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Research paper

Analytic and numerical solution for duffing equations

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Abstract

Daftardar Gejji and Hossein Jafari have proposed a new iterative method for solving many of the linear and nonlinear equations namely (DJM). This method proved already the effectiveness in solved many of the ordinary differential equations, partial differential equations and integral equations. The main aim from this paper is to propose the Daftardar-Jafari method (DJM) to solve the Duffing equations and to find the exact solution and numerical solutions. The proposed (DJM) is very effective and reliable, and the solution is obtained in the series form with easily computed components. The software used for the calculations in this study was MATHEMATICA® 9.0.

Keywords: *Duffing Equation; Damping Equation; Undamping Equation; Exact Solution; Iterative Method.*

1. Introduction

Duffing equation is a nonlinear differential equation, which is used in many sciences such as physical, engineering, and even biological problems. It is discovered by electrical engineer German Duffing in 1918 and has named after him.
It is worth to mention, this equation is founded both with Van der pol's equation, and Van der pol's considered as most well-known examples of nonlinear oscillation in research papers. Also, the Japanese scientist Ueda proposed excellent Poincare` maps of the Duffing equation which called Japanese attractor or Ueda attractor [1]. Duffing equation occurs as a result of the motion of a body subjected to a nonlinear spring power, linear sticky damping, and periodic powering. Oscillations of mechanical systems under the action of a periodic external force can be revealed given by using Duffing equation, see [2] and references therein.
The reason behind the importance of Duffing equations is used in many of the areas such as physical, engineering, and even biological problems [1], such as research large amplitude oscillation of centrifugal governor systems, nonlinear vibration of beams and plates [3, 4],], magnetic-pliancy mechanical systems [4], classical oscillator in chaotic systems [5], periodic orbit extraction, nonlinear mechanical oscillators and prediction of diseases [5, 6].. Duffing equation produces a helpful model for researching nonlinear oscillations and chaotic dynamical systems [3]. Both chaos and chaotic system are nonlinear by nature. In addition, we can find them in many natural and artificial systems, and differentiate by sensitivity to initial condition [7].
In this paper, the general form of Duffing equation [2] will be solved which given in the following form:

$$u''(x) + k_1 u'(x) + k_2 u(x) + k_3 u^3(x) = f(x). \qquad (1)$$

With initial conditions:

$u(0)=a$, and $u'(0)=b$.
where k_1, k_2, k_3, a and b are real constants. In this equation clearly is a nonlinear ordinary differential equation and it is from the second order [1]. The mean of, k_1 controls the size of damping, k_2 controls of size stiffness, k_3 controls of size amount of nonlinearity in the restoring force, r controls the amplitude of the periodic driving force, w controls frequency of the periodic driving force [8]. Duffing equation it resolves in many of new process in the open literature such as homotopy perturbation transform method [7], variational iteration method [9], an effective approach [6], modified differential transform method [4], a new approach method [10] and homotopy analysis method [1]. Some of these methods are used to find numerical, approximate and analytic solutions. It is worth to mention some of these methods is Adomain decomposition method (ADM) [11], this method needs to calculate the so-called Adomain polynomial for treat the nonlinear term.
In 2006, Daftardar and Jafari have proposed a new iterative method namely (DJM) [12]. This method solved many equations such as algebraic equations, ordinary differential equations, partial differential equations and integral equations. The main goal of this article is to apply the DJM for solving Duffing equations in two kind, damping and undamping. The present paper has been arranged as follows, in section 2, the basic idea of DJM. In section 3, solving Duffing equation is discussed. In section 4, some example for damping and undamping Duffing equations are solved and finally in section 5; the conclusion is presented.

2. Basic concept of DJM

Consider the following general functional equation [12]:

$$u = N(u) + f, \qquad (2)$$

Where N is nonlinear operator from a Banach space $B \rightarrow B$, u is an unknown function and f is a known function.
We must find the solution for u of Eq. (2) having the series from:

$$u = \sum_{i=0}^{\infty} u_i, \qquad (3)$$

The nonlinear operator N can be decomposed as:

$$N\left(\sum_{i=0}^{\infty}u_i\right) = N(u_0) + \sum_{I=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right) - N\left(\sum_{j=0}^{i-1}u_j\right)\right\}. \quad (4)$$

From
Eq. (3) and Eq. (4), Eq. (2) is equivalent to

$$\sum_{i=0}^{\infty}u_i = f + N(u_0) + \sum_{i=0}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right) - N\left(\sum_{j=0}^{i-1}u_j\right)\right\}. \quad (5)$$

We define the recurrence relation:

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0) \ldots \\ u_{m+1} = N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \\ m = 1, 2, \ldots . \end{cases} \quad (6)$$

Then

$$u = f + \sum_{i=0}^{\infty}u_i.$$

For the convergence of the DJM, we refer the reader to [13].

3. Solving duffing equation by using DJM

In this section, we apply the DJM for Duffing equation. Let us consider a form of Duffing equation given in Eq. (1).
Equation (1) can be written in an operator form as:

$$L_{xx}u(x) + k_1 L_x u(x) + k_2 u(x) + k_3 u^3(x) = f(x). \quad (8)$$

Where $L_{xx} = \frac{d^2}{dx^2}$, and $L_x = \frac{d}{dx}$.

We assume that the inverse operator L_x^{-1} and L_{xx}^{-1}, exist and it can be taken with respect x from 0 to x,
i.e.

$$L_{xx}^{-1}(.) = \int_0^x \int_0^x (.)dxdx, \text{and } L_x^{-1}(.) = \int_0^x (.)dx. \quad (9)$$

Then, by taking the invers operator L_{xx}^{-1}, to both sides of the equation (8) leads to:

$$L_{xx}^{-1}L_{xx}u(x) + k_1 L_{xx}^{-1}L_x u(x) + L_{xx}^{-1}\left(k_2 u(x) + k_3 u^3(x)\right) = L_{xx}^{-1}f(x), \quad (10)$$

Then, by applying the initial conditions, we obtain:

$$u(x) = a + k_1 ax + bx + g(x) - L_x^{-1}k_1 u(x) - L_{xx}^{-1}(k_2 u(x) + k_3 u^3(x)), \quad (11)$$

Where, $g(x) = \int_0^x \int_0^x f(t)dtdt$.

We convert L_{xx}^{-1} from double integral to single integral from the relation [14]:

$$\int_0^x \int_0^{x_1} \ldots \int_0^{x_{n-1}} u(x_n)dx_n dx_{n-1} \ldots dx_1 = \frac{1}{(n-1)!}\int_0^x (x - t)^{n-1}u(t)dt, \quad (12)$$

Then for double integral:

$$L_{xx}^{-1} = \int_0^x \int_0^x u(t)dt\, dt = \int_0^x (x - t)u(t)dx. \quad (13)$$

By using Eqs.(6)and (7),we obtain the following components:

$$u_0(x) = a + k_1 ax + bx + g(x), \quad (14)$$

$$u_1(x) = N(u_0) = -\int_0^x (k_1 u_0(t))dt - \int_0^x (x - t)\left(k_2 u_0(t) + k_3 u_0^3(t)\right)dt, \quad (15)$$

$$u_2(x) = N(u_0 + u_1) - u_1, \quad (16)$$

$$u_2(x) = -\int_0^x (k_1(u_0 + u_1)(t))dt - \int_0^x (x - t) \left(k_2(u_0 + u_1)(t) + k_3(u_0^3 + u_1^3)(t)\right)dt - u_1,$$

$$u_3(x) = -\int_0^x (k_1(u_0 + u_1 + u_2)(t)dt - \int_0^x (x - t) + \int_0^x (k_1(u_0 + u_1)(t)dt + \int_0^x (x - t)\left(k_2(u_0 + u_1)(t)\right) + k_3(u_0 + u_1)(t)) \quad (19)$$

and so on.
Continuing in this manner, the (n+1)th approximation of the exact solution for the unknown function u(x) can be achieved as:

$$u_{n+1}(x) = N(u_0 + u_1 + \cdots + u_n) - N(u_0 + u_1 + \cdots + u_{n-1}), \quad (20)$$

$$u_{n+1}(x) = -\int_0^x (k_1(u_0 + \cdots + u_n)(t)dt - \int_0^x (x - t)(k_2(u_0 + \cdots + u_n)(t) + k_3(u_0^3 + \cdots + u_n^3)(t)dt + \int_0^x (k_1(u_0 + .. + u_{n-1})(t)dt + \int_0^x (x - t)(k_2\begin{pmatrix}u_0 + \cdots \\ + u_{n-1}\end{pmatrix}(t) + k_3(u_0^3 + \cdots + u_{n-1}^3)(t))dt. \quad (21)$$

Then,

$$u(x) = \sum_{n=0}^{\infty}u_n. \quad (22)$$

4. Test examples

In this section, the applications of the DJM for the damping and undamping Duffing equations will be shown in some examples to assess the efficiency of the DJM. Some of these examples have exact solutions, and the others have numerical solutions. Also we calculate the error remainder with the maximal error remainder parameters and the approximate solution. The convenience function of the error remainder will be [15], [16]:

$$ER_n(x) = u_n''(x) + k_1 u_n'(x) + k_2 u_n + k_3 u_n^3 - f(x), \quad (23)$$

and the maximal error remainder parameters are:

$$MER_n = \max_{0 \le x \le 1}|ER_n(x)|. \quad (24)$$

4.1. Damping duffing equations

Example 1:
Consider the damping Duffing equation [17]:

$$u''(x) + 2u'(x) + u(x) + 8(u)^3(x) = 1 - 3x. \quad (25)$$

With initial conditions:

$u(0) = \frac{1}{2}$, and $u'(0) = -\frac{1}{2}$.

By using the initial conditions, we obtain:

$$u(x) = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} - 2\int_0^x u(t)dt - \int_0^x (x - t)\left(u(t) + 8u^3(t)\right)dt, \quad (26)$$

Following the algorithm given in section 2, we obtain:

$$u_0(x) = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2}, \tag{27}$$

$$u_{n+1}(x) = N\,(u_0 + u_1 + \cdots + u_n)$$

$$-N(u_0 + u_1 + \cdots + u_{n-1}), \tag{28}$$

$$u_{n+1} = -2\int_0^x (u_0 + \cdots + u_n)(t)dt - \int_0^x (x-t)\,((u_0 + \cdots + u_n)(t) + (8(u_0)^3 + \cdots + (u_n)^3)(t))dt + 2\int_0^x (u_0 + \cdots + u_{n-1})(t)dt + \int_0^x (x-t)\,(u_0 + \cdots + u_{n-1}) + 8((u_0)^3 + \cdots + (u_{n-1})^3)dt, \tag{29}$$

Then,

$$u_1(x) = N(u_0) = -2\int_0^x u_0(t)dt - \int_0^x (x-t)\,(u_0(t) + 8(u_0)^3(t))dt, \tag{30}$$

$$u_1(x) = -x - \frac{5x^2}{4} - \frac{11x^3}{12} - \frac{7x^4}{24} + \frac{7x^5}{40} + \cdots, \tag{31}$$

$$u_2(x) = N(u_0 + u_1) - N(u_0), \tag{32}$$

$$u_2(x) = -2\int_0^x (u_0 + u_1)(t)dt - \int_0^x (x-t)\,((u_0 + u_1)(t) - 8(u_0 + u_1)^3(t))dt - u_1, \tag{33}$$

$$u_2(x) = x^2 + 2x^3 + \frac{19x^4}{16} + \frac{31x^5}{80} - \frac{371x^6}{720} - \cdots, \tag{34}$$

and so on.
Continuing in this technique, the approximation of the exact solution for the unknown functions u(x) can be achieved as:

$$u(x) = u_0 + u_1 + u_2 + \cdots, \tag{35}$$

$$u(x) = \frac{1}{2}(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots), \tag{36}$$

This has the closed form:

$$u(x) = \frac{1}{2}\,e^{-x} \tag{37}$$

Which the exact solution for Eq. (25) [17].

Example 2:
Consider damping Duffing equation [10]:

$$u''(x)+u'(x)+u^3(x)=0. \tag{38}$$

With initial conditions:

u (0)=1, and u'(0)=1.

By applying the initial condition in the Eq. (38), we have:

$$u(x) = 1 + 2x - \int_0^x u(t)dt - \int_0^x (x-t)(u^3(t)\,dt, \tag{39}$$

According to the DJM, we obtain the following components:

$$u_0(x)=1+2x, \tag{40}$$

$$u_{n+1}(x) = -\int_0^x (u_0 + \cdots + u_n)(t)dt - \int_0^x (x-t)\,((u_0)^3 + \ldots$$

$$+(u_n)^3(t))dt + \int_0^x ((u_0 + \cdots + u_{n-1})(t))dt + \int_0^x (x-t)\,((u_0)^3 + \ldots + (u_{n-1})^3(t))dt, \tag{41}$$

$$u_1(x) = N(u_0) = -\int_0^x (u_0(t))dt - \int_0^x (x-t)\,(u_0)^3(t))dt, \tag{42}$$

$$u_1(x) = -x - \frac{3x^2}{2} - x^3 - x^4 - \frac{2x^5}{5}, \tag{43}$$

$$u_2(x) = N(u_0 + u_1) - N(u_0), \tag{44}$$

$$u_2(x) = -\int_0^x ((u_0 + u_1)(t))dt - \int_0^x (x-t)((u_0)^3 + (u_1)^3(t))\,dt$$

$$-u_1, \tag{45}$$

$$u_2(x) = \frac{x^2}{2} + x^3 + \frac{11x^4}{8} + \frac{23x^5}{20} + \frac{7x^6}{24} - \cdots, \tag{46}$$

and so on.
The solution in a series form is given by:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots, \tag{47}$$

$$u(x) = 1 + x - x^2 - \frac{x^3}{6} + \frac{x^4}{24} + \frac{11x^5}{40} + \cdots, \tag{48}$$

The approximate solution in Eq.(48) can be further examined by evaluating the maximal error remainder in Table 1. It can be clearly seen that from Figure (1) the points are lay on a straight lines which mean exponential rate of convergence is achieved.

Table 1: The Maximal Error Remainder for Eq. (38) Using DJM. Where N=1... 6.

n	MER
1	1.78881
2	0.157888
3	0.00756497
4	0.000246801
5	6.09094×10^{-6}
6	1.20778×10^{-7}

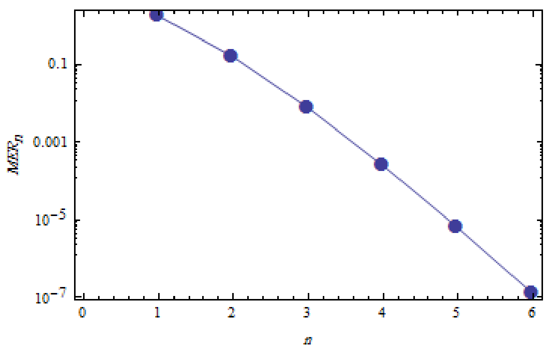

Fig. 1: Logarithmic plots of MER versus N Is 1 through 6.

4.2. Undamping duffing equations

Example 3:
Let us consider the undamping Duffing equation [10]:

$$u''(x) + 3u(x) - 2u^3(x) = cox(x)\,sin(2x). \tag{49}$$

With initial conditions:

u (0) =0, and u'(0)=1.

We apply the initial conditions in Eq. (49), and by taking the product of the series of both sin x and cos(x) (generates a power series expansion for both about the point x=0 to order 10), then we have:

$$u(x) = x + \frac{x^3}{3} - \frac{7x^5}{60} + \frac{61x^7}{2520} - \frac{547x^9}{181440} + \frac{703x^{11}}{2851200} - \int_0^x (x-t)\,(3u(t) + 2u^3(t))dt, \tag{50}$$

By using the DJM, we get the recurrence relation:

$$u_0(x) = x + \frac{x^3}{3} - \frac{7x^5}{60} + \frac{61x^7}{2520} - \frac{547x^9}{181440} + \frac{703x^{11}}{2851200}, \quad (51)$$

$$u_1(x) = N(u_0) = -\int_0^x (x-t)\left(3u_0 - 2(u_0)^3(t)\right)dt, \quad (52)$$

$$u_1(x) = \frac{-x^3}{2} + \frac{x^5}{20} + \frac{47x^7}{840} - \frac{89x^9}{60480} - \frac{2059x^{11}}{950400} + \cdots, \quad (53)$$

$$u_2(x) = N(u_0 + u_1) - N(u_0), \quad (54)$$

$$u_2(x) = -\int_0^x (x-t)\begin{pmatrix} 3(u_0+u_1)(t) \\ -2(u_0^3+u_0^3)(t)\end{pmatrix}dt - u_1, \quad (55)$$

$$u_2(x) = \frac{3x^5}{40} - \frac{3x^7}{40} - \frac{103x^9}{20160} + \frac{17273x^{11}}{2217600} + \cdots, \quad (56)$$

and so on. The solution in a series form is given by:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots, \quad (57)$$

$$u(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} - \frac{x^9}{4536} + \frac{691x^{11}}{1247400} - \cdots, \quad (58)$$

We get an infinite Taylor series:

$$u_N(x) = \sum_{n=0}^{N} u_n x^n, \quad (59)$$

u (x)=sin(x). (60)

Which the exact solution for Eq. (49) [6], obtained upon using the Taylor expansion for sin x in Eq. (59).

Example 4:
Consider undamming Duffing equation [11]:

$$u''(x) + u(x) + u^3(x) = 0. \quad (61)$$

With initial conditions:

u (0)=1, and u'(0)=5

We write the equation after apply the initial condition in Eq. (61):

$$u(x) = 1 + 5x - \int_0^x (x-t)\left(u(t) + u^3(t)\right), \quad (62)$$

Now, we apply the algorithm of DJM, then we have:

$$u_0(x) = 1 + 5x, \quad (63)$$

$$u_{n+1}(x) = -\int_0^x (x-t)\left((u_0 + \cdots + u_n)(t)\right) + ((u_0)^3 + \ldots + (u_n)^3(t))dt$$

$$+ \int_0^x (x-t)\left((2(u_0 + \cdots + u_{n-1})(t)\right) + ((u_0)^3 + \ldots + (u_{n-1})^3(t))dt, \quad (64)$$

$$u_1(x) = N(u_0) = -\int_0^x (x-t)\left(u(t) + u^3(t)\right)dt, \quad (65)$$

$$u_1(x) = -x^2 - \frac{10x^3}{3} - \frac{25x^4}{4} - \frac{25x^5}{4}, \quad (66)$$

$$u_2(x) = N(u_0 + u_1) - N(u_0), \quad (67)$$

$$u_2(x) = -\int_0^x (x-t)\left((u_0+u_1)(t) + (u_0^3 + u_0^3)(t)\right)dt - u_1, \quad (68)$$

$$u_2(x) = \frac{x^4}{3} + \frac{13x^5}{6} + \frac{197x^6}{30} + \frac{285x^2}{28} + \frac{5837x^8}{672} - \frac{455x^9}{864} - \frac{9625x^{10}}{864} - \cdots, \quad (69)$$

and so on. The solution in a series form is given by:

$$u(x) = u_0 + u_1 + u_2 + u_3 + \cdots, \quad (70)$$

$$u(x) = 1 + 5x - x^2 - \frac{10x^3}{3} - \frac{25x^4}{4} + \frac{25x^5}{4} + \cdots, \quad (71)$$

The obtained series solution in Eq.(71) can be used for numerical purposes. The more components that we determine the higher accuracy level that we can achieve. This can be clearly seen in Table 2 and Figure 2.

Table 2: The Maximal Error Remainder for Eq. (61) Using DJM. Where n = 1… 6.

n	MER
1	0.10778
2	0.000477836
3	8.78127×10^{-7}
4	8.75019×10^{-10}
5	5.44897×10^{-13}
6	8.88178×10^{-16}

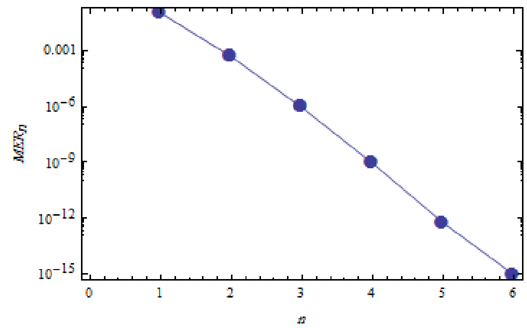

Fig. 2: Logarithmic plots of MER versus N Is 1 through 6.

5. Conclusion

In this paper, the DJM has been successfully applied for solving Duffing equations in two cases: damping and undamping. The analytic and approximate solutions are obtained without any restrictive assumptions for nonlinear terms as required by some existing techniques. Moreover, by solving some examples, it is seems that the DJM appears to be very accurate to employ with reliable results. The software used for the calculations in this study was MATHEMATICA® 9.0.

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