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ANALYTIC CONTINUATION BY THE FAST FOURIER TRANSFORM*

JOEL FRANKLIN†

Abstract. The ill-posed problem of analytic continuation is regularized by a prescribed bound. A simple computer algorithm is given that is based on the fast Fourier transform. The algorithm computes m complex values and a positive error bound with time complexity $O(m \log m)$. As a function of the data errors and the prescribed bound, the numerical error is shown to be consistent with that prescribed by the three-circles principle of Hadamard.

Key words. analytic continuation, fast Fourier transform, ill posed

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1. Introduction. Analytic continuation is an ill-posed problem because the solution depends discontinuously on the data.

Example 1. Let $f(z)$ be analytic for $1 \leqq |z| \leqq R$. Given that $|f(z)-z| \leqq \varepsilon$ for $|z|=1$, the problem is to compute $f(z)$ in the rest of the annulus:

If $N^{-1} < \varepsilon$, two possible solutions are $f(z) = z \pm N^{-1} z^N$. If $\varepsilon > 0$ and N is very large, the two solutions may differ greatly. Miller [11] has observed that analytic continuation can be regularized by using the three-circles theorem of Hadamard:

THEOREM. *Let $\phi(z)$ be analytic for $1 < |z| < R$ and continuous for $1 \leqq |z| \leqq R$. Let $\mu(\rho) = \max |\phi(z)|$ for $|z| = \rho$. Then $\log \mu(\rho)$ is a convex function of $\log \rho$. Thus, if $1 < r < R$ and if $\theta = (\log r)/\log R$, then*

$$\mu(r) \leqq \mu(1)^{1-\theta} \mu(R)^{\theta}. \tag{1.1}$$

Hardy proved an analogous theorem for an L^2 norm instead of the maximum norm $\mu(\rho)$.

Example 2. As before, let $f(z)$ be analytic for $1 \leqq |z| \leqq R$, and let $|f(z)-z| \leqq \varepsilon$ for $|z|=1$. Now assume $|f(z)| \leqq \beta$ for $|z| = R$. The problem is again to compute $f(z)$ for $1 < |z| < R$.

If $f_1(z)$ and $f_2(z)$ are two possible solutions and $\phi(z) = f_1(z) - f_2(z)$, then Hadamard's theorem implies, for $\theta = (\log r)/\log R$,

$$|\phi(z)| \leqq 2\varepsilon^{1-\theta} \beta^{\theta}. \tag{1.2}$$

Therefore, unless z is near the outer boundary, the difference between two possible solutions must be small.

The annulus is doubly connected. Consider, now, a simply connected region, D. Suppose $f(z)$ is analytic in D. Let $g(z)$ be given data such that

$$|g(z) - f(z)| \leqq \varepsilon \quad \text{for} \quad z \in S, \tag{1.3}$$

where S is an arc in D. The problem is to compute $f(z)$ as accurately as possible in $D - S$. Note that $D - S$ is doubly connected. By conformal mapping we can reduce this problem to the problem for the annulus and apply the three-circles theorem.

Example 3. Let $R > 1$. Let D be the elliptical domain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \tag{1.4}$$

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† Applied Mathematics 217-50, Firestone Laboratory, California Institute of Technology, Pasadena, California 91125.

where

$$a=\tfrac{1}{2}(R+R^{-1}),\qquad b=\tfrac{1}{2}(R-R^{-1}). \tag{1.5}$$

Let S be the interior arc $-1\leqq x\leqq 1$, and $y=0$. For z in S let $g(z)$ be given data. Let the unknown function $f(z)$ be analytic in $\bar{D}$ and assume $|g(z)-f(z)|\leqq \varepsilon$ for z in S. The problem is to compute $f(z)$ in $D-S$.

In its present form the problem is again ill posed and a bound $|f(z)|\leqq\beta$ for z on D is required to obtain a computational solution. If we use the conformal mapping $z=\frac{1}{2}(w+w^{-1})$, then $D-S$ corresponds to the annulus $1<|w|<R$ and the segment S corresponds to the unit circle $|w|=1$. If we set $F(w)=f(z)$, $G(w)=g(z)$, then we have

$$|G(w)-F(w)|\leqq\varepsilon\quad\text{for } |w|=1, \tag{1.6}$$

and

$$|F(w)|\leqq\beta\quad\text{for } |w|=R. \tag{1.7}$$

Now the solution $F(w)$ can be determined as in Example 2, and every two possible solutions satisfy

$$|F_1(w)-F_2(w)|\leqq 2\varepsilon^{1-\theta}\beta^{\theta}, \tag{1.8}$$

where $\theta=(\log|w|)/\log R$.

Miller's method for solving the regularized analytic-continuation problem depends on a general principle of least squares for ill-posed problems. His computer algorithm, SNAC, uses a finite-dimensional least-squares computation. Thus, to produce m solution values, his algorithm requires $O(m^3)$ arithmetic operations.

The present paper uses a different principle, which was motivated by some earlier work [5]. The computer algorithm uses the fast Fourier transform; see [4] and [8]. Thus, to produce m solution values, the algorithm requires $O(m\log m)$ arithmetic operations.

As Example 3 illustrates, the present method may depend on a preliminary conformal mapping of a doubly connected region into an annulus. Wegmann [16] has recently published an efficient computational method for the conformal mapping of doubly connected regions that can be used to implement the present method for analytic continuation. The original region D may be replaced by an approximate subregion D' for the purpose of regularization. If $f(z)$ is analytic in D, and if $|f(z)|\leqq\beta$ in D, then $f(z)$ is analytic and has the same bound in every subregion D'. One may choose D' to include the given arc S and to include as much of the rest of D as is conveniently possible.

Other approaches to numerical analytic continuation have been made by Bisshop [1], Niethammer [12], Stefanescu [14], and Reichel [13]. One may also state the problem as a Fredholm integral equation of the first kind, to which one may apply Tikhonov's method (see Tikhonov and Arsenin [15]). For a discussion of the error in Tikhonov's method see the section in [6] on harmonic continuation, which is equivalent to analytic continuation.

Analytic continuation from an arc is equivalent to the Cauchy problem for Laplace's equation, for which there exists a vast amount of literature. General references include the books by Hadamard [7], Tikhonov and Arsenin [15], Lavrentiev [10], and Carasso and Stone [3]. Logarithmic convexity and ill-posed problems are discussed by Knops [9].

Whereas the three-circles theorem proves logarithmic convexity for the maximum norm, Miller [11] used logarithmic convexity for a quadratic norm on the m-dimensional complex linear space; we shall do likewise. In the limit as $m \to \infty$, this norm becomes the continuous L^2 norm.

2. The problem for an annulus. We assume that an unknown function $f(z)$ has a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k \qquad (1 \leqq |z| \leqq R) \tag{2.1}$$

that is absolutely convergent on the bounding circles $|z| = 1$ and $|z| = R$. We are given numerical values g_j approximating $f(z)$ on the unit circle. Let m be a power of 2, and let $\omega = \exp(2\pi i/m)$. We assume

$$\frac{1}{m} \sum_{j=0}^{m-1} |g_j - f(\omega^j)|^2 \leqq \varepsilon^2, \tag{2.2}$$

where ε is a known positive bound for the data error. We are also given a positive bound β for a quadratic norm of f on the outer boundary:

$$\frac{1}{m} \sum_{j=0}^{m-1} |f(R\omega^j)|^2 \leqq \beta^2. \tag{2.3}$$

Finally, we are given a positive bound τ_m for the truncation error of the Laurent series. We assume

$$\sum_{k<-m/2} |c_k| R^{m/2-1} + \sum_{k \geqq m/2} |c_k| R^k \leqq \tau_m. \tag{2.4}$$

In summary, we are given the following: the integer m, where m is a power of 2 greater than 1; the complex numbers $g_0, g_1, \cdots, g_{m-1}$; the positive numbers ε, β, and τ_m; two radii, r and R, where $1 < r < R$. As a rule, the numbers ε and τ_m will be small; the number β will be moderate or large.

The problem is to compute the unknown $f(z)$ on the interior circle $|z| = r$. For $|z| = r$ we will compute complex numbers $b_0, \cdots, b_{m-1}$ approximating the unknowns $f(r\omega^j)$ $(j = 0, \cdots, m-1)$.

In the analysis of the algorithm, we will prove an inequality for the error norm μ defined by

$$\mu^2 = \frac{1}{m} \sum_{j=0}^{m-1} |b_j - f(r\omega^j)|^2. \tag{2.5}$$

We will show that the error norm μ satisfies

$$\mu \leqq \tau_m + 2\varepsilon^{1-\theta}(\beta + \varepsilon + \tau_m)^\theta, \tag{2.6}$$

where $\theta = (\log r)/\log R$. (Actually, we shall get a somewhat better result.) Thus, as a function of the data-error bound, ε, the solution-error bound, μ, is of the order $\varepsilon^{1-\theta}$. For example, if r is near 1, then μ behaves about like ε; if r is near $\sqrt{R}$, μ behaves like $\sqrt{\varepsilon}$; but if r is near R, we obtain $\mu \leqq \tau_m + 2(\beta + \varepsilon + \tau_m)$, which is of academic interest only.

For fixed r, the algorithm produces the positive error bound μ_1 and m complex numbers $b_0, \cdots, b_{m-1}$. Using the fast Fourier transform, the algorithm has time complexity of the order of $m \log m$.

3. The algorithm. As described in the last section, we are given the data

$$m;\ g_0, \cdots, g_{m-1};\ \varepsilon, \beta, \tau_m;\ r, R. \tag{3.1}$$

The algorithm will compute m complex numbers, $b_0, \cdots, b_{m-1}$, and a positive number, μ_1.

Notation. Let $\mathbf{u}$ be any vector with m complex components. By the equation

$$\mathbf{v} = F\mathbf{u} \tag{3.2}$$

we shall mean that $\mathbf{v}$ is the finite Fourier transform of $\mathbf{u}$:

$$v_j = \sum_{k=0}^{m-1} u_k \omega^{jk} \qquad (j = 0, \cdots, m-1) \tag{3.3}$$

where $\omega = \exp(2\pi i/m)$. Equivalently, we may write $\mathbf{u} = F^{-1}\mathbf{v}$, the inverse transform of $\mathbf{v}$:

$$u_k = \frac{1}{m} \sum_{j=0}^{m-1} v_j \omega^{-kj} \qquad (k = 0, \cdots, m-1). \tag{3.4}$$

The algorithm uses the data (3.1) and the auxiliary variables β_1, λ, θ, and $\mathbf{u}$ to compute the vector $\mathbf{b}$ and the positive number μ_1. (The components b_j approximate $f(r\omega^j)$; the number μ_1 is an upper bound for the error norm μ.)

Algorithm Analytic Continuation;
Begin

$\theta := (\log r)/\log R$;

$\beta_1 := \beta + \varepsilon + \tau_m$;

$\lambda := \dfrac{\varepsilon}{\beta_1} \dfrac{\theta}{1-\theta}$;

$\mathbf{u} := F^{-1}\mathbf{g}$; {inverse fft}

for $k := 0$ to $\dfrac{m}{2} - 1$ do

$\quad u_k := \dfrac{r^k}{1 + \lambda R^k} u_k$;

for $k := \dfrac{m}{2}$ to $m - 1$ do

$\quad u_k := r^{k-m} u_k$;

$\mathbf{b} := F\mathbf{u}$; {fft}

$\mu_1 := \tau_m + (\varepsilon + \lambda\beta_1)\lambda^{-\theta}$

end.

4. Analysis of the algorithm. Let $f(z)$ be an analytic function in the annulus $1 < |z| < R$ and r be a *fixed* radius satisfying $1 < r < R$. Given the integer m; positive numbers r, R, ε, β, τ_m; and complex numbers $g_0, \cdots, g_{m-1}$, the algorithm computes complex numbers $b_0, \cdots, b_{m-1}$ to approximate the unknown values $f(r\omega^j)$ ($j = 0, \cdots, m-1$), where $\omega = \exp(2\pi i/m)$. We assume m is a power of 2 greater than 1.

Time complexity. The algorithm uses the fast Fourier transform twice and $O(m)$ other operations. Therefore the algorithm has time complexity $T(m) = O(m \log m)$.

Error analysis. We now analyze the numerical error, $b_j - f(r\omega^j)$ $(j=0, \cdots, m-1)$. We will show that the error is bounded in terms of the given numbers ε, β, and τ_m, which are defined in the text that follows.

If $\mathbf{v}$ is a vector with complex components $v_0, \cdots, v_{m-1}$, then the L_2 norm is

$$\|\mathbf{v}\| = \left(m^{-1} \sum_{j=0}^{m-1} |v_j|^2 \right)^{1/2}. \tag{4.1}$$

Let x vary in the interval $1 \leqq x \leqq R$, and define the vector $\mathbf{f}(x)$ with m complex components $f(x\omega^j)$ $(j=0, 1, \cdots, m-1)$. Assume

$$\|\mathbf{f}(1) - \mathbf{g}\| \leqq \varepsilon, \tag{4.2}$$

where ε is a given positive bound for the data error.

On the outer circle, $|z| = R$, we assume the bound

$$\|\mathbf{f}(R)\| \leqq \beta, \tag{4.3}$$

which regularizes the ill-posed problem of analytic continuation. For a positive data error ε, the values of the unknown vector $\mathbf{f}(r)$ *must* depend on the outer bound β.

We assume that the unknown function $f(z)$ has an absolutely convergent Laurent series (2.1) and that the truncation error has the bound

$$\sum_{k<-m/2} |c_k| R^{m/2-1} + \sum_{k \geqq m/2} |c_k| R^k \leqq \tau_m. \tag{4.4}$$

This implies

$$\left| f(z) - \sum_{-n \leqq k < n} c_k z^k \right| \leqq \tau_m \tag{4.5}$$

in the closed annulus $1 \leqq |z| \leqq R$, where $n = m/2$.

THEOREM. *Under the preceding assumptions, define*

$$\theta = (\log r)/\log R, \qquad \beta_1 = \beta + \varepsilon + \tau_m \quad \textit{and} \quad \lambda = \frac{\varepsilon}{\beta_1} \frac{\theta}{1-\theta}, \tag{4.6}$$

which implies that $0 < \theta < 1$, $\beta_1 > 0$, $\lambda > 0$. *For* $k = -n, \cdots, n-1$ *define*

$$G_k = m^{-1} \sum_{j=0}^{m-1} g_j \omega^{-kj}. \tag{4.7}$$

For $j = 0, \cdots, m-1$ *define*

$$b_j = \sum_{-n \leqq k < 0} G_k r^k \omega^{jk} + \sum_{0 \leqq k < n} G_k r^k (1 + \lambda R^k)^{-1} \omega^{jk}. \tag{4.8}$$

Also define the constant $1 < C \leqq 2$ *by*

$$C = (1-\theta)^{-(1-\theta)} \theta^{-\theta}. \tag{4.9}$$

Then the numerical error satisfies

$$\|\mathbf{b} - \mathbf{g}(r)\| \leqq \tau_m + C\varepsilon^{1-\theta} \beta_1^\theta. \tag{4.10}$$

This completes the theorem whose proof will require the following elementary result.

LEMMA. *Assume* $x > 0$, $p_i > 0$, q_i *real. Then*

$$\log \sum_{k=1}^{N} p_k x^{q_k} \textit{ is a convex function of } \log x.$$

Proof. Set $x = e^t$ and call the sum $S(t)$. To prove $\log S(t)$ convex, it suffices to prove

$$S(t)^2 \leqq S(t-h) S(t+h) \tag{4.11}$$

for all real t and h. We have

$$S(t)=\sum p_k \exp(q_k t)$$
$$=\sum p_k[\exp \tfrac{1}{2}q_k(t-h)]\cdot[\exp \tfrac{1}{2}q_k(t+h)].$$

Since $p_k>0$, the inequality (4.11) follows from the Schwarz inequality. □

Proof of the Theorem. Define the m complex numbers

$$A_k=m^{-1}\sum_{j=0}^{m-1} f_j(1)\omega^{-kj} \qquad (-n\leqq k<n), \tag{4.12}$$

where, as usual, $n=m/2$. Similarly, define the numbers

$$G_k=m^{-1}\sum_{j=0}^{m-1} g_j\omega^{-kj} \qquad (-n\leqq k<n). \tag{4.13}$$

For $1\leqq x\leqq R$ and $j=0,\cdots,m-1$, define the following functions of x:

$$a_j(x)=\sum_{-n\leqq k<n} A_k x^k\omega^{jk}, \tag{4.14}$$

$$\phi_j(x)=\sum_{-n\leqq k<0} A_k x^k\omega^{jk}+\sum_{0\leqq k<n} A_k x^k(1+\lambda R^k)^{-1}\omega^{jk}, \tag{4.15}$$

$$b_j(x)=\sum_{-n\leqq k<0} G_k x^k\omega^{jk}+\sum_{0\leqq k<n} G_k x^k(1+\lambda R^k)^{-1}\omega^{jk}. \tag{4.16}$$

For the given $x=r$, we get the numbers $b_j(r)=b_j$ defined in (4.8). We wish to prove (4.10) for $\|\mathbf{b}-\mathbf{f}(r)\|$. First we will express the Fourier coefficients A_k in terms of the Laurent coefficients c_j. From (4.12) we have

$$f(\omega^j)=f_j(1)=\sum_{-n\leqq k<n} A_k\omega^{jk} \qquad (j=0,\cdots,m-1). \tag{4.17}$$

But the Laurent series gives

$$f(\omega^j)=\sum_{-\infty<k<\infty} c_k\omega^{jk} \qquad (j=0,\cdots,m-1). \tag{4.18}$$

Since $\omega^m=1$, we may write

$$f(\omega^j)=\sum_{-n\leqq k<n}\left(\sum_{-\infty<s<\infty} c_{k+sm}\right)\omega^{jk} \qquad (j=0,\cdots,m-1). \tag{4.19}$$

But the Fourier coefficients A_k are defined uniquely by (4.17). Therefore, (4.19) implies

$$A_k=\sum_{-\infty<s<\infty} c_{k+sm} \qquad (k=-n,\cdots,n-1). \tag{4.20}$$

Now we will determine a bound for $\|\mathbf{a}(x)-\mathbf{f}(x)\|$. From (4.14) and (4.20) we determine

$$a_j(x)=\sum_{-n\leqq k<n}\left(\sum_{-\infty<s<\infty} c_{k+sm}\right)x^k\omega^{jk} \qquad (j=0,\cdots,m-1). \tag{4.21}$$

If we define the unique residue k mod m in the set $-n,\cdots,n-1$, then from (4.21)

$$a_j(x)=\sum_{-\infty<k<\infty} c_k x^{(k \bmod m)}\omega^{jk} \qquad (j=0,\cdots,m-1). \tag{4.22}$$

Subtracting the Laurent series for $f_j(x)$, we obtain

$$a_j(x)-f_j(x)=\sum_{-\infty<k<\infty} c_k[x^{(k \bmod m)}-x^k]\omega^{jk}. \tag{4.23}$$

We have $(k \bmod m) = k$ for $-n \leqq k < n$, while

$$(k \bmod m) > k \quad \text{for} \quad k < -n,$$

and

$$(k \bmod m) < k \quad \text{for} \quad k \geqq n.$$

Since $1 \leqq x \leqq R$, equation (4.23) implies that

$$|a_j(x) - f_j(x)| \leqq \sum_{k<-n} |c_k| R^{n-1} + \sum_{k \geqq n} |c_k| R^k \leqq \tau_m \tag{4.24}$$

where the given bound τ_m satisfies (4.4). Therefore

$$\|\mathbf{a}(x) - \mathbf{f}(x)\| \leqq \tau_m \qquad (1 \leqq x \leqq R). \tag{4.25}$$

Next we will determine a bound for $\|\mathbf{b}(x) - \mathbf{a}(x)\|$ at $x = r$. To do so, we will determine a bound for $\|\mathbf{b}(x) - \mathbf{a}(x)\|$ at $x = 1$ and at $x = R$. We will then use the lemma, which implies that $\log \|\mathbf{b}(x) - \mathbf{a}(x)\|$ is a convex function of $\log x$, to show that

$$\|\mathbf{b}(r) - \mathbf{a}(r)\| \leqq \|\mathbf{b}(1) - \mathbf{a}(1)\|^{1-\theta} \|\mathbf{b}(R) - \mathbf{a}(R)\|^{\theta} \tag{4.26}$$

where $\theta = (\log r)/\log R$. The lemma is applicable because, by (4.14) and (4.16),

$$\|\mathbf{b}(x) - \mathbf{a}(x)\|^2 = \sum_{-n \leqq k < n} p_k x^{2k}, \tag{4.27}$$

where all p_k are positive.

First set $x = 1$. Then from (4.14) and (4.15),

$$\begin{aligned} \|\phi(1) - \mathbf{a}(1)\|^2 &= \sum_{0 \leqq k < n} |A_k \lambda R^k (1 + \lambda R^k)^{-1}|^2 \\ &\leqq \sum_{0 \leqq k < n} |A_k \lambda R^k|^2 \\ &\leqq \sum_{-n \leqq k < n} |A_k \lambda R^k|^2 = \lambda^2 \|\mathbf{a}(R)\|^2. \end{aligned} \tag{4.28}$$

But (4.25) implies, for $x = R$,

$$\|\mathbf{a}(R)\| \leqq \|\mathbf{f}(R)\| + \tau_m \leqq \beta + \tau_m, \tag{4.29}$$

where β is the given bound for $\|\mathbf{f}(R)\|$. Now (4.28) gives

$$\|\phi(1) - \mathbf{a}(1)\| \leqq \lambda(\beta + \tau_m). \tag{4.30}$$

From (4.15) and (4.16),

$$\begin{aligned} \|\mathbf{b}(1) - \phi(1)\|^2 &= \sum_{-n \leqq k < 0} |G_k - A_k|^2 + \sum_{0 \leqq k < n} |G_k - A_k|^2 (1 + \lambda R^k)^{-2} \\ &\leqq \sum_{-n \leqq k < n} |G_k - A_k|^2 = \|\mathbf{g} - \mathbf{f}(1)\|^2 \leqq \varepsilon^2. \end{aligned}$$

Thus, if ε is the given data-error bound, we have

$$\|\mathbf{b}(1) - \phi(1)\| \leqq \varepsilon. \tag{4.31}$$

Applying the triangle inequality to (4.30) and (4.31), we deduce

$$\|\mathbf{b}(1) - \mathbf{a}(1)\| \leqq \varepsilon + \lambda(\beta + \tau_m). \tag{4.32}$$

Now we will determine the bound for $\|\mathbf{b}(R)-\mathbf{a}(R)\|$. From (4.14) and (4.15) we get

$$\|\phi(R)-\mathbf{a}(R)\|^2=\sum_{0\leqq k<n}|A_kR^k\cdot \lambda R^k(1+\lambda R^k)^{-1}|^2$$
$$\leqq \sum_{-n\leqq k<n}|A_kR^k|^2=\|\mathbf{a}(R)\|^2.$$

From (4.29) we obtain

$$\|\phi(R)-\mathbf{a}(R)\|\leqq \beta+\tau_m. \tag{4.33}$$

From (4.15) and (4.16),

$$\|\mathbf{b}(R)-\phi(R)\|^2=\sum_{-n\leqq k<0}|G_k-A_k|^2R^{2k}+\sum_{0\leqq k<n}|G_k-A_k|^2R^{2k}(1+\lambda R^k)^{-2}$$
$$\leqq \sum_{-n\leqq k<0}|G_k-A_k|^2+\lambda^{-2}\sum_{0\leqq k<n}|G_k-A_k|^2.$$

Since

$$\sum_{-n\leqq k<n}|G_k-A_k|^2=\|\mathbf{g}-\mathbf{f}(1)\|^2\leqq \varepsilon^2,$$

we obtain

$$\|\mathbf{b}(R)-\phi(R)\|\leqq \varepsilon\cdot\max(1,\lambda^{-1})<\varepsilon(1+\lambda^{-1}). \tag{4.34}$$

Applying the triangle inequality to (4.33) and (4.34), we obtain

$$\|\mathbf{b}(R)-\mathbf{a}(R)\|\leqq \beta+\tau_m+\varepsilon(1+\lambda^{-1}). \tag{4.35}$$

Now we are ready to bring our results together. By (4.32) and (4.35), we have

$$\|\mathbf{b}(1)-\mathbf{a}(1)\|\leqq \varepsilon+\lambda\beta_1,\qquad \|\mathbf{b}(R)-\mathbf{a}(R)\|\leqq(\varepsilon+\lambda\beta_1)\lambda^{-1},$$

where $\beta_1=\beta+\varepsilon+\tau_m$. From the (4.26) we obtain

$$\|\mathbf{b}(r)-\mathbf{a}(r)\|\leqq(\varepsilon+\lambda\beta_1)\lambda^{-\theta}, \tag{4.36}$$

where r is the given radius satisfying $1<r<R$. Setting the variable x equal to r, we deduce from (4.25)

$$\|\mathbf{a}(r)-\mathbf{f}(r)\|\leqq \tau_m. \tag{4.37}$$

The triangle inequality yields

$$\|\mathbf{b}(r)-\mathbf{f}(r)\|\leqq \tau_m+(\varepsilon+\lambda\beta_1)\lambda^{-\theta}. \tag{4.38}$$

As a function of λ, the right-hand side is minimized by the value defined in (4.6). Then the proved (4.38) is the required inequality (4.10). This completes the proof of the theorem. □

5. Computer testing. It is easy to implement the algorithm described in § 3. Using an available fast Fourier transform (FFT) subroutine, a PASCAL program was written to test the algorithm for an example of analytic continuation from a line segment. The function

$$F(w)=\frac{1}{2-w} \tag{5.1}$$

was used and data $G(w)$ for $F(w)$ were given on the line segment $-1 \leqq w \leqq 1$. A data-error bound

$$|G(w)-F(w)| \leqq \varepsilon = 10^{-4} \tag{5.2}$$

was assumed. The data $G(w)$ are used to continue the supposedly unknown function $F(w)$ from the line segment into an ellipse with foci at ± 1.

Let E_R be the ellipse in the w-plane given by

$$\frac{u^2}{A^2}+\frac{v^2}{B^2}=1, \tag{5.3}$$

where $w=u+iv$, $A=\frac{1}{2}(R+R^{-1})$, $B=\frac{1}{2}(R-R^{-1})$. Assume $1<R<3.732$ so that the pole of $F(w)$ at $w=2$ lies outside the ellipse. A bound β is prescribed for $|F(w)|$ on the boundary E_R.

The conformal mapping $w=\frac{1}{2}(z+z^{-1})$ maps the annulus $1<|z|<R$ into the region bounded by the slit $-1 \leqq w \leqq 1$ and the ellipse E_R. For $1<r<R$ the circle $|z|=r$ is mapped into an interior confocal ellipse E_r. Values for $F(w)$ on the interior ellipse E_r are computed.

Set $F(w)=f(z)$, $G(w)=g(z)$. Thus $f(z)$ is the supposedly unknown function

$$f(z)=\frac{1}{2-\frac{1}{2}(z+z^{-1})}. \tag{5.4}$$

Let m be a large power of 2; typically, $m=256$. Let $\omega=\exp(2\pi i/m)$. The test simulates data $g(z)$ on the unit circle by computing

$$g(\omega^j)=f(\omega^j)+\varepsilon X_j \qquad (j=0,\cdots,m-1), \tag{5.5}$$

where $\varepsilon=10^{-4}$ and X_j is a computer-generated random number satisfying $-1 \leqq X_j \leqq 1$. Thus, εX_j is a simulated data error bounded by $\pm\varepsilon$.

As described in § 3, the algorithm requires an upper bound τ_m for the Laurent-series truncation error. The function $f(z)$ defined in (5.4) has the Laurent series

$$\sum_{k=-\infty}^{\infty} c_k z^k = \frac{\gamma_1}{z-z_1}+\frac{\gamma_2}{z-z_2}, \tag{5.6}$$

where z_1 and z_2 are the reciprocal poles $2\pm\sqrt{3}$. Therefore, if $n=m/2$,

$$\sum_{k<-n} |c_k| R^{n-1} + \sum_{k \geqq n} |c_k| R^k = O((2-\sqrt{3})^n R^n) \tag{5.7}$$

and $\tau_m = O((2-\sqrt{3})^n R^n)$. If $m \geqq 256$ and $R \leqq 3$, we have $\tau_m = O(7.28\times 10^{-13})$. Thus, within the limit of roundoff error, we may set $\tau_m=0$.

Table 1 gives the results of a numerical test. In accordance with (5.5), randomly perturbed data were given on the unit circle. The following values were fixed:

$$\varepsilon=10^{-4}, \qquad m=256, \qquad \tau_m=0, \qquad R=3, \qquad \beta=0.972. \tag{5.8}$$

TABLE 1
Precise $\beta=0.972$.

r	λ	μ_1	μ
1.25	2.62E−5	1.07E−3	5.81E−5
1.50	6.02E−5	5.72E−3	2.99E−4
1.75	1.07E−4	2.15E−2	1.14E−3
2.00	1.76E−4	6.34E−2	4.28E−3
2.25	2.90E−4	1.56E−1	1.32E−2
2.50	5.17E−4	3.32E−1	3.72E−2
2.75	1.20E−3	6.20E−1	1.02E−1

Thus, the data errors were bounded by $\pm 10^{-4}$. With the outer radius fixed at $R=3$, the norm $\|\mathbf{f}(R)\|$ is fixed at 0.972, and this value was used for β. (This upper bound may be replaced by a value twice as large without an appreciable change in the computations.)

The radius τ was given the seven values $1.25, 1.50, \cdots, 2.75$. For each r, the algorithm was applied to the randomly perturbed data $g_0, \cdots, g_{255}$ and the numbers λ, μ_1, and $b_0, \cdots, b_{255}$ were computed. For each r, using an IBM XT, the computation required approximately 4 seconds.

For each r the algorithm computed an upper bound μ_1 for the true solution error

$$\mu = \|\mathbf{b}-\mathbf{f}(r)\|. \tag{5.9}$$

In a separate computation, which used the function definition (5.4), the true error μ was computed. For the different values of r, the values of the true error μ appear in the last column of Table 1.

The effect of doubling the prescribed bound β for the norm $\|\mathbf{f}(R)\|$ on the outer circle is given in Table 2. The only surprise came for $r=2$. For that value the true error μ decreased: it went from 4.28×10^{-3} in Table 1 to 2.61×10^{-3} in Table 2. For the other tested values of r the true error increased with the use of the crude upper bound β. As a rough check of the computations, a naive analytic continuation by the FFT was performed with the parameter λ equal to zero. For inner radius $r=2$ the result was a true error $\mu = 7{\cdot}8\times 10^{32}$.

TABLE 2
Crude $\beta = 1.94$.

r	λ	μ_1	μ
1.25	1.31E−5	1.11E−3	6.72E−5
2.00	8.8E−5	6.72E−2	2.61E−3
2.75	6.0E−3	6.59E−1	1.59E−1

The algorithm was tested with other functions. $F(w)=e^w$ was continued from the interval $-1\leqq w\leqq 1$ into the rest of the complex plane. This example has the same form as the one given above, but it is easier because $F(w)$ has no singularity in the finite plane. As before, data $G(w)$ are given for $-1\leqq w\leqq 1$, with $|G(w)-e^w|\leqq 10^{-4}$. If the precise bound $\beta=5.71$ is prescribed for the norm $\|\mathbf{f}(5)\|$ on the outer ellipse E_5, the algorithm computes values b_j approximating the true values of e^w on the inner ellipse E_3 with two-digit accuracy. On E_3 the computed error bound is $\mu_1=0.33$, but the true error μ is smaller. A typical experiment with random data errors yielded the true error $\mu=\|\mathbf{b}-\mathbf{f}(3)\|=0.0125$.

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