ÖAW AUSTRIAN ACADEMY OF SCIENCES

Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

RICAM JOHANN-RADON-INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Analytic extention and reconstruction of obstacles from few measurements for elliptic second order operators

N. Honda, G. Nakamura, M. Sini

RICAM-Report 2008-05

Analytic extention and reconstruction of obstacles from few measurements for elliptic second order operators

N. Honda, G. Nakamura and M. Sini

February 18, 2008

Abstract

We deal with the inverse obstacle problem for general second order scalar operators with *analytic coefficients* near the obstacle. We assume that the boundary of the obstacle is a *non-analytic* hypersurface. We show that, when we impose Dirichlet boundary condition, one measurement is enough to reconstruct the obstacle while in the Neumann case, we need $(n-1)$ measurements associated to $(n-1)$ linearly independent inputs. Here n is the dimension of the space containing the obstacle. This is justified by investigating the analyticity properties of the zero set of real analytic functions for the Dirichlet case and the zero set of their normal derivatives for the Neumann case. In addition, we give a reconstruction proceedure to recover the shapes. We state the results for the scattering problem. However similar results are true for the associated boundary value problems.

1 The forward and inverse scattering problem

We suppose that D is a Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, such that $\mathbb{R}^n \setminus \overline{D}$ is connected. We assume that we know a smooth domain Ω containing D. Let $A := (a_{i.j})_{i,j=1,\dots,n}$ be a symmetric positive defined matrix values function with real valued C^1 regular entries in Ω and V be a a real valued bounded potential in Ω such that $A = I$ and $V(x) = 0$ in $\mathbb{R}^n \setminus \Omega$.

We define the expression $P := \nabla \cdot A\nabla + V$.

We are interested in the propagation of time-harmonic fields in the heterogeneous medium given by (A, V) with an obstacle D. This is governed by the equation

$$Pu + \kappa^2 u = 0, \quad \text{in } \mathbb{R}^n \setminus \overline{D}, \tag{1}$$

coupled with the Dirichlet boundary condition

$$u = 0, \quad \text{on } \partial D, \tag{2}$$

for modelling a sound soft obstacle or with the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial D, \tag{3}$$

for modelling a sound hard obstacle. The vector ν is the exterior unit normal to D and κ is the real positive *wave number*. In the electromagnetism scattering by orthotropic medium (where $V = 0$), A models the (non conductive, since ImA=0) electrical permittivity while V is the magnetic permeability, see [4, 7].

The associated scattering problem is to look for solutions of the form $u := u^i + u^s$, where u^i is an incident wave, solution of the free equation $(P + \kappa^2)u = 0$ in $\mathbb{R}^n$, and the *scattered field* u^s is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}(\frac{\partial u^s}{\partial r} - i\kappa u^s) = 0, \tag{4}$$

where $r = |x|$ and the limit is uniform with respect to all the directions $\hat{x} := \frac{x}{|x|}$. It is known, see [4, 7, 22], that a solution to this problem exists and it is unique. In addition, this scattered field satisfies the following asymptotic property,

$$u^s(x) \quad = \quad \frac{e^{i\kappa r}}{r^{\frac{n-1}{2}}} u^\infty(\hat{x}) + O(r^{-\frac{n+1}{2}}), \quad r \to \infty, \tag{5}$$

where the function $u^\infty(\cdot)$ defined on the $(n-1)-$unit sphere $\mathbb{S}^n$ is called the far-field associated to the incident field u^i.

Let us specify the kind of incident waves we will use. We set $u^i(x, d, P) := v_P(x, d) + e^{i\kappa d\cdot x}$, $d \in \mathbb{S}^n$, where $v_P(\cdot, d)$ satisfies the Sommerfeld radiation conditions. Hence v_P is the scattered wave by the known medium (A, V) of the plane wave $e^{i\kappa d\cdot x}$. In section 2.4, we give a way how to compute it knowing A and V. Remark that in the case where $A = I$ and $V = 0$, we end up with the plane waves $u^i(x, d) = e^{i\kappa d\cdot x}$.

Taking these particular incident fields given by the waves, $u^i(x, d)$, $d \in \mathbb{S}^n$, we define the far-field pattern $u^\infty(\hat{x}, d)$ for $(\hat{x}, d) \in \mathbb{S}^n \times \mathbb{S}^n$.

The problem we are concerned with is the following:

The Problem 1. *Suppose we know the background medium (A, V). Given $u^\infty(\cdot, d)$ on $\mathbb{S}^n$ for one incident wave or a few incident waves for the scattering problem (1) - (4) reconstruct the obstacle D.*

The results 1. *We need some additional conditions on the regularity of A and V. Precisely, we assume that entries of A and V are real analytic functions in an open set containing ∂D.*

With these regularity conditions, we obtain the following results:

A. The Dirichlet problem:

If ∂D is a Lipschitz and non-analytic hypersurface then one single incident wave is enough to reconstruct its shape.

B. The Neumann problem:

If ∂D is a C^1 and non-analytic hypersurface then $(n-1)$ incident waves with linearly independent directions of incidence are enough to reconstruct its shape.

A definition of the real analytic hypersurface will be given in section 2.1.

Remark 1.1. *We can consider the more general second order operator given by the Hamiltonian with electromagnetic fields:*

$$\sum_{j,k=1}^{n} \frac{1}{\sqrt{g}}(-i\frac{\partial}{\partial x_j} + B_j(x))\sqrt{g}g^{jk}(-i\frac{\partial}{\partial x_j} + B_k(x)) + V,$$

where g^{jk} is a metric, g its determinant and B_j defines electromagnetic potentials.

However, since we need the coefficients of the operator to be real analytic, see sections 2.2 and 2.3, as it is requiered in the previous section, we need to take $\mathrm{Re}B_j = 0$, $j = 1, 2, ..., n$. In the case where $\mathrm{Im}B_j \neq 0, j = 1, 2, ..., n$, we know that the well-posedness of the forward scattering problem is guaranteed if the wave number κ is away from a dicrete set in $\mathbb{R}$, see for example [22]. To avoid these complications, we assumed $B_j = 0$, $j = 1, 2, ..., n$. Hence this case reduces to an anisotropic acoustic operator.

This problem has a long history, see [7], [11] and [28]. The known results are related to the case $A = I$ and $V = 0$. The first uniqueness result is due to Colton and Sleemann [6] who considered the Dirichlet problem. They gave an estimate of the number of the incident plane waves that are needed to uniquely determine the obstacle. In particular, they show that if the size of the obstacle is small enough then one incident wave is enough. For the same Dirichlet problem, Stefanov and Uhlmann [30] show that one incident wave is enough to distinguish between two obstacles under the condition that they are sufficiently close. These two types of results have been generalized by Gintides [10] by weakning the smalleness and the closeness conditions. Some stability results related to these uniqueness results are given by Isakov in [12] and [13] for small obstacles and Sincich and Sini in [29] for small or close obstacles.

For other boundary conditions such as Neumann or Robin, related local results are not known so far.

If in addition we assume that the obstacles are polygonal or polyhedral then, in recent years, several results have been given by Cheng and Yamamoto [5], Elschner and Yamamoto [8], Alessandrini and Rondi [2], Rondi [28] and Liu and Zou [16] and [17].

In this paper, we assume that obstacles are Lipschitz or C^1-smooth but not analytic. Details will be given in the next section regarding the defintion of non-analytic hypersurfaces. For such regular obstacles, we prove global uniqueness results for both the Dirichlet and the Neumann cases. We note that we do not need any smalleness or closeness conditions. In this sense our results complete the ones mentioned above. We also mention a related result by Ramm [27] who considered non-analytic obstacles and proved similar uniqueness results by assuming that the obstacles are strictly convex and the operator P is taken to be the Laplace operator Δ. With the results proven in this paper, we remove, in particular, the convexity condition.

With these results at hand the open issue is to study this inverse obstacle scattering problem in case of obstacles with piecewise analytic boundaries.

To answer to our problem, we start by investigating the analyticity of a topological manifold contained in some real analytic set and that of a differentiable manifold tangent to analytic sections. Correspondingly, for the Dirichlet case, we show that the set of irregular points of real analytic functions are nowhere dense in their zero sets and their complementary in these zero sets are still real analytic subsets, see Theorem 2.2 and Corollary 2.3. For the Neumann case we show that if S is a non analytic hypersurface of $\mathbb{R}^n$, then we cannot find $n-1$ linearly independent real analytic functions with gradients tangent to S, see Theorem 2.6. Then we apply these arguments to prove that the total fields cannot be analyitically extendable across non-analytic obstacles with Dirichlet or Neumann boundary conditions respectively. This is due to the contrast between the real analytic regularity of the coefficients A and V which insures that total fields are analytic, in the domain where they are defined, and the non-analyticity of ∂D. This non-analytic extension reflects a high scattering effect which we can interprete mathematically by the unboundedness of the Taylor series for points near the hypersurface ∂D. This information is used to build up a reconstruction proceedure to detect the shape of the obstacle. Precisely, we compute reconstructively the Taylor coefficients of the scattered fields directly from the farfields. Similar ideas have been used by Potthast in [25] and Honda et al. in [9]. The argument of this paper is to use the blowup of the Taylor coefficients directly to reconstruct the obstacle. As a by product of this proceedure, we obtain the uniqueness results we have stated above.

The rest of the paper, contained in the section 2, is organized as follows. In subsection 2.1, 2.2 and 2.3, we state and prove the non-analytic extension results. In sections 2.4 and 2.5, we apply these results to the reconstruction of the obstacles.

2 Solution of the inverse problem for non-analytic obstacles

2.1 The definition of "non-analytic" boundary

We assume that the boundary ∂D of D is a closed hypersurface in Ω, that is, ∂D is a closed topological submanifold in Ω with its topological dimension being $n-1$. We say that ∂D is analytic at $p \in \partial D$ if there exist an open neighborhood V of p and a real analytic function φ in V such that $\partial D \cap V = \{x \in V;\ \varphi(x) = 0\}$.

DEFINITION 2.1. *∂D is non-analytic at $p \in \partial D$ if and only if ∂D is not analytic at p. It is said to be non-analytic if it is non-analytic at every point in ∂D.*

2.2 The sound soft obstacle case

Let $P(x, \partial_x)$ be an elliptic linear partial differential operator of the second order with real analytic coefficients in Ω. Let u designate a solution in Ω of $Qu = 0$. Hereafter Z designates the zero set of u.

For an analytic function f in Ω, the set of critical points of f (i.e. the set $\{x \in \Omega;\ \mathrm{grad}(f) = 0\}$) might contain the zero set of f completely. However a solution of an elliptic equation possesses the following good property:

THEOREM 2.2. *Let us define the analytic subset Z_{irr} as*

$$Z_{irr} = \{x \in Z;\ \mathrm{grad}(u)(x) = 0\}.$$

If u is not identically zero, then we have $\dim_{\mathbb{R}} Z_{irr} \leq n-2$. Here $\dim_{\mathbb{R}} Z_{irr}$ denotes the dimension of Z_{irr} as an analytic set (see the comment in the following proof).

Proof. It follows from Theorem 8.3.20, p.335 in the book [14] that there exists a stratification W_α of the the analytic subset Z_{irr}. Precisely, there exists a partition $Z_{irr} = \sqcup_\alpha W_\alpha$ where the family of the so-called strata W_α satisfies the following conditions 1., 2. and 3.:

1. Each W_α is connected, and Z_{irr} is a disjoint union of the family $(W_\alpha)_\alpha$. For every point $p \in W_\alpha$, W_α is a closed real analytic submanifold in an open neighborhood of p.
2. The family $(W_\alpha)_\alpha$ is locally finite, that is, for any compact set K the number of W_α intersecting with K is finite.
3. For any pair (α, β) such that $W_\alpha \cap \bar{W}_\beta$ is not empty, W_α is contained in $\bar{W}_\beta$ where $\bar{W}_\beta$ denotes the closure of W_β in $\mathbb{R}^n$. In particular, $W_\alpha \cap \bar{W}_\beta \neq \phi$ implies $\dim_{\mathbb{R}} W_\alpha < \dim_{\mathbb{R}} W_\beta$.

Note that $\dim_{\mathbb{R}} W_\alpha$ designates the dimension of W_α as a topological or a differentiable manifold (both are the same). For the dimension of an analytic set, although there are several ways to define it, it is known that

$$\dim_{\mathbb{R}} Z_{irr} = \max_\alpha \dim_{\mathbb{R}} W_\alpha.$$

Suppose that a component W_α with $\dim_{\mathbb{R}} W_\alpha = n-1$ exists. Since the operator P is elliptic, W_α is always non-characteristic for P. Moreover we have

$$u = 0, \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } W_\alpha$$

where $\dfrac{\partial u}{\partial \nu}$ is the normal derivative of u along W_α. By Cauchy-Kowalevski theorem, the solution u becomes identically zero, that contradicts the assumption. Hence we can conclude that the dimension of Z_{irr} is less than $n-1$. □

We have two corollaries.

Corollary 2.3. *Let u be a non-zero solution of $P(x, \partial_x)u = 0$ in Ω, and let ∂D be a closed topological submanifold in Ω with $\dim_{\mathbb{R}} \partial D = n-1$. If ∂D is contained in the zero set Z of u, then we have*

1. *$\partial D \cap Z_{irr}$ is nowhere dense in ∂D.*
2. *$\partial D \setminus Z_{irr}$ is real analytic smooth.*

Here we set $Z_{irr} = \{x \in Z; \operatorname{grad} u(x) = 0\}$.

Proof. Let $Z_{irr} = \underset{\alpha}{\sqcup} W_\alpha$ denote the partition that appeared in the proof of Theorem 2.2. Set for $k \in \mathbb{Z}$

$$\Lambda(k) = \{\beta; \dim_{\mathbb{R}} W_\beta \le k\}, \qquad Z_{irr}(k) = \underset{\beta \in \Lambda(k)}{\cup} W_\beta.$$

Then for any point $p \in W_\alpha$ with $\dim_{\mathbb{R}} W_\alpha = k$, we can find a open set $p \in V$ in Ω such that $V \cap Z_{irr}(k) = V \cap W_\alpha$. In fact, if we could not find such a V, then there exist a component W_β with $p \in \bar{W}_\beta$ ($\alpha \ne \beta \in \Lambda(k)$). Since $p \in \bar{W}_\beta \cap W_\alpha$ is not empty, by the condition 3. of the partition we have $Z_\alpha \subset \bar{Z}_\beta$. This implies $\dim_{\mathbb{R}} Z_\beta \ge k+1$, and that is impossible.

Now we suppose that Z_{irr} contains a not empty open set T of ∂D. Let k be a minimal integer such that $T \subset Z_{irr}(k)$, and let W_α be a component satisfying conditions $T \cap W_\alpha \ne \phi$ and $\dim_{\mathbb{R}}(W_\alpha) = k$. For a point $p \in T \cap W_\alpha$, we can take an open set $p \in V$ in Ω so that $Z_{irr}(k) \cap V = W_\alpha \cap V$. Then we have

$$p \in V \cap T \subset V \cap Z_{irr}(k) = V \cap W_\alpha.$$

Therefore W_α contains a subset which is homeomorphic to an open ball in $\mathbb{R}^{n-1}$. Since $\dim_{\mathbb{R}}(W_\alpha) < n-1$, this contradicts the fact that a topological manifold with the dimension less than $n-1$ never contains a subset which is homeomorphic to an open ball in $\mathbb{R}^{n-1}$ (see also the proof below). Hence $\partial D \cap Z_{irr}$ is nowhere dense in ∂D.

For any point $x_0 \in \partial D \setminus Z_{irr}$, Z is real analytic smooth near x_0, in particular, a topological submanifold near x_0 with $\dim_{\mathbb{R}} Z = n-1$. Therefore we can find an open neighborhood U (resp. V) of x_0 in ∂D (resp. Z) and a homeomorphic map $\phi_{\partial D} : U \to B$ (resp. $\phi_Z : V \to B$) for some open ball $B \subset \mathbb{R}^{n-1}$ respectively.

Now let us recall the famous Brouwer's invariance theorem of domain, that is, for any two homeomorphic subset W_1 and W_2 in a topological manifold M, W_1 is an open subset in M if and only if W_2 is open. Then since U and V are homeomorphic by the map $\phi_Z^{-1} \circ \phi_{\partial D}$ and V is an open subset in Z, it follows from the Brouwer's theorem that U is also open in Z. Hence ∂D and Z coincide near x_0, and ∂D is real analytic smooth near x_0. □

Note that one could not expect that ∂D is analytic everywhere even if ∂D is smooth everywhere.

Let u be the solution in $\Omega \setminus \bar{D}$ of the equation $Qu = 0$ that satisfies Dirichlet boundary condition $u|_{\partial D} = 0$.

Corollary 2.4. *If ∂D is non-analytic, then the solution v defined in $\Omega \setminus \bar{D}$ is never extended to D analytically across the boundary ∂D.*

2.3 The sound hard case

Let U be a connected open set in $\mathbb{R}^n$ and $f_1, f_2, \dots, f_{n-1}$ be $n-1$ real analytic functions in U. We give a weaker version of Corollary 2.3.

LEMMA 2.5. *Let S be a C^1 smooth hypersurface in U. If S is contained in an real analytic set Z in U, then points where S is real analytic smooth is dense in S.*

Proof. We consider a partition $Z = \sqcup_{\alpha} Z_\alpha$ satisfying the condition 1., 2. and 3. in the proof of Theorem 2.2, and we define the subset $\hat{Z}$ of Z as

$$\hat{Z} = \cup_{\dim Z_\alpha \le n-2} Z_\alpha.$$

Note that $S \setminus \hat{Z}$ is a dense subset in S because of $\dim_{\mathbb{R}} \hat{Z} < n-1$.

Let p be a point in $S \setminus \hat{Z}$. Then we can find a component Z_α with $p \in Z_\alpha$ and $\dim Z_\alpha = n-1$. By the same reasoning as in the proof of Corollary 2.3, there exists a neighborhood V of p satisfying $Z \cap V = Z_\alpha \cap V$, and we have

$$S \cap V \subset Z \cap V = Z_\alpha \cap V.$$

Since the topological dimensions of both sets $S \cap V$ and $Z_\alpha \cap V$ are the same, we conclude that $S \cap V = Z_\alpha \cap V$, and S is analytic smooth near p. □

Let S be a C^1 smooth hypersurface in U, and $f_1, f_2, \dots$ and f_{n-1} real analytic functions in U.

THEOREM 2.6. *Suppose that there exists a point $x^* \in U$ such that the $n-1$ vectors*

$$\operatorname{grad} f_1(x^*), \quad \operatorname{grad} f_2(x^*), \quad \dots, \operatorname{grad} f_n(x^*)$$

are linearly independent. If the relations

$$\frac{\partial f_1}{\partial \nu} = 0, \quad \frac{\partial f_2}{\partial \nu} = 0, \quad \dots, \frac{\partial f_{n-1}}{\partial \nu} = 0 \qquad \text{on } S$$

hold, then S is real analytic smooth at a dense subset of S.

Proof. First define $(n-1) \times n$ matrix

$$A := \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \dots \\ \nabla f_{n-1} \end{bmatrix}$$

and set

$$J_i := |\det B_i|^2$$

where B_i is the $(n-1) \times (n-1)$ matrix defined from A by deleting the i^{th} row. Then we define an analytic function ϕ as

$$\phi = \sum_{i=1}^{n-1} J_i.$$

Remark that by the assumptions on $f_1, f_2, \dots$, and f_{n-1}, the analytic functions ϕ is not identically zero.

We set

$$S_{\{\phi=0\}} = \{x \in S; \phi = 0\},$$
$$S_{\{\phi\neq 0\}} = \{x \in S; \phi \neq 0\}.$$

We denote by $\mathrm{Int}_S(S_{\{\phi=0\}}) \subset S$ the set of interior points of $S_{\{\phi=0\}}$ in S. Then since we have

$$S \setminus \left(\mathrm{Int}_S(S_{\{\phi=0\}}) \cup S_{\{\phi\neq 0\}}\right) = S_{\{\phi=0\}} \setminus \mathrm{Int}_S(S_{\{\phi=0\}}),$$

and clearly $S_{\{\phi=0\}} \setminus \mathrm{Int}_S(S_{\{\phi=0\}})$ contains no interior point in S, $\mathrm{Int}_S(S_{\{\phi=0\}}) \cup S_{\{\phi\neq 0\}}$ is an open dense subset in S. Taking the previous lemma into account, S is real analytic smooth at a dense point of $\mathrm{Int}_S(S_{\{\phi=0\}})$. Hence it is enough to show the theorem in $S_{\{\phi\neq 0\}}$. Since the problem is local, we assume $\phi \neq 0$ on S in what follows.

Then $n-1$ vectors

$$\mathrm{grad}(f_1), \quad \mathrm{grad}(f_2), \quad \dots, \mathrm{grad}(f_{n-1}),$$

are linearly independent over $\mathbb{R}$ at every point near S. Let us denote by B the real analytic vector bundle near S which is generated by the above $n-1$ vectors. The conditions

$$\frac{\partial f_1}{\partial \nu} = 0, \quad \frac{\partial f_2}{\partial \nu} = 0, \quad \dots, \frac{\partial f_{n-1}}{\partial \nu} = 0 \qquad \text{on } S$$

imply that S is tangent to the vector bundle B, that is, S is an integral manifold of B. Let $x_0 \in S$ and

$$X_1(x), \quad X_2(x), \dots, X_{n-1}(x)$$

be real analytic sections of the vector bundle B near x_0 which are linearly independent. Now for sufficiently small $\epsilon > 0$ we consider a solution

$$\gamma_1(t_1) : I = (-\epsilon, \epsilon) \to \mathbb{R}^n$$

of the differential equation

$$\frac{d}{dt_1}\gamma_1 = X_1(\gamma_1(t_1)),$$
$$\gamma_1(0) = x_0.$$

Then the solution $\gamma_1(t_1)$ is a real analytic function of the variable t_1, and $\gamma_1(I) \subset S$ holds because S is tangent to B. In the similar way, we will construct functions

$$\gamma_k(t_1, \dots, t_k) : I^k \to \mathbb{R}^n, \qquad k = 1, 2, \dots, n-1$$

by solving the differential equation

$$\frac{d}{dt_k}\gamma_k = X_k(\gamma_k(t_1, t_2, \dots, t_k)),$$
$$\gamma_k(t_1, \dots, t_{k-1}, 0) = \gamma_{k-1}(t_1, \dots, t_{k-1})$$

successively. Then $\gamma_k(t_1, \dots, t_k)$ is real analytic functions of the variables $t_1, \dots, t_k$ and $\gamma_k(I^k) \subset S$ is satisfied. It is easy to see

$$\det\left(\frac{\partial \gamma_{n-1}}{\partial t_1}, \frac{\partial \gamma_{n-1}}{\partial t_2}, \dots \frac{\partial \gamma_{n-1}}{\partial t_{n-1}}\right)(0) = \det\left(X_1(x_0), \dots, X_{n-1}(x_0)\right) \neq 0.$$

Therefore $\gamma_{n-1}(t_1, \dots, t_{n-1}) : I^{n-1} \to S$ becomes a real analytic coordinates function of S near x_0, and S is a real analytic manifold near x_0. □

Let S be a C^1 smooth hypersurface in U, and let us assume that $U \setminus S$ consists of two connected component U_+ and U_-. Let v_1, v_2, …and v_{n-1} be solutions in U_+ of the equation $Qv = 0$ that satisfy the sound hard conditions

$$\frac{\partial v_1}{\partial \nu} = 0, \quad \frac{\partial v_2}{\partial \nu} = 0, \quad \dots, \frac{\partial v_{n-1}}{\partial \nu} = 0 \qquad \text{on } S.$$

Corollary 2.7. *Assume S is non-analytic at any point in S, and v_1, v_2, ... and v_{n-1} satisfy the first assumption of Theorem 2.6 at some point $x_0 \in U$. Then for any point p in S at least one of the solutions v_1, v_2 ... or v_{n-1} can not be analytically continued across the boundary S near p.*

2.4 Computation of the Green's function for the unperturbed medium (A, V) in $\mathbb{R}^n$

The existence and uniqueness of the Green's function for our problem is well known, see for example [22] where this is shown for more general second order elliptic operators.

As we will see in the next section, we need to know the far field pattern of the Green's function of operator $P + \kappa^2$ on $\mathbb{R}^n$. In this section, we show how to compute it.

Let $\Phi(x, z)$ be the fundamental solution of the Helmholtz equation $\Delta + \kappa^2$.

I. The acoustic case: $A = I$, via the Lippman-Schwinger equation.

In this case the Green's function $G(x, z)$ of $\Delta + V + \kappa^2$ satisfies:

$$(\Delta + V + \kappa^2)G(x, z) = -\delta(x - z) \text{ in } \mathbb{R}^3, r^{\frac{n-1}{2}}(\frac{\partial}{\partial r}G - i\kappa G)(x, z) = o(1), (|x| \to \infty).$$

This Green's function G exists and it is the unique solution of the Lippmann-Schwinger equation:

$$-\int_\Omega \Phi(x, y)VG(y, z)dy + G(x, z) = \Phi(x, z), x, z \in \mathbb{R}^n. \tag{6}$$

We set $G^\infty(\hat{x}, z)$, $\hat{x} \in \mathbb{S}^n$ to be the far field of $G(x, z)$. Hence

$$G^\infty(\hat{x}, z) = C_n e^{-i\kappa\hat{x}\cdot z} + C_n \int_\Omega e^{-i\kappa\hat{x}\cdot y}VG(y, z)dy \tag{7}$$

where $C_n e^{-i\kappa\hat{x}\cdot z} = \Phi^\infty(\hat{x}, z)$, i.e the farfield of the fundamental solution $\Phi(\cdot, z)$ and C_n is a constant depending on the dimension n.

We can compute $G(x, z)$, $x, z \in \Omega$ by solving the integral equation (6). Hence combining (6) and (7) we can compute $G^\infty(\hat{x}, z)$, $\hat{x} \in \mathbb{S}^n$ and $z \in \Omega$.

II. The general case via the mixed reciprocity relations

Another way of computing the farfield pattern of $G(x, z)$, i.e. $G^\infty(\hat{x}, z)$, $\hat{x} \in \mathbb{S}^n$ and $z \in \Omega$, is to use the following mixed reciprocity relations:

$$G^\infty(\hat{x}, z) = C_n e^{-i\kappa z\cdot\hat{x}} + C_n v(z, -\hat{x}), \tag{8}$$

where $v(\cdot, \hat{x})$ is the scattered wave associated to the pane wave $e^{i\kappa z\cdot\hat{x}}$ with direction $\hat{x}$.

Let us give a justification to (8). Let $z \in \Omega$ and $d \in \mathbb{S}^n$, then we have

$$\int_{\partial\Omega} G(y, z)\frac{\partial u^s}{\partial\nu}(y, d) - \frac{\partial G}{\partial\nu}(y, z)u^s(y, d)ds(y) = 0. \tag{9}$$

Also replacing $u^s(y, d)$ by $\Phi(x, y)$, we obtain

$$\int_{\partial\Omega} G(y, z)\frac{\partial\Phi}{\partial\nu}(x, y) - \frac{\partial G}{\partial\nu}(y, z)\Phi(x, y)ds(y) = G(x, z), \text{ for } \forall z \in \Omega \text{ and } x \in \mathbb{R}^n \setminus \overline{\Omega}. \tag{10}$$

Hence taking the asymptotic, as $|x| \to \infty$, we obtain

$$G^\infty(\hat{x}, z) = C_n \int_{\partial\Omega} G(y, z)\frac{\partial e^{-i\kappa\hat{x}\cdot y}}{\partial\nu} - \frac{\partial G}{\partial\nu}(y, z)e^{-i\kappa\hat{x}\cdot y}ds(y), \text{ for } \forall z \in \Omega \text{ and } \hat{x} \in \mathbb{S}^n. \tag{11}$$

Multiplying (9) by C_n, replacing d by $-\hat{x}$ and adding the resulting equation to (11), we get

$$G^\infty(\hat{x}, z) = C_n \int_{\partial\Omega} G(y,z)\frac{\partial u}{\partial \nu}(y, -\hat{x}) - \frac{\partial G}{\partial \nu}(y,z)u(y,-\hat{x})ds(y), \forall z \in \Omega \text{ and } \hat{x} \in \mathbb{S}^n \tag{12}$$

which we can write as

$$G^\infty(\hat{x}, z) = C_n \int_{\partial\Omega} G(y,z)\frac{\partial u}{\partial \nu_A}(y, -\hat{x}) - \frac{\partial G}{\partial \nu_A}(y,z)u(y,-\hat{x})ds(y), \forall z \in \Omega \text{ and } \hat{x} \in \mathbb{S}^n \tag{13}$$

where we used the notation $\frac{\partial u^s(y,d)}{\partial \nu_A} := A\nabla u^s(y,d)\cdot\nu$ and ν is the outer unit normal of ∂D,

But using the Green's theorem the integral of (13) is no thing but $u(z, -\hat{x})$. This means that

$$G^\infty(\hat{x}, z) = C_n u(z, -\hat{x}), \ \forall z \in \Omega \text{ and } \hat{x} \in \mathbb{S}^n.$$

Hence to compute $G^\infty(\hat{x}, z)$, it is enough to solve the forward scattering problem and compute the values of $v(z, -\hat{x})$, $z \in \Omega := B_R(0)$.

For the practical case where $n = 2$ and $n = 3$, it is well known that the scattering problem is equivalent to the boundary value problem

$$\left\{ \begin{array}{l} (P+\kappa^2)v = 0, \text{ in } \Omega, \\ v|_{\partial\Omega} = M(v^s) \text{ on } \partial\Omega, \\ v := e^{i\kappa z\cdot\hat{x}} + v^s(z,\hat{x}), \end{array} \right. \tag{14}$$

where M is the following explicit operator:

$$M(u)(\cdot) := \sum_{m=K(n)}^{\infty} \sum_{j=-L(n)}^{L(n)} \beta_{|m|}(\kappa, R)\psi_{mj}(\cdot)\int_{\Gamma_R} \psi^*_{mj}(x_0)u(x_0)ds(x_0) \tag{15}$$

where the asterix denotes the complex conjugate and

$$K(n) = \left\{ \begin{array}{l} -\infty, \text{ if } n = 2, \\ 0 \text{ if } n = 3, \end{array} \right. \tag{16}$$

$$L(n) = \left\{ \begin{array}{l} 0, \text{ if } n = 2, \\ m \text{ if } n = 3, \end{array} \right. \tag{17}$$

and

$$\psi_{mj}(x) = \left\{ \begin{array}{l} \sqrt{\frac{1}{2\pi R}}e^{im\theta}, \text{ if } n = 2 \\ \sqrt{\frac{(2m+1)(n-|j|)}{4\pi R^2(n+|j|)}}P_n^{|j|}(\cos(\theta))e^{ij\phi} \text{ if } n = 3, \end{array} \right. \tag{18}$$

where $P_n^{|j|}$ is the associated Legendre function of degree n and order $|j|$. The coefficient $\beta_{|m|}(\kappa, R)$ is given by:

$$\beta_{|m|}(\kappa, R) \left\{ \begin{array}{l} \frac{\kappa(H^{(1)}_{|m|})'(\kappa R)}{H^{(1)}_{|m|}(\kappa R)}, \text{ if } n = 2, \\ \frac{\kappa(h^{(1)}_{|m|})'(\kappa R)}{h^{(1)}_{|m|}(\kappa R)} \text{ if } n = 3, \end{array} \right. \tag{19}$$

where $H^{(1)}_{|m|}$ is the Hankel function of the first kind of order $|m|$ and $h^1_{|m|}$ is the spherical Hankel function of the first kind of order $|m|$, see [15].

Another useful representation of $G^\infty(\hat{x}, z)$, $\hat{x} \in \mathbb{S}^n$ and $z \in \mathbb{R}^n$ is as follows: knowing $G(x, z)$, $x \in \Omega$, $z \in \mathbb{R}^n$, we state the exterior problem in $\mathbb{R}^n \setminus \overline{\Omega}$:

$$
\left\{
\begin{array}{l}
(\Delta + \kappa^2)G = 0, \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \\
\frac{\partial}{\partial \nu(\cdot)} G(\cdot, z) := k(\cdot, z), \text{ known on } \partial\Omega \\
G(\cdot, z) \text{ satisfies the Sommerfield radiation conditions}
\end{array}
\right. \tag{20}
$$

Then using an integral representation by single layer potential, we have

$$
G(x, z) = \int_{\partial\Omega} \Phi(x, y) f(y, z) ds(y) \tag{21}
$$

where $f(\cdot, z)$ is the unique solution of the integral equation of the second kind

$$
f(\cdot, z) - 2\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu(\cdot)}(\cdot, y) f(y, z) ds(y) = -2k(\cdot, z), \text{ on } \partial\Omega. \tag{22}
$$

From (21), we have

$$
G^\infty(\hat{x}, z) = C_n \int_{\partial\Omega} e^{-i\kappa \hat{x}\cdot y} f(y, z) ds(y). \tag{23}
$$

We can replace the Neumann boundary condition in (20) by the Dirichlet one and use the double layer potential to represent G.

The identity (23) will be of help when we consider the Herglotz wave operator in the next section.

2.5 Reconstructing the shape of the obstacle.

2.5.1 Computation of the Taylor coefficients of the scattered fields

The following identity

$$
u^\infty(\hat{x}, d) = \int_{\partial D} \left\{ \frac{\partial u^s(y, d)}{\partial \nu_A} G^\infty(\hat{x}, y) - \frac{\partial G^\infty(\hat{x}, y)}{\partial \nu_A} u^s(y, d) \right\} ds(y) \tag{24}
$$

follows from the Green's formula in $\mathbb{R}^n \setminus \overline{D}$ for $u^s(\cdot, d, p)$ and $G(\cdot, y)$ and their asymptotic behavior at infinity, see [7] where G is taken to be Φ, however similar arguments can be applied for G since on $\mathbb{R}^n \setminus \overline{D}$ both of G and u^s satisfy the same equation. Note also that since P is self-adjoint, because $A^T = A$, then G is symmetric, i.e. $G(x, z) = G(z, x)$, $x, z \in \mathbb{R}^n$.

Let $g \in L^2(\mathbb{S}^n)$, then

$$
\int_{\mathbb{S}^n} u^\infty(-\theta, d) g(\theta) ds(\theta) = \int_{\partial D} \left\{ \frac{\partial u^s(y, d)}{\partial \nu_A} v_g(y, d) - \frac{\partial v_g(y, d)}{\partial \nu_A} u^s(y, d) \right\} ds(y) \tag{25}
$$

where

$$
v_g(y) := \int_{\mathbb{S}^2} G^\infty(\hat{x}, y) g(\hat{x}) ds(\hat{x}). \tag{26}
$$

A bounded set B, $B \subset \mathbb{R}^n$, is said to be a non-vibrating domain for the differential operator $P + \kappa^2$ if B is of class C^2 and the operator given by $-P$ on B with Dirichlet boundary conditions does not have κ^2 as an eigenvalue.

We need the following known property.

LEMMA 2.8. *Let B be a non-vibrating domain. Then the operator $H : L^2(\mathbb{S}^n) \to L^2(\partial B)$ defined by*

$$H(g) := v_g := \int_{\mathbb{S}^2} G^\infty(\hat{x}, \cdot)g(\hat{x})ds(\hat{x})$$

is compact, injective and has a dense range.

Proof of Lemma 2.8. The proof of these properties is similar to the one given in [7], except that we need to replace Φ by G and to use the representation (23).

Using (23), we write, $\forall y \in \partial B$,

$$\int_{\mathbb{S}^2} G^\infty(\hat{x}, y)g(\hat{x})ds(\hat{x}) = C_n \int_{\mathbb{S}^2}\int_{\partial\Omega} e^{-i\kappa\hat{x}\cdot z} f(z,y)ds(z)g(\hat{x})ds(\hat{x})$$

$$= C_n \int_{\partial\Omega}\int_{\mathbb{S}^2} e^{-i\kappa\hat{x}\cdot z} g(\hat{x})ds(\hat{x})f(z,y)ds(z)$$

Hence

$$\int_{\mathbb{S}^n} G^\infty(\hat{x}, y)g(\hat{x})ds(\hat{x}) = C_n \int_{\partial\Omega} \mathbf{h}g(-z)f(z,y)ds(z)$$

i.e

$$H(g) = (M_f \circ \mathbf{h})g$$

where $M_f : L^2(\partial\Omega) \to L^2(\partial B)$,

$$\phi \in L^2(\partial\Omega) \to M_f\phi := C_n \int_{\partial\Omega} \phi(-z)f(z,y)ds(z)$$

and $\mathbf{h} : L^2(\mathbb{S}^n) \to L^2(\partial\Omega)$, $g \in L^2(\mathbb{S}^n) \to \mathbf{h}g(y) := \int_{\mathbb{S}^n} e^{i\kappa\hat{x}\cdot z}g(\hat{x})ds(\hat{x})$ is the usual Herglotz operator.

Since $\mathbf{h}$ is a compact operator and M_f is bounded then H is also compact.

Let us consider the denseness of the range of H. We have $H^*f = C_n \int_{\partial B} \overline{G^\infty(\hat{x}, y)}f(y)ds(y)$. Let $f \in L^2(\partial B)$ such that $H^*f = 0$. Then

$$\int_{\partial B} G^\infty(\hat{x}, y)\overline{f(y)}ds(y) = 0. \tag{27}$$

The function $v_f(x) := \int_{\partial B} G(x,y)\overline{f(y)}ds(y)$, $x \in \mathbb{R}^n \setminus \overline{B}$ is well defined and satisfies $(P+\kappa^2)v_f = 0$ in $\mathbb{R}^n \setminus \overline{B}$ and also the radiation conditions since G does. Its far field pattern is given by (27). From the Rellich Lemma, (27) implies that $v_f = 0$ in $\mathbb{R}^n \setminus \overline{B}$. Since B is non-vibrating, then $v_f = 0$ in B. Using the jump relation of the normal derivative of the single layer potential (which can be justified see [19] and [21], Appendix), we deduce that $f = 0$. Hence the kernel of H^* is trivial, i.e. $N(H^*) = \{0\}$. This means that the range of H is dense.

Similarly, we can prove that H is injective. □

Let z be a point in $\mathbb{R}^n \setminus D$ and set

$$\psi(x, z, \rho, \alpha) := \frac{\rho^{|\alpha|}}{\alpha!}\frac{\partial^\alpha}{\partial z^\alpha}G(x,z), \text{ for } x, z \in \mathbb{R}^n \tag{28}$$

where $\alpha = (\alpha_1, ...\alpha_n) \in \mathbb{Z}^n_+$ is a multi index, $\frac{\partial^\alpha}{\partial x^\alpha} := \frac{\partial^{\alpha_1}}{\partial x_1^\alpha} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^\alpha}$, $\alpha! := \alpha_1! \cdots \alpha_n!$ and $|\alpha| := \alpha_1 + \alpha_2 + ... + \alpha_n$.

We take B such that $\overline{D} \subset B$, $z \notin B$ and B is a non vibrating domain. For every $\rho > 0$ and $\alpha \in \mathbb{N}^n$, we take $g_m^{z,\rho,\alpha} \in L^2(\mathbb{S}^n)$ such that $v_{g_m^{z,\rho,\alpha}}$ tends to $\psi(\cdot, z, \rho, \alpha)$ in $C^1(\overline{B})$. This property is due to Lemma 2.8. Indeed, we approximate $\psi(\cdot, z, \alpha, \rho)$ on ∂B by $v_{g_m^{z,\rho,\alpha}}$. Since both of ψ and $v_{g_m^{z,\rho,\alpha}}$ satisfy the same problem in B then the well-posedness of the Dirichlet problem in B, via interior estimates, gives the desired property.

THEOREM 2.9. *Let $z \in \mathbb{R}^n \setminus \overline{D}$, $\rho > 0$ and $\alpha \in \mathbb{Z}_+^n$ be fixed. We can construct a sequence $(g_m^{z,\rho,\alpha})_m \subset L^2(\mathbb{S}^n)$ such that*

$$\lim_{m\to\infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d) g_m^{z,\rho,\alpha}(\theta) ds(\theta) = -\frac{\rho^{|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial z^\alpha} u^s(z, d). \tag{29}$$

Proof of Theorem 2.9.
From (25), we get:

$$\lim_{m\to\infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d) g_m^{z,\rho,\alpha}(\theta) ds(\theta) = \int_{\partial D} \left\{ \frac{\partial u^s(y, d)}{\partial \nu_A} \psi(y, z, \rho, \alpha) - \frac{\partial \psi(y, z, \rho, \alpha)}{\partial \nu_A} u^s(y, d) \right\} ds(y)$$

Using the Green's formula and due to the form of ψ, we have:

$$\lim_{m\to\infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d) g_m^{z,\rho,\alpha}(\theta) ds(\theta) = -\frac{\rho^{|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial z^\alpha} u^s(z, d) +$$

$$\int_{\partial B_{R'}} \left\{ \frac{d}{dr} u^s(y, d) \psi(y, z, \rho, \alpha) - \frac{d}{dr} \psi(y, z, \rho, \alpha) u^s(y, d) \right\} ds(y)$$

where $B_{R'}$ is a ball of radius R' large enough to contain $\overline{D}$.

We want to show that the integral over $B_{R'}$ tends to zero as R' tends to infinity. Indeed , we write this integral as

$$\int_{\partial B_{R'}} \left\{ (\frac{d}{dr} u^s(y, d) - i\kappa u^s(y, d)) \psi(y, z, \rho, \alpha) - (\frac{d}{dr} \psi(y, z, \rho, \alpha) - i\kappa \psi(y, z, \rho, \alpha)) u^s(y, d) \right\} ds(y)$$

We know that

$$\frac{d}{dr} u^s(y, d) - i\kappa u^s(y, d) = o(\frac{1}{|y|^{\frac{n-1}{2}}}) \text{ and } u^s(y, d) = O(\frac{1}{|y|^{\frac{n-1}{2}}}), \quad (|y| \to \infty).$$

It is then enough to show that

$$\frac{d}{dr} \psi(y, z, \rho, \alpha) - i\kappa \psi(y, z, \rho, \alpha) = o(\frac{1}{|y|^{\frac{n-1}{2}}}) \text{ and } \psi(y, z_q) = O(\frac{1}{|y|^{\frac{n-1}{2}}}), \quad (|y| \to \infty)$$

It is clear that $G(x, z) \in C^\infty(\partial\tilde{\Omega} \times B)$ where $B \subset\subset \Omega \subset\subset \tilde{\Omega}$. Hence from the integral equation of the second kind (22), we deduce that $f(y, z) \in C^\infty(B, L^2_y(\partial\tilde{\Omega}))$, i.e. for $z \in B$ fixed $f(\cdot, z)$ is in $L^2(\partial\tilde{\Omega}))$ and $|f(\cdot, z)|_{L^2(\partial\tilde{\Omega}))}$ is C^∞ with respect to z in B. Now, from (21), we have:

$$\psi(x, z, \rho, \alpha) = \frac{\rho^{|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial z^\alpha} G(x, z) = \frac{\rho^{|\alpha|}}{\alpha!} \int_{\partial\tilde{\Omega}} \Phi(x, y) \frac{\partial^\alpha}{\partial z^\alpha} f(y, z_q) ds(y) \tag{30}$$

$$(\frac{d}{dr} - i\kappa) \psi(x, z_q) = \frac{\rho^{|\alpha|}}{\alpha!} \int_{\partial\Omega} (\frac{d}{dr} - i\kappa) \Phi(x, y) \frac{\partial^\alpha}{\partial z^\alpha} f(y, z) ds(y).$$

However, we know that

$$\Phi(x, y) = O(\frac{1}{|x|^{\frac{n-1}{2}}}), (|x| \to \infty, y \in \partial\tilde{\Omega})$$

and

$$(\frac{d}{dr} - i\kappa) \Phi(x, y)) = o(\frac{1}{|x|^{\frac{n-1}{2}}}), (|x| \to \infty, y \in \partial\tilde{\Omega})$$

then for z, ρ and α fixed, we have

$$|\psi(x, z, \rho, \alpha)| = O(\frac{1}{|x|^{\frac{n-1}{2}}}), (|x| \to \infty, z \in B) \tag{31}$$

and

$$(\frac{d}{dr} - i\kappa)\psi(x, z, \rho, \alpha)) = o(\frac{1}{|x|^{\frac{n-1}{2}}}), (|x| \to \infty, z \in B) \tag{32}$$

Hence taking the limit with respect to R we deduce that

$$\lim_{m\to\infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d) g_m^{z,\rho,\alpha}(\theta) ds(\theta) = -\frac{\rho^{|\alpha|}}{\alpha!} \partial_z^\alpha u^s(z, d). \tag{33}$$

2.5.2 The sound soft case

The following lemma gives a quantitative version of the non-analytic extension of $u^s(z, d)$ near ∂D.

LEMMA 2.10. *If for some positive real number ρ, the set*

$$\{\rho^{|\alpha|} \frac{|\frac{\partial^\alpha}{\partial z^\alpha} u^s(z, d)|}{\alpha!}, \alpha \in \mathbb{Z}_+^n\} \tag{34}$$

is uniformly bounded, then $u^s(z, d)$ is analytically extendable into $B(z, \frac{\rho}{\sqrt{n}})$.

Proof of Lemma 2.10. From the boundedness of (34), say by M, we derive that the series

$$\sum_{\alpha\in\mathbb{Z}_+^n} \frac{1}{\alpha!}(x - z)^\alpha \frac{\partial^\alpha}{\partial z^\alpha} u^s(z, d) \tag{35}$$

has a majorant $\sum_{\alpha\in\mathbb{Z}_+^n} M \frac{|(x-z_0)^\alpha|}{\rho^{|\alpha|}}$. In addition, by setting $(x - z) := ((x - z)_1, ..., (x - z)_n)$, we have

$$\sum_{\alpha\in\mathbb{Z}_+^n} \frac{|(x - z)^\alpha|}{\rho^{|\alpha|}} = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \Pi_{k=1}^n |(x - z)_k|^\alpha \rho^{-|\alpha|}$$

We use the inquality $1 \leq \frac{|\alpha|!}{\alpha!}$ to get:

$$\sum_{\alpha\in\mathbb{Z}_+^n} \frac{|(x - z)^\alpha|}{\rho^{|\alpha|}} \leq \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{m!}{\alpha!} \Pi_{k=1}^n |(x - z_0)_k|^\alpha \rho^{-m} \leq \sum_{m=0}^{\infty} (\sum_{k=1}^{n} |(x - z)_k|)^m \rho^{-m}.$$

Using the inequality $\sum_{k=1}^n |(x - z_0)_k| \leq \sqrt{n}|x - z|$, we deduce that

$$|\sum_{\alpha\in\mathbb{Z}_+^n} \frac{1}{\alpha!}(x - z)^\alpha \frac{\partial^\alpha}{\partial z^\alpha} u^s(z, d)| \leq M \sum_{m=0}^{\infty} (\frac{\sqrt{n}|x - z|}{\rho})^m.$$

This series is absolutely convergent for $x \in B(z_0, \frac{\rho}{\sqrt{n}})$. Hence the Taylor series of $u^s(z, d)$ is absolutely convergent in $B(z, \frac{\rho}{\sqrt{n}})$ which means that $u^s(z, d)$ is analytically extendable into $B(z, \frac{\rho}{\sqrt{n}})$. □

We set

$$I(z, \rho, \alpha) := \lim_{m\to\infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d) g_m^{z,\rho,\alpha}(\theta) ds(\theta). \tag{36}$$

The following theorem gives a reconstructive characterization of the shape of the obstacle, ∂D. It is proof is a direct application of Lemma 2.10.

THEOREM 2.11. *We have the following two properties of* $I(z,\rho,\alpha)$

I. If

$$\limsup_{\rho\to 0}\lim_{|\alpha|\to\infty} I(z,\rho,\alpha)$$

is finite, then

$$d(z,\partial D)>0.$$

Precisely, if for some $\rho>0$, $I(z,\rho,\alpha)$ *is bounded with respect to* ρ, *then*

$$d(z,\partial D)\geq \frac{\rho}{\sqrt{n}}.$$

II. If

$$\limsup_{\rho\to 0}\lim_{|\alpha|\to\infty} I(z,\rho,\alpha)=\infty$$

then

$$d(z,\partial D)=0.$$

2.5.3 The sound hard case

In this case, we assume that A is, in addition to be analytic in Ω, of class C^1 across $\partial\Omega$. Let $d_1,...,d_{n-1}$ be $(n-1)$ directions of incidence which are linearly independent. Let $u_j := u_j(z,d_j) = u^s(z,d_j)+v_P(z,d_j)$ be the associated total fields, where $u^s(z,d_j)$ is the corresponding scattered fields and $v_P(z,d_j)$ is the incident field.

LEMMA 2.12. *Let* $z_0\in\Omega\setminus\overline{D}$ *and* $r_0>0$ *such that* $B(z_0,r_0)\subset\Omega\setminus\overline{D}$. *Suppose that*

$$\sum_{j=1}^{n-1}\beta_j\nabla \mathrm{Re}u(z,d_j)=0 \text{ in } B(z_0,r_0)$$

then $\beta_j=0,\ j=1,...,n-1$.

Proof of Lemma 2.12. Suppose that $\sum_{j=1}^{n-1}\beta_j\nabla\mathrm{Re}u(z,d_j)=0,$ in $B(z_0,r_0)$. Hence

$$\sum_{j=1}^{n-1}\beta_j\mathrm{Re}u(z,d_j)=C \text{ in } B(z_0,r_0),$$

where C is a constant. However $\sum_{j=1}^{n-1}\beta_j\mathrm{Re}u(\cdot,d_j)$ satisfies:

$$(P+\kappa^2)(\sum_{j=1}^{n}\beta_j\mathrm{Re}u(\cdot,d_j))=0 \text{ in } \mathbb{R}^n\setminus\overline{D}.$$

By the unique continuation property (which is justified since A is of class C^1 in $\mathbb{R}^n\setminus\overline{D}$), we deduce that $\sum_{j=1}^{n}\beta_j\mathrm{Re}u(\cdot,d_j)$ is constant in $\mathbb{R}^n\setminus\overline{D}$.

From the radiation conditions, we deduce that $\frac{\partial}{\partial r}u^s(z,d_j)\to 0, r\to\infty,\ r:=|x|$. Since $\sum_{j=1}^{n}\beta_j\mathrm{Re}u(\cdot,d_j)$ is constant in $\mathbb{R}^n\setminus\overline{D}$, then $\mathrm{Im}\sum_{k=1}^{n-1}\beta_k(d_k\cdot\hat{x})e^{i\kappa x\cdot d_k}\to 0$ when $r\to 0$, $\hat{x}:=\frac{x}{|x|}$. Recall that the used incident wave are given by $v_P(x,d)+e^{i\kappa x\cdot d}$ where v_P is actually propagating, due to the Sommerfeld radiation conditions. Hence we have also $\frac{\partial}{\partial r}v_P(z,d_j)\to 0, r\to\infty$.

Let e_1,e_2,...,e_n denote orthogonal unit vertors in $\mathbb{R}^n$. We can find an invertible matrix H satisfying $Hd_k = e_k$ $(k = 1, 2, \ldots, n-1)$. Then we have

$$\sum_{k=0}^{n-1} \beta_k(\hat{x} \cdot d_k) e^{i\kappa(x \cdot d_k)} = \sum_{k=0}^{n-1} \beta_k(\widehat{{}^tHy} \cdot H^{-1}e_k) e^{i\kappa(y \cdot e_k)}$$

where we set $y := {}^tH^{-1}x$. Note that $|x| \to \infty$ if and only if $|y| \to \infty$.

Now we will show that $\beta_l = 0$, $(l = 1, ..., n-1)$. By setting $y_m = \dfrac{(2m + \frac{1}{2})\pi}{\kappa} e_l$, $(m = 1, 2, \ldots)$, we have when $m \to \infty$

$$\operatorname{Im}\left(\sum_{k=0}^{n-1} \beta_k(\widehat{{}^tHy_m} \cdot H^{-1}e_k) e^{i\kappa(y_m \cdot e_k)}\right) = \sum_{k=0}^{n-1} \beta_k(e_l \cdot e_k) = \beta_l.$$

This implies $\beta_l = 0$.

□

From Lemma 2.12 and Corollary 2.7, we deduce that for every point $z \in \partial D$ at least one of the solutions $u^s(z, d_j)$ is not analytically extendable near z. As we did for the Dirichlet case, for each solution $u^s(z, d_j)$, we associate the functional:

$$I_j(z, \rho, \alpha) := \lim_{m \to \infty} \int_{\mathbb{S}^n} u^\infty(-\theta, d_j) g_m^{z,\rho,\alpha}(\theta) ds(\theta)$$

and we define the following functional for this Neumann problem:

$$J(z, \rho, \alpha) := \sum_{i=1}^{n-1} |I_j(z, \rho, \alpha)|. \tag{37}$$

Similar to Theorem 2.11, we have the following theorem which enables us to reconstruct non-analytic obstacles for the Neuman case by using $(n-1)$ linearly independent directions of incidence.

THEOREM 2.13. *We have the following two properties of $J(z, \rho, \alpha)$:*

I. If

$$\limsup_{\rho \to 0} \lim_{|\alpha| \to \infty} J(z, \rho, \alpha)$$

is finite, then

$$d(z, \partial D) > 0.$$

Precisely, if for some $\rho > 0$, $J(z, \rho, \alpha)$ is bounded with respect to ρ, then

$$d(z, \partial D) \geq \frac{\rho}{\sqrt{n}}.$$

II. If

$$\limsup_{\rho \to 0} \lim_{|\alpha| \to \infty} J(z, \rho, \alpha) = \infty$$

then

$$d(z, \partial D) = 0.$$

REMARK 2.14. *Combining Theorem 2.11 and Theorem 2.13, we can reconstruct the shape of a non-analytic obstacle by using $(n-1)$ linearly independent directions of incidence without knowing a-priori if on the boundary of the obstacle we have a Dircichlet or a Neumann condition.*

References

[1] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, *Optimal Stability for Inverse Elliptic Boundary Value Problems with Unknown Boundaries*, Ann. Sc. Norm. Super. Pisa - Scienze Fisiche e Matematiche - Serie IV. Vol XXIX. Fasc.4 (2000).

[2] G. Alessandrini; L. Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc. 133 (2005), no. 6, 1685–1691

[3] I. Bushuyev, *Stability of recovering the near-field wave from the scattering amplitude*, Inverse Problems 12 (1996), no. 6, 859–867.

[4] F. Cakoni, D. Colton, Qualitative Methods in Inverse Scattering Theory. Interaction of Mechanics and Mathematics, Springer, NewYork, 2006.

[5] J. Cheng; M. Yamamoto, *Corrigendum: "Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves"* [Inverse Problems 19 (2003), no. 6, 1361–1384]. Inverse Problems 21 (2005)

[6] D. Colton, B. D. Sleeman, *Uniqueness theorems for the inverse problem of acoustic scattering*, IMA J. Appl. Math. 31 (1983), no. 3, 253-259.

[7] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory. 2nd edition (Berlin-Springer) (1998).

[8] J. Elschner; M. Yamamoto, *Uniqueness in determining polygonal sound-hard obstacles with a single incoming wave*. Inverse Problems 22 (2006), no. 1, 355–364.

[9] N. Honda; R. Potthast; G. Nakamura; M. Sini, *The no-response approach and its relation to non-iterative methods for the inverse scattering* . Ann. Mat. Pura Appl. (4) 187 (2008)

[10] D. Gintides, *Local uniqueness for the inverse scattering problem in acoustics via the Faber-Krahn inequality* . Inverse Problems 21 (2005), no. 4, 1195–1205

[11] V. Isakov, Inverse Problems for Partial Differential Equations. Springer Series in Applied Math. Science. Berlin: Springer, **127**, (1998).

[12] V. Isakov, *Stability estimates for obstacles in inverse scattering*, J. Comp. Appl. Math. 42, (1991), 79-89.

[13] V. Isakov, *New stability results for soft obstacles in inverse scattering*, Inverse Problems 9, (1993), no. 5, 535–543.

[14] Kashiwara, M and Schapira, P. Sheaves on manifolds, Grundlehren der mathematischen Wissenschaften 292, Springer-Verlag (1990).

[15] J. B. Keller and D. Givoli, Exact non-reflecting boundary conditions. J. Computational Phys. V 82, 172-192, (1989)

[16] H. Liu; J. Zou, *On unique determination of partially coated polyhedral scatterers with far field measurements*. Inverse Problems 23 (2007), no. 1, 297–308.

[17] H. Liu; J. Zou, *Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers*. Inverse Problems 22 (2006), no. 2, 515–524.

[18] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University press, (2000).

[19] M. Mitrea, and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*. J. Funct. Anal. 176 , pp. 1-79, (2000).

[20] Narasimhan, R. Introduction to the theory of analytic spaces, Lecture Notes in Math. 25, Springer-Verlag (1966).

[21] G. Nakamura; R. Potthast; M. Sini, *Unification of the probe and singular sources methods for the inverse boundary value problem by the no-response test*. Comm. Partial Differential Equations 31 (2006), no. 10-12, 1505–1528.

[22] S. O'dell: *Inverse scattering for the Laplace-Beltrami operator with complex electromagnetic potential and embedded obstacles*. Inverse problems, V 22, 1579–1603, (2006).

[23] R. Potthast, *Point Sources and Multipoles in Inverse Scattering Theory*, Research Notes in Mathematics, Vol.427, Chapman-Hall/CRC, Boca Raton, Fl, 2001.

[24] R. Potthast, *Sampling and Probe Methods - An Algorithmical View. Computing*, 75, no. 2-3, 215–235, (2005).

[25] R. Potthast, On the convergence of the no response test. SIAM J. Math. Anal. 38, no. 6, 1808–1824, (2007)

[26] A. G. Ramm, Inverse Problems, Mathematical and Analytical Techniques with Applications to Engineering, Springer, 2004.

[27] A. G. Ramm, *Uniqueness of the solution to inverse obstacle scattering problem* Physics Letters A, 347, N4-6, (2005), 157-159.

[28] L. Rondi, *Unique determination of non-smooth sound-soft scatterers by finitely many far-field measurements*. Indiana Univ. Math. J. 52 (2003), no. 6, 1631–1662.

[29] E. Sincich and M. Sini. Local stability for soft obstacles by a single measurement. RICAM - Report No. 2007-23. Submitted.

[30] P. Stefanov, G. Uhlmann, *Local uniqueness for the fixed energy fixed angle inverse problem in obstacle scattering*, Proc. Amer. Math. Soc. 132 (2004), no. 5, 1351-1354.