WAVES†

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Abstract

The inviscid, unsteady flow in a two-dimensional channel with a sonic throat is analyzed using an asymptotic expansion for the velocity potential in terms of a small parameter E, which is a measure of the non-dimensional perturbation velocity. The analysis includes the case where shock waves exist in a channel with arbitrary wall shape, with arbitrary disturbances imposed at a given downstream location. Solutions for the first and second order pertubation potentials are derived. These outer solutions are not uniformly valid near the shock since they do not satisfy shock jump conditions of second and higher order, requiring the existence of an inner region with solutions which are matched asympotically to those in the outer region. Numerical results for the flow field in an accelerating nozzle flow with a shock wave, where the nozzle back pressure is oscillating sinusoidally, show the resulting shock wave motion and unsteady flow downstream of the shock wave. The analysis is extended to asymmetric channels with large radius of curvature.

I. Introduction

An analysis of unsteady transonic channel flow with a shock wave has many important applications to internal aerodynamic problems in air intakes and jet engines. Decelerating flows with shock waves may occur in the throat region of an inlet which has a sonic or near sonic throat, in internal nozzle flows, or in a transonic cascade of a jet engine compressor. In the turbine, where the flow is accelerating, shocks may or may not occur, depending on the pressure ratio across the transonic region. Unsteadiness of the flow in these applications can result from a variety of causes such as gusts, changes in engine power setting, bypass/bleed door actuation, and combustion/ignition pressure pulses associated with afterburner light-off or termination.

The study of two-dimensional transonic channel flows has generally been approached by searching for similarity solutions which give nozzle-like flows when applied to the transonic small-disturbance equation. Similarity solutions for steady flow without shock waves (Tomotika and Tamada (1)) have been extended to flows with shock waves by Sichel (2) and to unsteady transonic channel flows with shock waves by Adamson and Richey (3). The similarity transformations applied to inviscid transonic channel flow have also been extended to include longitudinal viscosity effects to study shocks with thickness of the order of the transonic region for steady flow by Sichel and for unsteady flow by Adamson and Richey.

Although similarity approaches have the advantage of yielding relatively simple solutions with minimum computational effort, and provide valuable insight into the nature of transonic channel flow, they do not provide a general solution to the "direct" problem of specifying arbitrary boundary and initial conditions to construct a specific solution. The solutions obtained are self-similar, satisfying special boundary conditions which may or may not correspond to a given physical problem. For unsteady flow with shock waves, Adamson and Richey point out that, with similarity solutions, only special wall shapes can be considered, and the unsteadiness is associated with unsteady motion of the channel walls. Also, with similarity solutions for those cases where the wall is instantaneously a streamline, the wall goes through a small change in slope where the oblique shock and the wall intersect. A smooth wall with a continuous slope violates the inviscid flow tangency condition at the shock. It is thus clear that, for direct applications, it is necessary to concentrate on procedures which can apply to flows where the wall shape is specified, and flow unsteadiness is induced by time-dependent disturbances imposed on the flow.

An approach to the direct problem for steady transonic channel flows has been discussed by Szaniawski (4), who expanded the perturbation velocity potential in an assumed power series in the transverse coordinate. Substitution of this power series into the general potential equation and boundary conditions is shown to yield recursion formulas for the terms in the series. Since it is a direct method, it should be advantageous with respect to the similarity approach. However, the applicability of the method to unsteady flow with shock waves was not clear, nor was it clear under what conditions the assumed power series would be valid.

In this paper it is shown that the Szaniawski power series can be derived in a systematic fashion, thereby indicating the limitations and regions of applicability of that solution. More importantly, a procedure is developed which is capable of treating unsteady flow with shock waves. The derived asymptotic expansions may not be uniformly valid as the nozzle throat region is approached, as discussed by Adamson, Messiter and Richey (5), thus requiring the existence of an inner region near the throat. Here it is shown that there must also be an inner region near the shock wave, required because the basic (outer) solutions do not satisfy shock jump conditions of second and higher order. Solutions are derived for unsteady flow in this inner region showing how the shock motion is related to time-dependent variations imposed on the flow.

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II. Derivation of Equations

The analysis of transonic channel flow with shock waves is treated by superimposing small time-dependent perturbations on a uniform, steady irrotational sonic flow. Figure 1 shows the coordinate system and notation used. Thermodynamic variables $\bar{p}, \bar{\rho}, \bar{T}$ and the local speed of sound $\bar{a}$ are made non-dimensional by referring them to the conditions in the parent flow $\bar{\rho}^*, \bar{p}^*, \bar{T}^*$ and $\bar{a}^*$ to form p, ρ, T and a. The enthalpy is referred to $(\bar{a}^*)^2$.

Non-dimensional space and time variables are defined as $X=\bar{X}/\bar{L}$, $Y=\bar{Y}/\bar{L}$ and $T=\bar{T}/(\bar{L}/\bar{a}^*)$, where $\bar{L}$ is the nozzle half-width at the throat of the two-dimensional channel, and $\bar{L}/\bar{a}^*$ is the time required for an acoustic disturbance to traverse this distance. Velocity components $\bar{U}$ and $\bar{V}$ in the $\bar{X}$ and $\bar{Y}$ directions are made dimensionless with respect to $\bar{a}^*$.

"Stretched" non-dimensional variables x, y, t are introduced to facilitate the study of various space and time regimes in transonic flow, and are related to X, Y, T by $X=\delta x$, $Y=\varepsilon y$, and $T=\tau t$ where $\delta=\bar{L}x/\bar{L}$, $\varepsilon=\bar{L}_y/\bar{L}$, and $\tau=Tch/(\bar{L}/\bar{a}^*)$. Here, x, y, t may be considered to be of order unity and Lx and Ly are the extent of the axial and transverse regions in the channel where the flow may be assumed to be nearly sonic, consistent with the small perturbation analysis employed. Time is stretched by τ which is gauged by the characteristic time (Tch) of an imposed disturbance on the flow.

The Reynolds number based on $\bar{L}$ is assumed to be large enough that viscous effects are negligible to the order of approximation considered here. In addition, since the parent flow is irrotational, no externally imposed vortical disturbances are considered, and the flow is transonic with weak shock waves, a velocity potential $\Phi(X,Y,T)=\bar{\Phi}/\bar{L}\bar{a}^*$ may be introduced. Everywhere in the transonic flow field of interest, the velocity components and thermodynamic variables are written in terms of asymptotic expansions in E, where E is a measure of the flow deviation from its sonic value so that for transonic flow, $E \ll 1$. Thus, because the flow field is considered as a perturbation from sonic flow, Φ is, in stretched variables,

$$\Phi(X,Y,T)=\delta[x+E\phi_1(x,y,t)+E^2\phi_2(x,y,t)+\cdots] \qquad \text{(II.1)}$$

Hence (subscripts denote partial differentiation),

$$U=\Phi_X=1+E\phi_{1x}+E^2\phi_{2x}+\cdots=1+Eu_1+E^2u_2+\cdots \qquad \text{(II.2)}$$

$$V=\Phi_Y=\frac{\delta}{\varepsilon}[E\phi_{1y}+E^2\phi_{2y}+\cdots]=\frac{\delta}{\varepsilon}[Ev_1+E^2v_2+\cdots],$$

and one can write, for example, for the temperature

$$T=1+ET_1+E^2T_2+\cdots, \qquad \text{(II.3)}$$

with similar expressions for p and ρ.

The governing equation for Φ is the "Gas-Dynamic Equation" (Guderly (6)), written in a stationary (laboratory) coordinate system. Thus,

$$(a^2-\Phi_X^2)\Phi_{XX}+(a^2-\Phi_Y^2)\Phi_{YY}-\Phi_{TT}-2\Phi_{XY}\Phi_X\Phi_Y \qquad \text{(II.4)}$$
$$-2\Phi_X\Phi_{XT}-2\Phi_Y\Phi_{YT}=0.$$

In this equation, the dimensionless speed of sound is obtained from the Bernoulli equation,

$$\Phi_T+\frac{a^2}{(\gamma-1)}+\frac{1}{2}(\Phi_X^2+\Phi_Y^2)=\frac{\gamma+1}{2(\gamma-1)} \qquad \text{(II.5)}$$

where γ is the ratio of specific heats, and the term on the right side of this equation, which in general is a function of time, is equal to $(\gamma+1)/2(\gamma-1)$ if a steady sonic stream is used for reference.

Using the asymptotic forms of Φ_X and Φ_Y given in equations (II.2), equations (II.4) and (II.5) are combined to give the following governing equation:

$$-\frac{E^2}{\delta}\phi_{1xx}[(\gamma+1)\phi_{1x}+(\gamma-1)\frac{\delta}{\tau}\phi_{1t}]+\frac{\delta}{\varepsilon^2}\cdot \qquad \text{(II.6)}$$
$$[E\phi_{1yy}+E^2\phi_{2yy}]$$
$$-(\gamma-1)\frac{\delta}{\varepsilon^2}E^2(\phi_{1x}+\frac{\delta}{\tau}\phi_{1t})\phi_{1yy}-\frac{\delta}{\tau^2}[E\phi_{1tt}+E^2\phi_{2tt}]$$
$$-2\frac{\delta}{\varepsilon^2}E^2\phi_{1y}\phi_{1xy}-\frac{2}{\tau}[E\phi_{1xt}+E^2(\phi_{2xt}+\phi_{1x}\phi_{1xt})]$$
$$-2\frac{\delta}{\varepsilon^2\tau}[E^2\phi_{1y}\phi_{1yt}]+\cdots=0,$$

where only terms to order E^2 are written to illustrate the method.

The stretching parameters δ, ε and τ may now be chosen to investigate various space and time variable regimes in transonic flow. In the similarity approach, Adamson and Richey (3) took $\delta/\varepsilon \ll 1$ which indicates that a region close to the throat was considered, since $x=O(1)$. Then, the first order perturbations are governed by the nonlinear transonic equation (see, for example, Sichel (2)) which has, as one class of solution, the similarity solutions discussed by Adamson (7) and Adamson and Richey (3).

In this analysis, however, the transonic channel flow is examined in a larger region. We choose $\varepsilon=O(1)$ for channels where the near-sonic region extends from wall to wall, and for $\delta=1$ we consider a region in the channel which is approximately square, i.e., the region under consideration has an axial length of the order of the throat height.

As in the case of the similarity solutions (7), there are three "distinguished limits" for equation (II.6) depending on whether $\tau \ll 1$, $\tau=O(1)$ or $\tau \gg 1$. From the definition of τ, these three cases correspond to those in which the characteristic time associated with disturbances imposed on the flow is much smaller than, on the order of, or much greater than the time required for a sonic disturbance to cross the transonic region $\bar{L}_x$ in the X direction. Thus the imposed disturbances could represent very high frequency oscillations ($\tau \ll 1$) or rather slowly varying phenomena ($\tau \gg 1$) such as gusts, starting and stopping processes in nozzles, changes in airflow demand in an inlet due to an engine power setting change, or the action of a flow bypass door. This latter regime of τ is called the "slowly varying" time regime for the disturbances and is the one nearest to steady state conditions. It would also be expected to yield the most meaningful results as the disturbances die out and the flow returns to a steady state condition as $t\to\infty$. This distinguished limit is considered here by taking $\tau=1/kE$ where $k=O(1)$.

With $\delta=\varepsilon=1$ and $\tau=1/kE$, equation (II.6) can be simplified, powers of E collected and equated to zero (8), to yield the following governing equations for ϕ_1, ϕ_2 and ϕ_3:

$$\phi_{1yy}=0 \tag{II.7}$$

$$\phi_{2yy}=2k\phi_{1xt}+(\gamma+1)\phi_{1x}\phi_{1xx}+2\phi_{1y}\phi_{1xy} \tag{II.8}$$

$$\phi_{3yy}=f(\phi_1,\ \phi_2)\ \text{(see reference 8).} \tag{II.9}$$

For symmetric channels where $V=0$ on the centerline $(y=0)$, equation (II.7) is integrated to give $\phi_1=\phi_1(x,t)$. Thus $\phi_{1y}=0$ and this may be used to simplify equations (II.8) and (II.9). Integrating equation (II.8) gives

$$\phi_2(x,y,t)=[(\gamma+1)\phi_{1x}\phi_{1xx}+2k\phi_{1xt}]\frac{y^2}{2} +h(x,t)+g(x,t)y \tag{II.10}$$

where $g(x,t)$ is zero since $V=0$ at $y=0$ for a symmetric channel. The term $h(x,t)$ is a function of integration, to be determined from boundary and initial conditions.

The wall shape can be written as $y_w=\pm(1+w_1 f(x,t) +w_2 f_{2_w}(x,t)+\cdots)$, *consistent with the assumption* of small velocity perturbations from a sonic stream. Applying the generalized wall tangency condition for inviscid flow, that the Eulerian derivative of $(y-y_w)$ vanish at the wall, one can again use the asymptotic expansions for the velocities with $\delta=\varepsilon=1$ and $\tau=1/kE$ to derive the following wall boundary conditions:

$$(\phi_{2y})_{y=y_W}=\partial f/\partial x,\ w_1=E^2 \tag{II.11}$$

$$(\phi_{3y})_{y=y_W}=\frac{\partial f_{2w}}{\partial x}+k\frac{\partial f}{\partial t}+(\phi_{1x})_w(\phi_{2y})_w. \tag{II.12}$$

A reduced form of equation (II.9), incorporating $\phi_1=\phi_1(x,t)$ and equation (II.10) is used to derive a governing equation for $h(x,t)$ from the wall tangency condition, equation (II.12). The details of this analysis are given by Richey (8). Equation (II.11) is the usual steady state boundary condition and indicates that, for the slowly varying time regime, the wall is instantaneously a streamline to first order, although the flow may be unsteady and the wall moving.

Equation (II.11) is combined with equation (II.10) to yield an expression for $\phi_1(x,t)=u_1$:

$$\frac{\gamma+1}{2}u_1^2+2k\phi_{1t}=f(x,t)+H(t). \tag{II.13}$$

Adamson, Messiter and Richey (5) have discussed this equation and point out that its x-derivative yields a first order non-linear equation in "characteristic form" with only one family of characteristics. This is because disturbances are carried downstream at a velocity $(U+a)\cong 2$ whereas the disturbances are carried upstream at a much slower rate $(U-a)<<1$. It is these latter disturbances which can be "tracked" in the slowly varying time regime; disturbances moving downstream cross the transonic region in a time which is short compared with the characteristic time considered in the analysis.

Equation (II.12) may be combined (8) with equations (II.10) and (II.9) to yield the following governing equation for $h(x,t)$ for symmetric *channels with fixed walls:*

$$\frac{2k}{\gamma+1}h_t+\phi_{1x}h_x=-\{\frac{1}{6}\phi_{1x}f_{xx}+(\frac{2\gamma-3}{6})\phi_{1x}^3 +\frac{k}{4}x\frac{\partial H}{\partial t}+k\frac{\gamma-1}{\gamma+1}\phi_{1t}\phi_{1x}\}+k\frac{(3-\gamma)}{4(\gamma+1)}\int_x(\phi_{1x}^2)_t dx+A(t)\,, \tag{II.14}$$

The integration function $A(t)$ can be absorbed into ϕ_2 *which is only known to within an arbitrary function* of time. Thus, equations (II.14) and the x-derivative of equation (II.13) are both in characteristic form and give the rate of change of u_1 or $h(x,t)$ along the characteristic

$$\frac{dt}{dx}=\frac{2k}{(\gamma+1)u_1}$$ in terms of the wall shape

$f(x)$. Equation (II.14) is considered further in the initial value problem in section V.

In deriving equation (II.14) the results from combining equations (II.13) and (II.10), that

$$u_2=\phi_{2x}=f_{xx}\frac{y^2}{2}+h_x \tag{II.15}$$

$$v_2=\phi_{2y}=f_x\ y \tag{II.16}$$

for symmetric channels, have been used.

Using the above procedures, higher order solutions may be similarly derived. It should also be noted that if the solutions in equations (II.13), (II.14) and (II.15) are written for steady state flow, they agree with the corresponding terms of the power series postulated by Szaniawski (4). The present method provides a systematic formulation of this series. Using the assumptions and limitations which have been imposed, one can now define the region of applicability of the Szaniawski solutions.

III. Regions of Validity

The form of the solutions for $\phi_1, \phi_2, \cdots$ found in section II for the asymptotic description of the velocity potential were derived without consideration of whether the flow region is that near a shock wave, or the channel throat where the solutions may not be uniformly valid.

The behavior of the outer solutions derived in section II near the throat $(x<<1)$ has been considered by Adamson, Messiter and Richey (5) and by Messiter and Adamson (9). The first order velocity perturbation in the inner region is shown to be governed by the nonlinear transonic equation, which has, as one family of solutions, the similarity solutions discussed in section I.

Another region of the channel flow where an inner region may exist is that near a shock when the shock is further downstream, i.e., $x=O(1)$. The necessity of an inner region near the shock is shown by applying the solutions of section II in steady state form to the shock polar equation (see, for example, reference (3)):

$$(V_dU_u - V_uU_d)^2 = [(U_u^2 + V_u^2 - U_dU_u - V_dV_u)^2 \cdot \frac{(U_dU_u + V_dV_u - 1)]}{\frac{2}{\gamma+1}(U_u^2+V_u^2) - (U_dU_u + V_dV_u - 1)} \quad \text{(III.1)}$$

where subscripts u and d denote conditions immediately upstream and downstream of the shock wave. Substitution of the asymptotic expansions for U and V into equation (III.1) yields the steady state shock jump conditions for the various orders of the velocity perturbations. These conditions are, for the first and second order terms:

$$u_{1d} = -u_{1u} \quad \text{(III.2)}$$

$$(v_{2d} - v_{2u})^2 = (\gamma+1)u_{1u}(u_{2u} + u_{2d} - u_{1u}^2) \quad \text{(III.3)}$$

Equation (III.2) is satisified by the steady state form of the solution for ϕ_{1x} (equation (II.13)) since it is double valued. Because the wall slope is continuous at the shock, equation (II.16) shows that $v_{2d} = v_{2u}$ and thus equation (III.3) becomes $u_{2d} = u_{1u}^2 - u_{2u}$ which, by using equation (II.13) and (II.15), may be written as:

$$h_{xd}(x) + h_{xu}(x) = \frac{2}{\gamma+1}[f(x=x_{sh}) - H] - [f_{xx}(x=x_{sh}^-) + f_{xx}(x=x_{sh}^+)]\frac{y^2}{2} .$$

With a continuous wall shape at $x = x_{sh}^-$ (just upstream of the shock) and $x = x_{sh}^+$ (just downstream), the coefficient of the y^2 term is non-zero and the above expression cannot be satisfied.

Thus, although the solutions derived in section II satisfy the equation of motion away from the shock, the wall boundary conditions and the first order jump condition, they do not satisfy the shock jump conditions of second (or higher) order. Therefore, these solutions should be regarded as outer solutions (not uniformly valid) to be matched with inner solutions applicable near the shock. This situation exists whether the flow is steady or unsteady. The proper formulation of the inner region is discussed in the next section.

IV. Inner Region Governing Equations Near The Shock

The inner region near the shock is defined by choosing a stretched x-variable of the form $x^* = (X - Xsh)/E^{1/2}$. The same form was found by Messiter and Adamson in their study of the corresponding steady flow problem. For unsteady flow, x^* is defined in terms of a particular shock position x_{o1} which might be the steady state position (the expansion of x_{sh} will be defined more precisely later). If perturbations from a steady shock position x_{o1} are considered, the variations in shock position will be of order E. Other problems can be considered where x_{o1} is time-dependent, allowing the unsteady shock to be in any position within the channel. Thus the choice for x_{o1} is set by the physical nature of the problem being considered. In many transonic channel problems, the shock motion due to imposed disturbances is fairly small, and an analysis where x_{o1} = const. is appropriate. The following matching conditions for the inner region are determined by expanding the outer solutions (equations (II.13, II.15, II.16)) away from the shock position x_{o1} and expressing the results in terms of the inner variables $x^* = (x - x_{o1})/E^{1/2}$, $y^* = y, t^* = t$:

$$\begin{aligned} U \sim 1 \pm E\eta(t) \pm E^{3/2}(x^*/2\eta)[2f_o' - 4k(\phi_{1tx})_o]/(\gamma+1) \\ + E^2\{\tfrac{1}{2}f_o''(y^*)^2 + (h_x)_o \pm (x^*)^2\tfrac{1}{4}[(2f_o'' - 4k(\phi_{1txx})_o) \\ /(\gamma+1)\eta(t) - (2f_o' - 4k(\phi_{1tx})_o)^2/(2(\gamma+1)\eta^3)]\} + \cdots \end{aligned} \quad \text{(IV.1)}$$

and,

$$V \sim E^2 f_o' y^* + E^{5/2} x^* f_o'' y^* + \frac{E^3 x^{*2}}{2} f_o'' y^* + \cdots . \quad \text{(IV.2)}$$

The upper (+) sign applies to flow upstream of the shock (matching for $x^* \to -\infty$), the lower (-) sign applies to flow downstream of the shock (matching for $x^* \to +\infty$), subscript o denotes evaluation at $x = x_{o1}$, and an independent variable subscript denotes partial differentiation. For convenience,

$$\eta^2(t) = [\frac{2}{\gamma+1}f(x) + \frac{2}{\gamma+1}H(t) - \frac{4k}{\gamma+1}\phi_{1t}]_{x=x_{o1}}$$

is introduced. The channel walls are considered fixed, described by $y_w = \pm(1 + E^2 f(x))$.

It is seen from the matching conditions in equations (IV.1) and (IV.2) that the inner solution must contain half-powers of E. There is no need for an $E^{1/2}$ term in the inner solution for U since there is no matching term. Also, since the inner region is constructed to allow satisfaction of the shock jump conditions in terms of orders higher than the first, the inner region expansions may be assumed to be of the form:

$$U \sim 1 + Eu_1^* + E^{3/2}u_{3/2}^* + E^2u_2^* + E^{5/2}u_{5/2}^* + \cdots \quad \text{(IV.3)}$$

$$V \sim E^2 v_{3/2}^* + E^{5/2}v_2^* + \cdots .$$

This form of the velocity expansions in the inner region may be used (8) to derive an expression for the shock position $X_{sh}(Y,T)$ and the wave velocity $\partial X_{sh}/\partial T$. In the "slowly varying" time regime the steady state shock jump conditions (in terms of relative velocities) can be applied at any instant, and the shape of the shock can be written as:

$$X_{sh} = X_o(T) + E^{3/2}X_\sigma(Y,T) + \cdots \quad \text{(IV.4)}$$

where, for a symmetric channel, $X_o(T)$ may be taken as the shock position at the centerline. The shock wave velocity in a direction normal to the curved shock front is determined by the vanishing of the Eulerian derivative of

$$S = X - X_{sh}(Y,T)$$

This normal wave velocity has x and y components U_{sh} and V_{sh} :

$$U_{sh}=\frac{\partial X_{sh}}{\partial T}\left(1+E^{3/2}\frac{\partial X\sigma}{\partial Y}\right)\left(1+O(E^{3/2})\right) \qquad \text{(IV.5a)}$$

$$V_{sh}=\frac{\partial X_{sh}}{\partial T}\left(1+E^{3/2}\frac{\partial X\sigma}{\partial Y}\right)\cdot O(E^{3/2}) \ . \qquad \text{(IV.5b)}$$

With $\partial X_{sh}/\partial T=\frac{1}{\tau}(\partial x_{sh}/\partial t)$, equation (IV.4) is substituted into equations (IV.5a, b) to give

$$U_{sh}=kE\ \frac{dx_o}{dt}+O(E^{5/2}) \qquad \text{(IV.6)}$$

and $$V_{sh}=kE^{5/2}\frac{dx_o}{dt} \ .$$

Using an asymptotic expansion for $X_o(T)$,

$$X_o(T)=x_{o1}+E^{\alpha}x_{o,\alpha+1}+\cdots \ ,$$

one finds that equation (IV.6) becomes

$$U_{sh}=kE\left(\frac{dx_{o1}}{dt}+E^{\alpha}x_{o,\ \alpha+1}+\ldots\right)+O(E^{5/2}) \ .$$

The velocity of the shock from rest is induced by perturbations in the outer region, transmitted through the inner region. Thus the orders of the shock velocity should be the same as the velocity *expansion in the inner region. It will be shown* later that $u^*_{3/2}$ goes to zero at the shock, and thus $\alpha=1$ to correspond to disturbances of order E^2. Therefore equations (IV.4) and (IV.6) become

$$X_{sh}(Y,T)=X_{o1}(t)+Ex_{o2}(t)+E^{3/2}\left(x_{\sigma}(y,t)+x_{o_{5/2}}\right)+\cdots \qquad \text{(IV.7)}$$

and

$$U_{sh}=kE\left(\frac{dx_{o1}}{dt}+E\frac{dx_{o2}}{dt}\right)+O(E^{5/2})+\ldots), \qquad \text{(IV.8)}$$

or

$$U_{sh}=kEu_{s1}+kE^2u_{s2}+O(E^{5/2})+\cdots$$

Equation (IV.8) is a general relation for the shock-wave velocity. As mentioned earlier, the physical problem chosen for study here is that flow field where the shock wave oscillates about a given steady state position, designated as x_{o1}. The perturbations from this shock position $(Ex_{o2}+\cdots)$ are time dependent. Thus $u_{s1}=dx_{o1}/dt=0$, and equation (IV.8) becomes $U_{sh}=kE^2u_{s2}+\cdots$.
Further, with $u_{s1}=0$, *the first order shock jump* condition is $u_{1d}+u_{1u}=0$ at $x=x_{sh}$. If the upstream flow were to be unsteady in first order, this would imply an artificial relationship across the shock. Steady upstream first order flow would imply that the downstream flow should be steady in first order. It is therefore clear that, consistent with x_{o1} = const $(u_{s1}=0)$, one must consider the case in which the first order velocity perturbation is steady throughout (i.e. $\phi_1=\phi_1(x)$). The second order velocity potential perturbation ϕ_2 may be unsteady in response to acoustic disturbances imposed on second order flow variables. This type of flow is termed "Second Order Unsteady Flow" and is the flow model which is treated in detail in the following sections.

The inner region velocity potential, based on *equation (IV.3), is,*

$$\phi^*=E^{1/2}[x^*+E\phi^*_1+E^{3/2}\phi^*_{3/2}+E^2\phi^*_2+\cdots]$$

with stretched inner region variables

$$X^*=E^{1/2}x^*, Y^*=y^*, T^*=t^*/kE \ .$$

It may be noted that even though the shock is *moving in the inner region, the inner region itself* does not move; the laboratory frame of reference is maintained.

The governing equations for $\phi^*_1, \phi^*_{3/2}, \phi^*_2\cdots$ derived from the gas dynamic and Bernoulli equations are,

$$-(\gamma+1)\phi^*_{1x^*}\phi^*_{1x^*x^*}+\phi^*_{1y^*y^*}=0 \qquad \text{(IV.9)}$$

$$-(\gamma+1)[\phi^*_{3/2x^*}\phi^*_{1x^*x^*}+\phi^*_{1x^*}\phi^*_{3/2x^*x^*}] \qquad \text{(IV.10)}$$
$$+\phi^*_{3/2y^*y^*}-2k\phi^*_{1x^*t^*}=0$$

and

$$-[(\gamma+1)\phi^*_{2x^*}+\frac{\gamma+1}{2}(\phi^*_{1x^*})^2]\phi^*_{1x^*x^*} \qquad \text{(IV.11)}$$
$$-(\gamma+1)(\phi^*_{3/2x^*}\phi^*_{3/2x^*x^*}+\phi^*_{1x^*}\phi^*_{2x^*x^*})$$
$$+\phi^*_{2y^*y^*}-(\gamma-1)\phi^*_{1x^*}\phi^*_{1y^*y^*}-2\phi^*_{1y^*}\phi^*_{1x^*y^*}$$
$$-2k\phi^*_{3/2x^*t^*}=0 \ .$$

It can be shown (8) that the instantaneous shock jump conditions will apply for the "slowly varying" time regime if one takes the position of an observer on the wave and uses inner region velocities relative to the shock $(\tilde{U}^*=U^*-U_{sh})$. However, since the shock (and the coordinate system attached to it) is moving in the inner region, the stagnation *enthalpy which is preserved across the wave is that* associated with the moving coordinate system rather than the stationary coordinate system. That is, one must account for the kinetic energy of the shock when deriving the shock polar equation. The modified form of the shock polar used to derive the shock jump conditions in the inner region is

$$[\tilde{V}^*_d\tilde{U}^*_u-\tilde{U}^*_d\tilde{V}^*_u]^2=\{(\tilde{U}^*_u)^2+(\tilde{V}^*_u)^2-\tilde{U}^*_d\tilde{U}^*_u \qquad \text{(IV.12)}$$
$$-\tilde{V}^*_d\tilde{V}^*_u\}^2\cdot\{[\tilde{U}^*_d\tilde{U}^*_u+\tilde{V}^*_d\tilde{V}^*_u-1+2(\frac{\gamma-1}{\gamma+1})kE^2u_{s_2}]$$
$$/[\frac{2}{\gamma+1}(\tilde{U}^{*2}_u+\tilde{V}^{*2}_u)-(\tilde{U}^*_d\tilde{U}^*_u+\tilde{V}^*_d\tilde{V}^*_u-1)-2(\frac{\gamma-1}{\gamma+1})kE^2u_{s_2}]\}$$

which applies not at $x^*=0$, but at x^*_{sh} (i.e. at the shock). Now, from the definition of x^* and *equation (IV.7), one can show that*

$$x-x_{sh}=E^{1/2}x^*-Ex_{o2}-\cdots$$

So that at the shock,

$$x^*_{sh}=E^{1/2}x_{o2}+\cdots$$

Since $x^*_{sh}=O(E^{1/2})$, one can expand the velocity components on either side of the shock about $x^*=0$ and derive equivalent jump conditions at $x^*=0$. In fact, as will be seen, the problem can finally be reduced to an equivalent steady flow problem with the shock at $x^*=0$. The jump conditions are then (at $x^*=0$),

$$u^*_{1u}+u^*_{1d}=0 \tag{IV.13}$$

$$u^*_{3/2u}+x_{o2}(u^*_{1x^*})_u+u^*_{3/2d}+x_{o2}(u^*_{1x^*})_d=0 \tag{IV.14}$$

$$\begin{aligned}[(v^*_{3/2})_d-(v^*_{3/2})_u]^2=2(\gamma+1)u^{*2}_{1u}\{u^*_{2u}+u^*_{2d} &+x_{02}[(u^*_{3/2x^*})_u+(u^*_{3/2x^*})_d]+\frac{x^2_{o2}}{2}[(u^*_{1x^*x^*})_u \\ &+(u^*_{1x^*x^*})_d]+(x_\sigma+x_{o_{5/2}})[(u^*_{1x^*})_u+(u^*_{1x^*})_d] \\ &-\frac{4k}{\gamma+1}u_{s_2}-(u^*_{1u})^2\}\end{aligned} \tag{IV.15}$$

where the subscripts u and d denote conditions at $x^*=0^-$ and $x^*=0^+$ (upstream and downstream of the shock) respectively.

Solutions to the flow in the inner region near the shock which are governed by equations (IV.9) to (IV.11), matching conditions from equations (IV.1) and (IV.2), appropriate wall tangency conditions, and shock jump conditions given by equations (IV.13) to (IV.15), are discussed in the next section.

V. Solutions For "Second Order Unsteady" Flow

With $\phi_1=\phi_1(x)$ in the outer region, the solution for $u_1(x)$ comes directly from equation (II.13) as

$$u_1=\pm\sqrt{\frac{2}{\gamma+1}f(x)+C_w} \tag{V.1}$$

where $H(t)=C_w$. If $C_w=0$, the throat of the channel is sonic with $f(o)=0$. In equation (V.1) the upper (+) sign corresponds to local supersonic flow (upstream of the shock) and the lower (-) sign corresponds to locally subsonic flow. At the shock,

$$u_1=\pm\sqrt{\frac{2}{\gamma+1}f(x_{o1})}=\pm C_u \quad \text{i.e., a constant.}$$

For second order unsteady flow ($\phi_{1t}=0$) the matching conditions for the inner solutions at large x^* given by equations (IV.1) and (IV.2) simplify to:

$$\begin{aligned}U\sim 1\pm EC_u\pm E^{3/2}x^*f'_o/(\gamma+1)\,C_u & \\ +E^2\Big\{\tfrac{1}{2}f''_o y^{*2}+h_x(x_{o1},t)\pm\frac{x^{*2}}{4}\Big[\frac{2f''_o}{(\gamma+1)C_u} &-\frac{(\frac{2}{\gamma+1}f'_o)^2}{2C_u^3}\Big]\Big\}+O(E^{5/2})\end{aligned} \tag{V.2}$$

and

$$V\sim E^2f'_oy^*+E^{5/2}x^*f''_oy^*+\dots \tag{V.3}$$

In the inner region upstream of the shock, it can be shown (8) that solutions for ϕ^*_1, $\phi^*_{3/2}$ and ϕ^*_2 which satisfy the governing equations (IV.9) to (IV.11), and match term by term with the outer solutions in equations (V.2) and (V.3) are, in fact, identical to those terms. That is, the inner solutions merely continue the outer solutions into the inner region, and thus the inner solutions upstream of the shock are not required in the composite solutions for U and V.

The solutions upstream of the shock ($x^*<0$) which are used in the shock jump conditions at $x^*=0^-$ are

$$\phi^*_{1x^*}=C_u, \qquad \phi^*_{3/2y^*}=f'_oy^*$$

$$\phi^*_{3/2x^*}=\frac{f'_o\,x^*}{C_u(\gamma+1)} \tag{V.4}$$

and

$$(\phi^*_2)_{x^*}=\frac{(x^*)^2}{4}\Big[\frac{2}{(\gamma+1)}\frac{f''_o}{C_u}-\frac{((2/\gamma+1)f'_o)^2}{2C_u^3}\Big]+\frac{f''_oy^{*2}}{2}+h_{xu}(x_{o1},t) \tag{V.5a}$$

$$(\phi^*_2)_{y^*}=f''_ox^*y^* \,. \tag{V.5b}$$

Downstream of the shock it can be shown (8) that the solutions for ϕ^*_1 and $\phi^*_{3/2}$ are also simply continuations of the outer solutions into the inner region. Thus, in the inner region,

$$\phi^*_{1x^*}=-C_u \qquad \phi^*_{3/2x^*}=\frac{-f'_ox^*}{C_u(\gamma+1)} \tag{V.6}$$

$$\phi^*_{3/2y^*}=f'_oy^*$$

However, since the second order jump conditions are not satisfied by the outer solutions ϕ^*_2 must be more than a continuation of these solutions. It is convenient to write ϕ^*_2 as a sum of the continuation solution and a function $\zeta^*_2(x^*,y^*,t^*)$ to be found. Then, (for $x^*>0$),

$$\phi^*_{2x^*}=u^*_2=\Big[\frac{(\frac{2}{\gamma+1}f'_o)^2}{2C_u^3}-\frac{\frac{2}{\gamma+1}f''_o}{C_u}\Big]\frac{x^{*2}}{4}+\frac{f''_oy^{*2}}{2}+h_{xd}(x_{o1},t)+((\gamma+1)C_u)^{-1/2}\zeta_{\tilde{x}} \,. \tag{V.7a}$$

$$\phi^*_{2y^*}=f''_ox^*y^*+\zeta_{\tilde{y}} \tag{V.7b}$$

where $\zeta_2^* = \zeta(\tilde{x},\tilde{y})$, with $\tilde{x} = x^*/\sqrt{(\gamma+1)C_u}$ and $\tilde{y} = y^*$ introduced for convenience. It can be shown (8) that the first term on the right hand side of equation (V.7b) (from the continuation of the outer solution) satisfies the flow tangency condition at the walls, $y_w = \pm 1$.

From equations (V.4) and (V.6), it is seen that $v_{3/2}^*$ does not change across the wave and that $u_{3/2u}^* = u_{3/2d}^* = 0$. Hence, jump conditions to order $E^{3/2}$ are satisfied (equation (IV.13) and (IV.14). Furthermore, the second order jump condition, equation (IV.15) reduces to

$$u_{2d}^* = u_{1u}^{*\,2} + \frac{4k}{\gamma+1} u_{s2} - u_{2u}^* . \tag{V.8}$$

Substituting equations (V.7a) and (V.7b) into the equations expressing the wall tangency condition, the jump conditions at $x^* = 0$ (equation V.8), the matching conditions as $x^* \to \infty$ (equation V.2 and V.3), and the governing equations (IV.11), one can show (8) that the calculation of ϕ_2^* reduces to consideration of the following boundary value problem of the Neumann type:

$$\zeta_{2\tilde{x}\tilde{x}} + \zeta_{2\tilde{y}\tilde{y}} = 0 \tag{V.9}$$

is the governing equation, with boundary conditions as follows:

$$\zeta_{2\tilde{y}}(\tilde{x},1,t^*) = 0 \tag{V.10a}$$

$$\zeta_{2\tilde{y}}(\tilde{x},o,t^*) = 0 \tag{V.10b}$$

$$\lim_{\tilde{x}\to\infty} \zeta_{2\tilde{x}} = 0 \tag{V.10c}$$

$$\zeta_{2\tilde{x}}(o,\tilde{y},t^*) = C_u^{\,2} + \frac{4k}{\gamma+1} u_{s2} - f_o''\tilde{y}^2 - (h_{xd} + h_{xu}) . \tag{V.10d}$$

Thus, we note that the derivatives are specified on the boundaries of a semi infinite strip. From the integral condition which must be met for a Neumann problem (10), it is easy to show that

$$\frac{4k}{\gamma+1} u_{s2}(t) = h_{xd}(t) + \frac{f_o''}{3} - C_u^{\,2} + h_{xu}(t) \tag{V.11}$$

so that equation (V.10d) becomes

$$\zeta_{2\tilde{x}}(o,\tilde{y}) = -f_o''(\tilde{y}^2 - \tfrac{1}{3}) . \tag{V.12}$$

Since the boundary conditions on the Neumann problem are not time-dependent, the mathematical problem described by equations (V.9, V.12), and (V.10a to V.10c), reduces to the problem solved in the equivalent steady state problem, (9).

The solution is as follows, written now in terms of x^* and y^*,

$$\zeta_{2x^*}^* = \frac{-2(\gamma+1)}{\Pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-n\Pi x^*/\sqrt{(\gamma+1)C_u}} \cos(n\Pi y^*) \tag{V.13a}$$

$$\zeta_{2y^*}^* = \frac{-4}{\Pi^2} f_o'' \sqrt{(\gamma+1)C_u} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-n\Pi x^*/\sqrt{(\gamma+1)C_u}} \sin(n\Pi y^*) \tag{V.13b}$$

Downstream of the shock wave a composite solution may be formed by adding the inner and outer solutions and subtracting the common terms (i.e. those used in matching the solutions). This composite solution is easily shown to be, to second order:

$$U = 1 + E\phi_{1x} + E^2(\phi_{2x} + \zeta_{2x^*}^*) + \cdots \tag{V.14a}$$

$$V = E^2(\phi_{2y} + \zeta_{2y^*}^*) + \cdots , \tag{V.14b}$$

where ϕ_{2x} and ϕ_{2y} are given by equations (II.15) and (II.16). Upstream of the shock, the same relations hold, but with $\zeta_{2x^*}^* = \zeta_{2y^*}^* = 0$. Downstream of the shock, $\zeta_{2x^*}^*$ decays exponentially with increasing x^* from its value at the shock, as illustrated in figure (2) for a symmetric channel which has a "nozzle type" contour.

It is seen from equation (V.13b) that although $(\zeta_2^*)_{y^*}(x^*,0) = 0$ and $(\zeta_2^*)_{y^*}(x^*,1) = 0$ as required by boundary conditions,

$$\zeta_{2y^*}^*(0,y^*) = \frac{-4}{\Pi^2} f_o'' \sqrt{(\gamma+1)C_u} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin(n\Pi y^*)$$

for any y^* at $x^* = 0^+$. This is the change in v_2 across the shock. Thus we see that there is a jump in v_2 everywhere along the shock except at $y=0$ and $y=1$ and so the shock actually has an inflection shape which will vary as f_o'' changes. The summation term in equation (V.13b) is positive so the jump in v_2 is proportional to $(-f_o'')$.

It should be noted that equation (V.11) allows a calculation of u_{s2} if the upstream and downstream values of $h(x_{o1},t)$ are known. Thus, as a variation in $(h_{xd} + h_{xu})$ is experienced at the shock, a shock velocity, u_{s2} is induced. The instantaneous shock position is $x_{sh} = x_{o1} + Ex_{o2} + \cdots$ where x_{o2} is found by integrating u_{s2} with respect to time.

Initial Value Problem for $h_x(x,t)$

The unsteady contribution to u_2 for fixed walls is given by $h(x,t)$ since the inner solutions are not time dependent for steady first order flow. The governing equation for $h(x,t)$ is equation (II.14) which, for $\phi_1 = \phi_1(x)$ simplifies to:

$$\frac{2k}{\gamma+1} h_t + u_1 h_x = A(t) - F(x) \tag{V.15}$$

where

$$F(x) = \tfrac{1}{6}[u_1 f_{xx} + (2\gamma-3)u_1^{\,3}] .$$

As noted in section II, equation (V.15) is in characteristic form. Thus one can define s and r variables such that equation (V.15) may be written as a total differential with respect to s. For the slowly varying time regime, there will be only one such "characteristic" as discussed in section II. Disturbances are carried upstream *relative to the flow along this family of characteristics.* Thus along lines of r=const, equation (V.15) can be written as

$$\left(\frac{\partial h}{\partial s}\right)_r = A(t) - F(x) \tag{V.16a}$$

where

$$\left(\frac{\partial x}{\partial s}\right)_r = u_1, \tag{V.16b}$$

and

$$\left(\frac{\partial t}{\partial s}\right)_r = 2k/(\gamma+1). \tag{V.16c}$$

Now equation (V.16a) may be integrated along a chosen r-characteristic from s=o (the initial condition) to the s-value at which we wish to evaluate $h(x,t) \equiv \hat{h}(r,s)$, as follows:

$$\hat{h}(r,s) = -\int_o^s F(x)ds + \int_o^s A(t)ds + \hat{h}_o(r,s=o) \tag{V.17}$$

where $\hat{h}_o(r,s=o)$ is the initial condition which is transmitted with increasing s (and t) along the *r-characteristic, modified as it is transmitted by* terms in equation (V.17). Note that x is a function of s since $\left(\frac{\partial x}{\partial s}\right)_r = u_1$.

Lines of constant s are chosen to be lines of constant t, i.e. horizontal in the x,t plane. $\left(\frac{\partial x}{\partial r}\right)_s \equiv \hat{g}(r,s)$ *can be shown* (8) *to be:*

$$\hat{g}(r,s) = \exp\left(\int_o^s \left(\frac{\partial u_1}{\partial x}\right)ds\right) \tag{V.18}$$

Other relationships between the x,t and r,s coordinates are:

$$\left(\frac{\partial s}{\partial x}\right)_t = 0,\ \left(\frac{\partial s}{\partial t}\right)_x = \frac{\gamma+1}{2k},\ \left(\frac{\partial r}{\partial x}\right)_t = \frac{1}{\hat{g}},\ \left(\frac{\partial r}{\partial t}\right)_x = -\frac{\gamma+1}{2k}\frac{u_1}{\hat{g}}. \tag{V.19}$$

Equations (V.16b) and (V.16c) are also integrated along a chosen r=const characteristic from s=o to give, since $u_1 = u_1(x)$, and for s=o at t=o;

$$t = \frac{2k}{\gamma+1}\, s \tag{V.20a}$$

$$x = \int_o^s u_1 ds + x_o(r,o) \tag{V.20b}$$

$$\int_{r=x}^{x} \frac{dx}{u_1} = \int_o^s ds = s, \tag{V.20c}$$

where we have set s=o *at* t=o.

With equations (V.20), the r=const characteristics can be determined in the x-t plane for a given $u_1(x)$ which is determined by the wall shape. Since the slope of the r-characteristics in the x,t plane is $\left(\frac{dx}{dt}\right)_r = \frac{\gamma+1}{2k}u_1$, the r-characteristics will have a positive slope upstream of the shock (u_1>o) and a negative slope downstream of the shock (u_1<o). A typical set of r=const characteristics for a parabolic channel described by $y_w = \pm(1+E^2C_o^2x^2)$ is shown in figure 3.

Equation (V.17) may be differentiated with respect to x by using the relationships between x,t and r,s given in equation (V.19) and applying Liebnitz' rule to the integrals (8). Thus,

$$\left(\frac{\partial h}{\partial x}\right)_t = -\frac{1}{\hat{g}}\int_o^s \left(\hat{g}\frac{dF}{dx}\right)ds + \frac{1}{\hat{g}}\left[\frac{\partial h_o(r,o)}{\partial r}\right]_{s=o} \tag{V.21}$$

where, again, the integrations are carried out along r=const curves.

As shown in figure 3, when disturbances beginning with the initial condition $\frac{\partial h_o(r,o)}{\partial r}$ are propagated upstream to $x=x_{o1}$ along r=const. characteristics *downstream of the shock,* the value of $h_x(x,t)$ at that point is denoted by $h_{xd} = h_x(x_{o1},t)$. *If* the upstream flow is unsteady in second order, there will be a value h_{xu} obtained from equation (V.21) by integrating along r=const upstream of the shock, or from a reduced form of equation (V.15) where h_t=o if the upstream flow is steady. At the shock, jump conditions expressed in expanded form as boundary conditions at x^*=o (equation IV.15) must be satisfied. Since the disturbance arriving at the shock creates an imbalance in h_x, the shock must move in second order with shock velocity u_{s2} given by equation (V.11). Thus it is seen that the shock will move only in response to a disturbance transmitted along an r-characteristic reaching $x=x_{o1}$ from either upstream or downstream in the channel.

As an example of initial conditions at t=o, we consider a flow which is initially steady but in which a disturbance from downstream has propagated upstream to some position in the channel, say to $x=\hat{x}$. Since, at t=o, x=r, and $\hat{g}$=1, the initial condition disturbance

$$\left(\frac{\partial h_o(r,o)}{\partial r}\right)_{s=o} = \left(\frac{\partial h(x,o)}{\partial x}\right)_{t=o}$$

is at $\hat{r}=\hat{x}$. The behavior of this "wave train" is then examined as it propagates upstream to the shock and induces shock oscillations. The disturbance is written in terms of a variation in h_x from its steady state solution, $(h_x)_{ss}$. For example, $h_r(r,o) = (h_x)_{ss} + \varepsilon \sin a(r-\hat{r})$ for $r>\hat{r}$. Such variations are equivalent to a perturbation in the channel back pressure of order E^2 since the static pressure in the channel can be written as

$$p = 1 - E\gamma\sqrt{\frac{2}{\gamma+1}f(x)+C_w} - E^2\gamma f''(x)\frac{y^2}{2} - E^2\gamma h_x(x,t)$$

for first order steady flow.

The solution for $(h_x)_{ss}$ is obtained from equation (V.15) where $A(t)$ is a constant, A, which will be different upstream and downstream of the shock. If $C_w=0$, Messiter and Adamson (9) show that $A=0$ upstream of the shock, while downstream of the shock, A can be determined (8) from the steady state shock jump conditions as

$$A_d=\frac{2}{3}\gamma(u_{1u})^3,$$

and thus the solutions for $(h_x)_{ss}$ are as follows:
Upstream of the shock; $(h_x)_{ss}=-\frac{F(x)}{u_1}=\frac{3-2\gamma u_1^2}{6}-\frac{1}{6}f''(x)$

while downstream of the shock,

$$(h_x)_{ss}=\frac{A_d-F(x)}{u_1}=\frac{3-2\gamma u_1^2}{6}-\frac{1}{6}f''(x)-\frac{2}{3}\frac{\gamma u_{1o}^3}{u_1(x)} \qquad (V.22)$$

where

$$u_1(x)=-\sqrt{\frac{2}{\gamma+1}f(x)} \text{ and } u_{1o}=\sqrt{\frac{2}{\gamma+1}f(x_{o1})} \quad .$$

For the initial condition, x is replaced by r.

Equation (V.22) can be applied at any point in the channel for $x>x_{sh}$. If it is applied at some x_b where the back pressure p_b is specified, one finds $h_x(x_b)$ and, from equation (V.22), determines $u_{1o}=u_1(x_{o1})$ just upstream of the shock. Thus the steady state shock position x_{o1} can be calculated. For a given wall contour, then one can find a curve of x_{o1} versus p_b. The initial value of x_{o2} is set by the order E^3 term in the series for the back pressure p_b. This is assumed to be chosen such that $x_{o2}=0$ at $t=0$.

The fact that the shock wave location is found here by prescribing terms of order E^2 might at first appear questionable, since for weak shocks, entropy changes are of third order. However, it can be shown (8, 9) that the important part of the third order terms, insofar as a shock location calculation is concerned, are those terms involving the cross products between first and second order velocity or pressure terms. Hence, the present result is valid.

VI. Symmetric Channel Solutions

For some wall shapes, the initial value problem for $h_x(x,t)$ discussed in section V can be solved analytically. For example, solutions for a parabolic wall channel $(y_w=\pm(1+E^2C_o^2x^2))$ are given by Richey (8). For other wall shapes, the calculations are performed by numerical integration (quadrature) of the expressions in equation (V.21) along r=constant characteristics. The paths of the r=const curves in the x,t plane are a function of the wall shape (through $u_1(x)$) and may be determined from equation (V.20c). In addition s is given for any t from equation (V.16a). Therefore, functions of x,t may be evaluated for chosen values of r and s and integrated along the r=const. curves. Equation (V.21) is used to calculate h_x for specific points of r and s, and thus at the corresponding x,t. Having computed $h_x(x,t)$ in the upstream and downstream regions, one can find $u_2(x,y,t)$ and $v_2(x,y)$ from equations (V.14a) and (V.14b).

If the upstream second order flow is unsteady, the shock jump conditions and shock velocity must be calculated at the same time, t_{sh}. *For a wave* train propagating upstream, we thus find the time where a given r-characteristic intersects the line $x=x_{o1}$ in the x,t plane and then determine the appropriate upstream characteristic which intersects the x_{o1} line at the same time. If the upstream second order flow is steady, there is no need to determine this particular upstream characteristic; the steady state equations are used directly. Following calculation of u_{s2} from equation (V.11), the perturbation in the shock position x_{o2} is given by the integral

$$x_{o2}(t)=\int_o^s u_{s2}dt.$$

As an example solution, figures 4 through 8 show the results for a sinusoidal wave train imposed downstream of the shock in a symmetric "nozzle-like" channel which terminates in a constant area section. The case of choked flow $(C_w=0)$ at the throat $(x=0)$ and supersonic flow ahead of the shock is considered. The geometry of the channel is shown in figure 4. The channel is parabolic to $x=1$, with constant area for $x\geq\bar{x}=2.0$, and a transition section in between.

With an initial shock location at $x_{o1}=1.5$, a "wave train" at $t=o$ beginning at $\hat{x}=\hat{r}=2.5$ described by $(h_x)_{t=o}=(h_x)_{ss}+\varepsilon\sin(a(r-\hat{r}))$ is imposed on the flow in the downstream region of the channel.

Figure 4 shows the lines of constant velocity $U=1+Eu_1+E^2u_2$ for the initial condition, $t=o$. In the region where the walls are curved, the curvature of these lines indicates the two-dimensional nature of the flow. The initial disturbance is considered to have already propagated to $x=2.5$, resulting in the pressure and velocity variation shown in the constant area section of the channel.

At $t=0.4$, figure 5, the wave train has moved upstream to $x=1.85$ (solution found by integrating along characteristic curves to $s=t=0.4$), resulting in a modification of the lines of constant velocity from the initial condition. The U =const lines account for the existence of the inner region downstream of the shock, which tend to make these lines slightly more vertical than if they were calculated by the outer solution alone.

In figure 6, $t=0.6$, and the wave train has reached the shock $(x=1.5)$, so the entire flow field downstream of the shock is influenced by the wave train and is time dependent. Therefore, at $t=0.6$, the shock begins to move for the first time. The wave train imposed initially gives a lower pressure downstream of the shock than that which existed for the steady state, and so the shock initially moves downstream, then back upstream as the wave front brings a region of increasing pressure near the shock. Notice that even though the channel should be diffusing (decelerating flow) with increasing x, the flow is accelerating behind the shock up to $x=2.4$ due to the presence of the wave in the channel. The wave train is modified by the presence of the channel as it approaches the shock; the initial expansion wave is weakened by the change in channel area (increasing pressure). A compression wave will tend to be reinforced as it approaches the shock.

The shock continues to move downstream in this example until $t=1.6$ where the minimum pressure point in the initial wave train reaches the shock, and the increasing pressure starts to cause the shock to move back upstream. Lines of constant velocity and the centerline pressure are shown in figure 7 at $t=1.6$ which corresponds to the maximum downstream shock position $x_{sh}=1.595$ with $x_{o2}=.95$. The compression wave downstream of the shock is now being reinforced by the channel shape.

The variation in shock velocity, u_{s2}, in response to the imposed unsteady disturbance is shown in figure 8. There is no shock motion until the wave train (i.e., the $\hat{f}$ characteristic) *reaches the shock at* $t=0.6$. *The shock starts to* move rapidly downstream but then slows up as it reaches the farthest downstream position at $t=1.6$. It then begins to return to the initial position (negative u_{s2}) and oscillates between $x=1.5$ and 1.6 as the oscillatory wave train is brought upstream to the shock. The time variant part of the shock position $x_{o2}(t)$, is determined by integrating $u_{s2}(t)$ with respect to time, and is shown in figure 9.

VII. Asymmetric Channels

An application of the analysis of transonic channel flow to asymmetric channels such as found in transonic compressor or turbine cascades, is of considerable importance.

Relative to the symmetric channel analysis discussed so far, we consider asymmetric channels which have a contour described by

$$Y_{wu}=1+E^2 f_u(x,t)$$

and

$$Y_{w\ell}=-1+E^2 f_\ell(x,t)$$

for the upper and lower walls, respectively.

In this case, it can be shown (8) that the asymmetry alters the boundary condition on the second order terms in the velocity expansion. One can also show (8) that if an arbitrary channel has a radius of curvature, R_1, of the order of E^{-2} and the ratio of the change in channel area to the throat area, say R_2, is of order E^2, so that the product of R_1 and R_2 is of order 1, then the asymmetry will not affect the form of the first order velocity terms, but will add an extra term to the second order velocity perturbations.

With an asymmetric channel as described above, the derivation of the governing equations is unchanged from section II. Combining the description of the channel walls into a shape function

$Y_w=K+KE^2 f_2(x,t)$ where $K=1$, $f_2=f_u$ for the upper wall, and $K=-1$, $f_2=-f_\ell$ for the lower wall, the wall tangency condition may be written (8) as

$$-KkE^3\frac{\partial f_2}{\partial t}-KE^2\frac{\partial f_2}{\partial x}+\dots-KE^3 u_1\frac{\partial f_2}{\partial x} \qquad \text{(VII.1)}$$

$$+Ev_1(x,y_w,t)+E^2 v_2(x,y_w,t)+E^3 v_3(x,y_w,t)=0.$$

The order E term requires that $\phi_1=\phi_1(x,t)$ as for the symmetric channel, while the order E^2 and E^3 terms modify the solution of equation (II.8) and the derivation of the equation for $h_x(x,t)$ as follows (8) for the outer region. From equation (II.8),

$$\phi_{2y}=[(\gamma+1)\phi_{1x}\phi_{1xx}+2k\phi_{1xt}]y+h_1(x,t) \qquad \text{(VII.2)}$$

and

$$\phi_{2x}=[(\gamma+1)\phi_{1x}\phi_{1xx}+2k\phi_{1xt}]_x\left(\frac{y^2}{2}\right)+h_{1x}y+h_x(x,t) \qquad \text{(VII.3)}$$

where h and h are functions of integration. Substituting equation (VII.2) for $v_2(x,y_w,t)$ into the $O(E^2)$ term in equation (VII.1) for both the upper and lower wall, and solving for $h_1(x,t)$, one can show that

$$h_1(x,t)=\frac{1}{2}[(f_u)_x+(f_\ell)_x] \qquad \text{(VII.4)}$$

and

$$\frac{2k\phi_{1xt}}{\gamma+1}+\phi_{1x}\phi_{1xx}=\frac{1}{2(\gamma+1)}[(f_u)_x-(f_\ell)_x]$$

which, for steady first order flow and fixed walls may be integrated immediately to yield

$$\phi_{1x}=\pm\sqrt{\frac{1}{\gamma+1}(f_u-f_\ell)+C_w} \qquad \text{(VII.5)}$$

It may be noted that if $f_\ell=-f_u$, *one recovers the* symmetric channel results and $h_1(x)=0$.

The equation for $h_x(x,t)$ is found by integrating equation (II.9) and substituting into the order E^3 terms in equation (VII.1) to yield, for fixed walls and steady first order flow,

$$\frac{2k}{\gamma+1}h_t+\phi_{1x}h_x=A(t)-\tilde{F}(x) \qquad \text{(VII.6)}$$

where

$$\tilde{F}(x)=\frac{1}{6}\left[\frac{\phi_{1x}}{2}(f_{ux}-f_{\ell x})_x+(2\gamma-3)\phi_{1x}^3\right].$$

If one defines $\tilde{f}(x)=f_u(x)-f_\ell(x)$, then the outer solution for the asymmetric channel (for the type of channel where $R_1R_2=O(1)$) is in *exactly* the same form as for the symmetric channel except that

u_2 and v_2 are modified by the $h_1(x)$ term in equations (VII.2) and (VII.3). Therefore, this type of asymmetric channel represents only a slight generalization of the symmetric channel problem; there is a slight increase in algebraic complexity but no new features are found.

The shock jump conditions in section IV were derived in general, and apply here. Equation (VII.5) satisfies the first order jump conditions without difficulty and the expression for u_2 is in the same form as for the symmetric channel except for an extra term. Thus one can anticipate that these outer solutions will not satisfy the second order jump conditions, and an inner region will be required downstream of the shock as in the symmetric channel case.

The analysis for the inner region follows that used for the symmetric channel very closely (8). Thus, one can again write composite solutions,

$$U=1+E\phi_{1x}+E^2(\phi_{2x}+\zeta^*_{2x^*})+\dots$$

$$V=E^2(\phi_{2y}+\zeta^*_{2y^*})+\dots$$

where $\zeta^*_{2x^*}=\zeta^*_{2y^*}=0$ upstream of the shock wave. Downstream of the shock wave, it can be shown (8) that

$$(\zeta^*_2)_{x^*}=\frac{4h_{1x}(x_{o1})}{\Pi^2}\sum_{n=1,3,5\dots}^{\infty}\Big|\Big[(-1)^{\frac{n+1}{2}}\exp\Big(-\frac{n\Pi}{2}x^*/\sqrt{(\gamma+1)C_u}\Big)\Big]\cdot\Big[n^{-2}\sin\Big(\frac{n\Pi}{2}y^*\Big)\Big]\Big| \qquad \text{(VII.7)}$$

$$-\frac{4f''_o}{\Pi^2}\sum_{n=2,4,6\dots}^{\infty}\Big|\Big[(-1)^{n/2}\Big(\frac{n}{2}\Big)^{-2}\Big]\cdot\Big[\exp\Big(-\frac{n\Pi}{2}x^*/\sqrt{(\gamma+1)C_u}\Big)\cos\Big(\frac{n\Pi y^*}{2}\Big)\Big]\Big|$$

and

$$(\zeta^*_2)_{y^*}=\frac{-4\sqrt{(\gamma+1)C_u}}{\Pi^2}h_{1x}(x_{o1})\sum_{n=1,3,5\dots}^{\infty}\Big|\Big[(-1)^{\frac{n+1}{2}}\Big(\frac{n}{2}\Big)^{-2}\Big]\cdot\Big[\exp\Big(-\frac{n\Pi}{2}x^*/\sqrt{(\gamma+1)C_u}\Big)\cos\Big(\frac{n\Pi y^*}{2}\Big)\Big]\Big| \qquad \text{(VII.8)}$$

$$-4\frac{(\gamma+1)}{\Pi^2}f''_o\sqrt{(\gamma+1)C_u}\sum_{n=2,4,6\dots}^{\infty}\Big|\Big[(-1)^{\frac{n}{2}}\Big(\frac{n}{2}\Big)^{-2}\Big]\cdot\Big[\exp\Big(-\frac{n\Pi}{2}x^*/\sqrt{(\gamma+1)C_u}\Big)\sin\Big(\frac{n\Pi y^*}{2}\Big)\Big]\Big|$$

where

$$\tilde{f}''_o=[f_{uxx}(x_{o1})-f_{\ell xx}(x_{o1})] \qquad \text{(VII.9)}$$

and

$$C_u=\sqrt{\frac{1}{(\gamma+1)}(f_u(x_{o1})-f_\ell(x_{o1}))}. \qquad \text{(VII.10)}$$

In addition, the shock velocity can be shown to be

$$\frac{4k}{\gamma+1}u_{s2}=h_{xd}+h_{xu}+\frac{1}{3}\tilde{f}''_o-C_u{}^2. \qquad \text{(VII.11)}$$

The analysis of the asymmetric channel proceeds exactly the same as for the symmetric channel in the case of "second order unsteady" flow. Disturbances are imposed as initial conditions which are variations in $h_x(x,t)$ from the steady state solutions. These steady state solutions are,

$$h_x(x<x_{sh})=-\frac{1}{6}\tilde{f}''+\frac{3-2\gamma}{6}\Big(\frac{2}{\gamma+1}\tilde{f}\Big)$$

$$h_x(x>x_{sh})=\frac{\frac{2}{3}\gamma\Big(\frac{2}{\gamma+1}\tilde{f}_o\Big)^{3/2}}{\Big(\frac{2}{\gamma+1}\tilde{f}\Big)^{\frac{1}{2}}}-\frac{1}{6}\tilde{f}''+\frac{3-2\gamma}{6}\Big(\frac{2}{\gamma+1}\tilde{f}\Big)$$

The application of these results to asymmetric channels (8) yields basically the same results as for symmetric channels, except that there is a difference in the pressure distribution along the upper and lower walls, and the lines of constant velocity are altered as would be expected. For the problem being considered, i.e. flow asymmetry affecting primarily the second order terms, the shock is still a straight line perpendicular to the X-axis, to the scale of the outer solution.

VIII. Conclusions

It is believed that the analytical procedures described in this paper give insight into the nature of unsteady transonic flow without recourse to more complicated numerical solutions of the equations of motion. The systematic derivation of the Szaniawski type of solutions places them in perspective and indicates their usefulness in analysis of transonic flow. The method described in this paper permits consideration of the direct transonic flow problem whereas similarity solutions employed heretofore cannot be used for arbitrary wall shapes and initial conditions.

In this paper the case where the shock is initially in a fixed location (x_{o1} =const) with unsteady second order flow has been considered. Other cases, where the flow is unsteady in first order and $x_{o1}=x_{o1}(t)$ should be analyzed. Also an investigation of moveable walls, for application to problems such as compressor blade vibration and flutter, should be considered.

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Figures

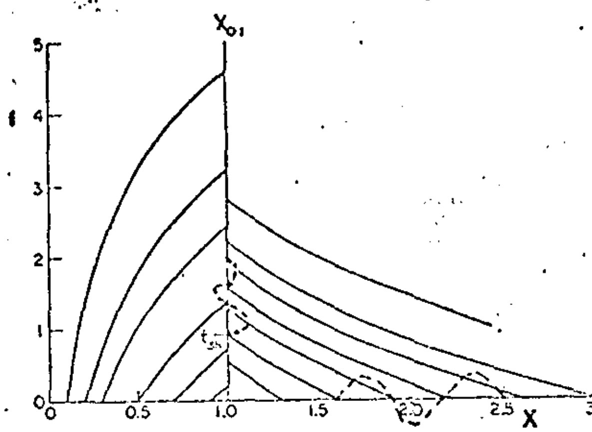

FIG. 3 X-t DIAGRAM SHOWING CHARACTERISTICS FOR PARABOLIC WALL. SHOCK WAVE AT X=1

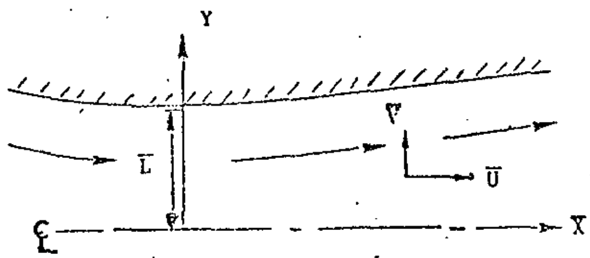

FIG. 1 SKETCH OF NOZZLE FLOW COORDINATE SYSTEM

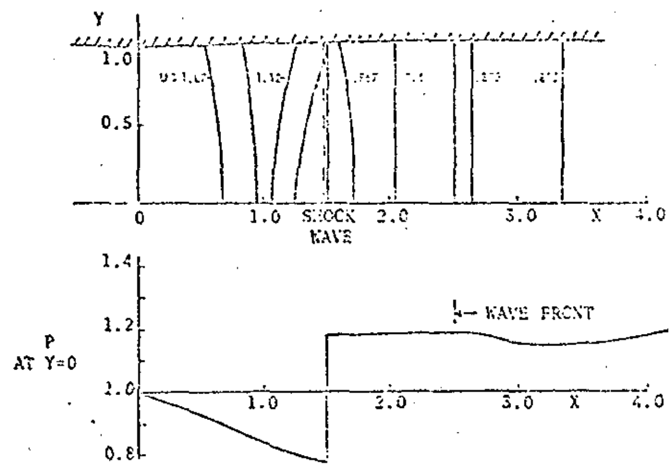

FIG. 4 SYMMETRIC CHANNEL FLOW AT t=0 CONSTANT AREA FOR X>2.0; ε=3, a=2, β=2.5

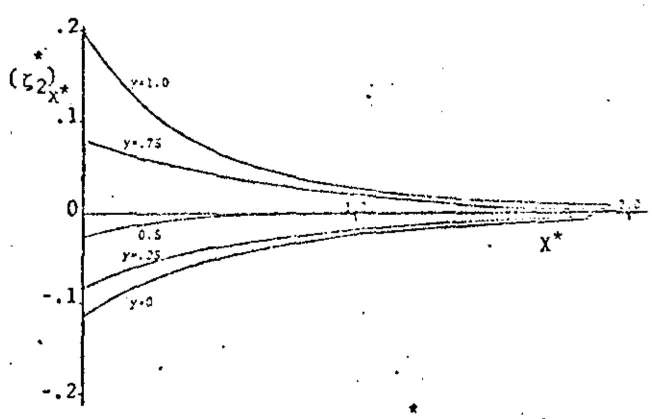

FIG. 2 VARIATION OF $(\zeta_2^*)_{x^*}$ IN THE INNER REGION DOWNSTREAM OF THE SHOCK AT $X_{0_1}=1.5$

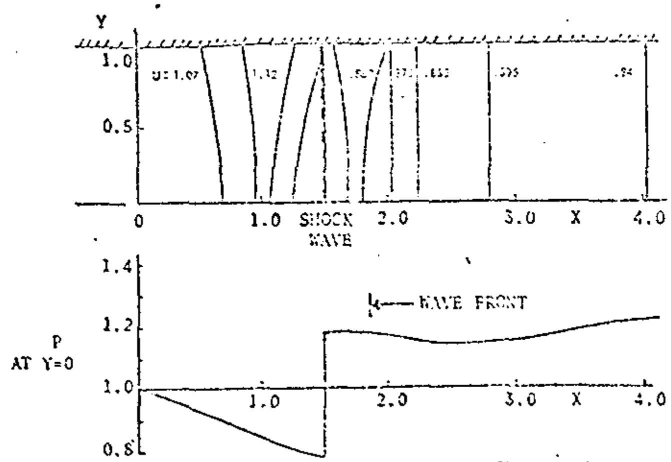

FIG. 5 SYMMETRIC CHANNEL FLOW AT t=0.4 CONSTANT AREA FOR X>2.0; ε=3, a=2, β=2.5

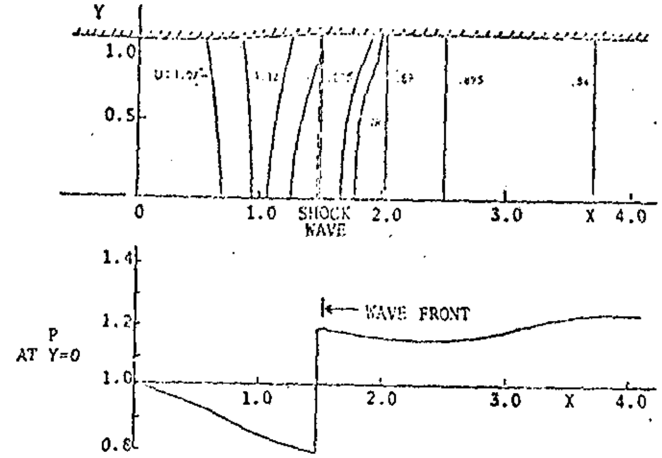

FIG. 6 SYMMETRIC CHANNEL FLOW AT t=.6
CONSTANT AREA FOR X>2.0; ε=3, a=2, f̂=2.5

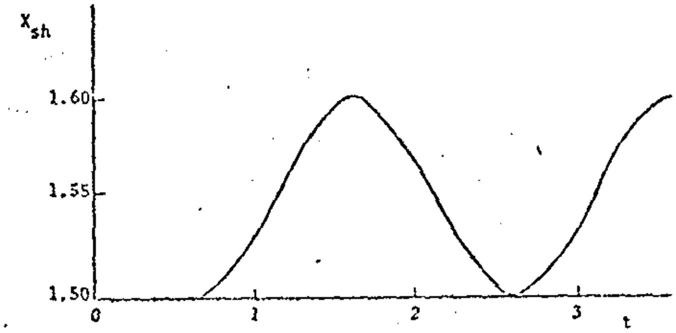

FIG. 9 SHOCK POSITION vs. TIME FOR SYMMETRIC CHANNEL. INITIAL SHOCK POSITION X_{o1}=1.5

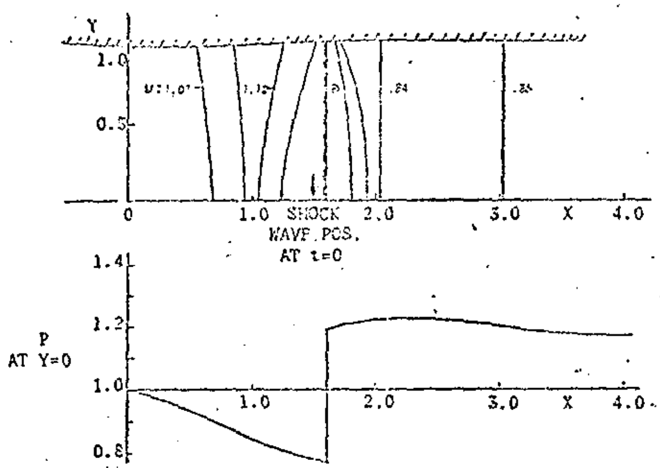

FIG. 7 SYMMETRIC CHANNEL FLOW AT t=1.6
CONSTANT AREA FOR X>2.0; ε=3, a=2, f̂=2.5

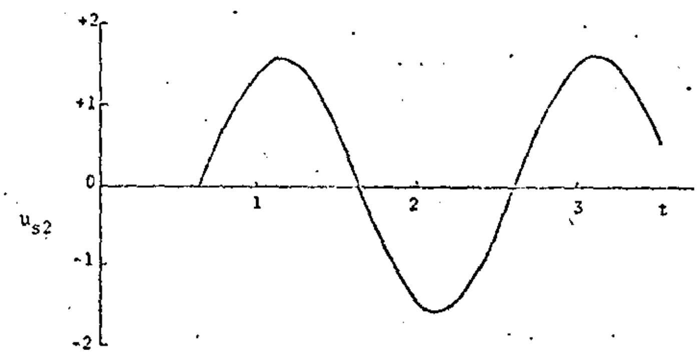

FIG. 8 SHOCK VELOCITY vs TIME FOR SYMMETRIC CHANNEL
ε=3, a=2, f̂=2.5, X_{o1}=1.5