

Analysis on Poisson and Gamma spaces

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Abstract

We study the spaces of Poisson, compound Poisson and Gamma noises as special cases of a general approach to non-Gaussian white noise calculus, see [KSS97]. We use a known unitary isomorphism between Poisson and compound Poisson spaces in order to transport analytic structures from Poisson space to compound Poisson space. Finally we study a Fock type structure of chaos decomposition on Gamma space.

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1 Introduction

The present paper elaborate the L^2 structure of compound Poisson spaces; we note that for all of this compound Poisson processes the results of [KSS97] immediately produce Gel'fand triples of test and generalized functions as well their characterizations and calculus.

The Analysis on pure Poisson spaces was developed in [CP90], [IK88], [NV95], [P95] and many others from different points of view. In [KSS97] we have developed methods for non Gaussian analysis based on generalized Appell systems. In the case of Poisson space, this coincide with the system of generalized Charlier polynomials, however the desirable extensions to compound Poisson and for example Gamma processes are trivial.

Let us describe this construction more precisely. We recall that the Poisson measure π_σ (with intensity measure σ which is a non-atomic Radon measure on \mathbb{R}^d) is defined by its Laplace transform as

$$l_{\pi_\sigma}(\varphi) = \int_{\mathcal{D}'} \exp(\langle \gamma, \varphi \rangle) d\pi_\sigma(\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) d\sigma(x)\right), \varphi \in \mathcal{D},$$

where \mathcal{D}' is the dual of $\mathcal{D} := \mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ (C^∞ -functions on \mathbb{R}^d with compact support). An additional analysis shows that the support of the measure π_σ consists of generalized functions of the form $\sum_{x \in \gamma} \varepsilon_x$, $\gamma \in \Gamma_{\mathbb{R}^d}$, where ε_x is the Dirac measure in x and $\Gamma_{\mathbb{R}^d}$ is the configuration space over \mathbb{R}^d , i.e.,

$$\Gamma_{\mathbb{R}^d} := \{\gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}.$$

The configuration space $\Gamma_{\mathbb{R}^d}$ can be endowed with its natural Borel σ -algebra $\mathcal{B}(\Gamma_{\mathbb{R}^d})$ and π_σ can be considered as a measure on $\Gamma_{\mathbb{R}^d}$.

Let us choose a transformation α on \mathcal{D} given by

$$\alpha(\varphi)(x) = \log(1 + \varphi(x)), \quad -1 < \varphi \in \mathcal{D}, \quad x \in \mathbb{R}^d.$$

Then the normalized exponential or Poisson exponential

$$e_{\pi_\sigma}^\sigma(\varphi; \gamma) = \exp(\langle \gamma, \log(1 + \varphi) \rangle - \langle \varphi \rangle_\sigma), \quad \gamma \in \Gamma_{\mathbb{R}^d}$$

is a real holomorphic function of φ on a neighborhood of zero \mathcal{U}_α on \mathcal{D} . Its Taylor decomposition (with respect to φ) has the form

$$e_{\pi_\sigma}^\sigma(\varphi; \gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma(\gamma), \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{U}'_\alpha \subset \mathcal{U}_\alpha, \quad \gamma \in \Gamma_{\mathbb{R}^d},$$

where $C_n^\sigma : \Gamma_{\mathbb{R}^d} \rightarrow \mathcal{D}'^{\otimes n}$. It follows from the above equality that for any $\varphi^{(n)} \in \mathcal{D}^{\otimes n}$, $n \in \mathbb{N}_0$ the system of functions

$$\Gamma_{\mathbb{R}^d} \ni \gamma \mapsto \langle C_n^\sigma(\gamma), \varphi^{(n)} \rangle$$

is a polynomial of order n on $\Gamma_{\mathbb{R}^d}$. This is precisely the system of generalized Charlier polynomials for the measure π_σ , see Subsection 2.3 for details.

This system can be used for a Fock realization. $L^2(\pi_\sigma)$ has a Fock realization analogous to Gaussian analysis, i.e.,

$$L^2(\pi_\sigma) \simeq \bigoplus_{n=0}^{\infty} \text{Exp}_n L^2(\sigma) = \text{Exp} L^2(\sigma),$$

where $\text{Exp}_n L^2(\sigma)$ denotes the n -fold symmetric tensor product of $L^2(\sigma)$.

The ‘‘Poissonian gradient’’ ∇^{P} on functions $f : \Gamma_{\mathbb{R}^d} \rightarrow \mathbb{R}$ which has specific useful properties on Poisson space, is introduced on a specific space of ‘‘nice’’ functions as a difference operator

$$(\nabla^{\text{P}} f)(\gamma; x) = f(x + \varepsilon_x) - f(x), \quad \gamma \in \Gamma_{\mathbb{R}^d}, x \in \mathbb{R}^d.$$

The gradient ∇^{P} appears from different points of view in many papers on conventional Poissonian analysis, see e.g. [IK88], [NV95], [KSS97] and references therein. We note also that the most important feature of the Poissonian gradient is that it produces (via a corresponding integration by parts formula) the orthogonal system of generalized Charlier polynomials on $(\Gamma_{\mathbb{R}^d}, \mathcal{B}(\Gamma_{\mathbb{R}^d}), \pi_\sigma)$, see Remark 2.14. In addition we mention here that as a tangent space to each point $\gamma \in \Gamma_{\mathbb{R}^d}$ we choose the same Hilbert space $L^2(\mathbb{R}^d, \sigma)$.

We conclude Section 2 with the expressions for the annihilation and creation operators on Poisson space. In terms of chaos decomposition Nualart and Vives [NV95] proved the analogous expression of the creation operator, in this paper we give an independent proof which is based on the results on absolute continuity of Poisson measures, see e.g. [Sk57] and [T90], details can be found in Subsection 2.5.

The analysis on compound Poisson space can be done with the help of the analysis derived from Poisson space described above. That possibility is based on the existence of an unitary isomorphism between compound Poisson space and Poisson space which allows us to transport the Fock structure from the Poisson space to the compound Poisson space. The above isomorphism has been identified before by K. Itô, [I56] and A. Dermoune, [De90]. All this is developed is Subsection 3.2.

The images of the annihilation and creation operators under the above isomorphism on compound Poisson space are worked out in Subsection 3.3.

The aim of Section 4 is to study in more details the previous analysis in a particular case of compound Poisson measure, the so called Gamma noise measure. Its Laplace transform is given by

$$l_{\mu_{\mathbb{G}}^{\sigma}}(\varphi) = \exp(-\langle \log(1 - \varphi) \rangle_{\sigma}), \quad 1 > \varphi \in \mathcal{D}.$$

This measure can be seen as a special case of compound Poisson measure $\mu_{\mathbb{CP}}$ for a specific choice of the measure ρ used in the definition of $\mu_{\mathbb{CP}}$, see Section 4 - (36) for details. From this point of view, of course, all structure may be implemented on Gamma space. The question that still remains is to find intrinsic expressions for all these operators on Gamma space as found in Poisson space.

The most intriguing feature of Gamma space we found is its Fock type structure. As in the Poisson case it is possible to choose a transformation α on \mathcal{D} such that the normalized exponentials $e_{\mu_{\mathbb{G}}^{\sigma}}^{\sigma}(\varphi; \cdot)$ produce a complete system of orthogonal polynomials, the so called system of generalized Laguerre polynomials. It leads to a Fock type realization of Gamma space as

$$L^2(\mu_{\mathbb{G}}^{\sigma}) \simeq \bigoplus_{n=0}^{\infty} \text{Exp}_n^{\mathbb{G}} L^2(\sigma) = \text{Exp}^{\mathbb{G}} L^2(\sigma),$$

where $\text{Exp}_n^{\mathbb{G}} L^2(\sigma) \subset \text{Exp}_n L^2(\sigma)$ is a quasi- n -particle subspace of $\text{Exp}^{\mathbb{G}} L^2(\sigma)$. The point here is that the scalar product in $\text{Exp}_n^{\mathbb{G}} L^2(\sigma)$ turns out to be different of the standard one given by $L^2(\sigma)^{\widehat{\otimes} n}$. As a result the space $\text{Exp}^{\mathbb{G}} L^2(\sigma)$ has a novel n -particle structure which is essentially different from traditional Fock picture.

2 Poisson analysis

Throughout this section we consider the measure space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \sigma)$ where $\sigma \in \mathcal{M}(\mathbb{R}^d)$ (the set of all positive Radon measures on $\mathcal{B}(\mathbb{R}^d)$) with density $\tau(x)$ with respect to the Lebesgue measure m on $\mathcal{B}(\mathbb{R}^d)$, moreover we assume that $\tau > 0$ m -a.e. and $\tau \in L^1_{loc}(\mathbb{R}^d, m)$. We denote the classical Schwartz space by $\mathcal{D} := \mathcal{D}(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^d)$ (C^∞ -functions on \mathbb{R}^d with compact supports).

2.1 The configuration space over \mathbb{R}^d

The configuration space $\Gamma_{\mathbb{R}^d} =: \Gamma$ over \mathbb{R}^d is defined as the set of all locally finite subsets (configurations) in \mathbb{R}^d , i.e.,

$$\Gamma := \{ \gamma \subset \mathbb{R}^d \mid |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \}.$$

Here $|A|$ denotes the cardinality of a set A .

We can identify any $\gamma \in \Gamma$ with the corresponding sum of Dirac measures, namely

$$\Gamma \ni \gamma \rightarrow \sum_{x \in \gamma} \varepsilon_x(dy) =: d\gamma(y) \in \mathcal{M}_p(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d), \quad (1)$$

where $\mathcal{M}_p(\mathbb{R}^d)$ denotes the set of all positive integer valued measures (or Radon point measures) over $\mathcal{B}(\mathbb{R}^d)$.

The space Γ can be endowed with the relative topology as a closed subset of the space $\mathcal{M}_p(\mathbb{R}^d)$ on $\mathcal{B}(\mathbb{R}^d)$ with the vague topology, i.e., a sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ converge in the vague topology to μ if and only if for any $f \in C_0(\mathbb{R}^d)$ (the set of all continuous functions with compact support) we have

$$\int_{\mathbb{R}^d} f(x) d\mu_n(x) \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}^d} f(x) d\mu(x).$$

Then for any $f \in C_0(\mathbb{R}^d)$ we have a continuous functional

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \langle f \rangle_\gamma = \int_{\mathbb{R}^d} f(x) d\gamma(x) = \sum_{x \in \gamma} f(x) \in \mathbb{R}.$$

Conversely, such functionals generate the topology of the space Γ .

Hence we have the following chain

$$\Gamma \subset \mathcal{M}(\mathbb{R}^d) \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^d).$$

The Borel σ -algebra on Γ , $\mathcal{B}(\Gamma)$, is generated by sets of the form

$$C_{\Lambda,n} = \{\gamma \in \Gamma \mid |\gamma \cap \Lambda| = n\}, \quad (2)$$

where $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ bounded, $n \in \mathbb{N}$, see e.g. [GGV75] and for any $\Lambda \in \mathcal{B}(\mathbb{R}^d)$ and all $n \in \mathbb{N}$ the set $C_{\Lambda,n}$ is a Borel set of Γ . Sets of the form (2) are called cylinder sets.

For any $B \subset \mathbb{R}^d$ we introduce a function $N_B : \Gamma \rightarrow \mathbb{N}$ such that

$$N(B)(\gamma) = |\gamma \cap B|, \gamma \in \Gamma.$$

Then $\mathcal{B}(\Gamma)$ is the minimal σ -algebra with which all the functions $\{N_B \mid B \in \mathcal{B}(\mathbb{R}^d) \text{ bounded}\}$ are measurable.

2.2 The Poisson measure and its properties

The Poisson measure π_σ (with intensity measure σ) on $(\Gamma, \mathcal{B}(\Gamma))$ may be defined in different ways, here we give two convenient characterizations of π_σ .

Definition 2.1 (Laplace transform) *The Laplace transform of π_σ is given by*

$$l_{\pi_\sigma}(\varphi) = \int_{\Gamma} \exp(\langle \varphi, \gamma \rangle) d\pi_\sigma(\gamma) = \exp\left(\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1) d\sigma(x)\right), \quad (3)$$

where $\varphi \in \mathcal{D}$, see e.g. [KMM78] and [GV68, Chap. III Sec. 4].

Remark 2.2 *The right hand side of (3) defines, via Minlos' theorem, the measure π_σ on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$, but an additional analysis shows that the support of the measure π_σ is $\Gamma \subset \mathcal{D}'$, see e.g. [Ka74], [Ka75] and [KMM78], hence π_σ can be considered as a measure on Γ .*

Let $f : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}$ be such that $f \geq 0$ and measurable. Define

$$\begin{aligned} F(\gamma) &:= \langle \gamma, f(\cdot, \gamma) \rangle \\ &= \int_{\mathbb{R}^d} f(x, \gamma) d\gamma(x) \\ &= \sum_{x \in \gamma} f(x, \gamma). \end{aligned}$$

Then π_σ is characterized by

$$\begin{aligned} \int_{\Gamma} F(\gamma) d\pi_\sigma(\gamma) &:= \int_{\Gamma} \int_{\mathbb{R}^d} f(x, \gamma) d\gamma(x) d\pi_\sigma(\gamma) \\ &= \int_{\mathbb{R}^d} \int_{\Gamma} f(x, \gamma + \varepsilon_x) d\pi_\sigma(\gamma) d\sigma(x). \end{aligned} \quad (4)$$

Equality (4) is known as Mecke identity, see e.g. [Me67] and [NZ76].

2.3 The Fock space isomorphism of Poisson space

Let us consider the following transformation on \mathcal{D} , $\alpha : \mathcal{D} \rightarrow \mathcal{D}$ defined by

$$\alpha(\varphi)(x) = \log(1 + \varphi(x)), \quad -1 < \varphi \in \mathcal{D}, \quad x \in \mathbb{R}^d.$$

As easily can be seen $\alpha(0) = 0$ and α is holomorphic in some neighborhood \mathcal{U}_α of zero. Using this transformation we introduce the normalized exponential $e_{\pi_\sigma}^\alpha(\cdot, \cdot)$ which is holomorphic on a neighborhood of zero $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha \subset \mathcal{D}$. For $\varphi \in \mathcal{U}'_\alpha$, $\gamma \in \Gamma$ we set

$$\begin{aligned} e_{\pi_\sigma}^\alpha(\varphi, \gamma) &:= \frac{\exp(\langle \gamma, \alpha(\varphi) \rangle)}{l_{\pi_\sigma}(\alpha(\varphi))} \\ &= \exp(\langle \gamma, \log(1 + \varphi) \rangle - \langle \varphi \rangle_\sigma), \end{aligned} \quad (5)$$

where $\langle \varphi \rangle_\sigma := \int_{\mathbb{R}^d} \varphi(x) d\sigma(x)$.

We use the holomorphy of $e_{\pi_\sigma}^\alpha(\cdot, \gamma)$ on a neighborhood of zero to expand it in power series which, with Cauchy's inequality, polarization identity and kernel theorem, give us the following result

$$e_{\pi_\sigma}^\alpha(\varphi, \gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\pi_\sigma, \alpha}(\gamma), \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{U}'_\alpha \subset \mathcal{U}_\alpha, \quad \gamma \in \Gamma, \quad (6)$$

where $P_n^{\pi_\sigma, \alpha} : \Gamma \rightarrow \mathcal{D}'^{\widehat{\otimes} n}$. $\{P_n^{\pi_\sigma, \alpha}(\cdot) =: C_n^\sigma(\cdot) \mid n \in \mathbb{N}_0\}$ is called the system of **generalized Charlier kernels** on Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$. From (6) it follows immediately that for any $\varphi^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$ the function

$$\Gamma \ni \gamma \mapsto \langle C_n^\sigma(\gamma), \varphi^{(n)} \rangle$$

is a polynomial of the order n on Γ . The system of functions

$$\left\{ C_n^\sigma(\varphi^{(n)})(\gamma) := \langle C_n^\sigma(\gamma), \varphi^{(n)} \rangle, \quad \forall \varphi^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, \quad n \in \mathbb{N}_0 \right\}$$

is called the system of **generalized Charlier polynomials** for the Poisson measure π_σ .

Proposition 2.3 For any $\varphi^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$ and $\psi^{(m)} \in \mathcal{D}^{\widehat{\otimes} m}$ we have

$$\int_{\Gamma} \langle C_n^{\sigma}(\gamma), \varphi^{(n)} \rangle \langle C_m^{\sigma}(\gamma), \psi^{(m)} \rangle d\pi_{\sigma}(\gamma) = \delta_{nm} n! (\varphi^{(n)}, \psi^{(n)})_{L^2(\sigma^{\otimes n})}.$$

Proof. Let $\varphi^{(n)}, \psi^{(m)}$ be given as in the proposition and such that $\varphi^{(n)} = \varphi^{\otimes n}, \psi^{(m)} = \psi^{\otimes m}$. Then for $z_1, z_2 \in \mathbb{C}$, and taking into account (3) and (5) we have

$$\begin{aligned} & \int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(z_1\varphi, \gamma) e_{\pi_{\sigma}}^{\alpha}(z_2\psi, \gamma) d\pi_{\sigma}(\gamma) \\ &= \exp(-\langle z_1\varphi + z_2\psi \rangle_{\sigma}) \int_{\Gamma} \exp(\langle \gamma, \log((1 + z_1\varphi)(1 + z_2\psi)) \rangle) d\pi_{\sigma}(\gamma) \\ &= \exp(-\langle z_1\varphi + z_2\psi \rangle_{\sigma}) \\ & \quad \cdot \exp\left(\int_{\mathbb{R}^d} (\exp(\log((1 + z_1\varphi)(1 + z_2\psi))) - 1) d\sigma\right) \\ &= \exp\left(z_1 z_2 (\varphi, \psi)_{L^2(\sigma)}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z_1^n z_2^n (\varphi^{\otimes n}, \psi^{\otimes n})_{L^2(\sigma^{\otimes n})}. \end{aligned} \tag{7}$$

On the other hand

$$\begin{aligned} & \int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(z_1\varphi, \gamma) e_{\pi_{\sigma}}^{\alpha}(z_2\psi, \gamma) d\pi_{\sigma}(\gamma) \\ &= \sum_{n,m=0}^{\infty} \frac{z_1^n z_2^m}{n!m!} \int_{\Gamma} \langle C_n^{\sigma}(\gamma), \varphi^{\otimes n} \rangle \langle C_m^{\sigma}(\gamma), \psi^{\otimes m} \rangle d\pi_{\sigma}(\gamma). \end{aligned} \tag{8}$$

Then a comparison of coefficients between (7) and (8), with polarization identity and linearity, gives the above result. \blacksquare

Remark 2.4 This proposition gives us the possibility to extend - in the $L^2(\pi_{\sigma})$ sense - the class of $\langle C_n^{\sigma}(\gamma), \varphi^{\otimes n} \rangle$ -functions to include kernels from the so-called n -particle Fock space over $L^2(\sigma)$.

We define the Fock space as the Hilbert sum

$$\text{Exp}L^2(\sigma) := \bigoplus_{n=0}^{\infty} \text{Exp}_n L^2(\sigma)$$

where $\text{Exp}_n L^2(\sigma) := L^2(\sigma)_{\mathbb{C}}^{\widehat{\otimes} n}$ and we put by definition $\text{Exp}_0 L^2(\sigma) := \mathbb{C}$.

For any $F \in L^2(\pi_\sigma)$ there exists a sequence $(f^{(n)})_{n=0}^\infty \in \text{Exp} L^2(\sigma)$ such that

$$F(\gamma) = \sum_{n=0}^{\infty} \langle C_n^\sigma(\gamma), f^{(n)} \rangle \quad (9)$$

and moreover

$$\|F\|_{L^2(\pi_\sigma)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{L^2(\sigma^{\otimes n})}^2, \quad (10)$$

where the r.h.s. of (10) coincides with the square of the norm in $\text{Exp} L^2(\sigma)$. And vice versa, any series of the form (9) with coefficients $(f^{(n)})_{n=0}^\infty \in \text{Exp} L^2(\sigma)$ gives a function from $L^2(\pi_\sigma)$. As a result we have the well-known isomorphism I_σ between $L^2(\pi_\sigma)$ and $\text{Exp} L^2(\sigma)$.

Now we introduce the action of the annihilation and creation operators in the Fock space $\text{Exp} L^2(\sigma)$, see e.g. [HKPS93, Appendix A.2] and [RS75]. Consider $f^{(n)} \in \text{Exp}_n L^2(\sigma)$ of the form

$$f^{(n)} = \widehat{\otimes}_{i=1}^n f_i, \quad f_i \in L^2(\sigma), \quad i = 1, \dots, n. \quad (11)$$

Then the action of the annihilation operator $a^-(\varphi)$, $\varphi \in \mathcal{D}$, on $f^{(n)}$ is defined as follows:

$$a^-(\varphi) f^{(n)} := \sum_{j=1}^n \langle \varphi, f_j \rangle \widehat{\otimes}_{\substack{i=1 \\ i \neq j}}^n f_i \in \text{Exp}_{n-1} L^2(\sigma).$$

This definition is independent of the particular representation of $f^{(n)}$ in (11), hence $a^-(\varphi) f^{(n)}$ is well-defined. Moreover this definition can be extended by linearity to a dense subspace of $\text{Exp}_n L^2(\sigma)$ consisting of finite linear combinations of elements of the form (11). One easily finds the following inequality for such elements

$$a^-(\varphi) f^{(n)} \leq \sqrt{n} |\varphi| |f^{(n)}|, \quad (12)$$

which shows that the extension of $a^-(\varphi)$ to $\text{Exp}_n L^2(\sigma)$ as a bounded operator exists. Consider the dense subspace $\text{Exp}^0 L^2(\sigma)$ of $\text{Exp} L^2(\sigma)$ consisting of those sequences $\{f^{(n)}, n \in \mathbb{N}_0\}$ which only have a finite numbers of non-vanishing entries. The bound (12) allow us to extend $a^-(\varphi)$, $\varphi \in \mathcal{D}$, component-wise to $\text{Exp}^0 L^2(\sigma)$ which, therefore, give us a densely defined operator on $\text{Exp} L^2(\sigma)$ denoted again by $a^-(\varphi)$. So the adjoint operator of

$a^-(\varphi)$ exists, which we denote by $a^+(\varphi)$, and call creation operator. The action of the creation operator on elements $f^{(n)} \in \text{Exp}_n L^2(\sigma)$ is given by

$$a^+(\varphi)f^{(n)} = \varphi \widehat{\otimes} f^{(n)} \in \text{Exp}_{n+1} L^2(\sigma).$$

For the creation operator we also have an estimate

$$|a^+(\varphi)f^{(n)}| \leq \sqrt{n+1} |\varphi| |f^{(n)}|.$$

As before, this estimate give us the possibility to deduce that in the same $a^+(\varphi)$ is densely defined on $\text{Exp} L^2(\sigma)$.

For latter use we introduce a vector $\text{Exp}\psi$, $\psi \in L^2(\sigma)$ as

$$\text{Exp}\psi = \left(\frac{1}{n!} \psi^{\otimes n} \right)_{n=0}^{\infty}$$

which is called the coherent state corresponding to the one-particle state ψ . For any set $\mathcal{L} \subset L^2(\sigma)$ which is total in $L^2(\sigma)$ the set of coherent states $\{\text{Exp}\psi \mid \psi \in \mathcal{L}\} \subset \text{Exp} L^2(\sigma)$ is also total in $\text{Exp} L^2(\sigma)$, see e.g. [BK88]. We note that $e_{\pi_\sigma}^\alpha(\psi, \cdot)$ is nothing as the coherent state in the Fock space picture, for any $\psi \in \mathcal{D}$, $\psi > -1$, we have

$$L^2(\pi_\sigma) \ni e_{\pi_\sigma}^\alpha(\psi, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma(\cdot), \psi^{\otimes n} \rangle \mapsto \text{Exp}\psi \in \text{Exp} L^2(\sigma).$$

The action of the annihilation operator $a^-(\varphi)$ on $\text{Exp}\psi$ is given by

$$a^-(\varphi) \text{Exp}\psi = \left(\frac{1}{(n-1)!} (\varphi, \psi)_{L^2(\sigma)} \psi^{\otimes(n-1)} \right)_{n=1}^{\infty}.$$

2.4 Annihilation operator on Poisson space

Let us introduce a set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ (dense in $L^2(\pi_\sigma)$) which consists of all functions of the form

$$f(\gamma) = F(\langle \gamma, \varphi_1 \rangle, \dots, \langle \gamma, \varphi_N \rangle), \quad \gamma \in \Gamma,$$

where the generating directions $\varphi_1, \dots, \varphi_N \in \mathcal{D}$, and F (generating function for f) is from $C_b^\infty(\mathbb{R}^N)$.

Definition 2.5 We define the **Poissonian gradient** ∇^{P} as a mapping

$$\nabla^{\text{P}} : \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \longrightarrow L^2(\pi_\sigma) \otimes L^2(\sigma)$$

given by

$$(\nabla^{\text{P}} f)(\gamma, x) = f(\gamma + \varepsilon_x) - f(\gamma), \quad \gamma \in \Gamma, \quad x \in \mathbb{R}^d.$$

Let us mention that the operation

$$\Gamma \ni \gamma \mapsto \gamma + \varepsilon_x \in \Gamma$$

is well-defined because of the property:

$$\pi_\sigma \{ \gamma \in \Gamma \mid x \in \gamma \} = 0, \quad \forall x \in \mathbb{R}^d.$$

The fact that $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \ni f \mapsto \nabla^{\text{P}} f \in L^2(\pi_\sigma) \otimes L^2(\sigma)$ arises from the use of the Hilbert space $L^2(\sigma)$ as a tangent space at any point $\gamma \in \Gamma$.

Remark 2.6 To produce differential structure we need linear structure where the measure have support Γ . If we consider π_σ on \mathcal{D}' then $\pi_\sigma(\xi + \varphi) \perp \pi_\sigma(\xi)$, see e.g. [GGV75], therefore integration by parts and adjoint of operators are not available. This is the reason why we embed Γ in \mathcal{D}' .

Proposition 2.7 For any $h \in \text{Dom}((\nabla^{\text{P}})^*)$ the following equality holds

$$((\nabla^{\text{P}})^* h)(\gamma) = \int_{\mathbb{R}^d} h(\gamma - \varepsilon_x, x) d\gamma(x) - \int_{\mathbb{R}^d} h(\gamma, x) d\sigma(x). \quad (13)$$

Proof. Let $f \in \text{Dom}(\nabla^{\text{P}})$ be given. Then we use the Mecke identity (4) to compute $(\nabla^{\text{P}} f, h)_{L^2(\pi_\sigma) \otimes L^2(\sigma)}$ as follows:

$$\begin{aligned} & (\nabla^{\text{P}} f, h)_{L^2(\pi_\sigma) \otimes L^2(\sigma)} \\ &= \int_{\mathbb{R}^d} \int_{\Gamma} (f(\gamma + \varepsilon_x) - f(\gamma)) h(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) \\ &= \int_{\mathbb{R}^d} \int_{\Gamma} f(\gamma + \varepsilon_x) h(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) \\ &\quad - \int_{\mathbb{R}^d} \int_{\Gamma} f(\gamma) h(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) \\ &= \int_{\Gamma} f(\gamma) \underbrace{\left[\int_{\mathbb{R}^d} h(\gamma - \varepsilon_x, x) d\gamma(x) - \int_{\mathbb{R}^d} h(\gamma, x) d\sigma(x) \right]}_{((\nabla^{\text{P}})^* h)(\gamma)} d\pi_\sigma(\gamma). \end{aligned}$$

Now we are going to give an internal description of the annihilation operator. ■

The directional derivative is then defined as

$$\begin{aligned} (\nabla_\varphi^{\mathbb{P}} f)(\gamma) &= ((\nabla^{\mathbb{P}} f)(\gamma, \cdot), \varphi(\cdot))_{L^2(\sigma)} \\ &= \int_{\mathbb{R}^d} (f(\gamma + \varepsilon_x) - f(\gamma)) \varphi(x) d\sigma(x) \end{aligned} \quad (14)$$

for any $\varphi \in \mathcal{D}$. Of course the operator

$$\nabla_\varphi^{\mathbb{P}} : \mathcal{FC}_b^\infty(\mathcal{D}, \Gamma) \longrightarrow L^2(\pi_\sigma)$$

is closable in $L^2(\pi_\sigma)$.

Proposition 2.8 *The closure of $\nabla_\varphi^{\mathbb{P}}$ coincide with the image under I_σ of the annihilation operator $a^-(\varphi)$ in $\text{Exp}L^2(\sigma)$, i.e., $I_\sigma a^-(\varphi) I_\sigma^{-1} = \nabla_\varphi^{\mathbb{P}}$.*

Proof. To prove this proposition it is enough to show this equality of operators in a total set in the core of the annihilation operator. Let $\psi \in \mathcal{U}'_\alpha$ be given, then having in mind (14) and (5) it follows that

$$\begin{aligned} (\nabla_\varphi^{\mathbb{P}} e_{\pi_\sigma}^\alpha(\psi; \cdot))(\gamma) &= \int_{\mathbb{R}^d} (e_{\pi_\sigma}^\alpha(\psi; \gamma + \varepsilon_x) - e_{\pi_\sigma}^\alpha(\psi; \gamma)) \varphi(x) d\sigma(x) \\ &= e_{\pi_\sigma}^\alpha(\psi; \gamma) \int_{\mathbb{R}^d} (\exp(\langle \varepsilon_x, \log(1 + \psi) \rangle) - 1) \varphi(x) d\sigma(x) \\ &= (\psi, \varphi)_{L^2(\sigma)} e_{\pi_\sigma}^\alpha(\psi; \gamma). \end{aligned} \quad (15)$$

On the other hand since $I_\sigma^{-1} e_{\pi_\sigma}^\alpha(\psi; \gamma) = \text{Exp}\psi$ it follows that

$$a^-(\varphi) \text{Exp}\psi = \left(\frac{1}{(n-1)!} (\varphi, \psi)_{L^2(\sigma)} \psi^{\otimes(n-1)} \right)_{n=1}^\infty.$$

Hence if we apply I_σ to this vector we just obtain the same result as (15) which had to be proven. ■

2.5 Creation operator on Poisson space

Proposition 2.9 For any $\varphi \in \mathcal{D}$, $g \in \text{Dom}(I_\sigma a^+(\varphi) I_\sigma^{-1})$, where $a^+(\varphi)$ is the creation operator in $\text{Exp}L^2(\sigma)$, the following equality holds

$$\begin{aligned} ((\nabla_\varphi^{\text{P}})^* g)(\gamma) &= \int_{\mathbb{R}^d} g(\gamma - \varepsilon_x) \varphi(x) d\gamma(x) - g(\gamma) \int_{\mathbb{R}^d} \varphi(x) d\sigma(x) \\ &= (g(\gamma - \varepsilon.), \varphi(\cdot))_{L^2(\gamma)} - g(\gamma) \langle \varphi \rangle_\sigma \end{aligned} \quad (16)$$

Remark 2.10 In terms of chaos decomposition of $g \in \text{Dom}((\nabla_\varphi^{\text{P}})^*)$ the equality (16) was established in [NV95]. We give an independent proof of (16), which is based on the results on absolute continuity of Poisson measures, see e.g. [Sk57] and [T90].

Proof. 1. First we give a version of the proof of (16) which uses the Mecke identity.

It follows from (14) that

$$\begin{aligned} (\nabla_\varphi^{\text{P}} f, g)_{L^2(\pi_\sigma)} &= \int_{\Gamma} ((\nabla^{\text{P}} f)(\gamma, \cdot), \varphi(\cdot))_{L^2(\sigma)} g(\gamma) d\pi_\sigma(\gamma) \\ &= ((\nabla^{\text{P}} f)(\cdot, \cdot), g(\cdot) \varphi(\cdot))_{L^2(\pi_\sigma) \otimes L^2(\sigma)}. \end{aligned} \quad (17)$$

Whence using Proposition 2.7 we obtain

$$\begin{aligned} ((\nabla_\varphi^{\text{P}})^* g)(\gamma) &= ((\nabla^{\text{P}})^* g \varphi)(\gamma) \\ &= \int_{\mathbb{R}^d} g(\gamma - \varepsilon_x) \varphi(x) d\gamma(x) - g(\gamma) \int_{\mathbb{R}^d} \varphi(x) d\sigma(x) \\ &= (g(\gamma - \varepsilon.), \varphi(\cdot))_{L^2(\gamma)} - g(\gamma) \langle \varphi \rangle_\sigma, \end{aligned}$$

which proves (16).

2. Alternatively we give an independent prove of (16) based on absolute continuity of Poisson measure.

Let $\eta \in \mathcal{D}$ be such that $\eta(x) > -1$, $\forall x \in \mathbb{R}^d$. Denote by σ_η the measure on \mathbb{R}^d having density with respect to σ ,

$$\frac{d\sigma_\eta}{d\sigma}(x) = 1 + \eta(x). \quad (18)$$

Lemma 2.11 *The Poisson measure π_σ and π_{σ_η} on $(\Gamma, \mathcal{B}(\Gamma))$ are mutually absolutely continuous and the Radon-Nikodym derivative $\frac{d\pi_{\sigma_\eta}}{d\pi_\sigma}(\gamma)$ coincides with the normalized exponential, i.e.,*

$$\frac{d\pi_{\sigma_\eta}}{d\pi_\sigma}(\gamma) = e_{\pi_\sigma}^\alpha(\eta; \gamma) = \exp(\langle \gamma, \log(1 + \eta) \rangle - \langle \eta \rangle_\sigma).$$

Proof. Let $\eta \in \mathcal{D}$ be such that $\eta(x) > -1, \forall x \in \mathbb{R}^d$. Then the Laplace transform of π_{σ_η} , given by (3),

$$\begin{aligned} & \int_\Gamma \exp(\langle \gamma, \varphi \rangle) d\pi_{\sigma_\eta}(\gamma) \\ &= \exp\left(\int_{\mathbb{R}^d} (e^{\varphi(x)} - 1)(1 + \eta(x)) d\sigma(x)\right) \\ &= e^{-\langle \eta \rangle_\sigma} \exp\left(\int_{\mathbb{R}^d} (e^{\varphi(x) + \log(1 + \eta(x))} - 1) d\sigma(x)\right) \\ &= \int_\Gamma \exp(\langle \gamma, \varphi \rangle) \underbrace{\exp(\langle \gamma, \log(1 + \eta) \rangle - \langle \eta \rangle_\sigma)}_{d\pi_{\sigma_\eta}(\gamma)} d\pi_\sigma(\gamma). \end{aligned}$$

■

In order to proof (16) it suffices to verify the equality

$$(\nabla_\varphi^P f, g)_{L^2(\pi_\sigma)} = \int_\Gamma f(\gamma) \left[(g(\gamma - \varepsilon \cdot), \varphi(\cdot))_{L^2(\gamma)} - g(\gamma) \langle \varphi \rangle_\sigma \right] d\pi_\sigma(\gamma) \quad (19)$$

for $f(\gamma) = e_{\pi_\sigma}^\alpha(\psi; \gamma)$, $g(\gamma) = e_{\pi_\sigma}^\alpha(\eta; \gamma)$, ψ, η belong to a neighborhood of zero $\mathcal{U} \subset \mathcal{D}$, because the coherent states $\text{Exp}\psi$, $\psi \in \mathcal{U}$ span a common core for the annihilation and creation operators.

Lemma 2.12 *For any $\varphi \in \mathcal{D}$ and for all ψ, η in a neighborhood of zero $\mathcal{U}'_\alpha \subset \mathcal{D}$, the following equality holds*

$$(\nabla_\varphi^P e_{\pi_\sigma}^\alpha(\psi; \cdot), e_{\pi_\sigma}^\alpha(\eta; \cdot))_{L^2(\pi_\sigma)} = (\psi, \varphi)_{L^2(\sigma)} \exp\left(\langle \psi, \eta \rangle_{L^2(\sigma)}\right). \quad (20)$$

Proof. Taking in account (15) we compute the right hand side of (20) to be

$$\begin{aligned} & (\nabla_\varphi^P e_{\pi_\sigma}^\alpha(\psi; \cdot), e_{\pi_\sigma}^\alpha(\eta; \cdot))_{L^2(\pi_\sigma)} \\ &= (\psi, \varphi)_{L^2(\sigma)} \exp(-\langle \psi + \eta \rangle_\sigma) \int_\Gamma \exp(\langle \gamma, \log((1 + \psi)(1 + \eta)) \rangle) d\pi_\sigma(\gamma) \end{aligned}$$

$$\begin{aligned}
&= (\psi, \varphi)_{L^2(\sigma)} \exp(-\langle \psi + \eta \rangle_\sigma) \\
&\quad \cdot \exp\left(\int_{\mathbb{R}^d} (\psi(x) + \eta(x) + \psi(x)\eta(x)) d\sigma(x)\right) \\
&= (\psi, \varphi)_{L^2(\sigma)} \exp\left((\psi, \eta)_{L^2(\sigma)}\right), \tag{21}
\end{aligned}$$

which proves the statement of the lemma. ■

Further, the r.h.s. of (19) can be rewritten as follows

$$\begin{aligned}
&\int_{\Gamma} e_{\pi_\sigma}^\alpha(\psi; \gamma) \left(\sum_{x \in \gamma} \exp(\langle \gamma - \varepsilon_x, \log(1 + \eta) \rangle) - \langle \eta \rangle_\sigma \right) \varphi(x) \\
&\quad - e_{\pi_\sigma}^\alpha(\eta; \gamma) \langle \varphi \rangle_\sigma \Big) d\pi_\sigma(\gamma) \\
&= \int_{\Gamma} e_{\pi_\sigma}^\alpha(\psi; \gamma) e_{\pi_\sigma}^\alpha(\eta; \gamma) \left\langle \frac{\varphi}{1 + \eta} \right\rangle_\gamma d\pi_\sigma(\gamma) \\
&\quad - \langle \varphi \rangle_\sigma \exp\left((\psi, \eta)_{L^2(\sigma)}\right). \tag{22}
\end{aligned}$$

Let us state the following useful lemma.

Lemma 2.13 1. $\langle \psi \rangle_{\sigma_\eta} = \langle \psi \rangle_\sigma + (\psi, \eta)_{L^2(\sigma)}$, $\forall \psi, \eta \in \mathcal{D}$.

2. $e_{\pi_{\sigma_\eta}}^\alpha(\psi; \gamma) = \exp\left(-(\psi, \eta)_{L^2(\sigma)}\right) e_{\pi_\sigma}^\alpha(\psi; \gamma)$, $\forall \psi \in \mathcal{U} \subset \mathcal{D}$.

3. $\left\langle \gamma, \frac{\psi}{1+\eta} \right\rangle = \left\langle C_1^{\sigma_\eta}(\gamma), \frac{\psi}{1+\eta} \right\rangle + \left\langle \frac{\psi}{1+\eta} \right\rangle_{\sigma_\eta}$.

Proof. The non-trivial step is 3. Let us denote for simplicity $\frac{\psi}{1+\eta} =: \xi$

$$\begin{aligned}
\langle C_1^{\sigma_\eta}(\gamma), \xi \rangle &= \left. \frac{d}{dt} e_{\pi_{\sigma_\eta}}^\alpha(t\xi; \gamma) \right|_{t=0} \\
&= \left. \frac{d}{dt} \exp\left(\langle \gamma, \log(1 + t\xi) \rangle - \langle t\xi \rangle_{\sigma_\eta}\right) \right|_{t=0} \\
&= \left. \frac{d}{dt} \sum_{x \in \gamma} \log(1 + t\xi(x)) - \langle t\xi \rangle_{\sigma_\eta} \right|_{t=0} \\
&= \langle \gamma, \xi \rangle - \langle \xi \rangle_{\sigma_\eta}.
\end{aligned}$$

Now the rest of the proof follows from the previous lemma and (22), i.e., ■

$$\begin{aligned}
& \int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(\psi; \gamma) e_{\pi_{\sigma}}^{\alpha}(\eta; \gamma) \left\langle \frac{\varphi}{1 + \eta} \right\rangle_{\gamma} d\pi_{\sigma}(\gamma) - \langle \varphi \rangle_{\sigma} \exp\left((\psi, \eta)_{L^2(\sigma)}\right) \\
&= \exp\left((\psi, \eta)_{L^2(\sigma)}\right) \left(\int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(\psi; \gamma) \left\langle \frac{\varphi}{1 + \eta} \right\rangle_{\gamma} d\pi_{\sigma}(\gamma) - \langle \varphi \rangle_{\sigma} \right) \\
&= \exp\left((\psi, \eta)_{L^2(\sigma)}\right) \left(\int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(\psi; \gamma) \left\langle C_1^{\sigma}(\gamma), \frac{\varphi}{1 + \eta} \right\rangle d\pi_{\sigma}(\gamma) \right. \\
&\quad \left. + \left\langle \frac{\varphi}{1 + \eta} \right\rangle_{\sigma} \int_{\Gamma} e_{\pi_{\sigma}}^{\alpha}(\psi; \gamma) d\pi_{\sigma}(\gamma) - \langle \varphi \rangle_{\sigma} \right) \\
&= \exp\left((\psi, \eta)_{L^2(\sigma)}\right) \left(\psi, \frac{\varphi}{1 + \eta} \right)_{L^2(\sigma)} \\
&= (\psi, \varphi)_{L^2(\sigma)} \exp\left((\psi, \eta)_{L^2(\sigma)}\right),
\end{aligned}$$

which is the same as (21). This completes the proof. ■

Remark 2.14 *The operator $(\nabla_{\varphi}^{\mathbb{P}})^*$ plays the role of creation operator since $a^+(\varphi)^n 1 = \varphi^{\otimes n}$, i.e.,*

$$((\nabla_{\varphi}^{\mathbb{P}})^{*n} 1)(\gamma) = \langle C_n^{\sigma}(\gamma), \varphi^{\otimes n} \rangle. \quad (23)$$

3 Compound Poisson measures

3.1 Definition and properties

This section is devoted to study the compound Poisson measures μ_{CP} on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$. Having in mind the full description of Lévy-Khinchine representation of all possible generalized white noise measures on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$, see e.g. [GV68] and [AW95], we take into account that such a measure is in general the convolution of a Gaussian and non Gaussian measures. We will be interested in the non Gaussian part of this class.

Let ρ be a measure on $\mathbb{R} \setminus \{0\}$ (finite or σ -finite) having all moments finite and satisfying the analyticity property

$$\exists C > 0 : \forall n \in \mathbb{N} \int_{\mathbb{R} \setminus \{0\}} |s|^n d\rho(s) < C^n n! \quad (24)$$

and σ a σ -finite non-atomic measure on \mathbb{R}^d .

We denote

$$\psi_\rho(u) := \int_{\mathbb{R}} (e^{su} - 1) d\rho(s), \quad s \in \mathbb{R}.$$

Definition 3.1 *A measure μ_{CP} on $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$ is called a **compound Poisson measure** with Lévy characteristic ψ_ρ if its Laplace transform is given by, as e.g. [GGV75]*

$$\begin{aligned} l_{\mu_{\text{CP}}}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle \omega, \varphi \rangle) d\mu_{\text{CP}}(\omega) \\ &= \exp\left(\int_{\mathbb{R}^d} \psi_\rho(\varphi(x)) d\sigma(x)\right) \\ &= \exp\left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} (e^{s\varphi(x)} - 1) d\sigma(x) d\rho(s)\right), \quad \varphi \in \mathcal{D}. \end{aligned} \quad (25)$$

Proposition 3.2 *1. μ_{CP} has an analytic Laplace transform.*

2. Let $\rho(\mathbb{R} \setminus \{0\}) < \infty$. Then

$$\mu_{\text{CP}}(\Omega) := \mu_{\text{CP}}\left(\left\{\sum_{x_k \in \gamma} s_k \varepsilon_{x_k} \in \mathcal{D}' \mid s_k \in \text{supp}\rho, \gamma \in \Gamma\right\}\right) = 1.$$

3. Let $\rho(\mathbb{R} \setminus \{0\}) = \infty$. Then

$$\mu_{\text{CP}}(\Omega) := \mu_{\text{CP}} \left(\left\{ \sum_{x_k \in \tilde{\gamma}} s_k \varepsilon_{x_k} \in \mathcal{D}' \mid s_k \in \text{supp} \rho, \tilde{\gamma} \in \tilde{\Gamma} \right\} \right) = 0, \quad (26)$$

where $\tilde{\gamma}$ is locally countable configuration in \mathbb{R}^d and $\tilde{\Gamma}$ stands for the set of all locally countable configurations in \mathbb{R}^d .

Proof. 1. By (24) the Lévy characteristic ψ_ρ is holomorphic on some neighborhood of $0 \in \mathbb{C}$. Then by (25) the Laplace transform $l_{\mu_{\text{CP}}}$ of μ_{CP} is holomorphic in some neighborhood of zero $\mathcal{U} \subset \mathcal{D}_{\mathbb{C}}$.

2, 3. Let $d = 1$. Then μ_{CP} is generated by a compound Poisson process and statements 2, 3 follow immediately from the properties of the paths of this process. Namely, almost every paths ξ_t of compound Poisson process is right continuous step function with the jumps from $\text{supp} \rho$. If ρ is finite measure, then any finite interval contains only finite number of the points of discontinuities of ξ_t ; for infinite measure ρ the set of discontinuities of ξ_t is locally countable, see e.g. [Ta67].

For $d > 1$ the statements 2, 3 follow from the analogous results of the theory of random measures, see e.g. [Ka74], [Ka75] and [KMM78]. \blacksquare

3.2 The isomorphism between Poisson and compound Poisson spaces

Let us define the measure $\hat{\sigma}$ on $(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}))$ as the product of the measures ρ and σ , i.e.,

$$d\hat{\sigma}(\hat{x}) := d\rho(s) d\sigma(x), \quad \hat{x} = (s, x) \in \mathbb{R} \times \mathbb{R}^d.$$

Denote by $\hat{\Gamma}$ the set of the locally finite configurations $\hat{\gamma} \subset \mathbb{R}^{d+1}$ such that

$$\hat{\gamma} = \sum_{\hat{x}_i \in \hat{\gamma}} \varepsilon_{\hat{x}_i}, \quad \hat{x}_i = (s_i, x_i) \in \mathbb{R} \times \mathbb{R}^d, \quad x_i \neq x_j, \quad i \neq j$$

and define the Poisson measure $\pi_{\hat{\sigma}}$ with intensity measure $\hat{\sigma}$ on $(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ via its Laplace transform

$$\begin{aligned} l_{\pi_{\hat{\sigma}}}(\hat{\varphi}) &= \int_{\hat{\Gamma}} \exp(\langle \hat{\gamma}, \hat{\varphi} \rangle) d\pi_{\hat{\sigma}}(\hat{\gamma}) \\ &= \exp \left(\int_{\mathbb{R}^{d+1}} (e^{\hat{\varphi}(\hat{x})} - 1) d\hat{\sigma}(\hat{x}) \right), \quad \hat{\varphi} \in \mathcal{D}(\mathbb{R}^{d+1}). \end{aligned} \quad (27)$$

It follows from (24) that the Laplace transform $l_{\pi_{\hat{\sigma}}}$ is well defined for $\widehat{\varphi}(s, x) = p(s) \varphi(x)$ where $p(s) = \sum_{k=0}^m p_k s^k$ ($p_0 \neq 0$ for finite ρ and $p_0 = 0$ for infinite ρ) is a polynomial and $\varphi \in \mathcal{D}$ (cf. [LRS95]). Let us put $\widehat{\varphi}(s, x) = s\varphi(x)$, $\varphi \in \mathcal{D}$ in (27). Then by (25) we obtain

$$l_{\mu_{\text{CP}}}(\varphi) = l_{\pi_{\hat{\sigma}}}(s\varphi), \quad \varphi \in \mathcal{D}.$$

Then using (26) it follows that the compound Poisson measure μ_{CP} is the image of $\pi_{\hat{\sigma}}$ under the transformation $\Sigma : \widehat{\Gamma} \rightarrow \Sigma\widehat{\Gamma} = \Omega \subset \mathcal{D}'$ given by

$$\widehat{\Gamma} \ni \widehat{\gamma} \mapsto (\Sigma\widehat{\gamma})(\cdot) = \Sigma \left(\sum_{\widehat{x}_i \in \widehat{\gamma}} \varepsilon_{\widehat{x}_i} \right) (\cdot) := \sum_{(s_i, x_i) \in \widehat{\gamma}} s_i \varepsilon_{x_i} (\cdot) \in \Omega \subset \mathcal{D}', \quad (28)$$

i.e., $\forall B \in \mathcal{B}(\mathcal{D}')$

$$\mu_{\text{CP}}(B) = \mu_{\text{CP}}(B \cap \Omega) = \pi_{\hat{\sigma}}(\Sigma^{-1}(B \cap \Omega)),$$

where $\Sigma^{-1}\Delta$ is the pre-image of the set Δ .

The latter equality may be rewritten in the following form

$$\int_{\mathcal{D}'} \mathbb{1}_B(\omega) d\mu_{\text{CP}}(\omega) = \int_{\Omega} \mathbb{1}_B(\omega) d\mu_{\text{CP}}(\omega) = \int_{\widehat{\Gamma}} \mathbb{1}_B(\Sigma\widehat{\gamma}) d\pi_{\hat{\sigma}}(\widehat{\gamma}),$$

which is analogous to the well known change of variable formula for the Lebesgue integral. Namely, for any $h \in L^1(\mathcal{D}', \mu_{\text{CP}}) = L^1(\Omega, \mu_{\text{CP}})$ the function $h \circ \Sigma \in L^1(\widehat{\Gamma}, \pi_{\hat{\sigma}})$ and

$$\int_{\Omega} h(\omega) d\mu_{\text{CP}}(\omega) = \int_{\widehat{\Gamma}} h(\Sigma\widehat{\gamma}) d\pi_{\hat{\sigma}}(\widehat{\gamma}). \quad (29)$$

Remark 3.3 *It is worth noting that there exists on Ω an inverse map $\Sigma^{-1} : \Omega \rightarrow \widehat{\Gamma}$. And we obtain that $\pi_{\hat{\sigma}}$ on $\widehat{\Gamma}$ is the image of μ_{CP} on Ω under the map Σ^{-1} , i.e., $\forall \widehat{C} \in \mathcal{B}(\widehat{\Gamma})$, $\pi_{\hat{\sigma}}(\widehat{C}) = \mu_{\text{CP}}(\Sigma\widehat{C})$ or after rewriting*

$$\int_{\widehat{\Gamma}} \mathbb{1}_{\widehat{C}}(\widehat{\gamma}) d\pi_{\hat{\sigma}}(\widehat{\gamma}) = \int_{\Omega} \mathbb{1}_{\Sigma\widehat{C}}(\omega) d\mu_{\text{CP}}(\omega) = \int_{\Omega} \mathbb{1}_{\widehat{C}}(\Sigma^{-1}\omega) d\mu_{\text{CP}}(\omega).$$

As before we easily can write the corresponding change of variables formula, namely $\forall \widehat{f} \in L^1(\widehat{\Gamma}, \pi_{\hat{\sigma}})$ the function $\widehat{f} \circ \Sigma^{-1} \in L^1(\Omega, \mu_{\text{CP}})$ and

$$\int_{\widehat{\Gamma}} \widehat{f}(\widehat{\gamma}) d\pi_{\hat{\sigma}}(\widehat{\gamma}) = \int_{\Omega} \widehat{f}(\Sigma^{-1}\omega) d\mu_{\text{CP}}(\omega).$$

So we construct a unitary isomorphism U_Σ between the Poisson space $L^2(\pi_{\hat{\sigma}}) = L^2(\widehat{\Gamma}, \pi_{\hat{\sigma}})$ and the compound Poisson space $L^2(\mu_{\mathbb{C}\mathbb{P}}) = L^2(\Omega, \mu_{\mathbb{C}\mathbb{P}})$. Namely,

$$L^2(\Omega, \mu_{\mathbb{C}\mathbb{P}}) \ni h \mapsto U_\Sigma h := h \circ \Sigma \in L^2(\widehat{\Gamma}, \pi_{\hat{\sigma}})$$

and

$$L^2(\widehat{\Gamma}, \pi_{\hat{\sigma}}) \ni \hat{f} \mapsto (U_\Sigma^{-1} \hat{f})(\omega) = \hat{f} \circ \Sigma^{-1} \in L^2(\Omega, \mu_{\mathbb{C}\mathbb{P}}).$$

The isometry of U_Σ and U_Σ^{-1} follows from (29).

As a result we have established the following proposition.

Proposition 3.4 *The map U_Σ is a unitary isomorphism between the Poisson space and the compound Poisson space.*

Remark 3.5 *In the space $L^2(\pi_{\hat{\sigma}})$ we have a basis of generalized Charlier polynomials, annihilation and creation operators etc. Now we can use the unitary isomorphism U_Σ in order to transport the Fock structure from $L^2(\pi_{\hat{\sigma}})$ to $L^2(\mu_{\mathbb{C}\mathbb{P}})$.*

3.3 Annihilation and creation operators on compound Poisson space

Let $\nabla_{\hat{\varphi}}^{\mathbb{P}}, (\nabla_{\hat{\varphi}}^{\mathbb{P}})^*, \hat{\varphi} \in \mathcal{D}(\mathbb{R}^{d+1})$ be the annihilation and creation operators on Poisson space $L^2(\pi_{\hat{\sigma}})$. Their images under U_Σ

$$U_\Sigma^{-1} \nabla_{\hat{\varphi}}^{\mathbb{P}} U_\Sigma, \quad U_\Sigma^{-1} (\nabla_{\hat{\varphi}}^{\mathbb{P}})^* U_\Sigma \quad (30)$$

play the role of annihilation and creation operators in compound Poisson space $L^2(\mu_{\mathbb{C}\mathbb{P}})$. Let us calculate the actions of (30).

The set of smooth cylinder functions $\mathcal{F}C_b^\infty(\mathcal{D}, \Omega)$, (dense in $L^2(\mu_{\mathbb{C}\mathbb{P}})$) consists of all functions of the form

$$\begin{aligned} h(\omega) &= H(\langle \omega, \varphi_1 \rangle, \dots, \langle \omega, \varphi_N \rangle) \\ &= H(\langle \Sigma^{-1} \omega, s\varphi_1 \rangle, \dots, \langle \Sigma^{-1} \omega, s\varphi_N \rangle), \end{aligned} \quad (31)$$

where (generating directions) $\varphi_1, \dots, \varphi_N \in \mathcal{D}$ and H (generating function for h) is from $C_b^\infty(\mathbb{R}^N)$. Whence it follows that

$$\mathcal{F}C_b^\infty(\mathcal{D}, \Omega) = U_\Sigma^{-1} \mathcal{F}C_b^\infty(\mathcal{D}(\mathbb{R}^{d+1}), \widehat{\Gamma}).$$

By (14) for any $\widehat{f} \in \mathcal{FC}_b^\infty(\mathcal{D}(\mathbb{R}^{d+1}), \widehat{\Gamma})$ we have

$$\left(\nabla_{\widehat{\varphi}}^{\mathbb{P}} \widehat{f}\right)(\widehat{\gamma}) = \int_{\mathbb{R}^{d+1}} \left(\widehat{f}(\widehat{\gamma} + \varepsilon_{\widehat{x}}) - \widehat{f}(\widehat{\gamma})\right) \widehat{\varphi}(\widehat{x}) d\widehat{\sigma}(\widehat{x}). \quad (32)$$

Proposition 3.6 *For any $h \in \mathcal{FC}_b^\infty(\mathcal{D}, \Omega)$ the operator $U_\Sigma^{-1} \nabla_{\widehat{\varphi}}^{\mathbb{P}} U_\Sigma$ has the following form*

$$\left(U_\Sigma^{-1} \nabla_{\widehat{\varphi}}^{\mathbb{P}} U_\Sigma h\right)(\omega) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} (h(\omega + s\varepsilon_x) - h(\omega)) \widehat{\varphi}(s, x) d\rho(s) d\sigma(x).$$

Proof. Let $h \in \mathcal{FC}_b^\infty(\mathcal{D}, \Omega)$ be given and denote $U_\Sigma h = h \circ \Sigma =: \widehat{h}$ and $\Sigma^{-1}\omega =: \widehat{\gamma}$. Taking into account (30) and (31) we obtain

$$\begin{aligned} \left(U_\Sigma^{-1} \nabla_{\widehat{\varphi}}^{\mathbb{P}} U_\Sigma h\right)(\omega) &= \left(\nabla_{\widehat{\varphi}}^{\mathbb{P}} \widehat{h}\right)(\widehat{\gamma}) \\ &= \int_{\mathbb{R}^{d+1}} \left(\widehat{h}(\widehat{\gamma} + \varepsilon_{\widehat{x}}) - \widehat{h}(\widehat{\gamma})\right) \widehat{\varphi}(\widehat{x}) d\widehat{\sigma}(\widehat{x}). \end{aligned} \quad (33)$$

Now we use the definition of \widehat{h} , the additivity of the map Σ and the obvious equality $\Sigma\varepsilon_{\widehat{x}} = s\varepsilon_x$ for $\widehat{x} = (s, x)$; with this (33) turns out to be

$$\begin{aligned} &\int_{\mathbb{R}^{d+1}} (h(\Sigma(\widehat{\gamma} + \varepsilon_{\widehat{x}})) - h(\Sigma\widehat{\gamma})) \widehat{\varphi}(\widehat{x}) d\widehat{\sigma}(\widehat{x}) \\ &= \int_{\mathbb{R}^{d+1}} (h(\omega + s\varepsilon_x) - h(\omega)) \widehat{\varphi}(\widehat{x}) d\widehat{\sigma}(\widehat{x}). \end{aligned} \quad (34)$$

The result of the proposition follows then by definition of $\widehat{\sigma}$. ■

Putting $\widehat{\varphi}(\widehat{x}) = \phi(s) \varphi(x)$ in (34) we obtain

$$\begin{aligned} &\left(U_\Sigma^{-1} \nabla_{\phi\varphi}^{\mathbb{P}} U_\Sigma h\right)(\omega) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} (h(\omega + s\varepsilon_x) - h(\omega)) \phi(s) d\rho(s) \right) \varphi(x) d\sigma(x). \end{aligned}$$

Let us note that by (24) we can admit not only bounded functions $\phi(s)$ but also polynomials. For finite ρ and $\phi \equiv 1$ we have the following formula for the **annihilation operator** $\nabla_{\varphi}^{\text{CP}}$ in compound Poisson space $L^2(\Omega, \mu_{\text{CP}})$:

$$\begin{aligned} \left(\nabla_{\varphi}^{\text{CP}} h\right)(\omega) &:= \left(U_\Sigma^{-1} \nabla_{\varphi}^{\mathbb{P}} U_\Sigma h\right)(\omega) \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} (h(\omega + s\varepsilon_x) - h(\omega)) d\rho(s) \right) \varphi(x) d\sigma(x). \end{aligned} \quad (35)$$

Example 3.7 1. Let $\rho = \varepsilon_1$, then $\mu_{\text{CP}} = \pi_{\widehat{\sigma}}$ and, of course, (35) coincides with (14).

2. Let $\rho = \frac{1}{2}(\varepsilon_{-1} + \varepsilon_1)$ (for $d = 1$ μ_{CP} is generated by the so called telegraph process) then the annihilation operator $\nabla_{\varphi}^{\text{CP}}$ has the form

$$(\nabla_{\varphi}^{\text{CP}} h)(\omega) = \int_{\mathbb{R}} \left(\frac{1}{2} h(\omega + \varepsilon_x) + \frac{1}{2} h(\omega - \varepsilon_x) - h(\omega) \right) \varphi(x) d\sigma(x).$$

Example 3.8 Let $h \in \mathcal{FC}_b^{\infty}(\mathcal{D}, \Omega)$ be given by

$$\begin{aligned} h(\omega) &= \exp \left(\langle \omega, \log(1 + \eta) \rangle - \langle \eta \rangle_{\sigma} \int_{\mathbb{R}} s d\rho(s) \right) \\ &= \exp \left(\langle \omega, \log(1 + \eta) \rangle - \langle \eta \rangle_{\sigma} m_1(\rho) \right) \end{aligned}$$

for $\mathcal{D} \ni \eta > -1$. Then the annihilation operator $\nabla_{\varphi}^{\text{CP}}$ applied to h can be computed to be

$$\begin{aligned} (\nabla_{\varphi}^{\text{CP}} h)(\omega) &= \left(\nabla_{\varphi}^{\text{P}} \widehat{h} \right) (\widehat{\gamma}) \\ &= \int_{\mathbb{R}^{d+1}} \left(\widehat{h}(\widehat{\gamma} + \varepsilon_{(s,x)}) - \widehat{h}(\widehat{\gamma}) \right) \varphi(x) d\widehat{\sigma}(s, x) \\ &= \int_{\mathbb{R}^{d+1}} (h(\omega + s\varepsilon_x) - h(\omega)) \varphi(x) d\widehat{\sigma}(s, x) \\ &= h(\omega) \int_{\mathbb{R}^{d+1}} ((1 + \eta(x))^s - 1) \varphi(x) d\widehat{\sigma}(s, x) \\ &= \langle ((1 + \eta) \cdot - 1) \varphi \rangle_{\widehat{\sigma}} h(\omega). \end{aligned}$$

Now we proceed to compute an expression for the creation operator on compound Poisson space.

Proposition 3.9 Let $g \in L^2(\Omega, \mu_{\text{CP}})$ be such that $U_{\Sigma} g \in \text{Dom}(I_{\widehat{\sigma}} a^+(\widehat{\varphi}) I_{\widehat{\sigma}}^{-1})$ and $\widehat{\varphi} \in \mathcal{D}(\mathbb{R}^{d+1})$. Then the operator $U_{\Sigma}^{-1} (\nabla_{\widehat{\varphi}}^{\text{P}})^* U_{\Sigma}$ has the following representation

$$(U_{\Sigma}^{-1} (\nabla_{\widehat{\varphi}}^{\text{P}})^* U_{\Sigma} g)(\omega) = \int_{\mathbb{R}^{d+1}} g(\omega - s\varepsilon_x) \widehat{\varphi}(s, x) d\widehat{\gamma}(s, x) - g(\omega) \langle \widehat{\varphi} \rangle_{\widehat{\sigma}}.$$

Proof. We know from (16) that for any $\widehat{g} \in \text{Dom}(I_{\widehat{\sigma}} a^+(\widehat{\varphi}) I_{\widehat{\sigma}}^{-1})$ the creation operator $(\nabla_{\widehat{\varphi}}^{\text{P}})^*$ on Poisson space $L^2(\widehat{\Gamma}, \pi_{\widehat{\sigma}})$ has the form

$$((\nabla_{\widehat{\varphi}}^{\text{P}})^* \widehat{g})(\widehat{\gamma}) = \int_{\mathbb{R}^{d+1}} \widehat{g}(\widehat{\gamma} - \varepsilon_{\widehat{x}}) \widehat{\varphi}(\widehat{x}) d\widehat{\gamma}(\widehat{x}) - \widehat{g}(\widehat{\gamma}) \langle \widehat{\varphi} \rangle_{\widehat{\sigma}}.$$

On the other hand,

$$\begin{aligned} (U_{\Sigma}^{-1} (\nabla_{\widehat{\varphi}}^{\text{P}})^* U_{\Sigma} g)(\omega) &= ((\nabla_{\widehat{\varphi}}^{\text{P}})^* \widehat{g})(\widehat{\gamma}) \\ &= \int_{\mathbb{R}^{d+1}} \widehat{g}(\widehat{\gamma} - \varepsilon_{\widehat{x}}) \widehat{\varphi}(\widehat{x}) d\widehat{\gamma}(\widehat{x}) - \widehat{g}(\widehat{\gamma}) \langle \widehat{\varphi} \rangle_{\widehat{\sigma}}, \\ &= \int_{\mathbb{R}^{d+1}} g(\omega - s\varepsilon_x) \widehat{\varphi}(s, x) d\widehat{\gamma}(s, x) - g(\omega) \langle \widehat{\varphi} \rangle_{\widehat{\sigma}}, \end{aligned}$$

which proves the result of the proposition. \blacksquare

As before if we choose $\widehat{\varphi} = 1\varphi$, in the case when ρ is finite, then we have the following form for the **creation operator** $(\nabla_{\varphi}^{\text{CP}})^*$ in compound Poisson space $L^2(\Omega, \mu_{\text{CP}})$:

$$\begin{aligned} ((\nabla_{\varphi}^{\text{CP}})^* g)(\omega) &:= (U_{\Sigma}^{-1} (\nabla_{\varphi}^{\text{P}})^* U_{\Sigma} g)(\omega) \\ &= \int_{\mathbb{R}^{d+1}} g(\omega - s\varepsilon_x) \varphi(x) d\widehat{\gamma}(s, x) - g(\omega) \rho(\mathbb{R}) \langle \varphi \rangle_{\sigma}. \end{aligned}$$

Remark 3.10 *The generalized Charlier polynomials in $L^2(\pi_{\widehat{\sigma}})$, according to (23), have the following representation*

$$((\nabla_{\widehat{\varphi}}^{\text{P}})^{*n} 1)(\widehat{\gamma}) = \langle C_n^{\widehat{\sigma}}(\widehat{\gamma}), \widehat{\varphi}^{\otimes n} \rangle.$$

Their images under U_{Σ}^{-1} have the following form

$$\begin{aligned} (U_{\Sigma}^{-1} \langle C_n^{\widehat{\sigma}}(\cdot), \widehat{\varphi}^{\otimes n} \rangle)(\omega) &= \langle C_n^{\widehat{\sigma}}(\Sigma^{-1}\omega), \widehat{\varphi}^{\otimes n} \rangle \\ &= (U_{\Sigma}^{-1} (\nabla_{\widehat{\varphi}}^{\text{P}})^{*n} U_{\Sigma} 1)(\omega). \end{aligned}$$

In particular for finite measure ρ and $\widehat{\varphi} = \varphi$ we obtain

$$((\nabla_{\varphi}^{\text{CP}})^{*n} 1)(\omega) = \langle C_n^{\widehat{\sigma}}(\Sigma^{-1}\omega), \varphi^{\otimes n} \rangle.$$

4 Gamma analysis

4.1 Definition and properties

In this section we consider the classical (real) Schwartz triple

$$\mathcal{D}(\mathbb{R}^d) =: \mathcal{D} \subset L^2(\mathbb{R}^d) \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^d).$$

Definition 4.1 We call **Gamma noise** the measure $\mu_{\mathbb{G}}^{\sigma}$ on the measure space $(\mathcal{D}', \mathcal{B}(\mathcal{D}'))$ determined via its Laplace transform

$$\begin{aligned} l_{\mu_{\mathbb{G}}^{\sigma}}(\varphi) &= \int_{\mathcal{D}'} \exp(\langle \omega, \varphi \rangle) d\mu_{\mathbb{G}}^{\sigma}(\omega) \\ &= \exp(-\langle \log(1 - \varphi) \rangle_{\sigma}), \quad 1 > \varphi \in \mathcal{D}. \end{aligned}$$

Remark 4.2 In order to apply the Minlos' theorem we must verify that $l_{\mu_{\mathbb{G}}^{\sigma}}$ define a positive definite functional on \mathcal{D} . Indeed $\mu_{\mathbb{G}}^{\sigma}$ is a special case of $\mu_{\mathbb{CP}}$ for the choice of ρ as follows

$$\rho(\Delta) = \int_{\Delta \cap]0, \infty[} \frac{e^{-s}}{s} ds, \quad \Delta \in \mathcal{B}(\mathbb{R}). \quad (36)$$

Whence by Minlos' theorem $\mu_{\mathbb{G}}^{\sigma}$ is well-defined and, of course $l_{\mu_{\mathbb{G}}^{\sigma}}$ is an analytic function.

Remark 4.3 Let us explain the term “Gamma noise”. If $d = 1$, then for any $t > 0$ the value of the Laplace transform

$$l_{\mu_{\mathbb{G}}^{\sigma}}(\lambda \mathbb{1}_{[0,t]}) = \exp(-t \log(1 - \lambda)), \quad \lambda < 1$$

coincides with the Laplace transform $l_{\xi(t)}(\lambda)$ of a random variable $\xi(t)$ having Gamma distribution, i.e., the density of $\xi(t)$ has the form

$$p_t(x) = \frac{1}{2} \frac{|x|^{t-1} e^{-|x|}}{\Gamma(t)}, \quad t > 0.$$

The process $\{\xi(t), t > 0; \xi(0) := 0\}$ is known as Gamma process, see e.g. [Ta67, Section 19]. Thus the triple $(\mathcal{D}', \mathcal{B}(\mathcal{D}'), \mu_{\mathbb{G}}^{\sigma})$ is a direct representation of the generalized stochastic process $\{\xi(t), t \geq 0\}$ (detailed information on generalized stochastic process can be found in [GV68]) which is a distributional derivative of the Gamma process $\{\xi(t), t \geq 0\}$. So the term “Gamma noise” is natural for $\mu_{\mathbb{G}}^{\sigma}$.

4.2 Chaos decomposition of Gamma space

Let us now consider a function $\alpha : \mathcal{D} \rightarrow \mathcal{D}$ defined by

$$\alpha(\varphi)(x) = \frac{\varphi(x)}{\varphi(x) - 1}, \quad \varphi \in \mathcal{D}, \quad x \in \mathbb{R}^d.$$

We stress that α is a holomorphic function on a neighborhood of zero $\mathcal{U}_\alpha \subset \mathcal{D}$, in other words $\alpha \in \text{Hol}_0(\mathcal{D}, \mathcal{D})$.

Because of the holomorphy of $l_{\mu_\sigma^\alpha}$ and $l_{\mu_\sigma^\alpha}(0) = 1$, there exists a neighborhood of zero $\mathcal{U}'_\alpha \subset \mathcal{U}_\alpha$ such that the normalized exponential $e_{\mu_\sigma^\alpha}^\alpha(\varphi; \omega)$ is holomorphic for any $\varphi \in \mathcal{U}'_\alpha$ and $\omega \in \mathcal{D}'$. Then

$$\begin{aligned} e_{\mu_\sigma^\alpha}^\alpha(\varphi; \omega) &:= \frac{\exp(\langle \omega, \alpha(\varphi) \rangle)}{l_{\mu_\sigma^\alpha}(\alpha(\varphi))} \\ &= \exp\left(\left\langle \omega, \frac{\varphi}{\varphi - 1} \right\rangle - \langle \log(1 - \varphi) \rangle_\sigma\right), \quad \varphi \in \mathcal{U}'_\alpha. \end{aligned} \quad (37)$$

We use the holomorphy of $\varphi \mapsto e_{\mu_\sigma^\alpha}^\alpha(\varphi; \omega)$ to expand it in a power series which, with Cauchy's inequality, polarization identity and kernel theorem, give us

$$e_{\mu_\sigma^\alpha}^\alpha(\varphi; \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu_\sigma^\alpha, \alpha}(\omega), \varphi^{\otimes n} \rangle, \quad \varphi \in \mathcal{U}'_\alpha, \quad \omega \in \mathcal{D}', \quad (38)$$

where $P_n^{\mu_\sigma^\alpha, \alpha} : \mathcal{D}' \rightarrow \mathcal{D}'^{\widehat{\otimes} n}$. $\{P_n^{\mu_\sigma^\alpha, \alpha}(\cdot) =: L_n^\sigma(\cdot) \mid n \in \mathbb{N}_0\}$ is called the system of **generalized Laguerre kernels** on Gamma space $(\mathcal{D}', \mathcal{B}(\mathcal{D}'), \mu_\sigma^\alpha)$. From (38) it follows immediately that for any $\varphi^{(n)} \in \mathcal{D}'^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$ the function

$$\mathcal{D}' \ni \omega \mapsto \langle L_n^\sigma(\omega), \varphi^{(n)} \rangle$$

is a polynomial of the order n on \mathcal{D}' . The system of functions

$$\left\{ L_n^\sigma(\varphi^{(n)})(\omega) := \langle L_n^\sigma(\omega), \varphi^{(n)} \rangle, \quad \forall \varphi^{(n)} \in \mathcal{D}'^{\widehat{\otimes} n}, \quad n \in \mathbb{N}_0 \right\}$$

is called the system of **generalized Laguerre polynomials**¹ for the Gamma measure μ_σ^α .

¹In one-dimensional case this system coincides with the system of Laguerre polynomials, see e.g. [Boas64].

Now we proceed establishing the following result. Let $\varphi, \psi \in \mathcal{U}'_\alpha$ be given, then using (37) follows that for $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned}
& \int_{\mathcal{D}'} e_{\mu_\sigma^\alpha}^\alpha(\lambda_1\varphi; \omega) e_{\mu_\sigma^\alpha}^\alpha(\lambda_2\psi; \omega) d\mu_\sigma^\alpha(\omega) \\
&= \exp(\langle -\log(1 - \lambda_1\varphi) - \log(1 - \lambda_2\psi) \rangle_\sigma) \\
&\quad \cdot \int_{\mathcal{D}'} \exp\left(\left\langle \omega, \frac{\lambda_1\varphi}{\lambda_1\varphi - 1} + \frac{\lambda_2\psi}{\lambda_2\psi - 1} \right\rangle\right) d\mu_\sigma^\alpha(\omega) \\
&= \exp(\langle -\log(1 - \lambda_1\varphi) - \log(1 - \lambda_2\psi) \rangle_\sigma) \\
&\quad \cdot \exp\left(-\left\langle \log\left(1 - \frac{\lambda_1\varphi}{\lambda_1\varphi - 1} - \frac{\lambda_2\psi}{\lambda_2\psi - 1}\right) \right\rangle_\sigma\right) \\
&= \exp(-\langle \log(1 - \lambda_1\varphi\lambda_2\psi) \rangle_\sigma) \\
&= l_{\mu_\sigma^\alpha}(\lambda_1\lambda_2\varphi\psi). \tag{39}
\end{aligned}$$

Since $l_{\mu_\sigma^\alpha} \in \mathcal{M}_a(\mathcal{D}')$, then (39) turns out to be an analytic function on λ_1 and λ_2 , hence

$$l_{\mu_\sigma^\alpha}(\lambda_1\lambda_2\varphi\psi) = \sum_{n=0}^{\infty} \frac{1}{n!} (\lambda_1\lambda_2)^n (\varphi^{\otimes n}, \psi^{\otimes n})_{\text{Exp}_n^{\mathfrak{G}}L^2(\sigma)}, \tag{40}$$

where the coefficients $(\varphi^{\otimes n}, \psi^{\otimes n})_{\text{Exp}_n^{\mathfrak{G}}L^2(\sigma)}$ are given by

$$(\varphi^{\otimes n}, \psi^{\otimes n})_{\text{Exp}_n^{\mathfrak{G}}L^2(\sigma)} = \left. \frac{d^n}{dt^n} \exp(-\langle \log(1 - t\varphi\psi) \rangle_\sigma) \right|_{t=0}$$

and $\text{Exp}_n^{\mathfrak{G}}L^2(\sigma)$ stands for a quasi- n -particle subspace of $\text{Exp}^{\mathfrak{G}}L^2(\sigma)$ defined by (44) below.

By using the formula, see e.g. [B58] and [GR81],

$$\begin{aligned}
& \frac{d^n}{dt^n} e^{f(t)} \\
&= \sum_{\substack{i_1+2i_2+\dots+ki_k=n \\ i_1, i_2, \dots, i_k \in \mathbb{N}_0}} \frac{n!}{i_1! \dots i_k!} \left(\frac{f^{(1)}(t)}{1!}\right)^{i_1} \left(\frac{f^{(2)}(t)}{2!}\right)^{i_2} \dots \left(\frac{f^{(k)}(t)}{k!}\right)^{i_k} e^{f(t)}
\end{aligned}$$

follows

$$\begin{aligned}
& (\varphi^{\otimes n}, \psi^{\otimes n})_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)} \\
= & \sum_{\substack{i_1+2i_2+\dots+k i_k=n \\ i_1, i_2, \dots, i_k \in \mathbb{N}_0}} \frac{n!}{i_1! i_2! \dots i_k!} \frac{1}{2^{i_2} \dots k^{i_k}} \\
& \cdot \left(\int_{\mathbb{R}^d} \varphi(x) \psi(x) d\sigma(x) \right)^{i_1} \left(\int_{\mathbb{R}^d} \varphi^2(x) \psi^2(x) d\sigma(x) \right)^{i_2} \\
& \dots \left(\int_{\mathbb{R}^d} \varphi^k(x) \psi^k(x) d\sigma(x) \right)^{i_k}. \tag{41}
\end{aligned}$$

On the other hand

$$\begin{aligned}
& \int_{\mathcal{D}'} e_{\mu_{\mathfrak{G}}^{\sigma}}^{\alpha}(\lambda_1 \varphi; \omega) e_{\mu_{\mathfrak{G}}^{\sigma}}^{\alpha}(\lambda_2 \psi; \omega) d\mu_{\mathfrak{G}}^{\sigma}(\omega) \\
= & \sum_{n, m=0}^{\infty} \frac{\lambda_1^n \lambda_2^m}{n! m!} \int_{\mathcal{D}'} \langle L_n^{\sigma}(\omega), \varphi^{\otimes n} \rangle \langle L_m^{\sigma}(\omega), \psi^{\otimes m} \rangle d\mu_{\mathfrak{G}}^{\sigma}(\omega). \tag{42}
\end{aligned}$$

Then a comparison of coefficients between (40) and (42) gives us

$$\int_{\mathcal{D}'} \langle L_n^{\sigma}(\omega), \varphi^{\otimes n} \rangle \langle L_m^{\sigma}(\omega), \psi^{\otimes m} \rangle d\mu_{\mathfrak{G}}^{\sigma}(\omega) = \delta_{nm} n! (\varphi^{\otimes n}, \psi^{\otimes n})_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)},$$

which shows the orthogonality property of the system $\{L_n^{\sigma}(\cdot) | n \in \mathbb{N}_0\}$.

Since $(\cdot, \cdot)_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)}$ is n -linear we can extend it by polarization, linearity and continuity to general smooth kernels $\varphi^{(n)}, \psi^{(n)} \in \text{Exp}_n^{\mathfrak{G}} L^2(\sigma)$. To this end we proceed as follows.

First we consider a partition of the numbers $I_n := \{1, 2, \dots, n\}$ in

$$I_n = \bigcup_{\alpha} I_{\alpha} =: \mathcal{I}^{(n)}.$$

Then for each such partition $\mathcal{I}^{(n)}$, we define i_k by

$$i_k := \# \{I_{\alpha} | |I_{\alpha}| = k\}, \quad 1 \leq k \leq n.$$

Finally we define the contraction of the kernel $\varphi^{(n)}$ w.r.t. $\mathcal{I}^{(n)}$ as

$$\varphi_{\mathcal{I}^{(n)}}^{(n)}(x_1, x_2, \dots, x_n) := \varphi^{(n)}(x_{i_1}, x_{i_2}, \dots, x_{i_k}),$$

where $x_{i_m} = (x_m, x_m, \dots, x_m)$ m -times, $1 \leq m \leq k$.

Hence the inner product is given by

$$\begin{aligned}
& (\varphi^{(n)}, \psi^{(n)})_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)} \\
&= \sum_{\mathcal{I}^{(n)}} n! \left(\prod_{k=1}^n \frac{1}{i_k! k^{i_k}} \right) \left(n! \prod_{k=1}^n \frac{1}{(k!)^{i_k} i_k!} \right)^{-1} \\
&\quad \cdot \int_{\mathbb{R}^{dn}} \varphi_{\mathcal{I}^{(n)}}^{(n)}(x_1, \dots, x_n) \psi_{\mathcal{I}^{(n)}}^{(n)}(x_1, \dots, x_n) d\sigma^{\otimes n}(\vec{x}) \\
&= \sum_{\mathcal{I}^{(n)}} \prod_{k=1}^n ((k-1)!)^{i_k} \int_{\mathbb{R}^{dn}} \varphi_{\mathcal{I}^{(n)}}^{(n)}(x_1, \dots, x_n) \psi_{\mathcal{I}^{(n)}}^{(n)}(x_1, \dots, x_n) d\sigma^{\otimes n}(\vec{x}), \tag{43}
\end{aligned}$$

where the sum extends over all possible partition $\mathcal{I}^{(n)}$ of I_n .

Hence we have established the proposition.

Proposition 4.4 *Let $\varphi, \psi \in \mathcal{D}$ be given. Then the system of generalized Laguerre polynomials verifies the following orthogonality property*

$$\int_{\mathcal{D}'} \langle L_n^\sigma(\omega), \varphi^{(n)} \rangle \langle L_m^\sigma(\omega), \psi^{(m)} \rangle d\mu_{\mathfrak{G}}^\sigma(\omega) = \delta_{nm} n! (\varphi^{(n)}, \psi^{(n)})_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)},$$

where $(\varphi^{(n)}, \psi^{(n)})_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)}$ is defined by (43) above.

As a consequence of the last proposition we have established the following isomorphism

$$I : L^2(\mu_{\mathfrak{G}}^\sigma) \longrightarrow \bigoplus_{n=0}^{\infty} \text{Exp}_n^{\mathfrak{G}} L^2(\sigma) =: \text{Exp}^{\mathfrak{G}} L(\sigma). \tag{44}$$

Therefore for any $F \in L^2(\mu_{\mathfrak{G}}^\sigma)$ there is a sequence $(f^{(n)})_{n=0}^{\infty} \in \text{Exp}^{\mathfrak{G}} L^2(\sigma)$ such that

$$F(\omega) = \sum_{n=0}^{\infty} \langle L_n^\sigma(\omega), f^{(n)} \rangle,$$

moreover

$$\|F\|_{L^2(\mu_{\mathfrak{G}}^\sigma)}^2 = \sum_{n=0}^{\infty} n! |f^{(n)}|_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)}^2.$$

Remark 4.5 *Hence we see that the Gamma noise does not produce the standard Fock type isomorphism since the inner product $(\cdot, \cdot)_{\text{Exp}_n^{\mathfrak{G}} L^2(\sigma)}$ do not coincide with the inner product in the n -particle subspace, $L^2(\mathbb{R}^d)^{\widehat{\otimes} n}$.*

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