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ANALYTIC CAPACITY AND LINEAR MEASURE

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INTRODUCTION

As usual, we shall denote by $\mathbb{C}$ the set of all complex numbers which will be identified with the Euclidean plane $\mathbb{R}^2$. For $M \subset \mathbb{C}$ we shall denote by $\overline{M}$ and diam M the closure and the diameter of M, respectively. Given $\varepsilon > 0$ we put

$$\mathscr{H}^1_\varepsilon(M) = \inf \sum_{n=1}^{\infty} \operatorname{diam} M_n\,,$$

where the infimum is taken over all sequences of sets $M_n \subset \mathbb{C}$ with diam $M_n \leqq \varepsilon$ such that

$$M \subset \bigcup_{n=1}^{\infty} M_n\,.$$

The linear measure $(=$ length$)$ of M is defined by

$$\mathscr{H}^1(M) = \lim_{\varepsilon \to 0+} \mathscr{H}^1_\varepsilon(M)\,.$$

If $K \subset \mathbb{C}$ is a compact set, then $A(K, 1)$ will stand for the class of all holomorphic functions φ on $\mathbb{C} \setminus K$ with

$$|\varphi| \leqq 1\,, \quad \varphi(\infty) \equiv \lim_{z \to \infty} \varphi(z) = 0\,.$$

For any $\varphi \in A(K, 1)$ the derivative

$$\varphi'(\infty) = \lim_{z \to \infty} z\, \varphi(z)$$

is available and the analytic capcity of K is defined by

$$\gamma(K) = \sup \{|\varphi'(\infty)|;\ \varphi \in A(K, 1)\}\,.$$

This quantity plays an important role in a number of investigations in complex function theory (cf. [1]–[7]) and much research has been done on its relations to various measures of K and, in particular, to $\mathscr{H}^1(K)$ (cf. [8]–[10] where further references may be found). If K is situated on a straight line, then the equality

$$\gamma(K) = \tfrac{1}{4}\,\mathscr{H}^1(K)$$

holds by a result of POMMERENKE (cf. [11], Satz 3, p. 272; see also [8], th. 6.2 on p. 29). For general K the estimate $\gamma(K) \leqq \mathscr{H}^1(K)$ yields the implication

$$\mathscr{H}^1(K) = 0 \Rightarrow \gamma(K) = 0$$

which also follows from a classical result of PAINLEVÉ [12]. The converse of this implication does not hold and examples were exhibited by VITUŠKIN [13] and GARNETT [14] (compare also [16], pp. 346–348), showing that $\gamma(K) = 0$ is possible for disconnected K with $\mathscr{H}^1(K) > 0$. For compact sets K situated on sufficiently smooth curves such a situation cannot occur, because $\gamma(K)$ can be estimated from below by a multiple of $\mathscr{H}^1(K)$; general smoothness restrictions on the curve (stronger than the mere existence of a continuous tangent) sufficient for such estimates have been established by IVANOV [15] (compare also [17]). The assertion that $\gamma(K) > 0$ for every compact set K with $\mathscr{H}^1(K) > 0$, K situated on a rectifiable curve, is known as the Denjoy conjecture (cf. [8], p. 36). It was shown by DAVIE [18] that the validity of the Denjoy conjecture for C^1 curves would imply its validity for general rectifiable curves.*) On the other hand MATYSKA has shown in [30] by modifying the method of Vituškin [13] that there exists a non rectifiable curve $y = f(x)$, with f satisfying a Hölder condition for every exponent less than 1, carrying a compact set K with $\gamma(K) = 0$ and $\mathscr{H}^1(K) > 0$.

In the present paper we shall be concerned with geometric conditions on plane continua Q (which need not be smooth, in general) guaranteeing the validity of an estimate of the form

$$\gamma(K) \geqq \text{const}\ \mathscr{H}^1(K)$$

for all compact sets $K \in Q$. In order to be able to formulate our main result we shall first introduce the following

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*) Added in October, 1977: It has recently been proved by Calderón [31] that the singular integral operator $f \to \varphi$ on a C^1 curve K, given by the Cauchy integral

$$\varphi(\tau) = P.V. \int_K \frac{f(t)\,\mathrm{d}t}{t-\tau},$$

is bounded in $L^p(K)$ for $p > 1$. This in combination with earlier results of HAVIN and HAVINSON [10] (cf. p. 791) and Havin [32] (cf. p. 512) implies the validity of the Denjoy conjecture.

The authors are indebted to L. I. HEDBERG for the reference [31].

Notation. Let

$$\Gamma = \{\zeta \in \mathbb{C};\ |\zeta| = 1\}$$

be the unit circumference. Given $z \in \mathbb{C}$ we denote by

$$\pi_z : \zeta \to \frac{\zeta - z}{|\zeta - z|}$$

the projection of $\mathbb{C} \setminus \{z\}$ onto Γ. For $M \subset \mathbb{C}$ the symbol χ_M is used to denote the characteristic (= indicator) function of M. If $Q \subset \mathbb{C}$ is compact, we define for $\theta \in \Gamma$

$$N_z^Q(\theta) = \sum \chi_Q(u)\,, \quad u \in Q \setminus \{z\}\,, \quad \pi_z(u) = \theta$$

(with the sum extended over all $u \in \pi_z^{-1}(\theta)$).

Thus $N_z^Q(\theta)$ ($0 \leqq N_z^Q(\theta) \leqq +\infty$) denotes the total number (possibly infinite) of all points in the intersection of Q with the half-line $\{z + t\theta;\ t > 0\}$. It is well known that the function

$$N_z^Q : \theta \mapsto N_z^Q(\theta)$$

(which is called the Banach indicatrix of the mapping π_z) is Borel measurable (cf. [19], p. 217) and we may adopt the following

Definition. *If $Q \in \mathbb{C}$ is compact, we define for any $z \in \mathbb{C}$*

$$v^Q(z) = \int_\Gamma N_z^Q(\theta)\,\mathrm{d}\mathscr{H}^1(\theta)\,.$$

Further we put

$$V(Q) = \sup_{\zeta \in Q} v^Q(\zeta)\,. \tag{1}$$

Our main result may now be formulated as follows.

Theorem. *If $Q \subset \mathbb{C}$ is a fixed continuum (or, more generally, a compact set having only a finite number of components), then for all compact sets $K \subset Q$ the following estimate holds:*

$$\gamma(K) \geqq \frac{1}{2} \cdot \frac{1}{V(Q) + \pi}\,\mathscr{H}^1(K)\,. \tag{2}$$

Of course, (2) is of interest only if

$$V(Q) < \infty\,. \tag{3}$$

If Q is a straight line segment, then $V(Q) = 0$ and (2) reduces to

$$\gamma(K) \geqq \frac{1}{2\pi}\,\mathscr{H}^1(K)\,.$$

Let us note that (3) can be fulfilled also for curves Q that are not smooth and contain many angular points. On the other hand, (3) is not fulfilled for many arcs $Q \equiv Q(f)$ with the equation

$$y = f(x), \quad 0 \leqq x \leqq 1,$$

where $f : \langle 0, 1 \rangle \to \mathbb{R}^1$ is continuously differentiable. If $C^1(\langle 0, 1 \rangle)$ is the Banach space of all continuously differentiable functions f on $\langle 0, 1 \rangle$ vanishing at 0 equipped with the norm

$$\|f\| = \max_{0 \leqq x \leqq 1} |f'(x)|,$$

then the set

$$\{f \in C^1(\langle 0, 1 \rangle);\ v^{Q(f)}(\zeta) = \infty \ \text{ for all } \zeta \in Q(f)\}$$

is residual in $C^1(\langle 0, 1 \rangle)$ (cf. [20]).

The fact that the above theorem holds not only for arcs, but also for continua Q submitted to (3), is based on Ważewski's deep characterization of rectifiable continua [21] (a formulation of Ważewski's result is given below in the proof of lemma 1.6).

We first prove in section 1 that continua Q satisfying (3) are rectifiable. In section 2 we establish a "maximum principle" for the function $v^Q(\cdot) : \mathbb{C} \to \mathbb{R}^1_+$ and finally, in section 3, we give the proof of the main theorem and present several corollaries.

1

We shall start with the following

1.1. Proposition. *Let us suppose that the points $z_1, z_2, z_3 \in \mathbb{C}$ are not situated on a single straight line. If $Q \subset \mathbb{C}$ is a continuum such that $v^Q(z_j) < \infty$ for $j = 1, 2, 3$, then $\mathscr{H}^1(Q) < \infty$.*

Proof. If $z \in Q$, then at least one of the straight lines determined by a couple of the points z_j does not contain z. In view of the compactness of Q it is sufficient to establish the following lemma.

1.2. Lemma. *Let $Q \subset \mathbb{C}$ be a continuum and suppose that the points z_1, z_2 are different and*

$$v^Q(z_1) + v^Q(z_2) < \infty. \tag{4}$$

If L denotes the straight line passing through z_1, z_2, then every point $z \in Q \setminus L$ has an open neighborhood $U \subset \mathbb{C}$ such that $\mathscr{H}^1(U \cap Q) < \infty$.

Proof. By a compact arc we shall always mean a homeomorphic image of a non-degenerate compact interval. If C is a compact arc, then C^0 will denote the open arc obtained by removing the end-points of C.

Let us now fix compact arcs $\Gamma_1, \Gamma_2 \subset \Gamma$ with the end-points $\gamma_j \in \Gamma_1$ and $\delta_j \in \Gamma_2$ $(j = 1, 2)$ such that the following conditions (i)–(iv) hold:

(i) $\Gamma_1 \cap \Gamma_2 = \emptyset$,

(ii) $\pi_{z_j}(z) \in \Gamma_j^0$ $(j = 1, 2)$,

(iii) $K = \pi_{z_1}^{-1}(\Gamma_1) \cap \pi_{z_2}^{-1}(\Gamma_2)$ is a compact set disjoint with L,

(iv) Q has a finite (possibly void) intersection with each of the half-lines $\pi_{z_1}^{-1}(\gamma_j)$, $\pi_{z_2}^{-1}(\delta_j)$ $(j = 1, 2)$.

Let us note that (iv) can be satisfied according to the condition (4) which guarantees that each of the sets

$$(5) \qquad \{\theta \in \Gamma;\ N_{z_j}^Q(\theta) < \infty\} \quad (j = 1, 2)$$

is dense in Γ.

We are going to prove that

$$\mathscr{H}^1(K \cap Q) < \infty\,.$$

For this purpose it is sufficient to show that there is a constant k such that, for any $\varepsilon > 0$,

$$(6) \qquad H_\varepsilon^1(K \cap Q) \leqq k[v^Q(z_1) + v^Q(z_2)]\,.$$

Let us fix $\varepsilon > 0$ and divide the arcs Γ_1 and Γ_2 into a finite number of non-overlapping compact subarcs $\Gamma_1^1, \ldots, \Gamma_1^n$ and $\Gamma_2^1, \ldots, \Gamma_2^m$ by means of the points $\gamma^1 = \gamma_1, \gamma^2, \ldots, \gamma^{n+1} = \gamma_2$ and $\delta^1 = \delta_1, \delta^2, \ldots, \delta^{m+1} = \delta_2$, respectively, in such a way that

$$\operatorname{diam}\,[\pi_{z_1}^{-1}(\Gamma_1^r) \cap \pi_{z_2}^{-1}(\Gamma_2^s)] \leqq \varepsilon$$

and each of the half-lines

$$\pi_{z_1}^{-1}(\gamma^r),\ \pi_{z_2}^{-1}(\delta^s)$$

meets Q in a finite (possibly void) set $(r = 1, \ldots, n;\ s = 1, \ldots, m)$. This is again possible because the sets (5) are dense in Γ. Every set

$$(7) \qquad \pi_{z_1}^{-1}(\Gamma_1^r) \cap \pi_{z_2}^{-1}(\Gamma_2^s) \cap Q\,,$$

considered as a subset of the space Q, has a finite relative boundary B^{rs} $(1 \leqq r \leqq n$, $1 \leqq s \leqq m)$.

Let us now recall a classical result of Janiszewski (cf. [22], p. 112):

If A is a proper closed subset of a continuum Q and C is a component of A, then $C \cap \overline{Q \setminus A} \neq \emptyset$, i.e. C has a non-void intersection with the relative boundary of A in Q.

Hence it follows that each of the sets (7) has only a finite number of components Γ_p^{rs}, $p = 1, \ldots, n_{rs}$. Let us denote by χ_p^{rs} the characteristic function of $\pi_{z_1}(\Gamma_p^{rs})$ on Γ.

Then

$$N^Q_{z_1}(\theta) \geqq \sum_{r,s} \sum_{p=1}^{n_{rs}} \chi^{rs}_p(\theta) \quad \text{for} \quad \theta \in \Gamma \setminus \bigcup_{r,s} \pi_{z_1}(B^{rs}),$$

whence

(8₁) $$v^Q(z_1) = \int_\Gamma N^Q_{z_1}(\theta)\, \mathrm{d}\mathscr{H}^1(\theta) \geqq \sum_{r,s} \sum_{p=1}^{n_{rs}} H^1(\pi_{z_1}(\Gamma^{rs}_p)).$$

Analogously

(8₂) $$v^Q(z_2) \geqq \sum_{r,s} \sum_{p=1}^{n_{rs}} \mathscr{H}^1(\pi_{z_2}(\Gamma^{rs}_p)).$$

Now we shall use the following simple geometric fact whose proof may be found in [23], lemma 1.29:

For every compact set K disjoint with L there exists a constant k (depending on K and on the mutual position of L and K) such that, for every couple of points $\zeta_1, \zeta_2 \in K$,

(9) $$|\zeta_1 - \zeta_2| \leqq k[|\pi_{z_1}(\zeta_1) - \pi_{z_1}(\zeta_2)| + |\pi_{z_2}(\zeta_1) - \pi_{z_2}(\zeta_2)|].$$

Employing (iii) and (9) and using the connectivity of Γ^{rs}_p we obtain the estimate

$$\begin{aligned} \operatorname{diam} \Gamma^{rs}_p &\leqq k[\operatorname{diam} \pi_{z_1}(\Gamma^{rs}_p) + \operatorname{diam} \pi_{z_2}(\Gamma^{rs}_p)] \leqq \\ &\leqq k[\mathscr{H}^1(\pi_{z_1}(\Gamma^{rs}_p)) + \mathscr{H}^1(\pi_{z_2}(\Gamma^{rs}_p))] \end{aligned}$$

which together with (8₁), (8₂) gives

$$\sum_{r,s} \sum_{p=1}^{n_{rs}} \operatorname{diam} \Gamma^{rs}_p \leqq k[v^Q(z_1) + v^Q(z_2)].$$

Since $\operatorname{diam} \Gamma^{rs}_p \leqq \varepsilon (1 \leqq r \leqq n,\ 1 \leqq s \leqq m,\ 1 \leqq p \leqq n_{rs})$, we have

$$\mathscr{H}^1_\varepsilon(K \cap Q) \leqq k[v^Q(z_1) + v^Q(z_2)]$$

and the proof is complete.

Remark. Ideas similar to those employed in the above proof appear in [24].

1.3. Notation and remarks. Let $J \subset \mathbb{R}^1$ be an interval and consider continuous mapping $\psi : J \to \mathbb{C}$. It is well-known that for every $z \in \mathbb{C} \setminus \psi(J)$ there exists a continuous real-valued function $\vartheta^\psi_z(\cdot)$ on J such that

$$\psi(t) - z = |\psi(t) - z| \exp i\, \vartheta^\psi_z(t), \quad t \in J.$$

This continuous single-valued argument ϑ^ψ_z is determined up to an additive constant; if $J = \langle a, b \rangle$ is compact, then the increment

$$\vartheta^\psi_z(b) - \vartheta^\psi_z(a)$$

is independent of the choice of that constant and represents a harmonic function of the variable $z \in \mathbb{C} \setminus \psi(J)$. If, besides that, $\psi(b) = \psi(a)$, then the function

(10) $$z \mapsto \vartheta_z^{\psi}(b) - \vartheta_z^{\psi}(a)$$

is constant on each component of $\mathbb{C} \setminus \psi(J)$.

Suppose now that $C \subset \mathbb{C}$ is a compact arc and $\psi : \langle a, b \rangle \to C$ is the corresponding homeomorphism. Then $|\vartheta_z^{\psi}(b) - \vartheta_z^{\psi}(a)|$ does not depend on the choice of the homeomorphism ψ and we are justified to introduce the notation

$$\Delta_C \arg(z) = |\vartheta_z^{\psi}(b) - \vartheta_z^{\psi}(a)| \ (z \in \mathbb{C} \setminus C)$$

for this quantity which depends on C and z only. The function

(11) $$z \mapsto \Delta_C \arg(z)$$

is continuous and subharmonic on $\mathbb{C} \setminus C$.

If $\zeta \in C^0$, then there are disjoint open sets G_1, G_2 contained in $\mathbb{C} \setminus C$ such that $\bar{G}_j \cap C$ is a neighborhood of ζ in C, each of the functions (10), (11) is uniformly continuous on G_j $(j = 1, 2)$ and $\bar{G}_1 \cup \bar{G}_2$ is a neighborhood of ζ in $\mathbb{C}$.

To see this it is sufficient to place the arc C on a Jordan curve $\tilde{C}$ (which is always possible by [22], p. 381) or, which is just the same, to extend ψ from $\langle a, b \rangle$ to a continuous mapping $\tilde{\psi} : \langle a, b + 1 \rangle \to \mathbb{C}$ in such a way that $\tilde{\psi}(b + 1) = \tilde{\psi}(a)$ and $\tilde{\psi}(u) \neq \tilde{\psi}(v)$ whenever $0 < |u - v| < b + 1 - a$, $u, v \in \langle a, b + 1 \rangle$. By the Jordan theorem, the complement of $\tilde{C} = \tilde{\psi}(\langle a, b + 1 \rangle)$ consists precisely of two components G, E with $\bar{E} \cap \bar{G} = \tilde{C}$, $\bar{G} \cup \bar{E} = \mathbb{C}$. Since the function

$$z \mapsto [\vartheta_z^{\psi}(b) - \vartheta_z^{\psi}(a)] + [\vartheta_z^{\tilde{\psi}}(b + 1) - \vartheta_z^{\tilde{\psi}}(b)]$$

remains constant on both G and E and the function

(12) $$z \mapsto [\vartheta_z^{\tilde{\psi}}(b + 1) - \vartheta_z^{\tilde{\psi}}(b)]$$

is continuous on $\mathbb{C} \setminus \tilde{\psi}(\langle b, b + 1 \rangle)$, it is sufficient to fix $\varrho > 0$ less than the distance of ζ from $\tilde{\psi}(\langle b, b + 1 \rangle)$ and put

$$G_1 = \{z \in G;\ |z - \zeta| < \varrho\}, \quad G_2 = \{z \in E;\ |z - \zeta| < \varrho\}.$$

Then (12) is uniformly continuous on $\bar{G}_1 \cup \bar{G}_2 = \{z \in \mathbb{C};\ |z - \zeta| \leqq \varrho\}$ and, consequently, the function (10) (and the function (11) as well) is uniformly continuous on each of the sets G_1, G_2.

We have thus seen that (11), (10) are continuously extendable to any point $\zeta \in C^0$ "from both sides of C". In particular, the function (10) (and the function (11) as well) has at most two limit values at any $\zeta \in C^0$ and these depend continuously on ζ. Consequently,

(13) $$\zeta \mapsto \limsup_{z \to \zeta,\, z \in \mathbb{C} \setminus C} \Delta_C \arg(z)$$

is a continuous function of the variable $\zeta \in C^0$.

If f is a (real- or complex-valued) function and J is an interval in the domain of f, then var $[f; J]$ denotes the variation of f on J.

1.4. Lemma. *Let $C \subset \mathbb{C}$ be a compact arc and let $\psi : \langle a, b \rangle \to C$ be the corresponding homeomorphism. Then*

$$v^C(z) = \sum_J \operatorname{var} [\vartheta_z^{\psi_J}; J],$$

where J runs over all components of $\langle a, b\rangle \setminus \psi^{-1}(z)$ and $\vartheta_z^{\psi_J}$ is a continuous single-valued argument of $\psi - z$ on J.

Proof. This follows easily from the Banach theorem on variation of a continuous function (see lemma 2.2 in [25]).

1.5. Lemma. *If $Q \subset C$ is a continuum fulfilling (3), then $\mathscr{H}^1(Q) < \infty$.*

Proof. If Q is contained in a straight line, then $\mathscr{H}^1(Q) = \operatorname{diam} Q < \infty$. In the opposite case we may pick up three points $z_1, z_2, z_3 \in Q$ that are not situated on a single straith line and apply proposition 1.1.

1.6. Lemma. *If $Q \subset \mathbb{C}$ is a continuum with $\mathscr{H}^1(Q) < \infty$, then there is an increasing sequence of sets K_n, each of them being a union of finitely many disjoint compact arcs, such that*

$$\bigcup_n K_n = Q \setminus Z, \quad \mathscr{H}^1(Z) = 0.$$

Proof. If $Q_1, \ldots, Q_k$ are disjoint continua contained in Q, then

$$\sum_{j=1}^{k} \operatorname{diam} Q_j \leqq \sum_{j=1}^{k} \mathscr{H}^1(Q_j) \leqq \mathscr{H}^1(Q).$$

We see that

$$W = \sup \sum_j \operatorname{diam} Q_j < \infty,$$

where the supremum is taken over all finite disjoint systems of continua $Q_j \subset Q$. In other words, Q is rectifiable in the sense of Ważewski [21]. Ważewski proved that then there exist a mapping

$$\psi : \langle 0, 2W \rangle \to Q$$

onto Q and a sequence of open arcs*) $C_n \subset Q$ such that the set

$$T = \langle 0, 2W \rangle \setminus \psi^{-1}\Bigl(\bigcup_n C_n\Bigr)$$

*) By an open arc we mean a homeomorphic image of (0, 1).

has linear measure zero and ψ fulfils the Lipschitz condition

$$0 \leqq t < u \leqq 2W \Rightarrow |\psi(t) - \psi(u)| \leqq |t - u| .$$

Consequently, $\mathscr{H}^1(\psi(T)) = 0$ and, in view of the inclusion

$$Z = Q \setminus \bigcup_n C_n \subset \psi(T) ,$$

we have $\mathscr{H}^1(Z) = 0$. Each open arc C_n can be expressed as a union of an increasing sequence of compact arcs C_n^k $(k = 1, 2, \ldots)$ and the sets

$$K_n = \bigcup_{j=1}^{n} C_j^n$$

have all the required properties.

1.7. Proposition. *Let $Q \subset \mathbb{C}$ be a compact set with $\mathscr{H}^1(Q) < +\infty$, having only a finite number of components. If $z \in \mathbb{C} \setminus Q$, then*

$$v^Q(z) = \sup \sum_{j=1}^{n} \Delta_{C_j} \arg(z) , \tag{14}$$

where the supremum is taken over all finite systems of mutually disjoint compact arcs $C_1, \ldots, C_n \subset Q$.

Proof. Let us fix $z \in \mathbb{C} \setminus Q$. Given a system of dijoint compact arcs $C_j \subset Q$ $(j = 1, \ldots, n)$ defined by the corresponding homeomorphisms $\psi_j : \langle a_j, b_j \rangle \to C_j$, we have by lemma 1.4

$$v^{C_j}(z) = \operatorname{var}\,[\vartheta_z^{\psi_j}; \langle a_j, b_j \rangle] \geqq \Delta_{C_j} \arg(z) ,$$

whence we get writing $C = \bigcup_{j=1}^{n} C_j$

$$v^Q(z) \geqq v^C(z) = \sum_{j=1}^{n} v^{C_j}(z) \geqq \sum_{j=1}^{n} \Delta_{C_j} \arg(z) .$$

Fix now an arbitrary number

$$d < v^Q(z) . \tag{15}$$

By lemma 1.6 there is an increasing sequence of compact sets $K_n \subset Q$, each consisting of a finite number of disjoint compact arcs, such that (as $n \to \infty$)

$$K_n \nearrow Q \setminus Z , \quad \mathscr{H}^1(Z) = 0 .$$

Consequently, $\mathscr{H}^1(\pi_z(Z)) = 0$ and for $\theta \in \Gamma \setminus \pi_z(Z)$ we have

$$N_z^{K_n}(\theta) \nearrow N_z^Q(\theta) ,$$

whence

$$v_z^{K_n}(z) = \int_\Gamma N_z^{K_n}(\theta)\, \mathrm{d}\mathscr{H}^1(\theta) \nearrow \int_\Gamma N_z^Q(\theta)\, \mathrm{d}\mathscr{H}^1(\theta) = y^Q(z)\,.$$

We can thus fix a natural number m with

$$v^{K_m}(z) > d\,.$$

If K_m consists of disjoint compact arcs C_j $(j = 1, \ldots, k)$ defined by the homeomorphisms $\psi_j : \langle a_j, b_j \rangle \to C_j$, $\bigcup_{j=1}^{k} C_j = K_m$, then by lemma 1.4

$$\sum_{j=1}^{k} \operatorname{var}\,[\vartheta_z^{\psi_j}; \langle a_j, b_j \rangle] = \sum_{j=1}^{k} v^{C_j}(z) = v^{K_m}(z) > d\,.$$

We may thus fix numbers $d_j < \operatorname{var}\,[\vartheta_z^{\psi_j}; \langle a_j, b_j \rangle]$ such that

$$\sum_{j=1}^{k} d_j \geqq d\,.$$

For every j there are disjoint non-degenerate intervals

$$\langle a_j^1, b_j^1 \rangle, \ldots, \langle a_j^{n_j}, b_j^{n_j} \rangle \subset \langle a_j, b_j \rangle$$

such that

$$\sum_{r=1}^{n_j} \left|\vartheta_z^{\psi_j}(b_j^r) - \vartheta_z^{\psi_j}(a_j^r)\right| > d_j\,.$$

Defining $C_j^r = \psi_j(\langle a_j^r, b_j^r \rangle)$ we get

$$\sum_{j=1}^{k} \sum_{r=1}^{n_j} \Delta_{C_j^r} \arg(z) > \sum_{j=1}^{k} d_j \geqq d$$

and the arcs C_j^r are mutually disjoint. This completes the proof of the equality (14).

2

In the introduction we have associated with every compact set $Q \subset \mathbb{C}$ and every $z \in \mathbb{C}$ the quantity $v^Q(z)$ (which is sometimes called the cyclic variation of Q at z). Estimates of the function $v^Q(\cdot)$ on $\mathbb{C} \setminus Q$ in terms of its supremum (1) on Q are useful in various investigations in potential theory (cf. [26]). In § 3 we shall need a precise form of this "maximum principle" in the following formulation.

2.1. Proposition. *Let $Q \subset \mathbb{C}$ be a compact set having only a finite number of components and define $V(Q)$ by (1). Then for any $z \in \mathbb{C}$ the estimate*

$$v^Q(z) \leqq \pi + V(Q) \tag{16}$$

holds.

Before going into the proof of this proposition we shall recall several known auxiliary results.

2.2. Remarks. Let $\psi : \langle a, b\rangle \to \mathbb{C}$ be a homeomorphism, $\psi(\langle a, b\rangle) = C$, and fix $\xi \in (a, b)$, $\psi(\xi) = \zeta$. We shall denote by $\vartheta^{\psi}_{\zeta+}(t)$ and $\vartheta^{\psi}_{\zeta-}(t)$ a continuous single-valued argument of $\psi(t) - \zeta$ on $(\xi, b\rangle$ and on $\langle a, \xi)$, respectively.

According to lemma 1.4

(17) $$v^C(\zeta) = \operatorname{var}[\vartheta^{\psi}_{\zeta+}; (\xi, b\rangle] + \operatorname{var}[\vartheta\gamma^{\psi}_{\zeta-}; \langle a, \xi)],$$

so that the assumption $v^C(\zeta) < \infty$ implies the existence of the limits

$$\lim_{t\to\xi+} \vartheta^{\psi}_{\zeta+}(t) = \vartheta^{\psi}_{\zeta}(\xi+), \quad \lim_{t\to\xi-} \vartheta^{\psi}_{\zeta-}(t) = \vartheta^{\psi}_{\zeta}(\xi-)$$

and, in particular, the existence of half-tangent vectors

$$\tau^{\psi}_{+}(\zeta) = \lim_{t\to\xi+} \frac{\psi(t) - \zeta}{|\psi(t) - \zeta|} = \exp i\, \vartheta^{\psi}_{\zeta}(\xi+),$$

$$\tau^{\psi}_{-}(\zeta) = -\lim_{t\to\xi-} \frac{\psi(t) - \zeta}{|\psi(t) - \zeta|} = -\exp i\, \vartheta^{\psi}_{\zeta}(\xi-).$$

Under the assumption

(18) $$V(C) \equiv \sup_{z\in C} v^C(z) < \infty$$

(which will always be fulfilled below) the half-tangent vectors $\tau^{\psi}_{+}(\zeta)$, $\tau^{\psi}_{-}(\zeta)$ are thus available for all $\zeta \in C^0$.

We shall say that z is an angular point of C if either z is an end-point of C or else $z \in C^0$ and $\tau^{\psi}_{+}(z) \neq \tau^{\psi}_{-}(z)$. It is easily seen that the set of all angular points of C is at most countable (cf. [27], p. 464). Consequently, the set of those $\zeta \in C^0$ at which a unique tangent vector $\tau^{\psi}(\zeta) \equiv \tau^{\psi}(\zeta+) = \tau^{\psi}(\zeta-)$ exists is dense in C. [Of course, this follows also from the known fact that a rectifiable arc C has a unique tangent $\mathscr{H}^1$ – almost everywhere on C.]

Let us now suppose that $\zeta = \psi(\xi)$ is not an angular point of C and put $v = i\,\tau(\zeta)$ [here i denotes the imaginary unit],

$$A(\zeta) = \int_{\langle a,\zeta)} d\vartheta^{\psi}_{\zeta-} + \int_{(\zeta,b\rangle} d\vartheta^{\psi}_{\zeta+}.$$

In accordance with 1.3 we denote by $\vartheta^{\psi}_{z}(t)$ a continuous single-valued argument of $\psi(t) - z$ on $\langle a, b\rangle$ whenever $z \in \mathbb{C} \setminus C$. Then

(19_1) $$\lim_{r\to0+} [\vartheta^{\psi}_{\zeta+rv}(b) - \vartheta^{\psi}_{\zeta+rv}(a)] = A(\zeta) + \pi,$$

(19_2) $$\lim_{r\to0+} [\vartheta^{\psi}_{\zeta-rv}(b) - \vartheta^{\psi}_{\zeta-rv}(a)] = A(\zeta) - \pi,$$

as it follows from [28], th. 2.11 (cf. also 1.1 and 1.5). We have already seen in 1.3 that the function (10) has at most two limit values at ζ. Since the limits (19_1) and (19_2) are different, we conclude that $\{|A(\zeta) + \pi|, |A(\zeta) - \pi|\}$ is just the set of all limit values of the function (11) at ζ. Hence we obtain

2.3. Lemma. *Let $C \subset \mathbb{C}$ be a compact arc satisfying* (18) *and suppose that $\zeta \in C$ is not an angular point of C. Then*

$$\limsup_{z \to \zeta,\, z \in \mathbb{C} \setminus C} \Delta_C \arg(z) \leqq v^C(\zeta) + \pi . \tag{20}$$

In particular, the set of those $\zeta \in C$ for which (20) *holds is dense in C.*

Proof. We have just seen that

$$\limsup_{z \to \zeta,\, z \in \mathbb{C} \setminus C} \Delta_C \arg(z) = \max\{|A(\zeta) + \pi|, |A(\zeta) - \pi|\} . \tag{21}$$

Employing (17) we get

$$|A(\zeta)| \leqq \operatorname{var}[\vartheta^{\psi}_{\zeta-}; \langle a, \xi)] + \operatorname{var}[\vartheta^{\psi}_{\zeta+}; (\xi, b\rangle] = v^C(\zeta)$$

which together with (21) yields (20).

Now we are in position to present the following

2.4. Proof of proposition 2.1. We may clearly suppose that $z \in \mathbb{C} \setminus Q$ and (3) holds. Fix an arbitrary $d < v^Q(z)$. By proposition 1.7 there is a finite system of mutually disjoint compact arcs $C_1, \ldots, C_n$ contained in Q such that

$$d < \sum_{j=1}^{n} \Delta_{C_j} \arg(z) . \tag{22}$$

The function

$$q : \zeta \mapsto \sum_{j=1}^{n} \Delta_{C_j} \arg(\zeta)$$

is continuous and subharmonic on the complement of $K = \bigcup_{j=1}^{n} C_j$ and

$$\lim_{\zeta \to \infty} q(\zeta) = 0 . \tag{23}$$

Besides that, $q(\zeta) \leqq \sum_{j=1}^{n} v^{C_j}(\zeta)$ is bounded on $\mathbb{C} \setminus K$ by proposition 1.5 in [28].

If $\eta \in C_1$, then the function

$$\zeta \mapsto \sum_{k=2}^{n} \Delta_{C_k} \arg(\zeta)$$

is continuous in the vicinity of η. Defining

$$w(\eta) = \limsup_{\zeta \to \eta,\, \zeta \in \mathbb{C} \setminus K} \Delta_{C_1} \arg(\zeta) + \sum_{k=2}^{n} \Delta_{C_k} \arg(\eta),$$

we have thus

$$\limsup_{\zeta \to \eta,\, \zeta \in \mathbb{C} \setminus K} q(\zeta) \leqq w(\eta). \tag{24}$$

As we have observed in 1.3., w is a continuous function of the variable $\eta \in C_1^0$. If $\eta \in C_1^0$ is not an angular point of C_1, then lemma 2.3 gives

$$w(\eta) \leqq \pi + v^{C_1}(\eta) + \sum_{k=2}^{n} \Delta_{C_k} \arg(\eta) \leqq \pi + \sum_{k=1}^{n} v^{C_j}(\eta) = \pi + v^K(\eta) \leqq \pi + V(Q).$$

Since the inequality

$$w \leqq \pi + V(Q) \tag{25}$$

holds on a dense subset of C_1, we infer from the continuity of w that (25) holds everywhere on C_1^0. According to (24) we have

$$\limsup_{\zeta \to \eta,\, \zeta \in \mathbb{C} \setminus K} q(\zeta) \leqq \pi + V(Q) \tag{26}$$

for all $\eta \in C_1^0$. Of course, the same inequality holds for $\eta \in C_j^0$ for any $j = 1, \ldots, n$. We see that (26) holds for all but a finite number of points $\eta \in K$. This together with (23) and the boundedness of q permits us to conclude on account of the maximum principle for subharmonic functions that

$$q \leqq \pi + V(Q) \quad \text{on} \quad \mathbb{C} \setminus K. \tag{27}$$

Combining (27) and (22) we get

$$d < \pi + V(Q).$$

Since d was an arbitrary number satisfying $d < v^Q(z)$ we arrive at (16).

3

Now we shall supply the proof of our main result formulated in the introduction.

3.1. Proof of the theorem. Let $Q \in \mathbb{C}$ be a compact set with $V(Q) < \infty$ consisting of finitely many components. Let us consider an arbitrary compact set $H \subset Q$ with $\mathscr{H}^1(H) > 0$ and fix a $\delta \in (0, 1)$. Let $K_n \nearrow Q \setminus Z$ be a sequence of compact sets with the properties described in lemma 1.6, $\mathscr{H}^1(Z) = 0$. We have then for suitable $K = K_n$

$$\mathscr{H}^1(K \cap H) \geqq \delta\, \mathscr{H}^1(H).$$

Let $K = \bigcup_{j=1}^{m} C_j$, where $C_j = \psi_j(\langle 0, 1\rangle)$ are disjoint compact arcs and $\psi_j : \langle 0, 1\rangle \to C_j$ are the corresponding homeomorphisms $(j = 1, \ldots, m)$. Since $\mathscr{H}^1(Q) < \infty$ by lemma 1.5, each ψ_j must have bounded variation on $\langle 0, 1\rangle$ and the same holds of real-valued functions $\operatorname{Im} e^{i\alpha}\psi_j$, $\alpha \in \langle -\pi, \pi\rangle$. The identifinite variations of the functions ψ_j, $\operatorname{Im} e^{i\alpha}\psi_j$ determine in the usual way Borel measures on $\langle 0, 1\rangle$ which will be denoted by $\operatorname{var} \psi_j$, $\operatorname{var} \operatorname{Im} e^{i\alpha}\psi_j$, respectively. Put $H_j = \psi_j^{-1}(H \cap C_j)$. Then there is a real-valued Baire function f_j on $\langle 0, 1\rangle$ such that

$$|f_j| \leqq 1, \quad f_j(\langle 0, 1\rangle \setminus H_j) = \{0\},$$

$$\int_0^1 f_j \, \mathrm{d} \operatorname{Im} \psi_j = \operatorname{var} \operatorname{Im} \psi_j(H_j).$$

Let us define for $z \in \mathbb{C} \setminus H$

$$\Phi(z) = \sum_{j=1}^{m} \int_0^1 \frac{f_j(t)}{\psi_j(t) - z} \, \mathrm{d}\psi_j(t).$$

Note that $f_j = 0$ outside $H_j = \psi_j^{-1}(H)$, so that Φ is holomorphic on $\mathbb{C} \setminus H$; besides that,

$$\lim_{z \to \infty} \Phi(z) = 0.$$

If $\vartheta_z^{\psi_j}$ denotes a continuous single-valued argument of $\psi_j - z$ on $\langle 0, 1\rangle$, then we get from lemma 1.4

$$|\operatorname{Im} \Phi(z)| = \left| \sum_{j=1}^{m} \int_0^1 f_j \, \mathrm{d}\vartheta_z^{\psi_j} \right| \leqq \sum_{j=1}^{m} \operatorname{var} [\vartheta_z^{\psi_j}; \langle 0, 1\rangle] = \sum_{j=1}^{m} v^{C_j}(z) \leqq v^Q(z),$$

which together with (16) implies

$$|\operatorname{Im} \Phi(z)| \leqq \pi + V(Q). \tag{28}$$

Next we obtain

$$|\Phi'(\infty)| = \lim_{z \to \infty} |z \, \Phi(z)| = \tag{29}$$

$$= \left| \sum_{j=1}^{m} \int_0^1 f_j \, \mathrm{d}\psi_j \right| \geqq \left| \sum_{j=1}^{m} \int_0^1 f_j \, \mathrm{d} \operatorname{Im} \psi_j \right| = \sum_{j=1}^{m} \operatorname{var} \operatorname{Im} \psi_j(H_j).$$

The inequality (28) permits us to conclude that the function

$$F = \frac{1 - \exp \dfrac{\pi\Phi}{2(V(Q) + \pi)}}{1 + \exp \dfrac{\pi\Phi}{2(V(Q) + \pi)}}$$

belongs to $A(H, 1)$ and (29) results in

$$|F'(\infty)| = \frac{\pi}{4(V(Q) + \pi)} |\Phi'(\infty)| \geqq \frac{\pi}{4(V(Q) + \pi)} \sum_{j=1}^{m} \text{var Im } \psi_j(H_j).$$

Consequently, by the definition of the analytic capacity,

$$\gamma(H) \geqq \frac{\pi}{4(V(Q) + \pi)} \sum_{j=1}^{m} \text{var Im } \psi_j(H_j).$$

Since the analytic capacity is invariant with respect to rotations, we have also for any $\alpha \in \langle -\pi, \pi \rangle$

$$\gamma(H) \geqq \frac{\pi}{4(V(Q) + \pi)} \sum_{j=1}^{m} \text{var Im } e^{i\alpha} \psi_j(H_j).$$

Using the well-known formula

$$\frac{1}{4} \int_{-\pi}^{\pi} \text{var Im } e^{i\alpha} \psi_j(H_j) \, d\alpha = \mathscr{H}^1(H \cap C_j)$$

(cf. [33], lemma 13 on p. 59 and also the definition of the so-called linear variation on p. 17) we get

$$\gamma(H) \geqq \frac{1}{2(V(Q) + \pi)} \sum_{j=1}^{m} \mathscr{H}^1(H \cap C_j) = \frac{1}{2(V(Q) + \pi)} \mathscr{H}^1(H).$$

Since $\delta \in (0, 1)$ was arbitrarily chosen, we arrive at

$$\gamma(H) \geqq \frac{1}{2(V(Q) + \pi)} \mathscr{H}^1(H)$$

and the proof is complete.

3.2. Corollary. *Let $Q \subset \mathbb{C}$ be a compact set with* (3) *consisting of finitely many components. Then, for any compact set $H \subset Q$, the inequalities*

$$\frac{1}{2(V(Q) + \pi)} \mathscr{H}^1(H) \leqq \gamma(H) \leqq \mathscr{H}^1(H) \tag{30}$$

are valid; in particular,

$$\gamma(H) = 0 \Leftrightarrow \mathscr{H}^1(H) = 0.$$

Proof. The first inequality occurring in (30) has been proved in 3.1, while the second inequality (which can be further improved) is known – it follows from the elementary fact that $\gamma(H) \leqq r_1 + \ldots + r_n$ if H can be covered by circular discs of radii $r_1, \ldots, r_n$ (cf. [4]).

3.3. Corollary. *Let $Q \subset \mathbb{C}$ be a compact set with $V(Q) < \infty$ consisting of finitely many components. Then for each couple of compact sets $H_j \subset Q$ $(j = 1, 2)$ the inequality*

$$\gamma(H_j \cup H_2) \leqq 2(V(Q) + \pi)\,[\gamma(H_1) + \gamma(H_2)]$$

is true.

Proof. This follows at once from the inequalities established in 3.2.

3.4. Remark. The above corollary shows that the analytic capacity γ is semi-additive on subsets of Q provided Q has the properties described in 3.3. Further comments on the semi-additivity property of the analytic capacity may be found in [29].

References

[1] *L. Ahlfors:* Bounded analytic functions, Duke Math. J. *14* (1947), 1—11.

[2] *L. Ahlfors, A. Beurling:* Conformal invariants and function theoretic null sets, Acta Mathematica *83* (1950), 101—129.

[3] *А. Г. Витушкин:* Аналитическая емкость и некоторые её свойства, Доклады АН СССР *123* (1958), 778—781.

[4] *А. Г. Витушкин:* Аналитическая емкость в задачах теории приближений, Успехи матем. наук *22* (1967), 142—199.

[5] *L. Sario, M. Nakai:* Classification theory of Riemann surfaces, Springer-Verlag 1970.

[6] *L. Zalcman:* Analytic capacity and rational approximation, Lecture Notes in Math. vol. 50, Springer-Verlag 1968.

[7] *T. W. Gamelin:* Uniform algebras, Prentice-Hall 1969.

[8] *J. Garnett:* Analytic capacity and measure, Lecture Notes in Math. vol. 217, Springer-Verlag 1972.

[9] *С. Я. Хавинсон:* О стирании особенностей Литовский математический сборник *III* (1963), 271—287.

[10] *В. П. Хавин, С. Я. Хавинсон:* Некоторые оценки аналитической емкости, Доклады АН СССР *138* (1961), 789—792.

[11] *Ch. Pommerenke:* Über die analytische Kapazität, Archiv der Math. *11* (1960), 270—277.

[12] *P. Painlevé:* Sur les lignes singulières des fonctions analytiques, Ann. Fac. Sci. Toulouse, Sci. Math. et Sci. Phys. *2* (1888), 1—130.

[13] *А. Г. Витушкин:* Пример множества положительной длины, но нулевой аналитической емкости, Доклады АН СССР *127* (1959), 246—249.

[14] *J. Garnett:* Positive length but zero analytic capacity, Proc. Amer. Math. Soc. *24* (1970), 696—699.

[15] *Л. Д. Иванов:* О гипотезе Данжуа, Успехи матем. наук *18* (1964), 147—149.

[16] *Л. Д. Иванов:* Вариации множеств и функций, Изд. „Наука", Москва 1975.

[17] *Н. А. Широков:* Аналитическая емкость множеств, близких к гладкой кривой, Вестник Ленинград. Ун-та *4* (1973), № 19, 73—78, 153.

[18] *A. M. Davie:* Analytic capacity and approximation problems, Trans. Amer. Math. Soc. *171* (1972), 409—444.

[19] *T. Radó, P. V. Reichelderfer:* Contincous transformations in Analysis, Springer-Verlag 1955.

[20] *J. Král:* Hladké funkce s nekonečnou cyklickou variací, Čas. pěst. mat. *93* (1968), 178—185.

[21] *T. Ważewski:* Kontinua prostowalne w związku z funkcjami i odwzorowaniami absolutnie ciagłemi, Ann. Soc. Polon. Math. *3*, Suppl. (1927), 9—49.

[22] *K. Kuratowski:* Topologie II, Warszawa 1952.

[23] *J. Štulc, J. Veselý:* Souvislost cyklické a radiální variace cesty s její délkou a ohybem, Čas. pěst. mat. *93* (1968), 80—116.

[24] *Г. Е. Перевалов:* О мере множеств, лежащих на плоских континуумах, Сибир. матем. жур. *III* (1962), 573—581.

[25] *J. Král:* Some inequalities concerning the cyclic and radial variations of a plane path-curve, Czechoslovak Math. J. *14* (1964), 271—280.

[26] *J. Král:* Teorie potenciálu I, Stát. ped. nakl. Praha 1965.

[27] *J. Král:* The Fredholm radius in potential theory, Czechoslovak Math. J. *15* (1965), 454—473, 565—588.

[28] *J. Král:* Non-tangential limits of the logarithmic potential, Czechoslovak Math. J. *14* (1964), 455—482.

[29] *S. Jacobson:* Pointwise bounded approximation and analytic capacity of open sets, Trans. Amer. Math. Soc. *218* (1976), 261—283.

[30] *J. Matyska:* An example of removable singularities for bounded holomorphic functions, to appear.

[31] *A. P. Calderón:* Cauchy integrals on Lipschitz curves and related operators, Proc. Natl. Acad. Sci. USA *4* (*74*) 1977, 1324—1327.

[32] *В. П. Хавин:* Граничные свойства интегралов типа Коши и гармонически сопряженных функций в областях со спрямляемой границей, Матем. сборник т. *68* (*110*) 1965, 499--517

[33] *А. Г. Витушкин:* О многомерных вариациях, Москва 1955.

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