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Analytic calculation of the parallel mean free path of heliospheric cosmic rays

I. Dynamical magnetic slab turbulence and random sweeping slab turbulence

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Abstract. The parallel mean free path of cosmic ray particles in partially turbulent electromagnetic fields is a key input parameter for cosmic ray transport. Here the parallel mean free paths of cosmic ray protons, electrons and positrons are calculated for two particular turbulence models: slab-like dynamical and random sweeping turbulence. After outlining the general quasilinear formalism for deriving the pitch-angle Fokker-Planck coefficient in weak turbulence from the particle's equation of motion, the rigidity dependence and the absolute value of the mean free path for these specific turbulence models are calculated. Approximations for the mean free path for realistic Kolmogorov-type turbulence power spectra which include the steepening at high wavenumbers due to turbulence dispersion and/or dissipation are given.

Key words. cosmic rays – plasmas – turbulence – diffusion – Sun: particle emission

1. Introduction

Besides field line random walk, drifts and non-resonant interactions, resonant wave-particle interaction in the partially random heliospheric magnetic field is regarded as one of the important mechanisms of cosmic ray transport in the heliosphere. In the presence of low-frequency magnetohydrodynamic electromagnetic field fluctuations, whose magnetic field components are much larger than their electric field components, the particle's phase space density adjusts rapidly to a quasi-equilibrium through pitch-angle diffusion, which is characterized by a nearly isotropic distribution. The isotropic part of the phase space distribution function $F(\boldsymbol{x}, p, t)$ obeys the diffusion-convection equation including as dominant terms spatial diffusion in the partially irregular magnetic field as well as spatial convection and adiabatic deceleration in the expanding solar wind plasma. Since the pioneering work of Parker (1965), Axford & Gleeson (1967) and Jokipii & Parker (1969) this diffusion-convection transport equation has been the theoretical basis to describe the modulation of galactic cosmic rays by the Sun. In these studies the heliosphere is regarded as residing in a constant, isotropic bath of galactic cosmic rays, and the electromagnetic fluctuations carried by the outflowing solar wind act to partially exclude these particles from the inner heliosphere in phase with solar activity. Then, the solar modulation may be regarded as a balance between the inward random walk or diffusion, the outward convection by the solar wind, gradient and curvature drifts caused by the large-scale structure of the heliospheric magnetic field, and the adiabatic cooling to the radial expansion of the solar wind (Jokipii 1983). Besides analytical solutions to the transport equation for idealized flow and diffusion coefficient variations (e.g. Stawicki et al. 2000) sophisticated fully three-dimensional numerical solutions of the transport equation (Kota & Jokipii 1983; Burger & Hattingh 1995) have been developed, incorporating the interplanetary distribution of the solar wind and its entrained magnetic field, which were successfully applied not only to the modulation of galactic cosmic rays but also to the modulation of anomalous cosmic rays (for review see Fichtner 2001) and Jovian electrons (Ferreira et al. 2001a, 2001b).

A key input parameter for the cosmic ray transport is the parallel spatial diffusion coefficient $\kappa_\parallel = v\lambda/3$ which is conventionally expressed in terms of the mean free path λ along the background magnetic field and the particle speed v. In many studies the parallel mean free path also controls the perpendicular spatial diffusion coefficient $\kappa_\perp = \alpha\kappa_\parallel$, which, due to the lack of a rigorous theory of perpendicular diffusion, is assumed to be proportional to $\kappa_\parallel$. Numerical simulations of perpendicular transport (e.g. Giacalone & Jokipii 1999) indicate values of $\alpha = 0.02$–0.04.

Within quasilinear theory the parallel mean free path results from the pitch-angle-cosine ($\mu = p_\parallel/p$) average of the inverse of the pitch-angle Fokker-Planck coefficient $D_{\mu\mu}$ as (Jokipii 1966; Hasselmann & Wibberenz 1968; Earl 1974)

$$\lambda = \frac{3v}{8}\int_{-1}^{1} \mathrm{d}\mu\, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}. \quad (1)$$

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The pitch-angle Fokker-Planck coefficient is calculated from the ensemble-averaged first-order corrections to the particle orbits in the weakly turbulent magnetic field (Hall & Sturrock 1968)

$$D_{\mu\mu}(\mu) = Re \int_0^\infty \mathrm{d}\xi < \dot{\mu}(t)\dot{\mu}^*(t+\xi) > \quad (2)$$

and depends on the nature and statistical properties of the electromagnetic turbulence and the turbulence-carrying background medium.

It is the purpose of this paper to calculate the parallel mean free path of cosmic ray protons, electrons and positrons in two particular turbulence models: slab-like dynamical and random sweeping turbulence (Bieber et al. 1994). In Sect. 2 we present the general quasilinear formalism for deriving the pitch-angle Fokker-Planck coefficient in weak turbulence from the particle's equation of motion. In Sect. 3 we specify to the parallel propagating slab-like dynamical and random sweeping turbulence models: analytical expressions for the parallel mean free paths are derived for realistic Kolmogorov-type turbulence power spectra which include the steepening at high wavenumbers due to turbulence dispersion and/or dissipation. The general formalism given in Sect. 2 allows us to generalise to more general (than slab-like) turbulence geometries which will be the subject of forthcoming work.

2. Calculation of the cosmic ray Fokker-Planck coefficient $D_{\mu\mu}$

To obtain an equation for $D_{\mu\mu}$ we must calculate the derivative of the pitch angle cosine μ. To do this we start with the equation of motion

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e\left[\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right]\cdot \quad (3)$$

Now we split the fields in a background and a turbulent component:

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_0 + \delta\boldsymbol{B} \\ \boldsymbol{E} &= \delta\boldsymbol{E} \end{aligned} \quad (4)$$

assuming magnetic turbulence and an ordered magnetic field:

$$\begin{aligned} \delta\boldsymbol{E} &= 0 \\ \delta\boldsymbol{B} &\neq 0 \\ \boldsymbol{B}_0 &= B_0\boldsymbol{e}_z. \end{aligned} \quad (5)$$

We obtain for the individual components in Cartesian coordinates:

$$\begin{aligned} \frac{\mathrm{d}p_x}{\mathrm{d}t} &= \frac{\Omega}{B_0}\left[p_y(B_0+\delta B_z) - p_z\delta B_y\right] \\ \frac{\mathrm{d}p_y}{\mathrm{d}t} &= \frac{\Omega}{B_0}\left[-p_x(B_0+\delta B_z) + p_z\delta B_x\right] \\ \frac{\mathrm{d}p_z}{\mathrm{d}t} &= \frac{\Omega}{B_0}\left[p_x\delta B_y - p_y\delta B_x\right] \end{aligned} \quad (6)$$

with the gyrofrequency

$$\Omega = \frac{eB_0}{mc\gamma} \quad (7)$$

where m is the particle mass and γ is the Lorentz factor. Using spherical momentum coordinates

$$\begin{aligned} p_x &= p\sqrt{1-\mu^2}\cos\Phi \\ p_y &= p\sqrt{1-\mu^2}\sin\Phi \\ p_z &= p\mu \end{aligned} \quad (8)$$

(with the pitch angle cosine μ) we obtain

$$\dot{\mu} = \frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\Omega\sqrt{1-\mu^2}}{B_0}\left[\delta B_y\cos\Phi - \delta B_x\sin\Phi\right] \quad (9)$$

with

$$\begin{aligned} \cos\Phi &= \tfrac{1}{2}\left[\mathrm{e}^{i\Phi} + \mathrm{e}^{-i\Phi}\right] \\ \sin\Phi &= \tfrac{1}{2i}\left[\mathrm{e}^{i\Phi} - \mathrm{e}^{-i\Phi}\right] \end{aligned} \quad (10)$$

and using left-handed and right-handed components for the turbulence field

$$\begin{aligned} \delta B_\mathrm{L} &= \tfrac{1}{\sqrt{2}}\left(\delta B_x + i\delta B_y\right) \\ \delta B_\mathrm{R} &= \tfrac{1}{\sqrt{2}}\left(\delta B_x - i\delta B_y\right) \end{aligned} \quad (11)$$

we find

$$\dot{\mu} = \frac{i\Omega}{\sqrt{2}B_0}\sqrt{1-\mu^2}\left[\delta B_\mathrm{R}(x(t))\mathrm{e}^{i\Phi} - \delta B_\mathrm{L}(x(t))\mathrm{e}^{-i\Phi}\right]. \quad (12)$$

Now we are using the quasilinear approximation:

$$\dot{\mu} = \frac{i\Omega}{\sqrt{2}B_0}\sqrt{1-\mu^2}\left[\delta B_\mathrm{R}(\bar{x}(t))\mathrm{e}^{i\bar{\Phi}} - \delta B_\mathrm{L}(\bar{x}(t))\mathrm{e}^{-i\bar{\Phi}}\right] \quad (13)$$

with the unperturbed orbit $(\bar{x}, \bar{y}, \bar{z})$, which is the orbit with $\delta B = 0$, i.e.

$$\begin{aligned} \bar{x} &= -\frac{v}{\Omega}\sqrt{1-\mu^2}\sin(\Phi_0 - \Omega t) \\ \bar{y} &= +\frac{v}{\Omega}\sqrt{1-\mu^2}\cos(\Phi_0 - \Omega t) \\ \bar{z} &= v_\parallel t \quad \text{with} \quad v_\parallel = v\mu \end{aligned} \quad (14)$$

for the orbit and

$$\begin{aligned} \dot{p} = 0 &\Rightarrow \bar{p} = p_0 = \text{const} \\ \dot{\mu} = 0 &\Rightarrow \bar{\mu} = \mu_0 = \text{const} \\ \dot{\Phi} = -\Omega &\Rightarrow \bar{\Phi} = \Phi_0 - \Omega t \end{aligned} \quad (15)$$

for the momentum. Introducing the Fourier-transforms of the turbulent magnetic fields

$$\delta B_{L,R}(\bar{\boldsymbol{x}}, t) = \int \mathrm{d}^3k\,\delta B_\mathrm{L,R}(\boldsymbol{k}, t)\exp\left[i\boldsymbol{k}\cdot\bar{\boldsymbol{x}}\right] \quad (16)$$

with spherical coordinates for the wave vector

$$\begin{aligned} k_x &= k_\perp\cos\Psi = k\sin\Theta\cos\Psi \\ k_y &= k_\perp\sin\Psi = k\sin\Theta\sin\Psi \\ k_z &= k_\parallel = k\cos\Theta \end{aligned} \quad (17)$$

we derive

$$\boldsymbol{k}\cdot\bar{\boldsymbol{x}} = \frac{k_\perp v_\perp}{\Omega}\sin(\Phi_1) + k_\parallel v_\parallel t \quad (18)$$

with

$$\Phi_1 = \Psi - \Phi_0 + \Omega t. \quad (19)$$

Using the identity with the Bessel functions J_n

$$e^{iz\sin\Phi_1} = \sum_{n=-\infty}^{+\infty} J_n(z)e^{in\Phi_1} \tag{20}$$

we obtain

$$\dot{\mu} = \frac{i\Omega}{\sqrt{2}B_0}\sqrt{1-\mu^2}\int d^3k \sum_{n=-\infty}^{+\infty} e^{i(k_\parallel v_\parallel + n\Omega)t + in(\Psi-\Phi_0)} \times \left[\delta B_R(\boldsymbol{k},t)J_{n+1}\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{i\Psi} - \delta B_L(\boldsymbol{k},t)J_{n-1}\left(\frac{k_\perp v_\perp}{\Omega}\right)e^{-i\Psi}\right]. \tag{21}$$

Now we are able to calculate the Fokker-Planck-coefficient (see Eq. (2)). To do this, we introduce the correlation tensor of the magnetic field fluctuations

$$\langle \delta B_\alpha(\boldsymbol{k},t)\delta B_\beta(\boldsymbol{k}',t+\xi)\rangle = \delta(\boldsymbol{k}-\boldsymbol{k}')P_{\alpha\beta}(\boldsymbol{k},\xi) \tag{22}$$

where we have made the assumption that Fourier components at different wave vectors are uncorrelated. So, we obtain for $D_{\mu\mu}$:

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{2B_0^2} Re \sum_{n=-\infty}^{+\infty}\int_0^\infty d\xi \int d^3k e^{-i(k_\parallel v_\parallel + n\Omega)\xi} \times\left[J_{n+1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{RR}(\boldsymbol{k},\xi) + J_{n-1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{LL}(\boldsymbol{k},\xi) - J_{n+1}\left(\frac{k_\perp v_\perp}{\Omega}\right)J_{n-1}\left(\frac{k_\perp v_\perp}{\Omega}\right)\left(P_{RL}(\boldsymbol{k},\xi)e^{+2i\Psi} + P_{LR}(\boldsymbol{k},\xi)e^{-2i\Psi}\right)\right] \tag{23}$$

where we also have averaged over the initial phase Φ_0. To do the time-integration ξ we have to specify the time dependence of the tensor $P_{\alpha\beta}$. Using

$$P_{\alpha\beta}(\boldsymbol{k},\xi) = P_{\alpha\beta}^0(\boldsymbol{k})F(\xi,\boldsymbol{k}) \tag{24}$$

we obtain

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{2B_0^2}\sum_{n=-\infty}^{+\infty}\int d^3k \times\left[J_{n+1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{RR}^0(\boldsymbol{k}) + J_{n-1}^2\left(\frac{k_\perp v_\perp}{\Omega}\right)P_{LL}^0(\boldsymbol{k}) - J_{n+1}\left(\frac{k_\perp v_\perp}{\Omega}\right)J_{n-1}\left(\frac{k_\perp v_\perp}{\Omega}\right)\left(P_{RL}^0(\boldsymbol{k})e^{+2i\Psi} + P_{LR}^0(\boldsymbol{k})e^{-2i\Psi}\right)\right]\cdot R_i \tag{25}$$

with

$$R_i = Re\int_0^\infty d\xi e^{-i(k_\parallel v_\parallel + n\Omega)\xi}\cdot F_i(\xi,\boldsymbol{k}). \tag{26}$$

To proceed we must specify the function $F_i(\xi,\boldsymbol{k})$ and the tensor $P_{\alpha\beta}^0(\boldsymbol{k})$.

3. Slab-like dynamical and random sweeping tubulence

3.1. Equations for $D_{\mu\mu}$

For F_i we examine two different models. This first model is called the damping model of dynamical turbulence, the second model is called the random sweeping model.

3.1.1. Damping model of dynamical turbulence

In this model we use (Bieber et al. 1994)

$$F_1(\xi) = e^{-\xi/q_D} \tag{27}$$

with

$$q_D = \frac{1}{\alpha v_A \mid k_\parallel \mid}, \quad 0 \le \alpha \le 1. \tag{28}$$

According to Eq. (26) we must solve the integral

$$R_1 = \Re\int_0^\infty d\xi e^{-i(k_\parallel v_\parallel + n\Omega)\xi - \xi/q_D} = \frac{q_D}{1 + q_D^2(k_\parallel v_\parallel + n\Omega)^2}. \tag{29}$$

3.1.2. Random sweeping model

In this model we use (Bieber et al. 1994)

$$F_2(\xi) = e^{-(\xi/q_D)^2} \tag{30}$$

with the same q_D as for the damping model of dynamical turbulence (Eq. (28)). In this case we find for the integral (26)

$$R_2 = Re\int_0^\infty d\xi e^{-i(k_\parallel v_\parallel + n\Omega)\xi - (\xi/q_D)^2} = \frac{\sqrt{\pi}}{2}q_D e^{-(k_\parallel v_\parallel + n\Omega)^2 q_D^2/4}. \tag{31}$$

3.1.3. Slab turbulence and Kolmogorov-type turbulence spectrum

Next we also need to specify the geometry and polarisation of the field fluctuations (the tensor $P_{\alpha\beta}^0$). We use slab-turbulence, i.e. that the wave vectors of the fluctuations are all parallel or antiparallel to the background magnetic field (e.g. Jaekel & Schlickeiser 1992):

$$P_{lm} = \begin{cases} g(k_\parallel)\frac{\delta(k_\perp)}{k_\perp}\left[\delta_{lm} + i\sigma\epsilon_{lm3}\right] \\ 0 \quad \text{for} \quad l,m = 3. \end{cases} \tag{32}$$

So we obtain for the turbulence tensor components:

$$\begin{aligned} P_{RR}^0 &= (1-\sigma)g(k_\parallel)\frac{\delta(k_\perp)}{k_\perp} \\ P_{LL}^0 &= (1+\sigma)g(k_\parallel)\frac{\delta(k_\perp)}{k_\perp} \\ P_{RL}^0 &= P_{LR}^0 = 0. \end{aligned} \tag{33}$$

With Eq. (25) and using $J_n(0) = \delta_{n0}$ we obtain for the Fokker-Planck-coefficient

$$D_{\mu\mu} = \frac{\pi\Omega^2\left(1-\mu^2\right)}{B_0^2}\int_{-\infty}^{+\infty} dk_\parallel g\left(k_\parallel\right) \times\left[(1-\sigma)R(n=-1) + (1+\sigma)R(n=+1)\right] \tag{34}$$

with the damping model of dynamical turbulence (DT) and $g(k_\parallel) = g(\mid k_\parallel \mid)$

$$D_{\mu\mu}(DT) = \frac{2\pi\Omega^2(1-\mu^2)}{B_0^2} \times\int_0^\infty dk_\parallel g(\mid k_\parallel \mid)q_D\left[\frac{1}{1+q_D^2\left(k_\parallel v_\parallel - \Omega\right)^2} + \frac{1}{1+q_D^2\left(k_\parallel v_\parallel + \Omega\right)^2}\right] \tag{35}$$

and for the random-sweeping model (RS)

$$D_{\mu\mu}(RS) = \frac{\pi^{3/2}\Omega^2\left(1-\mu^2\right)}{B_0^2}\int_0^\infty \mathrm{d}k_\parallel g\left(\mid k_\parallel \mid\right) q_\mathrm{D} \times\left[\mathrm{e}^{-(k_\parallel v_\parallel+\Omega)^2 q_\mathrm{D}^2/4}+\mathrm{e}^{-\left(k_\parallel v_\parallel-\Omega\right)^2 q_\mathrm{D}^2/4}\right]. \quad (36)$$

For $g(k_\parallel)$ we assume a Kolmogorov-type turbulence spectrum (see Fig. 1):

$$g(k_\parallel) = \begin{cases} 0 & \text{for} \quad \mid k_\parallel \mid \le k_\mathrm{min} \\ g_0 \mid k_\parallel \mid^{-s} & \text{for} \quad k_\mathrm{min} \le \mid k_\parallel \mid \le k_d \\ g_1 \mid k_\parallel \mid^{-p} & \text{for} \quad \mid k_\parallel \mid \ge k_d \end{cases}$$

with $g_1 = g_0 k_d^{p-s}$, $1 < s < 2$, $2 < p$ and $k_\mathrm{min} \ll k_d$. For g_0 we can use the total fluctuating magnetic field strength:

$$\begin{aligned}(\delta B)^2 &= \sum_{m=1}^{3}(\delta B_m)^2 = \int \mathrm{d}^3k \int \mathrm{d}^3k' \delta B_m(\boldsymbol{k})\delta B_m^*(\boldsymbol{k}')\mathrm{e}^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}} \\ &= \sum_{m=1}^{3}\int \mathrm{d}^3k P_{mm}(\boldsymbol{k}) = 8\pi g_0\left[\int_{k_\mathrm{min}}^{k_d}\mathrm{d}k k^{-s}+\frac{g_1}{g_0}\int_{k_d}^{\infty}\mathrm{d}k k^{-p}\right] \\ &= \frac{8\pi g_0}{s-1}k_\mathrm{min}^{1-s}\left[1-\left(\frac{k_\mathrm{min}}{k_d}\right)^{s-1}+\frac{s-1}{p-1}\frac{g_1}{g_0}\frac{k_\mathrm{min}^{s-1}}{k_d^{p-1}}\right] \\ &\approx \frac{8\pi g_0}{s-1}k_\mathrm{min}^{1-s}\end{aligned} \quad (37)$$

with $q_\mathrm{D} = 1/\alpha v_\mathrm{A} \mid k_\parallel \mid$ we derive

$$\begin{aligned}D_{\mu\mu}(DT) &= \frac{(s-1)\Omega^2\left(1-\mu^2\right)}{4\alpha v_\mathrm{A}}\left(\frac{\delta B}{B_0}\right)^2 k_\mathrm{min}^{s-1} \\ &\times\left\{\int_{k_\mathrm{min}}^{k_d}\mathrm{d}k\, k^{-s-1}\left[\frac{1}{1+\left(\frac{kv\mu-\Omega}{\alpha v_\mathrm{A}k}\right)^2}+\frac{1}{1+\left(\frac{kv\mu+\Omega}{\alpha v_\mathrm{A}k}\right)^2}\right]\right. \\ &\left.+k_d^{p-s}\int_{k_d}^{\infty}\mathrm{d}k\, k^{-p-1}\left[\frac{1}{1+\left(\frac{kv\mu-\Omega}{\alpha v_\mathrm{A}k}\right)^2}+\frac{1}{1+\left(\frac{kv\mu+\Omega}{\alpha v_\mathrm{A}k}\right)^2}\right]\right\}.\end{aligned} \quad (38)$$

Now it is useful to introduce the following parameters:

$$\begin{aligned}\epsilon &= \frac{v_\mathrm{A}}{v} \\ a &= \frac{1}{\alpha\epsilon} = \frac{v}{\alpha v_\mathrm{A}} \\ b &= \frac{1}{2\alpha\epsilon} = \frac{v}{2\alpha v_\mathrm{A}} = \frac{a}{2} \\ R_L &= \frac{v}{\Omega} = \frac{pc}{B_0 \mid q \mid} = \frac{r}{B_0} \\ R &= R_\mathrm{L}k_\mathrm{min} = r\frac{k_\mathrm{min}}{B_0} \\ Q &= R_\mathrm{L}k_d = r\frac{k_d}{B_0}\end{aligned} \quad (39)$$

with the rigidity $r = pc/\mid q \mid$. Rewriting Eq. (38) as

$$\begin{aligned}D_{\mu\mu}(DT) &= \frac{(s-1)\Omega^2\left(1-\mu^2\right)}{4\alpha v_\mathrm{A}}\left(\frac{\delta B}{B_0}\right)^2 k_\mathrm{min}^{s-1} \\ &\times\left\{\int_{k_\mathrm{min}}^{\infty}\mathrm{d}k\, k^{-s-1}\left[\frac{1}{1+\left(\frac{kv\mu-\Omega}{\alpha v_\mathrm{A}k}\right)^2}+\frac{1}{1+\left(\frac{kv\mu+\Omega}{\alpha v_\mathrm{A}k}\right)^2}\right]\right. \\ &-\int_{k_d}^{\infty}\mathrm{d}k\, k^{-s-1}\left[\frac{1}{1+\left(\frac{kv\mu-\Omega}{\alpha v_\mathrm{A}k}\right)^2}+\frac{1}{1+\left(\frac{kv\mu+\Omega}{\alpha v_\mathrm{A}k}\right)^2}\right] \\ &\left.+k_d^{p-s}\int_{k_d}^{\infty}\mathrm{d}k\, k^{-p-1}\left[\frac{1}{1+\left(\frac{kv\mu-\Omega}{\alpha v_\mathrm{A}k}\right)^2}+\frac{1}{1+\left(\frac{kv\mu+\Omega}{\alpha v_\mathrm{A}k}\right)^2}\right]\right\}\end{aligned} \quad (40)$$

and substituting $x = k_\mathrm{min}/k$ in the first integral and $x = k_d/k$ in the second and third integral we find

$$\begin{aligned}D_{\mu\mu}(DT) &= \frac{(s-1)v\left(1-\mu^2\right)a}{4k_\mathrm{min}R_\mathrm{L}^2}\left(\frac{\delta B}{B_0}\right)^2 \\ &\times\left\{\int_0^1\mathrm{d}x\, x^{s-1}\left[\frac{1}{1+a^2(\mu-x/R)^2}+\frac{1}{1+a^2(\mu+x/R)^2}\right]\right. \\ &-\frac{R^s}{Q^s}\int_0^1\mathrm{d}x\, x^{s-1}\left[\frac{1}{1+a^2(\mu-x/Q)^2}+\frac{1}{1+a^2(\mu+x/Q)^2}\right] \\ &\left.+\frac{R^s}{Q^s}\int_0^1\mathrm{d}x\, x^{p-1}\left[\frac{1}{1+a^2(\mu-x/Q)^2}+\frac{1}{1+a^2(\mu+x/Q)^2}\right]\right\}.\end{aligned} \quad (41)$$

Now we do the same calculation for the RS - model:

$$\begin{aligned}D_{\mu\mu}(RS) &= \frac{\sqrt{\pi}(s-1)\Omega^2\left(1-\mu^2\right)}{8\alpha v_\mathrm{A}}\left(\frac{\delta B}{B_0}\right)^2 k_\mathrm{min}^{s-1} \\ &\times\left\{\int_{k_\mathrm{min}}^{k_d}\mathrm{d}k\, k^{-s-1}\left[\mathrm{e}^{-\left(k_\parallel v_\parallel-\Omega\right)^2 q_D^2/4}+\mathrm{e}^{-\left(k_\parallel v_\parallel+\Omega\right)^2 q_D^2/4}\right]\right. \\ &\left.+k_d^{p-s}\int_{k_d}^{\infty}\mathrm{d}k\, k^{-p-1}\left[\mathrm{e}^{-\left(k_\parallel v_\parallel-\Omega\right)^2 q_D^2/4}+\mathrm{e}^{-\left(k_\parallel v_\parallel+\Omega\right)^2 q_D^2/4}\right]\right\}\end{aligned} \quad (42)$$

what can be written as

$$\begin{aligned}D_{\mu\mu}(RS) &= \frac{\sqrt{\pi}(s-1)v(1-\mu^2)b}{4k_\mathrm{min}R_\mathrm{L}^2}\left(\frac{\delta B}{B_0}\right)^2 \\ &\times\left\{\int_0^1\mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2(\mu-x/R)^2}+\mathrm{e}^{-b^2(\mu+x/R)^2}\right]\right. \\ &-\frac{R^s}{Q^s}\int_0^1\mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2(\mu-x/Q)^2}+\mathrm{e}^{-b^2(\mu+x/Q)^2}\right] \\ &\left.+\frac{R^s}{Q^s}\int_0^1\mathrm{d}x\, x^{p-1}\left[\mathrm{e}^{-b^2(\mu-x/Q)^2}+\mathrm{e}^{-b^2(\mu+x/Q)^2}\right]\right\}.\end{aligned} \quad (43)$$

Note that both expressions (41) and (43) for $D_{\mu\mu}$ do not depend of the charge sign. So we obtain the same λ for electrons

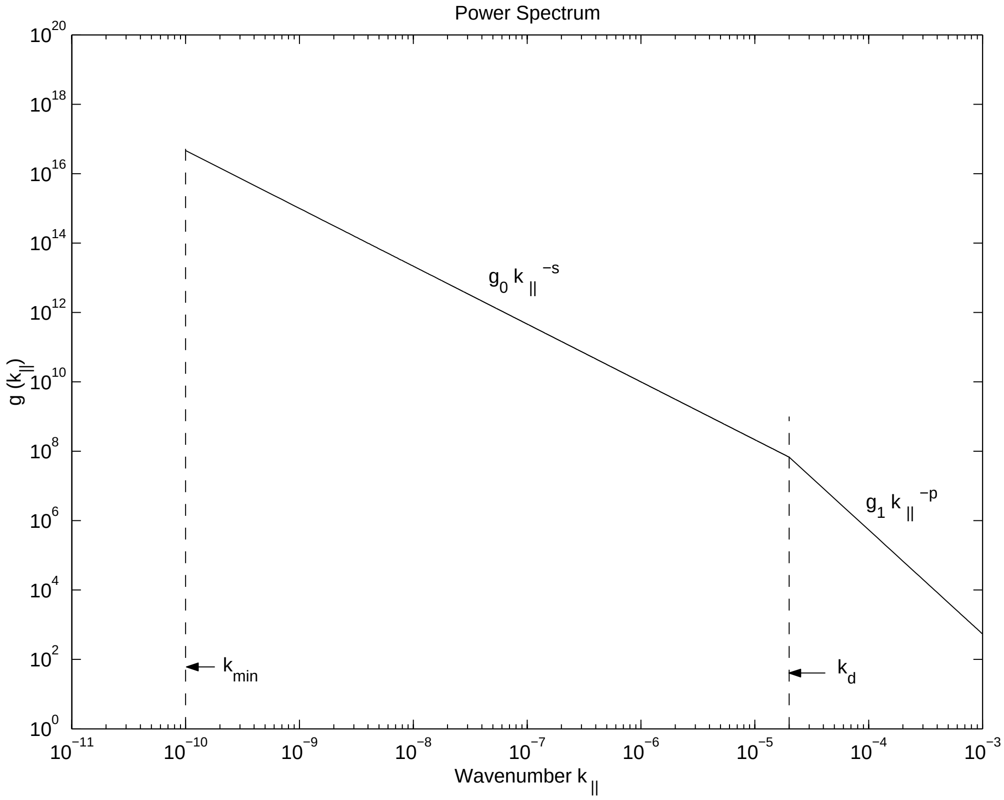

Fig. 1. Power spectrum of slab model used in our calculations. For $k_{\|} \leq k_{\min}$ we set $g(k_{\|}) = 0$.

and positrons. Moreover, we notice that the $D_{\mu\mu}$ are symmetric functions of μ:

$$D_{\mu\mu}(-\mu) = D_{\mu\mu}(+\mu). \tag{44}$$

3.1.4. The damping model of dynamical turbulence

Here $D_{\mu\mu}(DT)$ can be written as

$$D_{\mu\mu}(DT) = \frac{(s-1)v(1-\mu^2)a}{4k_{\min}R_{\mathrm{L}}^2}\left(\frac{\delta B}{B_0}\right)^2 \cdot I(\mu) \tag{45}$$

with

$$I(\mu) = A + \left(\frac{R}{Q}\right)^s [C - B] \tag{46}$$

where

$$A = \int_0^1 \mathrm{d}x x^{s-1}\left[\frac{1}{1+a^2(\mu - x/R)^2} + \frac{1}{1+a^2(\mu + x/R)^2}\right]$$
$$B = \int_0^1 \mathrm{d}x x^{s-1}\left[\frac{1}{1+a^2(\mu - x/Q)^2} + \frac{1}{1+a^2(\mu + x/Q)^2}\right]$$
$$C = \int_0^1 \mathrm{d}x x^{p-1}\left[\frac{1}{1+a^2(\mu - x/Q)^2} + \frac{1}{1+a^2(\mu + x/Q)^2}\right]\cdot \tag{47}$$

So we must solve integrals of the type

$$M = \int_0^1 \mathrm{d}x x^{s-1}\left[\frac{1}{1+a^2/R^2(\mu R - x)^2} + \frac{1}{1+a^2/R^2(\mu R + x)^2}\right]\cdot \tag{48}$$

As shown in Appendix A these integrals can be solved analytically in different pitch-angle regimes and we obtain

$$A(\mu R \gg 1) = \frac{2}{s}\frac{1}{1+a^2\mu^2}$$
$$A(\mu R \ll 1, a\mu \gg 1) = \pi\frac{R^s}{a}\mu^{s-1} - \frac{2}{2-s}\frac{R^2}{a^2}$$
$$A(\mu R \ll 1, a\mu \ll 1, a/R \gg 1) = \frac{\pi}{\sin(\frac{\pi s}{2})}\frac{R^s}{a^s} - \frac{2}{2-s}\frac{R^2}{a^2}$$
$$A(\mu R \ll 1, a\mu \ll 1, a/R \ll 1) = 2/s$$
$$B(\mu Q \gg 1) = \frac{2}{s}\frac{1}{1+a^2\mu^2}$$
$$B(\mu Q \ll 1, a\mu \gg 1) = \pi\frac{Q^s}{a}\mu^{s-1} - \frac{2}{2-s}\frac{Q^2}{a^2}$$
$$B(\mu Q \ll 1, a\mu \ll 1, a/Q \gg 1) = \frac{\pi}{\sin(\frac{\pi s}{2})}\frac{Q^s}{a^s} - \frac{2}{2-s}\frac{Q^2}{a^2}$$
$$B(\mu Q \ll 1, a\mu \ll 1, a/Q \ll 1) = 2/s$$
$$C(\mu Q \gg 1) = \frac{2}{p}\frac{1}{1+a^2\mu^2}$$
$$C(\mu Q \ll 1, a\mu \gg 1) = \frac{2}{p-2}\frac{Q^2}{a^2} + \pi\frac{Q^p}{a}\mu^{p-1}$$
$$C(\mu Q \ll 1, a\mu \ll 1, a/Q \gg 1) = \frac{2}{p-2}\frac{Q^2}{a^2}$$
$$C(\mu Q \ll 1, a\mu \ll 1, a/Q \ll 1) = 2/p. \tag{49}$$

With Eq. (46) we obtain 8 different cases for I (and therefore 8 different cases for $D_{\mu\mu}$) which are shown in Table 1.

3.1.5. The random sweeping model

In the RS-model $D_{\mu\mu}$ can be written as

$$D_{\mu\mu}(RS) = \frac{\sqrt{\pi}(s-1)v\left(1-\mu^2\right)b}{4k_{\min}R_{\mathrm{L}}^2}\left(\frac{\delta B}{B_0}\right)^2 \cdot I(\mu) \tag{50}$$

with

$$I(\mu) = A + \left(\frac{R}{Q}\right)^s [C - B] \tag{51}$$

where

$$\begin{aligned} A &= \int_0^1 \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2(\mu-x/R)^2} + \mathrm{e}^{-b^2(\mu+x/R)^2}\right] \\ B &= \int_0^1 \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2(\mu-x/Q)^2} + \mathrm{e}^{-b^2(\mu+x/Q)^2}\right] \\ C &= \int_0^1 \mathrm{d}x\, x^{p-1}\left[\mathrm{e}^{-b^2(\mu-x/Q)^2} + \mathrm{e}^{-b^2(\mu+x/Q)^2}\right]. \end{aligned} \tag{52}$$

The integrals of the type

$$\begin{aligned} M &= \int_0^1 \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2[\mu-x/R]^2} + \mathrm{e}^{-b^2[\mu+x/R]^2}\right] \\ &= \int_0^1 \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-b^2/R^2[\mu R-x]^2} + \mathrm{e}^{-b^2/R^2[\mu R+x]^2}\right] \end{aligned} \tag{53}$$

will be solved by approximations for special cases again. This is done in Appendix B and we obtain

$$\begin{aligned}
A\,(\mu R \gg 1) &= \frac{2}{s}\mathrm{e}^{-\mu^2 b^2} \\
A\,(\mu R \ll 1, b/R \ll 1) &= \frac{2}{s} \\
A\,(\mu R \ll 1, b/R \gg 1, b\mu \ll 1) &= \frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})}\frac{R^s}{b^s} \\
A\,(\mu R \ll 1, b/R \gg 1, b\mu \gg 1) &= \frac{\pi\Gamma(s)}{2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}(b\mu)^{s-1}\frac{R^s}{b^s} \\
B\,(\mu Q \gg 1) &= \frac{2}{s}\mathrm{e}^{-\mu^2 b^2} \\
B\,(\mu Q \ll 1, b/Q \ll 1) &= \frac{2}{s} \\
B\,(\mu Q \ll 1, b/Q \gg 1, b\mu \ll 1) &= \frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})}\frac{Q^s}{b^s} \\
B\,(\mu Q \ll 1, b/Q \gg 1, b\mu \gg 1) &= \frac{\pi\Gamma(s)}{2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}(b\mu)^{s-1}\frac{Q^s}{b^s} \\
C\,(\mu Q \gg 1) &= \frac{2}{p}\mathrm{e}^{-\mu^2 b^2} \\
C\,(\mu Q \ll 1, b/Q \ll 1) &= \frac{2}{p} \\
C\,(\mu Q \ll 1, b/Q \gg 1, b\mu \ll 1) &= \frac{\sqrt{\pi}\Gamma(p)}{2^{p-1}\Gamma(\frac{p+1}{2})}\frac{Q^p}{b^p} \\
C\,(\mu Q \ll 1, b/Q \gg 1, b\mu \gg 1) &= \frac{\pi\Gamma(p)}{2^{p-1}\Gamma(\frac{p}{2})\Gamma(\frac{p+1}{2})}(b\mu)^{p-1}\frac{Q^p}{b^p}\cdot
\end{aligned} \tag{54}$$

With Eq. (51) we obtain 8 different cases for I (and therefore for $D_{\mu\mu}$) which are shown in Table 2.

3.2. Equations for the mean free path

With the above equations for $D_{\mu\mu}(DT)$ and $D_{\mu\mu}(RS)$ we are able to calculate the parallel mean free path (1).

3.2.1. Damping model of dynamical turbulence

With Eqs. (1) and (44) we obtain

$$\lambda = \frac{3v}{8}\int_{-1}^{+1}\mathrm{d}\mu\frac{\left(1-\mu^2\right)^2}{D_{\mu\mu}(\mu)} = \frac{3v}{4}\int_0^{+1}\mathrm{d}\mu\frac{\left(1-\mu^2\right)^2}{D_{\mu\mu}(\mu)}\cdot \tag{55}$$

With

$$\lambda_0 = \left(\frac{B_0}{\delta B}\right)^2 \tag{56}$$

and

$$D_{\mu\mu} = \frac{(s-1)k_{\min}av}{4R^2}\left(\frac{\delta B}{B_0}\right)^2\left(1-\mu^2\right)I(\mu) \tag{57}$$

we derive

$$\frac{\lambda}{\lambda_0} = \frac{3}{s-1}\frac{R^2}{k_{\min}\cdot a}\cdot K \tag{58}$$

with the integral

$$K = \int_0^1 \mathrm{d}\mu\frac{1-\mu^2}{I(\mu)}\cdot \tag{59}$$

For K we obtain the 12 different cases listed in Table 3. With Eq. (58) these yield the analytical approximations for the mean free path. With the parameters defined in Eq. (39) it is very easy to simplify the results for $1 < s < 2$, $2 < p$ and $R \ll Q$, using approximations like

$$K = \int_0^1 \mathrm{d}\mu\frac{1-\mu^2}{I(\mu)} \approx \int_0^{1/a}\mathrm{d}\mu\frac{1-\mu^2}{I(a\mu \ll 1)} + \int_{1/a}^1\mathrm{d}\mu\frac{1-\mu^2}{I(a\mu \gg 1)}\cdot \tag{60}$$

The first three cases of Table 3 apply to large particle rigidities ($R = R_{\mathrm{L}}k_{\min} \gg 1$). If we calculate the mean free path for typical heliospheric parameters, we find that the value of the mean free path formally becomes larger than the size of the heliospere in which case the diffusion approximation to cosmic ray transport no longer is justified. The reason for this is the sharp cut-off of the turbulence power spectrum at $k_{\min}$. The second three cases are not relevant for typical heliospheric parameters. So only the last six cases are important for calculating the mean free path and fulfil the restriction $R_{\mathrm{L}}k_{\min} < 1$. To demonstrate what happens at high rigidities, we consider this limit for special parameters in Appendix C.

3.2.2. Random sweeping model

For this model we can do similar approximations. But now we have

$$D_{\mu\mu} = \frac{\sqrt{\pi}(s-1)k_{\min}bv}{4R^2}\left(\frac{\delta B}{B_0}\right)^2\left(1-\mu^2\right)I(\mu) \tag{61}$$

Table 1. This table shows the the function $I(\mu)$ for the DT-model, where we have introduced the functions $f_1(s,p) = \frac{2}{p-2} + \frac{2}{2-s}$ and $f_2(s) = \frac{\pi}{\sin(\frac{\pi s}{2})}$.

case	$I(\mu)$
$\mu R \gg 1, \mu Q \gg 1$	$\frac{2}{s} \frac{1}{1+a^2\mu^2}$
$\mu R \ll 1, \mu Q \gg 1, a\mu \gg 1$	$\pi \frac{R^s}{a} \mu^{s-1}$
$\mu R \ll 1, \mu Q \ll 1, a\mu \gg 1$	$f_1 \frac{R^s Q^{2-s}}{a^2} + \pi \frac{R^s Q^{p-s}}{a} \mu^{p-1}$
$\mu R \ll 1, \mu Q \gg 1, a\mu \ll 1, a/R \gg 1$	$f_2 \frac{R^s}{a^s}$
$\mu R \ll 1, \mu Q \ll 1, a\mu \ll 1, a/R \gg 1, a/Q \gg 1$	$f_1 \frac{R^s Q^{2-s}}{a^2}$
$\mu R \ll 1, \mu Q \ll 1, a\mu \ll 1, a/R \gg 1, a/Q \ll 1$	$f_2 \frac{R^s}{a^s}$
$\mu R \ll 1, \mu Q \gg 1, a\mu \ll 1, a/R \ll 1$	$2/s$
$\mu R \ll 1, \mu Q \ll 1, a\mu \ll 1, a/R \ll 1, a/Q \ll 1$	$2/s$

Table 2. This table shows the function $I(\mu)$ for the RS-model.

case	$I(\mu)$
$\mu R \gg 1, \mu Q \gg 1$	$\frac{2}{s} \mathrm{e}^{-\mu^2 b^2}$
$\mu R \ll 1, b/R \ll 1, \mu Q \gg 1$	$2/s$
$\mu R \ll 1, b/R \ll 1, \mu Q \ll 1, b/Q \ll 1$	$2/s$
$\mu R \ll 1, b/R \gg 1, b\mu \ll 1, \mu Q \gg 1$	$\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})} \frac{R^s}{b^s}$
$\mu R \ll 1, b/R \gg 1, b\mu \ll 1, \mu Q \ll 1, b/Q \ll 1$	$\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})} \frac{R^s}{b^s}$
$\mu R \ll 1, b/R \gg 1, b\mu \ll 1, \mu Q \ll 1, b/Q \gg 1$	$\frac{\sqrt{\pi}\Gamma(p)}{2^{p-1}\Gamma(\frac{p+1}{2})} \frac{Q^{p-s} R^s}{b^p}$
$\mu R \ll 1, b/R \gg 1, b\mu \gg 1, \mu Q \gg 1$	$\frac{\pi\Gamma(s)}{2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})} \frac{(R\mu)^s}{b\mu}$
$\mu R \ll 1, b/R \gg 1, b\mu \gg 1, \mu Q \ll 1, b/Q \gg 1$	$\frac{\pi\Gamma(p)}{2^{p-1}\Gamma(\frac{p}{2})\Gamma(\frac{p+1}{2})} \frac{R^s \mu^{p-1}}{Q^{s-p} b}$

so that

$$\frac{\lambda}{\lambda_0} = \frac{3}{\sqrt{\pi}(s-1)} \frac{R^2}{b k_{\min}} \cdot K \tag{62}$$

with the integral

$$K = \int_0^1 \mathrm{d}\mu \frac{1-\mu^2}{I(\mu)}. \tag{63}$$

For K we obtain the 12 cases listed in Table 4, where we have used the same approximations as for the damping model. We can use these approximations for values of $1 < s < 2$, $2 < p$ and $R \ll Q$.

For the random sweeping model we also notice that for the first three cases of Table 4 ($R = R_{\mathrm{L}} k_{\min} \gg 1$) the mean free path is unphysically large. The second three cases are not relevant for typical heliosperic parameters again. Like for the DT-model, only the last six cases are important for calculating the mean free path in the limit $R_{\mathrm{L}} k_{\min} < 1$.

3.3. Calculating the mean free path for special parameters

Here we calculate λ for electrons, positrons and protons for special parameters and compare it with numerical solutions to test the approximations we have done. We use the following set of parameters appropriate for interplanetary conditions (Bieber et al. 1994):

$$\begin{aligned} B_0 &= 4.12 \text{ nT} \\ k_{\min} &= 10^{-10} \text{ m}^{-1} \\ k_d &= 2 \cdot 10^{-5} \text{ m}^{-1} \\ s &= 5/3 \\ p &= 3 \\ v_A &= 33.5 \text{ km s}^{-1} \\ \alpha &= 1. \end{aligned} \tag{64}$$

With these parameters and with Eq. (39) it is very easy to calculate R, Q, a and b as functions of the rigidity r, and to derive the parallel mean free path for the damping model of dynamical turbulence (using Table 3) and for the random sweeping model (using Table 4). For these special parameters, the restriction $R_{\mathrm{L}} k_{\min} < 1$ corresponds to rigidities $r < 1.23 \times 10^4 MV \simeq 10^4 MV$, but for the dynamical turbulence model we have also calculated the mean free path at high rigidities in Appendix C.

Table 3. These are the expressions for K and therefore the formulas for the mean free path for the DT-model. The functions f_1 and f_2 are defined in Table 1.

case	K
$1 \ll a \ll R \ll Q$	$\frac{s}{15}a^2$
$1 \ll R \ll Q \ll a$	$\frac{s}{15}a^2$
$1 \ll R \ll a \ll Q$	$\frac{s}{15}a^2$
$a \ll 1 \ll R \ll Q$	$\frac{s}{15}a^2 + \frac{s}{2R}$
$a \ll R \ll Q \ll 1$	$\frac{s}{3}$
$a \ll R \ll 1 \ll Q$	$\frac{s}{3}$
$R \ll Q \ll 1 \ll a$	$\frac{a^2}{f_1 R^s Q^{2-s}} \left\{ {}_2F_1\left(1, \frac{1}{p-1}, \frac{p}{p-1}, -\frac{\pi a}{f_1} Q^{p-2}\right) - \frac{1}{3}\, {}_2F_1\left(1, \frac{3}{p-1}, \frac{p+2}{p-1}, -\frac{\pi a}{f_1} Q^{p-2}\right)\right\}$
$R \ll Q \ll a \ll 1$	$\frac{2}{3f_1} \frac{a^2}{R^s Q^{2-s}}$
$R \ll 1 \ll a \ll Q$	$\frac{1}{\pi}\left[\frac{1}{2-s} - \frac{1}{4-s}\right] \frac{a}{R^s}$
$R \ll 1 \ll Q \ll a$	$\frac{1}{\pi}\left[\frac{1}{2-s} - \frac{1}{4-s}\right] \frac{a}{R^s} + \frac{a^2}{f_1 R^s Q^{3-s}}\, {}_2F_1\left(1, \frac{1}{p-1}, \frac{p}{p-1}, -\frac{\pi a}{f_1 Q}\right)$
$R \ll a \ll 1 \ll Q$	$\frac{2}{3f_2} \frac{a^s}{R^s}$
$R \ll a \ll Q \ll 1$	$\frac{2}{3f_2} \frac{a^s}{R^s}$

Table 4. These are the expressions for K and therefore the formulas for the mean free path for the RS-model.

case	K
$1 \ll b \ll R \ll Q$	$\frac{\sqrt{\pi}s}{4ib}\left(1 - \frac{1}{2b^2}\right)\mathrm{erf}(ib) + \frac{s}{4b^2}\mathrm{e}^{b^2}$
$1 \ll R \ll Q \ll b$	$\frac{\sqrt{\pi}s}{4ib}\left(1 - \frac{1}{2b^2}\right)\mathrm{erf}(ib) + \frac{s}{4b^2}\mathrm{e}^{b^2}$
$1 \ll R \ll b \ll Q$	$\frac{\sqrt{\pi}s}{4ib}\left(1 - \frac{1}{2b^2}\right)\mathrm{erf}(ib) + \frac{s}{4b^2}\mathrm{e}^{b^2}$
$b \ll 1 \ll R \ll Q$	$\frac{\sqrt{\pi}s}{4ib}\left(1 - \frac{1}{2b^2}\right)\mathrm{erf}(ib) + \frac{s}{4b^2}\mathrm{e}^{b^2}$
$b \ll R \ll Q \ll 1$	$s/3$
$b \ll R \ll 1 \ll Q$	$s/3$
$R \ll Q \ll 1 \ll b$	$\frac{2^{p-1}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(p)}\left[1 + \frac{\Gamma(p/2)}{(p-2)\sqrt{\pi}}\right]\frac{b^{p-1}}{Q^{p-s}R^s}$
$R \ll Q \ll b \ll 1$	$\frac{2^p\Gamma(\frac{p+1}{2})}{3\sqrt{\pi}\Gamma(p)}\frac{b^p}{Q^{p-s}R^s}$
$R \ll 1 \ll b \ll Q$	$\frac{2^{s-1}\Gamma(s/2)\Gamma(\frac{s+1}{2})}{\pi\Gamma(s)}\left[\frac{1}{2-s} - \frac{1}{4-s}\right]\frac{b}{R^s}$
$R \ll 1 \ll Q \ll b$	$\frac{2^{p-1}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}\Gamma(p)}\left[1 + \frac{\Gamma(p/2)}{\sqrt{\pi}(p-2)}\right]\frac{b^{p-1}}{R^s Q^{p-s}} + \frac{2^{s-1}\Gamma(\frac{s+1}{2})\Gamma(s/2)}{\pi\Gamma(s)}\left[\frac{1}{2-s} - \frac{1}{4-s}\right]\frac{b}{R^s}$
$R \ll b \ll 1 \ll Q$	$\frac{2^s\Gamma(\frac{s+1}{2})}{3\sqrt{\pi}\Gamma(s)}\frac{b^s}{R^s}$
$R \ll b \ll Q \ll 1$	$\frac{2^s\Gamma(\frac{s+1}{2})}{3\sqrt{\pi}\Gamma(s)}\frac{b^s}{R^s}$

3.3.1. Damping model of dynamical turbulence

For protons we obtain two different ranges of λ:

$$\frac{\lambda}{\lambda_0}\left(10^{-1}MV \ll r \ll 10^4 MV\right) \approx 0.0106 \text{ AU} \cdot \left(\frac{r}{MV}\right)^{1/3}$$
$$\frac{\lambda}{\lambda_0}\left(r \ll 10^{-1}MV\right) \approx 0.0062 \text{ AU} \cdot \left(\frac{r}{MV}\right) \qquad (65)$$

with $r_0 = 938\ MV$. For electrons and positrons we also obtain two ranges:

$$\frac{\lambda}{\lambda_0}\left(10^{-1}MV \ll r \ll 10^4 MV\right) \approx 0.0106 \text{ AU} \times \left(\left(\frac{r}{MV}\right)^{1/3} + \frac{3.57}{\left(\left(\frac{r_0}{MV}\right)^2 + \left(\frac{r}{MV}\right)^2\right)^{1/4}}\right)$$

$$\frac{\lambda}{\lambda_0}\left(r \ll 10^{-1}MV\right) \approx 0.0337 \text{ AU} \times \left\{\left[1 + \left(\frac{0.003MV}{r}\right)^2\right]\arctan\left(\frac{r}{0.003MV}\right) - \frac{0.003MV}{r}\right\}$$

with $r_0 = 0.511\ MV$. Figure 2 shows that the approximations agree very well with the exact numerically integrated results (crosses) for small and medium rigidities.

3.3.2. Random sweeping model

For protons we obtain two ranges of λ:

$$\frac{\lambda}{\lambda_0}\left(10^{-1}MV \ll r \ll 10^4 MV\right) \approx 0.0107 \text{ AU} \cdot \left(\frac{r}{MV}\right)^{1/3}$$
$$\frac{\lambda}{\lambda_0}\left(r \ll 10^{-1}MV\right) \approx 0.0123 \text{ AU} \cdot \left(\frac{r}{MV}\right) \qquad (66)$$

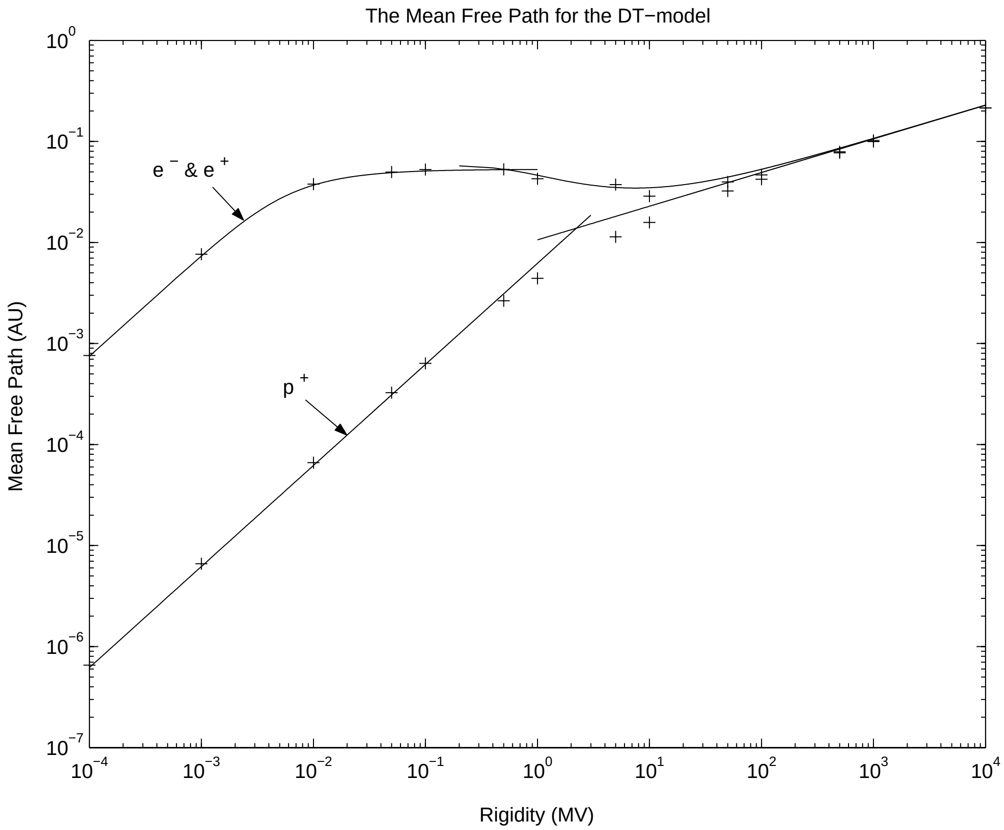

Fig. 2. Parallel mean free paths for electrons, positrons and protons in the damping model of dynamical turbulence. The crosses are the numerical results, the lines are our approximations.

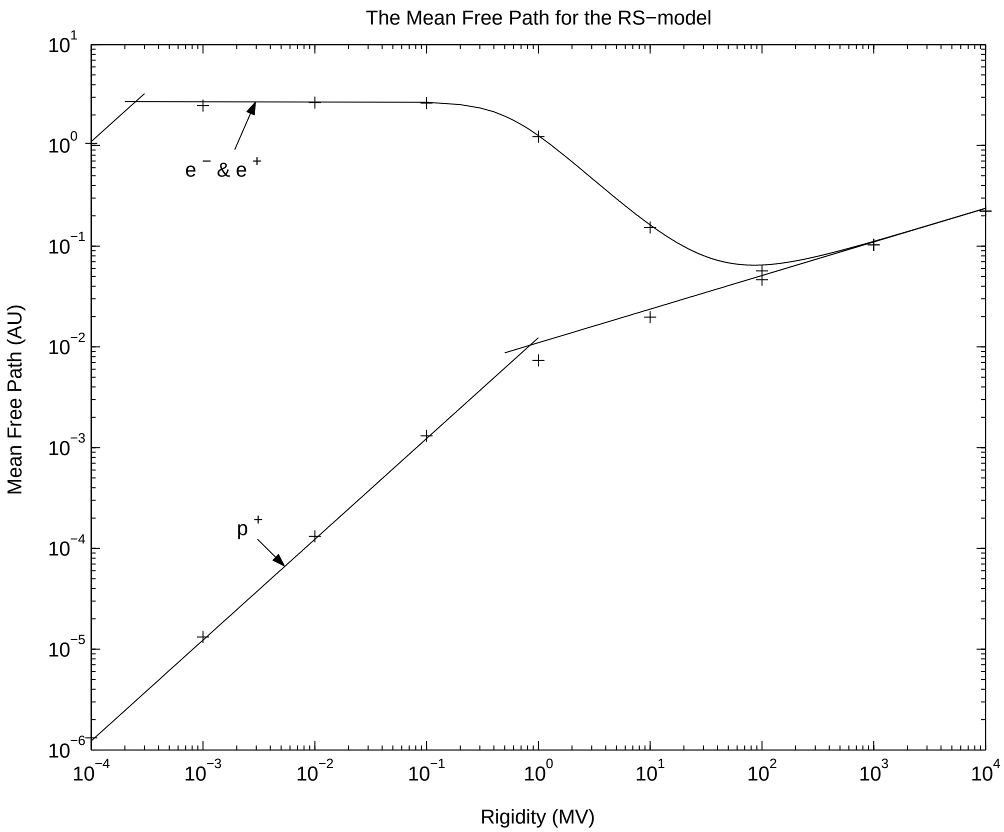

Fig. 3. Parallel mean free paths for electrons, positrons and protons in the random sweeping model. The crosses are the numerical results, the lines are our approximations.

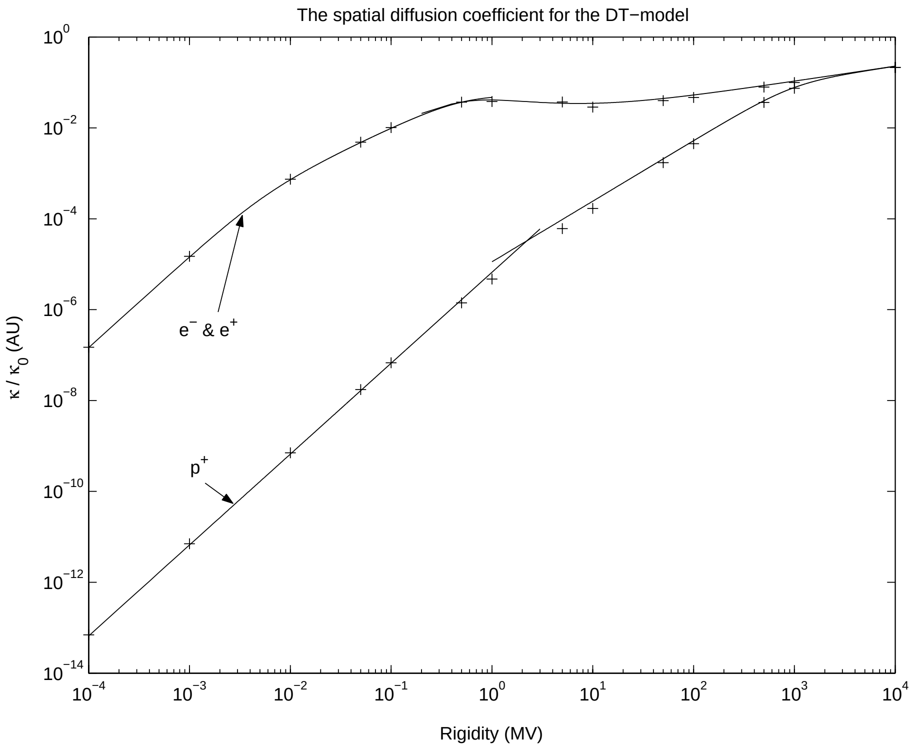

Fig. 4. The parallel spatial diffusion coefficient $\kappa_\|$ for the damping model of dynamical turbulence.

with $r_0 = 938\ MV$. For electrons we also obtain the two ranges:

$$\frac{\lambda}{\lambda_0}\left(10^{-4}MV \ll r \ll 10^{4}MV\right) \approx \frac{1.36\ \mathrm{AU}}{\sqrt{\left(\frac{r_0}{MV}\right)^2+\left(\frac{r}{MV}\right)^2}} + 0.0107\ \mathrm{AU}\cdot\left(\frac{r}{MV}\right)^{1/3}$$

$$\frac{\lambda}{\lambda_0}\left(r \ll 10^{-4}MV\right) \approx 10.4\times 10^{3}\cdot\left(\frac{r}{MV}\right) \quad (67)$$

with $r_0 = 0.511\ MV$. Figure 3 shows the analytic approximations in comparison with numerical results (crosses) for small and medium rigidities. For both models (DT and RS) we have very good agreement between the approximations and the numerical results for the parallel mean free path. To obtain the approximations for the mean free path we have only made the assumptions that $k_{\min}$ is much smaller than k_d, $1 < s < 2$, $2 < p$ and $R_L k_{\min} < 1$.

3.3.3. The parallel spatial diffusion coefficient

In this section we calculate the parallel spatial diffusion coefficient $\kappa_\|$. With the equations for the mean free path and

$$\kappa_\| = \frac{v}{3}\lambda \quad (68)$$

we can write the parallel spatial diffusion coefficient as

$$\frac{\kappa_\|}{\kappa_0} = \frac{r}{\sqrt{r_0^2+r^2}}\frac{\lambda}{\lambda_0} \quad (69)$$

with

$$\kappa_0 = \frac{c\lambda_0}{3}. \quad (70)$$

Figures 4 and 5 show $\kappa_\|/\kappa_0$ for the DT- and the RS-model, respectively.

If we would adopt as in many previous studies the simple relation $\kappa_\perp = \alpha\kappa_\|$ with constant α, Figs. 4 and 5 also show the rigidity dependence of the perpendicular spatial diffusion coefficient for the dynamical turbulence and random sweeping turbulence model.

4. Summary and conclusion

The parallel mean free path of cosmic ray particles in partially turbulent electromagnetic fields is a key input parameter for cosmic ray transport. In this work we have calculated the parallel mean free path of cosmic ray protons, electrons and positrons in two particular turbulence models: slab-like dynamical and random sweeping turbulence. After outlining the general quasilinear formalism for deriving the pitch-angle Fokker-Planck coefficient in weak turbulence from the particle's equation of motion, we explicitly determine the rigidity dependence and the absolute value of the mean free path for different cosmic ray particles for these specific turbulence models. Besides illustrating the results we also derive approximations for the mean free path for realistic Kolmogorov-type turbulence power spectra which include the steepening at high wavenumbers due to turbulence dispersion and/or dissipation.

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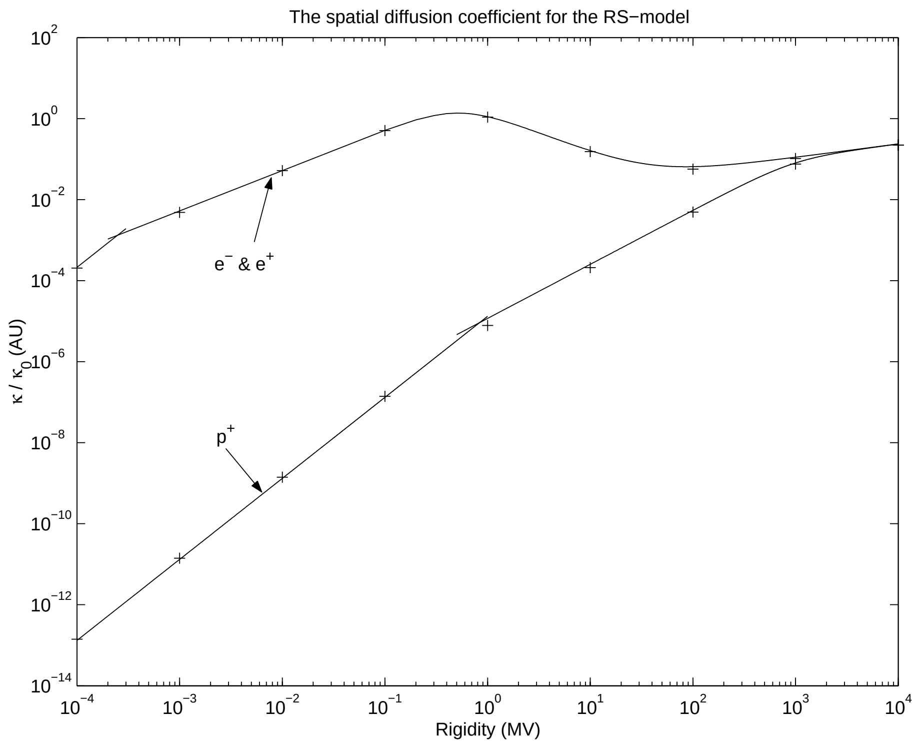

Fig. 5. The parallel spatial diffusion coefficient $\kappa_\|$ for the random sweeping model.

Appendix A: Solving the integrals for the damping model of dynamical turbulence

For the damping model of dynamical turbulence we must solve integrals of the type

$$M = \int_0^1 \mathrm{d}x x^{s-1}\left[\frac{1}{1+a^2/R^2\,(\mu R-x)^2}+\frac{1}{1+a^2/R^2\,(\mu R+x)^2}\right]\cdot \quad (A.1)$$

While this integral can be solved exactly, for the μ-integration we must employ approximations for three special cases.

A.1. The case $\mu R \gg 1$

In this case we can use $\mu R \gg x$

$$M_1 \approx 2\int_0^1 \mathrm{d}x\, x^{s-1}\frac{1}{1+a^2\mu^2} = \frac{2}{s}\frac{1}{1+a^2\mu^2}\cdot \quad (A.2)$$

A.2. The case $\mu R \ll 1$ and $a/R \ll 1$

In this case we can use $a\mu \ll 1$ so that

$$\frac{1}{1+a^2/R^2\,[\mu R \mp x]^2} \approx 1. \quad (A.3)$$

Hence we obtain

$$M_2 \approx 2\int_0^1 \mathrm{d}x\, x^{s-1} = \frac{2}{s}\cdot \quad (A.4)$$

A.3. The case $\mu R \ll 1$ and $a/R \gg 1$

In the last case we write down the exact solution of the integral M

$$M_3 = \frac{1}{s}\left\{\frac{1}{1-ia\mu}\,{}_2F_1\left(1,\frac{s}{2},\frac{s+2}{2},-\frac{a^2}{R^2}\frac{1}{(1-ia\mu)^2}\right) + \frac{1}{1+ia\mu}\,{}_2F_1\left(1,\frac{s}{2},\frac{s+2}{2},-\frac{a^2}{R^2}\frac{1}{(1+ia\mu)^2}\right)\right\} \quad (A.5)$$

in terms of confluent hypergeometric functions ${}_2F_1$. To continue with the calculations, we look at three special cases of the integral M_3. The first case is $a\mu \ll 1$. Here we find

$$M_3 \approx \frac{2}{s}\,{}_2F_1\left(1,\frac{s}{2},\frac{s+2}{2},-\frac{a^2}{R^2}\right) \quad (A.6)$$

what can be approximated as

$$M_3\,(a\mu \ll 1, 1 < s < 2) \approx \frac{\pi}{\sin\left(\frac{\pi s}{2}\right)}\frac{R^s}{a^s} - \frac{2}{2-s}\frac{R^2}{a^2}$$
$$M_3\,(a\mu \ll 1, 2 < s) \approx \frac{2}{s-2}\frac{R^2}{a^2}\cdot \quad (A.7)$$

In the other case we transform the hypergeometric functions to obtain

$$M_3\,(a\mu \gg 1) \approx \pi\frac{R^s}{a}\mu^{s-1} + \frac{2}{s-2}\frac{R^2}{a^2}\cdot \quad (A.8)$$

Appendix B: Solving the integrals for the random sweeping model

For the RS-model we must solve integrals of the type

$$M = \int_0^1 \mathrm{d}x x^{s-1}\left[\mathrm{e}^{-b^2[\mu-x/R]^2}+\mathrm{e}^{-b^2[\mu+x/R]^2}\right]$$
$$= \int_0^1 \mathrm{d}x x^{s-1}\left[\mathrm{e}^{-b^2/R^2[\mu R-x]^2}+\mathrm{e}^{-b^2/R^2[\mu R+x]^2}\right]. \quad \text{(B.1)}$$

The integrals can not be solved exactly, but we can again employ approximations for three special cases.

B.1. The case $\mu R \gg 1$

In this case we can use $\mu R \gg x$ so that

$$M_1 \approx 2\int_0^1 \mathrm{d}x x^{s-1}\mathrm{e}^{-\mu^2 b^2} = \frac{2}{s}\mathrm{e}^{-\mu^2 b^2}. \quad \text{(B.2)}$$

B.2. The case $\mu R \ll 1$ and $b/R \ll 1$

In this case we can use $\mu b \ll 1$ so that

$$\mathrm{e}^{-b^2[\mu\mp x/R]^2} = \mathrm{e}^{-[b\mu\mp xb/R]^2} \approx 1 \quad \text{(B.3)}$$

and we obtain for the integral

$$M_2 \approx 2\int_0^1 \mathrm{d}x x^{s-1} = \frac{2}{s}\cdot \quad \text{(B.4)}$$

B.3. The case $\mu R \ll 1$ and $b/R \gg 1$

For $x = 1$ we have

$$\mathrm{e}^{-b^2[\mu\mp x/R]^2} = \mathrm{e}^{-b^2/R^2[\mu R\mp x]^2} \approx \mathrm{e}^{-b^2/R^2}. \quad \text{(B.5)}$$

So we can use

$$M_3 = \frac{R^s}{b^s}\int_0^{b/R} \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-[b\mu-x]^2}+\mathrm{e}^{-[b\mu+x]^2}\right]$$
$$\approx \frac{R^s}{b^s}\int_0^{\infty} \mathrm{d}x\, x^{s-1}\left[\mathrm{e}^{-[b\mu-x]^2}+\mathrm{e}^{-[b\mu+x]^2}\right]. \quad \text{(B.6)}$$

The integrals can be solved in terms of parabolic cylinderfunctions D_{-s}

$$\frac{R^s}{b^s}\mathrm{e}^{-b^2\mu^2}\int_0^{\infty}\mathrm{d}x\, x^{s-1}\cdot \mathrm{e}^{-x^2}\cdot \mathrm{e}^{\pm 2b\mu x} = \frac{R^s}{b^s}\frac{\Gamma(s)}{2^{s/2}}\mathrm{e}^{-b^2\mu^2/2}D_{-s}(\mp\sqrt{2}b\mu) \quad \text{(B.7)}$$

what can be expressed as Kummer's functions:

$$\frac{R^s}{b^s}\mathrm{e}^{-b^2\mu^2}\int_0^{\infty}\mathrm{d}x x^{s-1}\cdot \mathrm{e}^{-x^2}\cdot \mathrm{e}^{\pm 2b\mu x} = \frac{R^s}{b^s}\frac{\sqrt{\pi}\Gamma(s)}{2^s\Gamma(\frac{s+1}{2})}\mathrm{e}^{-b^2\mu^2}$$
$$\times\left[{}_1F_1\left(\frac{s}{2},\frac{1}{2};b^2\mu^2\right) \pm \frac{2\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})}b\mu\, {}_1F_1\left(\frac{s+1}{2},\frac{3}{2};b^2\mu^2\right)\right]. \quad \text{(B.8)}$$

So we approximately obtain for the integral M_3

$$M_3 \approx \frac{R^s}{b^s}\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})}\mathrm{e}^{-b^2\mu^2}\ {}_1F_1\left(\frac{s}{2},\frac{1}{2};b^2\mu^2\right). \quad \text{(B.9)}$$

To continue the calculations we must look at two special cases of the integral M_3. The first case is $b\mu \ll 1$. Here we can use the approximation

$${}_1F_1\left(\frac{s}{2},\frac{1}{2};b^2\mu^2\right) \approx 1. \quad \text{(B.10)}$$

In the case $b\mu \gg 1$ we can use

$${}_1F_1\left(\frac{s}{2},\frac{1}{2};b^2\mu^2\right) \approx \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{s}{2})}\mathrm{e}^{b^2\mu^2}(b\mu)^{s-1} \quad \text{(B.11)}$$

so that we obtain:

$$M_3(b\mu \ll 1) \approx \frac{R^s}{b^s}\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})}\mathrm{e}^{-b^2\mu^2} \approx \frac{R^s}{b^s}\frac{\sqrt{\pi}\Gamma(s)}{2^{s-1}\Gamma(\frac{s+1}{2})}$$
$$M_3(b\mu \gg 1) \approx \frac{R^s}{b^s}\frac{\pi\Gamma(s)}{2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}(b\mu)^{s-1}. \quad \text{(B.12)}$$

Appendix C: The mean free path for high rigidities

If we calculate the mean free path for $R \gg 1$ and for the damping model of dynamical turbulence we find with Table 3

$$\frac{\lambda}{\lambda_0}(R \gg 1) \approx \frac{3}{s-1}\frac{R^2}{k_{\min}a}\frac{s}{15}a^2 \quad \text{(C.1)}$$

and for the special parameters of Sect. 3.3 we find for protons, positrons and electrons

$$\frac{\lambda}{\lambda_0}\left(r \gg 10^4 MV\right) \simeq 1.96\times 10^{-6}\ \mathrm{AU}\cdot\left(\frac{r}{MV}\right)^2\cdot \quad \text{(C.2)}$$

Figure C.1 shows the analytical results in comparison with the numerically integrated results (crosses). Using $\delta B \approx 0.33 B_0$ so that $\lambda_0 = 10$, Eq. (C.2) becomes

$$\lambda\left(r \gg 10^4 MV\right) \simeq 2\times 10^{-5}\ \mathrm{AU}\left(\frac{r}{MV}\right)^2 \quad \text{(C.3)}$$

and we note that for rigidities larger than $10^4\ MV$, Eq. (C.3) implies values of the mean free path $\lambda > 2000$ AU which is much larger than the size of the heliosphere ($L \simeq 100$ AU). Although formally mathematicaly correct, such large values of the mean free path are unphysical because the diffusion approximation of cosmic ray transport breaks down when the condition $\lambda \ll L$ is violated. In the random sweeping model we obtain even larger values of the mean free path of high rigidities.

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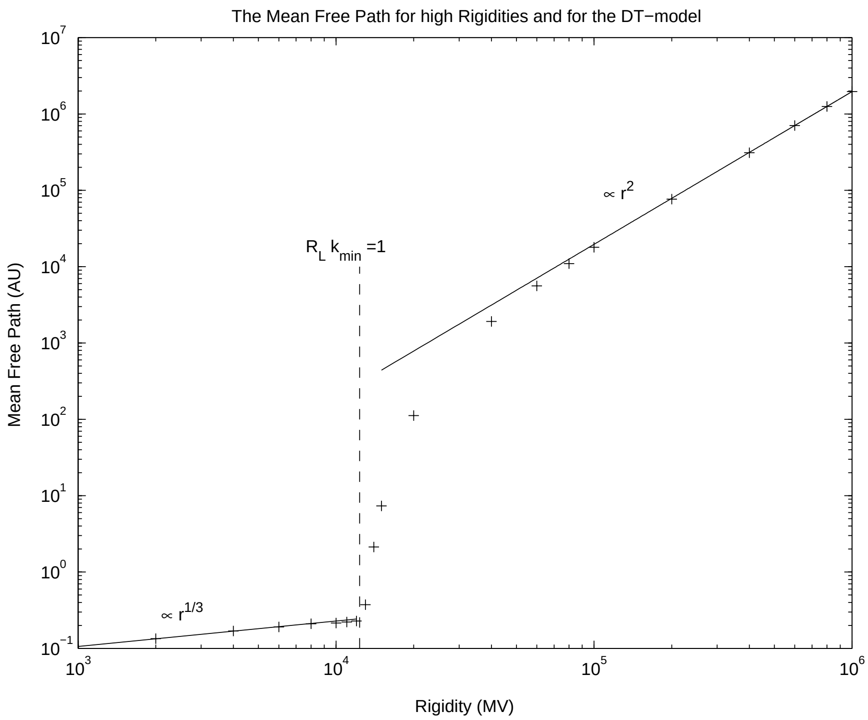

Fig. C.1. Parallel mean free paths for electrons, positrons and protons in the damping model of dynamical turbulence for high rigidities. The crosses are the numerical results, the lines are our approximations.

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