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ANALYSIS ON THEORETICAL BOUNDS FOR APPROXIMATING DOMINATING SET PROBLEMS*

XIAOFENG GAO

Department of Computer Science
University of Texas at Dallas, 800 West Campbell Road
Richardson, Texas 75080, USA
xiaofeng.gao@student.utdallas.edu

YUEXUAN WANG

Institute for Theoretical Computer Science
Tsinghua University, Beijing, 100084, P. R. China
wangyuexuan@tsinghua.edu.cn

XIANYUE LI†

School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu, 730000, P. R. China
lixianyue@lzu.edu.cn

WEILI WU

Department of Computer Science
University of Texas at Dallas, 800 West Campbell Road
Richardson, Texas 75080, USA
weiliwu@utdallas.edu

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Connected Dominating Set is widely used as virtual backbone in wireless networks to improve network performance and optimize routing protocols. Based on special characteristics of ad-hoc and sensor networks, we usually use *unit disk graph* to represent the corresponding geometrical structures, where each node has a unit transmission range and two nodes are said to be adjacent if the distance between them is less than 1. Since every Maximal Independent Set (MIS) is a dominating set and it is easy to construct, we can firstly find an MIS and then connect it into a Connected Dominating Set (CDS). Therefore, the ratio to compare the size of an MIS with a minimum CDS becomes a theoretical upper bound for approximation algorithms to compute CDS. In our paper,

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†This work was done while this author visited at University of Texas at Dallas.

with the help of Voronoi diagram and Euler's formula, we improved this upper bound, so that improved the approximations based on this relation.

Keywords: Connected dominating set; maximal independent set; unit disk graph.

Mathematics Subject Classification 2000: 68W25

1. Introduction

Wireless ad-hoc and sensor network can be widely used in many civilian application areas, including healthcare applications, environment and habitat monitoring, home automation, and traffic control [10, 6]. Due to the special characteristics of such networks, we usually use *Unit Disk Graph* (UDG) to represent their geometrical structures (assuming that each wireless node has the same transmission range). A UDG can be formally defined as follows: Given an undirected graph $G = (V, E)$, each vertex v has a transmission range with radius 1. An edge $(v_1, v_2) \in E$ means the distance between vertex v_1 and v_2 is less than or equal to 1, say, $dist(v_1, v_2) \leq 1$.

Compared with traditional computer networks, wireless ad-hoc and sensor networks have no fixed or pre-defined infrastructure as hierarchical structure, resulting the difficulty to achieve scalability and efficiency [2]. To better improve the performance and increase efficiency of routing protocols, a *Connected Dominating Set* (CDS) is selected to form a virtual network backbone. The formal definition of CDS can be shown as follows: Given a graph $G = (V, E)$, a *Dominating Set* (DS) is a subset $C \subseteq V$ such that for every vertex $v \in V$, either $v \in C$, or there exist an edge $(u, v) \in E$ and $u \in C$. If the graph induced from C ($G[C]$) is connected, then C is called a *Connected Dominating Set* (CDS). Since CDS plays a very important role in routing, broadcasting and connectivity management in wireless ad-hoc and sensor networks, it is desirable to find a minimum CDS (MCDS) of a given set of nodes.

Clark *et al.* [3] proved that computing MCDS is NP-hard in UDG, and a lot of approximation algorithms for MCDS can be found in literatures [8,7,1,5]. It is well known that in graph theory, a Maximal Independent Set (MIS) is also a Dominating Set (DS). MIS can be defined formally as follows: Given a graph $G = (V, E)$, an Independent Set (IS) is a subset $I \in V$ such that for any two vertex $v_1, v_2 \in I$, they are not adjacent, say, $(v_1, v_2) \notin E$. An IS is called a Maximal Independent Set (MIS) if we add one more arbitrary vertex to this set, the new set will not be an IS any more. Compared with CDS, MIS is much easier to be constructed. Therefore, people usually construct the approximation for CDS with two steps. The first step is to find a MIS, and the second step is to make this MIS connected. As a result, the performance of these approximations highly depends on the relationship between the size of MIS ($mis(G)$) and the size of minimum CDS ($mcds(G)$) in graph G. Such a relation, say, $\frac{mis(G)}{mcds(G)}$ is also called the theoretical bound to approximate CDS.

In our paper, we will give a better theoretical bound to approximate CDS, which is $mis(G) \leq 3.399 \cdot mcds(G) + 4.874$, if there are no holes in the area constructed by the MCDS. The rest of this paper is organized as follows. In Sec. 2 we introduce the preliminaries and relation between $mis(G)$ and $cds(G)$, including related works. In Sec. 3 with the help of Voronoi division, we divide the plane into several convex polygons and calculate the area for each polygon under different situations. In Sec. 4 we use Euler's formula to calculate a better bound for $\frac{mis(G)}{mcds(G)}$, and finally Sec. 5 gives the conclusion and future works.

2. Preliminary and Related Works

As mentioned in Sec. 1, we use two steps to approximate a CDS in graph G. The first step is to select a MIS and the second step is to connect this MIS. Let $mis(G)$ be the size of selected MIS, $connect(G)$ be the size of disks that are used to connect this MIS, and $mcds(G)$ be the size of minimum CDS. Then, the approximation ratio for such algorithm is

$$\frac{mis(G) + connect(G)}{mcds(G)} = \frac{mis(G)}{mcds(G)} + \frac{connect(G)}{mcds(G)}.$$

For the connecting part, Min *et al.* [9] developed a steiner tree based algorithm to connect a MIS, with $\frac{connect(G)}{mcds(G)} \leq 3$, which becomes the best result to connect a MIS. On the other hand, for selecting MIS part, Wan *et al.* [12] constructed a distributed algorithm which can select a MIS in graph G with size $mis(G) \leq 4 \cdot mcds(G)+1$. Later, Wu and her cooperators [13] improved this result into $mis(G) \leq 3.8 \cdot mcds(G) + 1.2$. Funke *et al.* [4] discussed the relation between $mis(G)$ and $mcds(G)$ and gave a theorem saying that $mis(G) \leq 3.453 \cdot mcds(G) + 8.291$, but the proof lacks evidences. In this paper we give a better bound for $mis(G)$ and $mcds(G)$, with a detailed analysis for the approximation ratio.

Actually, $mis(G)$ and $mcds(G)$ have a really close relationship. Given an UDG $G = (V, E)$, let M be the set of disks forming MCDS. If we increase the radius of disks in M from 1 to 1.5, and decrease the radius of the rest disks in $V \backslash M$ from 1 to 0.5, then we can construct a new graph G'. It is easy to know that all the disks in V are located insides the area formed by M. (For disks in M, obviously they are located insides themselves, and for disks in $V \backslash M$, e.g. v_1, since M is a MCDS, there exists a disk $v_2 \in M$ dominating v_2. Therefore $dist(v_1, v_2) \leq 1$. Besides, the radius of v_1 is 0.5, while the radius of v_2 is 1.5, so v_1 must locate inside v_2's disk.) If we select a MIS for G, then based on the definition of UDG, the distance between any two disks from MIS should be greater than 1. And since the radius of disks in $V \backslash M$ for G' is 0.5, any of two disks from MIS will not intersect each other. (To simplify the conception, we can consider the radius of the disks in both MIS and M as 0.5.) Then we can get the conclusion that the sum of maximum area for MIS should be less than the area of MCDS, which is a rough bound for $\frac{mis(G)}{mcds(G)}$. The following theorem gives this bound.

Theorem 2.1. *The rough bound for* $mis(G)$ *and* $mcds(G)$ *is* $mis(G) \leq 3.748 \cdot mcds(G) + 5.252$.

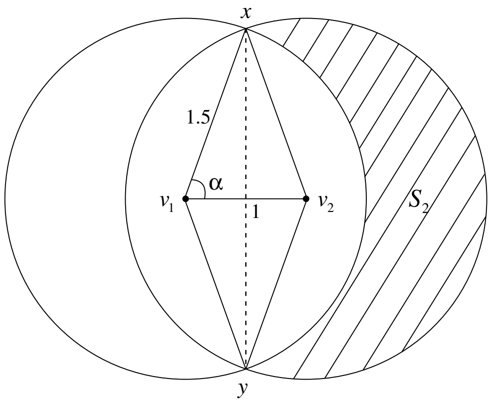

Fig. 1. Two disks in MCDS.

Proof. Consider two disks v_1, v_2 in MCDS set M. Both of them have radius 1.5, and $\max(dist(v_1, v_2)) = 1$. If we set v_1 and then add v_2, then the newly covered area will be at most S_2, just shown as the shadow in Fig. 1.

Let $area(xv_1y)$ be the area of sector xv_1y, and $area(\triangle xv_1y)$ be the area of triangle xv_1y. Besides, $\cos\alpha = \frac{1}{3}$. Then, the area of S_2 should be:

$$\begin{aligned} area(S_2) &= \pi \cdot 1.5^2 - 2 \cdot (area(xv_1y) - area(\triangle xv_1y)) \\ &= 2.25\pi - 2\left(\arccos\frac{1}{3} \cdot 1.5^2 - \frac{1}{2} \cdot \frac{1}{2} \cdot 2\sqrt{2}\right) \\ &\approx 2.25\pi - 4.1251. \end{aligned}$$

If we mimic the growth of a spanning tree for MCDS, then the maximum number of MIS should less than the total areas induced from M divide the area for a small disk with radius 0.5. Consequently, we can get the following inequations.

$$\begin{aligned} mis(G) &\leq \frac{\pi \cdot 1.5^2 + (mcds(G) - 1) \cdot S_2}{\pi \cdot 0.5^2} \\ &= \frac{4 \cdot S_2}{\pi} \cdot mcds(G) + \frac{4 \cdot 4.1251}{\pi} \\ &\approx 3.748 \cdot mcds(G) + 5.252. \end{aligned}$$

Thus we proved the theorem. □

3. Voronoi Division

Based on Theorem 2.1 we get an upper bound for $\frac{mis(G)}{mcds(G)}$. Now let us analyze the relationship between $mis(G)$ and $mcds(G)$ more specifically. Before our discussion, we firstly introduce the definition of Voronoi Division, which can be referred from [11].

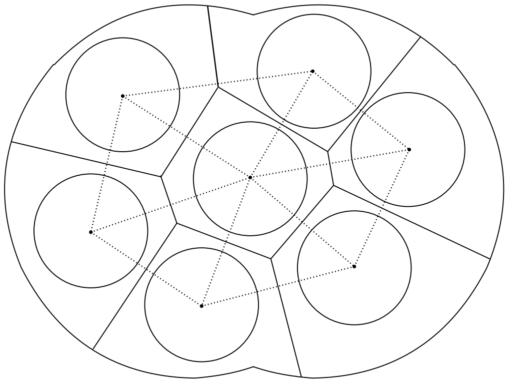

Fig. 2. Example for Voronoi diagram.

Definition 3.1. Let S be a set of n sites in Euclidean space. For each site p_i of S, the Voronoi cell $V(p_i)$ of p_i is the set of points that are closer to p_i than to other sites of S, say,

$$V(p_i) = \bigcap_{1 \le j \le n,\ j \ne i} \{p : |p - p_i| \le |p - p_j|\}.$$

The Voronoi diagram $V(S)$ is the space partition induced by Voronoi cells.

Similarly, for graph G', let S be the set of selected MIS, then for each disk $w_i \in S$, we can find the corresponding Voronoi cell (the outer boundary is the boundary for MCDS). Figure 2 gives an example with $mcds(G') = 2$ and $mis(G') = 7$. It is easy to know that each non-boundary Voronoi cell is a convex polygon, and the area is greater than a disk with radius 0.5. Next let us analyze the area for each kind of polygons under densest situations. For these boundary Voronoi cells, we also consider them as a special kind of polygons with one arc edge.

3.1. *Triangle*

Assume that we have a Voronoi cell C_i as a triangle including disk w_i. Then the area of C_i is smaller if w_i is its inscribed circle. Besides, among those triangles, the area of equilateral triangle is the smallest. The following lemma gives proof for this conclusion.

Lemma 3.2. *The equilateral triangle has the smallest area among other triangles with w_i as its inscribed circle.*

Proof. Let a, b, c be the lengths of three edges for triangle C_i, w_i be its inscribed circle, and $r = 0.5$ be the radius of this circle. Then based on Heron's formula, we have

$$area(C_i) = \frac{1}{2}(a + b + c) \cdot r = s \cdot r = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter. Since r is fixed, the smallest area comes when s is smallest. Therefore we have the following model.

$$\begin{cases} \min\ s = \dfrac{1}{2}(a+b+c) \\ s.t.\ \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}} = r = \dfrac{1}{2}. \end{cases} \tag{3.1}$$

Based on Lagrange's formula, let

$$F(a,b,c) = (a+b+c) - \lambda\left(\sqrt{\frac{(b+c-a)(a+c-b)(a+b-c)}{a+b+c}} - 1\right),$$

then (3.1) can be changed into $\min F(a,b,c)$, and the extreme value comes out when the following partial derivative holds:

$$\begin{cases} \partial F/\partial a = 0 \\ \partial F/\partial b = 0 \\ \partial F/\partial c = 0 \\ \partial F/\partial \lambda = 0. \end{cases} \tag{3.2}$$

Then we get that when $a = b = c = f(\lambda, s)$, (3.2) holds. Therefore the equilateral triangle has the smallest area. Let P_3 denote such kind of triangle, just shown in Fig. 3(a). □

Similarly, if C_i is a boundary cell, then the one with smallest area should be an equilateral triangle with one side cut by an arc from disks in MCDS at one of its tangency point. An example can be seen from Fig. 3(b). Let E_3 denote such pseudo triangle. It is easy to know that $area(P_3) = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \approx 1.299$. To compute the area of E_3, we will use integral. According to Fig. 4, $area(E_3) = area(P_3) - 2 \cdot S_3$, where S_3 is the shadow formed by the boundary arc and two edges of P_3. Therefore,

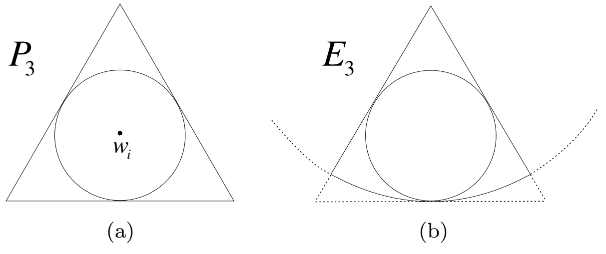

Fig. 3. Example for triangle cells.

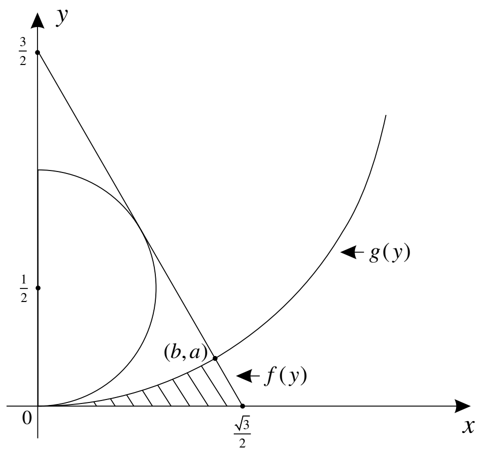

Fig. 4. Compute area for E_3.

we have that

$$\begin{aligned} S_3 &= f(y) - g(y) \\ &= \int_0^a \left\{ \left(\frac{y}{\tan\frac{2\pi}{3}} + \frac{1}{2}\tan\frac{\pi}{3} \right) - \sqrt{\frac{9}{4} + \left(y - \frac{3}{2} \right)^2} \right\} dy \\ &\approx 0.0605, \end{aligned}$$

where $f(y)$ is the function for intersecting edge of triangle and $g(y)$ is the function for the arc of ICMS. As a consequence, $area(E_3) = 1.1781$.

3.2. *Quadrangle, pentagon and hexagon*

If a non-boundary Voronoi cell C_i has four edges, then using similar conclusion, we can get that a square with w_i as its inscribed circle has the smallest area. Let P_4 be such kind of polygon, just shown as Fig. 5(a). If C_i is a boundary Voronoi cell, then under two conditions C_i will have the minimum area. The first condition is when boundary arc cuts off one angle of P_3, just shown as Fig. 5(b), we name it as A_4; and the second condition is when boundary arc cuts off one edge of P_4, shown as

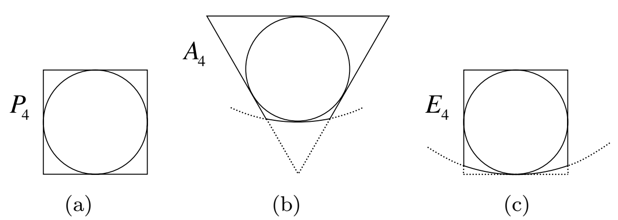

Fig. 5. Example for quadrangle cells.

Fig. 5(c), we name it as E_4. Using similar approach as triangles, we can calculate the area for these quadrangles, and give the result that

$$area(P_4) = 1, \quad area(A_4) \geq 1.1357, \quad area(E_4) = 0.9717.$$

Repeat the above step for C_i as Pentagon and Hexagon, we can have the following conclusion:

$$area(P_5) = 0.9082, \quad area(A_5) \geq 0.9499, \quad area(E_5) = 0.8968,$$
$$area(P_6) = 0.8661, \quad area(A_6) \geq 0.8855, \quad area(E_6) = 0.8546.$$

Figure 6 shows examples for pentagons and hexagons. After our calculation, we can get the conclusion that $area(A_i) \geq area(E_i)$ for $i \geq 3$. Therefore, in the next section, we will use E_i as the smallest boundary Voronoi Cell as i pseudo polygon.

3.3. *Heptagon and others*

For a non-boundary Voronoi cell C_i, if C_i is a heptagon or n-polygon, $n \geq 7$, we will have the following lemma.

Lemma 3.3. *The area of non-boundary n-polygon C_i $(n \geq 7)$ is greater then $area(P6)$.*

Proof. Firstly, it is easy to know that C_i with 6 adjacent neighbors is the densest situation if any two small disks does not intersect each other, just shown in Fig. 7(a). Next, if C_i has 7 or more neighbors, then there must exist at least one disk w_j which does not touch w_i (w_i is the inner disk for C_i). Hence, the edge for C_i created by w_i and w_j is not the tangent line for w_i. As a consequence, the area covered by C_i is greater than $area(P_6)$. An example of P_7 can be shown in Fig. 7(b). If $n > 7$, then the area of C_i will be bigger. Therefore, any Voronoi cell whose edges are more than 6 will have bigger area then P_6. □

However, for boundary Voronoi heptagon C_i, when boundary arc cuts off one angle of P_6, the area will become minimum. Such pseudo heptagon is A_7 (see Fig. 8).

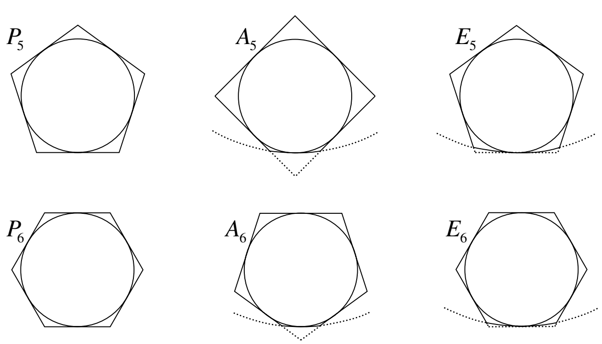

Fig. 6. Examples for pentagon and hexagon cells.

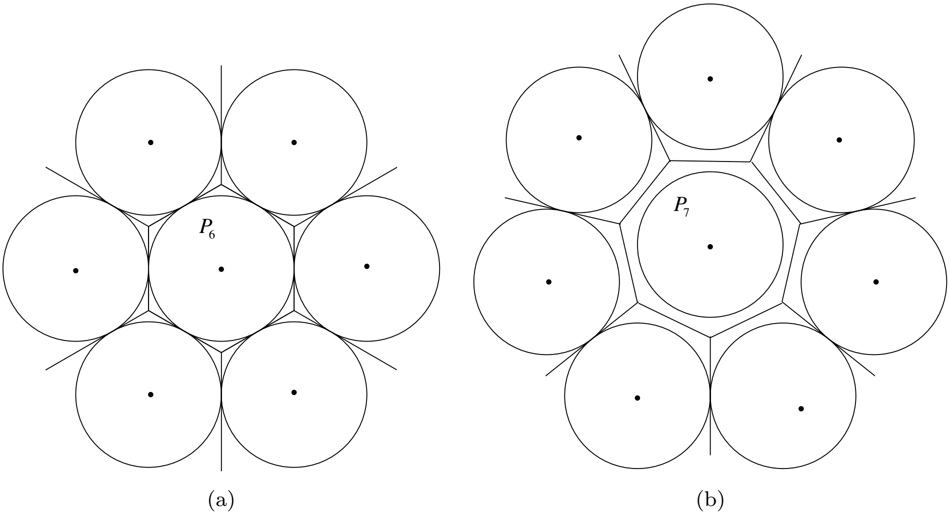

Fig. 7. Compare P_6 and P_7.

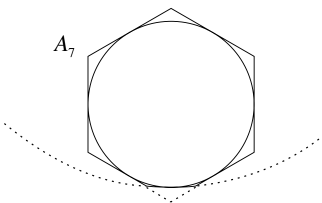

Fig. 8. Example for heptagon cells.

After calculation, we have that $area(A_7) = 0.8525$. Similar as Lemma 3.3, the boundary n-polygon C_i will have bigger area than $area(A_7)$ if $n > 7$.

3.4. *Updated upper bound*

As mentioned above, A_7 is the smallest type of Voronoi cells. Then we can have a better bound for $\frac{mis(G)}{mcds(G)}$.

Theorem 3.4. $mis(G) \leq 3.453 \cdot mcds(G) + 4.839$.

Proof. Similarly as proof for Lemma 3.2, we have

$$\begin{aligned} mis(G) &\leq \frac{\pi \cdot 1.5^2 + (mcds(G) - 1) \cdot S_2}{area(A_7)} \\ &= \frac{S_2}{0.8525} \cdot mcds(G) + \frac{4.1251}{0.8525} \\ &\approx 3.453 \cdot mcds(G) + 4.839, \end{aligned}$$

which is almost the same as [4]. □

4. Computing New Upper Bound

In this section, we will compute a better upper bound for $\frac{mis(G)}{mcds(G)}$ using Voronoi division and Euler's formula. Firstly, we give some notations. Let s_i be the minimum area of the non-boundary cell(i-polygon cell) and s'_i that of the boundary cell. From Sec. 3, we have that

$$s_3 \geq s_4 \geq s_5 \geq s_6 \leq s_7 \leq s_8 \dots \quad \text{and} \quad s'_3 \geq s'_4 \geq s'_5 \geq s'_6 \geq s'_7 \leq s'_8 \leq s'_9 \dots.$$

For convenience, we set $s_i = s_6$ when $i \geq 7$ and $s'_i = s'_7$ when $i \geq 8$. Hence, we get the following equations.

$$s_3 = 1.299, \quad s_4 = 1, \quad s_5 = 0,9082, \quad s_6 = s_7 = \cdots = 0.8661. \tag{4.1}$$

$$\begin{gathered} s'_3 = 1.1781, \quad s'_4 = 0.9717, \quad s'_5 = 0,8968, \quad s'_6 = 0.8546, \\ s'_7 = s'_8 = \cdots = 0.8525. \end{gathered} \tag{4.2}$$

4.1. *3-regularization*

To simplify our calculation, in the subsection we will modify the Voronoi division such that any vertex of v in Voronoi division has degree exactly 3. For every vertex v, it is easy to see that $d(v) \geq 3$. For any vertex v whose $d(v) = d > 3$, let $u_0, u_1, \dots, u_{d-1}$ be its neighbors in clockwise ordering. Replace this vertex with $d-2$ new vertices $v_1, \dots, v_{d-2}$ such that the distance between any v_i and v_j is not more than ε. Then, connect every u_i and v_i and add two edges u_0v_1 and $u_{d-1}v_{d-2}$. Figure 9 gives an illustration when $d(v) = 5$.

After the regularization, we can see that every vertex in Voronoi division has degree of exactly 3. Furthermore, if we choose ε sufficiently small, the area of every Voronoi cell will almost remain the same and the number of edges of new Voronoi cell is no less than that of original Voronoi cell. Hence, Eqs. (4.1) and (4.2) are also held.

4.2. *Euler's formula*

Let ∂f_{out} be the outer boundary of the area constructed by the MCDS. It is trivial that the inside part of ∂f_{out} together with ∂f_{out} form graph G'. Note that there

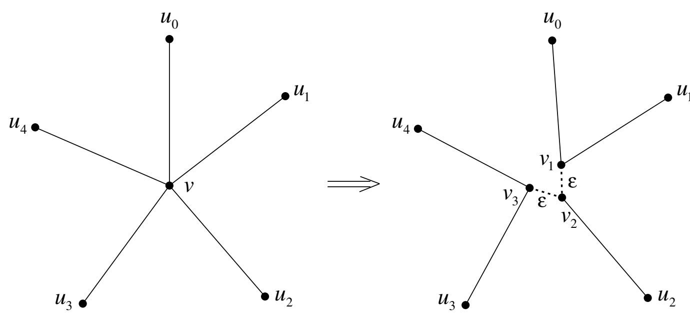

Fig. 9. Regularization when $d(v) = 5$.

may exist some holes in G', where each hole means a connected area inside the ∂f_{out}, but not within the area constructed by the MCDS. In this subsection, we firstly suppose there are no holes in G', which means that the wireless transmission range will cover the plane we discuss. Let f_i and f'_i be the number of non-boundary and boundary Voronoi cells with exactly i edges, respectively. Then using Euler's formula, we have $\sum_i (f_i + f'_i) + 1 - m + n = 2$. Since G' is a cubic graph, $2m = 3n$. Hence,

$$\sum_i (f_i + f'_i) + 1 - \frac{1}{2}n = 2. \tag{4.3}$$

Let $|\partial f_{out}|$ be the number of edges in the outer face. Since every edge is exactly in two faces,

$$\sum_i (i(f_i + f'_i)) + |\partial f_{out}| = 2m = 3n. \tag{4.4}$$

For any boundary Voronoi cell, it must have at least one edge belonging to the outer face. Hence,

$$\sum_i f'_i \le |\partial f_{out}|. \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\sum_i i f_i + \sum_i (i+1) f'_i - 3n \le 0. \tag{4.6}$$

Then we combine Euler's formula and (4.6) together. Let $-1 \times (4.6) + 6 \times (4.3)$, we have

$$3f_3 + 2f'_3 + 2f_4 + f'_4 + f_5 - f'_6 - f_7 - 2f'_7 - \cdots \ge 6. \tag{4.7}$$

Since all Voronoi cells are contained in the area constructed by the MCDS, consider this area and combining (4.1) and (4.2), we have

$$\begin{aligned}\sum_i (s_i f_i + s'_i f'_i) &= 1.299 f_3 + 1.178 f'_3 + f_4 + 0,972 f'_4 + 0.9082 f_5 + 0.8968 f'_5 \\ &\quad + 0.866(f_6 + f_7 + \cdots) + 0.8546 f'_6 + 0.8525(f'_7 + f'_8 + \cdots) \\ &\le 2.9435 \cdot mcds(G) + 4.1251. \end{aligned} \tag{4.8}$$

Then, $-0.0114 \times (4.7) + (4.8)$, we obtain

$$\begin{aligned} & 1.2648 f_3 + 1.1402 f'_3 + 0.9672 f_4 + 0.9492 f'_4 + 0.8853 f_5 + 0.8968 f'_5 \\ & \quad + 0.866 f_6 + 0.8974 f_7 + \cdots + 0.866 f'_6 + 0.8753 f'_7 + \cdots \\ & \le 2.9435 \cdot mcds(G) + 4.2205. \end{aligned} \tag{4.9}$$

From (4.9), since $mis(G) = \sum_i (f_i + f'_i)$, we have

$$0.866 \cdot mis(G) = 0.866 \sum_i (f_i + f'_i) \le 2.9435 \cdot mcds(G) + 4.2205.$$

Hence, $mis(G) \le 3.399 \cdot mcds(G) + 4.874$. Consequently, we have the following theorem.

Theorem 4.1. *For any unit disk graph* G, *let* $mis(G)$ *and* $mcds(G)$ *be the number of disks in any maximal independent set and minimum connected dominating set, respectively. If there are no holes in the area constructed by the MCDS, then* $mis(G) \leq 3.399 \cdot mcds(G) + 4.874$.

4.3. *Discussion with holes*

Actually, in the real world there may exist some place where the wireless signal cannot reach, and some holes in the area constructed by the MCDS. Therefore, in this subsection we will discuss G' with holes in the following. Let k be the number of the holes in G' and $|\partial f_{hole}|$ be the number of edges in all holes. Equations (4.3) and (4.4) alter as

$$\sum_i (f_i + f_i') + 1 + k - \frac{1}{2}n = 2.$$

$$\sum_i (i(f_i + f_i')) + |\partial f_{out}| + |\partial f_{hole}| = 2m = 3n.$$

For any boundary Voronoi cell, it must have at least one edge belonging to the outer face or one hole. Hence,

$$\sum_i f_i' \leq |\partial f_{out}| + |\partial f_{hole}|.$$

Calculate them by the same strategy as Sec. 4.2, we can obtain that

$$\begin{aligned}
&1.2648 f_3 + 1.1402 f_3' + 0.9672 f_4 + 0.9492 f_4' + 0.8853 f_5 + 0.8968 f_5' \\
&\quad + 0.886 f_6 + 0.8974 f_7 + \cdots + 0.866 f_6' + 0.8753 f_7' + \cdots \\
&\quad \leq 2.9435 \cdot mcds(G) + 0.0684k + 4.2205. \qquad (4.10)
\end{aligned}$$

Then we have,

$$mis(G) \leq 3.399 \cdot mcds(G) + 0.0790k + 4.874.$$

It is easy to see that $k \leq mcds(G)$. Next we can obtain the following theorem.

Theorem 4.2. *For any unit disk graph* G, *let* $mis(G)$ *and* $mcds(G)$ *be the number of disks in any maximal independent set and minimum connected dominating set, respectively. Then* $mis(G) \leq 3.478 \cdot mcds(G) + 4.874$.

Besides, after analyzing the relation between disks in MCDS and based on the characteristics for CDS, we can have the following lemma.

Lemma 4.3. *For any unit disk graph* G, *let MCDS be a minimum connected dominating set. To form a hole, there need at least 6 connect vertices in MCDS. Figure* 10 *is an example for a hole.*

Proof. Let h be a point in a hole and $m_1, \ldots, m_t$ be the vertices in MCDS which can form the hole including h and can induce a connect graph. By the definition

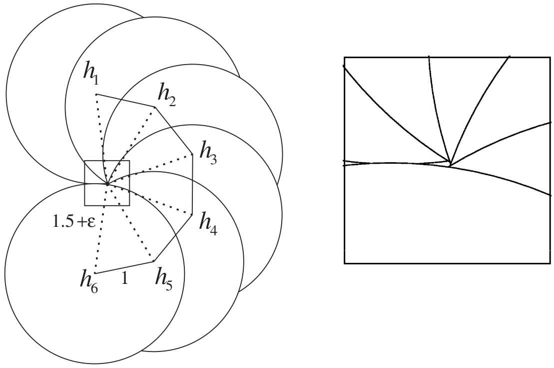

Fig. 10. Example for a hole.

of a hole, h can not be covered by any disk from MCDS with radius 1.5. Hence, choosing h as the center and draw a disk D with radius 1.5, any vertex m_i will lie outside this disk D. It is easy to see that if we form a hole with minimum number of vertices, the graph induced by $m_1, \ldots, m_t$ is a path and m_i is sufficiently close to disk D. Let hm_i intersect disk D at h_i. Then the radians of the central angle $\angle h_i h h_{i+1}$ should be

$$\angle h_i h h_{i+1} \leq 2 \arcsin \frac{1/2 h_i h_{i+1}}{h h_i} = 2 \arcsin \frac{1}{3}.$$

Furthermore, since $m_1, \ldots, m_t$ form a hole, the distance between m_1 and m_t is less than 3. Hence, the central angle $\angle h_1 h h_t$ is more than π and $t \geq \lceil \frac{\pi}{2 \arcsin \frac{1}{3}} \rceil + 1 = 6$. □

5. Conclusion

In this paper, we presented a better upper bound to compare MIS and MCDS in a given UDG G with the help of Voronoi Division and Euler's Formula. If the area covered by MCDS has no holes, then the best upper bound for MIS and MCDS should be $mis(G) \leq 3.399 \cdot mcds(G) + 4.874$. If there exist some uncovered holes, then the bound will become $mis(G) \leq 3.478 \cdot mcds(G) + 4.874$ by Euler's formula, and $mis(G) \leq 3.453 \cdot mcds(G) + 4.839$ by the comparison of area for MCDS and area for smallest Voronoi Cell. Actually, based on the discussion for Lemma 4.3, we guess that the relation between k and $mcds(G)$ can be $k \leq \frac{1}{3} mcds(G)$, and so comes the result that $mis(G) \leq 3.425 \cdot mcds(G) + 4.839$. The detailed proof becomes a future work which needs thorough discussion.

References

[1] B. S. Baker, Approximation algorithms for NP-complete problems on planar graphs, *J. ACM* **41** (1994) 153–180.

[2] J. Blum, M. Ding, A. Thaeler and X. Z. Cheng, Connected dominating set in sensor networks and MANETs, in *Handbook of Combinatorial Optimization* (2004), pp. 329–369.
[3] B. N. Clark, C. J. Colbourn and D. S. Johnson, Unit disk graphs, *Discrete Math.* **86** (1990) 165–177.
[4] S. Funke, A. Kesselman and U. Meyer, A simple improved distributed algorithm for minimum CDS in unit disk graphs, *ACM Trans. Sensor Net.* **2** (2006) 444–453.
[5] D. S. Hochbaum and W. Maass, Approximation schemes for covering and packing problems in image processing and VLSI, *J. ACM* **32** (1985) 130–136.
[6] G. T. Huang, Casting the wireless sensor net, *Technol. Rev.* (2003), pp. 50–56.
[7] H. B. Hunt III, M. V. Marathe, V. Radhakrishnan, S. S. Ravi, D. J. Rosenkrantz and R. E. Stearns, NC-approximation schemes for NP- and PSPACE-hard problems for geometric graphs, *J. Algorithms* **26** (1998) 238–274.
[8] M. V. Marathe, H. Breu, H. B. Hunt, III, S. S. Ravi and D. J. Rosenkrantz, Simple heuristics for unit disk graphs, *Networks* **25** (1995) 59–68.
[9] M. Min, H. W. Du, X. H. Jia, C. X. Huang, S. Ch. Huang and W. L. Wu, Improving construction for connected dominating set with Steiner tree in wireless sensor networks, *J. Global Optim.* **35** (2006) 111–119.
[10] H. Salem and N. Mohamed, Middleware challenges and approaches for wireless sensor networks, *IEEE Distrib. Syst. Online* **7** (2006), art. no. 0603-o3001.
[11] G. M. Voronoi, Nouvelles applications des paramètres continus à la théorie des formes quadratiques. deuxième Mémoire: Recherches sur les parallélloèdres primitifs, *J. Reine Angew. Math.* **134** (1908) 198–287.
[12] P. J. Wan, K. M. Alzoubi and O. Frieder, Distributed construction of connected dominating set in wireless ad hoc networks, in *Proc. 3rd ACM Int. Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications* (1999), pp. 7–14.
[13] W. L. Wu, H. W. Du, X. H. Jia, Y. S. Li and S. Ch. Huang, Minimum connected dominating sets and maximal independent sets in unit disk graphs, *Theor. Comput. Sci.* **352** (2006) 1–7.