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ANALYTIC-FUNCTION BASES FOR MULTIPLY-CONNECTED REGIONS

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Abstract. Let E be a nonempty (not necessarily bounded) region of finite connectivity, whose boundary consists of a finite number of nonintersecting analytic Jordan curves. Work of J. L. Walsh is utilized to construct an absolute basis $(Q_n, n=0, \pm 1, \pm 2, \ldots)$ of rational functions for the space $H(E)$ of functions analytic on E, with the topology of compact convergence; or the space $H(\mathrm{Cl}\,(E))$ of functions analytic on $\mathrm{Cl}\,(E)=$the closure of E, with an inductive limit topology. It is shown that $\sum_{n=0}^{\infty} Q_n(z)Q_{-n-1}(w)=1/(w-z)$, the convergence being uniform for z and w on suitable subsets of the plane. A sequence $(P_n, n=0, \pm 1, \pm 2, \ldots)$ of elements of $H(E)$ (resp. $H(\mathrm{Cl}\,(E))$) is said to be absolutely effective on E (resp. $\mathrm{Cl}\,(E)$) if it is an absolute basis for $H(E)$ (resp. $H(\mathrm{Cl}\,(E))$) and the coefficients arise by matrix multiplication from the expansion of (Q_n). Conditions for absolute effectivity are derived from W. F. Newns' generalization of work of J. M. Whittaker and B. Cannon. Moreover, if $(P_n, n=0, 1, 2, \ldots)$ is absolutely effective on a certain simply-connected set associated with E, the sequence is extended to an absolutely effective basis $(P_n, n=0, \pm 1, \pm 2, \ldots)$ for $H(E)$ (or $H(\mathrm{Cl}\,(E))$) such that $\sum_{n=0}^{\infty} P_n(z)P_{-n-1}(w)=1/(w-z)$. This last construction applies to a large class of orthogonal polynomials.

1. **Interpolation bases.** For any subset S of the extended complex plane, let $H(S)$ be the set of functions analytic on S (that is, f is in $H(S)$ if and only if f is analytic on some open set containing S), and zero at infinity if the point at infinity is in S. The convergence of a sequence of elements (f_n) of $H(S)$ will be said to be *compact-open* on S if and only if the sequence converges uniformly on compact subsets of some open set containing S.

Throughout the paper, let E be a nonempty region of finite connectivity, whose boundary consists of a finite number of nonintersecting analytic Jordan curves. If E is bounded, divide the boundary curves into two mutually disjoint sets, let $L(0)$ and $L(1)$ be the respective unions of the curves in each set, and suppose that the curve exterior to all the others is a subset of $L(1)$. Let F be harmonic in E and continuous on the closure of E, and take on the values 0 and 1 on $L(0)$ and $L(1)$ respectively. Then there exist points $z_1, z_2, z_3, \ldots$, in the components of the complement of E bounded by $L(0)$, and points $w_1, w_2, w_3, \ldots$, in the components of

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the complement of E bounded by $L(1)$, such that

$$\lim_{n\to\infty}\left|\frac{(z-z_1)(z-z_2)\cdots(z-z_n)}{(z-w_1)(z-w_2)\cdots(z-w_n)}\right|^{1/n} = K_1\exp(K_2F(z)) \tag{1.1}$$

where K_1 and K_2 are positive constants, and the convergence is compact-open in E [4, pp. 209–211]. For each real number x such that $0\leqq x\leqq 1$ let $L(x)$ be the set of all z such that $F(z)=x$; let $E(x)$ be the set of all z such that either $F(z)<x$ or z is in a component of the complement of E bounded by a subset of $L(0)$; and let $e(x)$ be the set of all z such that either $F(z)>x$ or z is in a component of the complement of E bounded by a subset of $L(1)$. We consider the point at infinity to be a member of $e(x)$ for each x.

If E is unbounded, let F be the Green's function with pole at infinity for E. Then there exist points $z_1, z_2, z_3, \ldots$, in the complement of E, such that

$$\lim_{n\to\infty}|(z-z_1)(z-z_2)\cdots(z-z_n)|^{1/n} = K\exp(F(z)) \tag{1.2}$$

where K is a positive constant called the capacity or transfinite diameter of the complement of E, and the convergence is compact-open in E [4, pp. 72–73, 157–158]. For each nonnegative extended real number x let $L(x)$ be the set of all z such that $F(z)=x$; let $E(x)$ be the set of all z such that either $F(z)<x$ or z is in the complement of E; and let $e(x)$ be the set of all z such that $F(z)>x$. We consider the point at infinity to be a member of $e(x)$ for each x.

In the bounded case, let

$$\begin{gathered} Q_n(z) = \frac{(z-z_1)\cdots(z-z_n)}{(z-w_1)\cdots(z-w_n)} \quad \text{for } n>0, \qquad Q_0(z)=1, \\ Q_{-1}(z) = 1/(z-z_1), \qquad Q_{-n}(z) = \frac{z_n-w_{n-1}}{(z-z_n)(z-w_{n-1})Q_{n-1}(z)} \quad \text{for } n>1. \end{gathered} \tag{1.3}$$

In the unbounded case, let

$$\begin{aligned} &Q_n(z) = (z-z_1)\cdots(z-z_n) && \text{for } n>0, \\ &Q_0(z)=1, \qquad Q_{-n}(z) = 1/Q_n(z) && \text{for } n>0. \end{aligned} \tag{1.4}$$

In either case, let $g(z,w)=\sum_{n=0}^{\infty} Q_n(z)Q_{-n-1}(w)$ wherever the series converges. If f is analytic on $E(x)$ then for all z in $E(x)$,

$$f(z) = (1/2\pi i)\int_{L(y)} g(z,t)f(t)\,dt$$

where, for each z in $E(x)$, $L(y)$ is a curve, contained in the region of analyticity of f, such that z is in $\mathrm{Cl}\,(E(y_1))$ for some $y_1<y$. ([4, pp. 190, 193] for the bounded case, [pp. 159–160] for the unbounded case.) From (1.1) or (1.2), $g(z,t)$ converges absolutely and uniformly for z on $\mathrm{Cl}\,(E(y_1))$ and t on $L(y)$. In particular, for fixed w let $f(z)=1/(w-z)$. If z is in the interior of $E(x)$ and w is on $L(y)$, with $x<y$,

then, because of the uniform convergence,

$$\begin{aligned} 1/(w-z) &= (1/2\pi i)\int_{L(x)} (g(z,t)/(w-t))\,dt \\ &= \sum_{n=0}^{\infty} Q_n(z)(1/2\pi i)\int_{L(x)} (Q_{-n-1}(t)/(w-t))\,dt. \end{aligned}$$

Now if $y<y_1$ then

$$\int_{L(y_1)} (Q_{-n-1}(t)/(w-t))\,dt = 0$$

for all n, since this integral is independent of y_1 as long as w is on $L(y)$ and $y<y_1$; so we may take $y_1=1$ in the bounded case with $n\neq 0$; we may replace $L(y_1)$ by a circle with center at the origin and radius approaching infinity in the bounded case with $n=0$; and we may let y_1 approach infinity in the unbounded case. Thus if z is in the interior of $E(x)$ and w is on $L(y)$, with $x<y<y_1$, then, for every n,

$$\begin{aligned} \int_{L(x)} (Q_{-n-1}(t)/(w-t))\,dt &= \int_{L(y_1)} (Q_{-n-1}(t)/(t-w))\,dt - \int_{L(x)} (Q_{-n-1}(t)/(t-w))\,dt \\ &= 2\pi i Q_{-n-1}(w). \end{aligned}$$

Thus $1/(w-z)=g(z,w)=\sum_{n=0}^{\infty} Q_n(z)Q_{-n-1}(w)$, where the convergence is absolute and uniform for z on any compact subset of the interior of $E(x)$; and w on any compact subset of $e(x)\cap E(1)$ in the bounded case or w on any compact subset of $e(x)$ in the unbounded case.

Let f be any function analytic in $e(x)\cap E(y)$, where $x<y$. Let z be any point in $e(x)\cap E(y)$, and choose r and s such that $x<r<s<y$ and z is in $e(r)\cap E(s)$. Then

$$\begin{aligned} f(z) &= (1/2\pi i)\int_{L(s)} (f(t)/(t-z))\,dt - (1/2\pi i)\int_{L(r)} (f(t)/(t-z))\,dt \\ &= \sum_{n=0}^{\infty} Q_n(z)(1/2\pi i)\int_{L(s)} f(t)Q_{-n-1}(t)\,dt \\ &\quad + \sum_{n=0}^{\infty} Q_{-n-1}(z)(1/2\pi i)\int_{L(r)} f(t)Q_n(t)\,dt, \end{aligned}$$

and so

$$f(z) = \sum_{n=-\infty}^{\infty} B_n Q_n(z) \tag{1.5}$$

where

$$B_n = (1/2\pi i)\int_L f(t)Q_{-n-1}(t)\,dt \quad \text{for } n = 0, \pm 1, \pm 2, \ldots; \tag{1.6}$$

$L=$ any $L(s)$ between z and $L(y)$ for nonnegative n; $L=$ any $L(r)$ between z and $L(x)$ for negative n; and the convergence is compact-open in $e(x)\cap E(y)$. If also $f(z)=$

$\sum_{n=-\infty}^{\infty} b_n Q_n(z)$ where the convergence is compact-open in $e(x) \cap E(y)$ then (1.6) implies that $b_n = B_n$ for all n, since

$$(1/2\pi i) \int_L Q_k(t) Q_{-n-1}(t)\, dt = \delta_{nk}$$

where $\delta_{nk}=1$ if $n=k$, and $=0$ otherwise. If $f_0(z)=\sum_{n=0}^{\infty} B_n Q_n(z)$ and $f_1(z)=\sum_{n=-1}^{-\infty} B_n Q_n(z)$ then f_0 is analytic in $E(y)$, while f_1 is analytic in $e(x)$ and zero at infinity, and $f=f_0+f_1$. If f is any function analytic on $\mathrm{Cl}\,(e(x) \cap E(y))$, where $0<x<y<1$ in the bounded case or $0<x<y$ in the unbounded case, then there exist r and s such that $r<x$, $y<s$, and f is analytic on $e(r) \cap E(s)$. Thus (1.5) holds, the convergence being compact-open on $\mathrm{Cl}\,(e(x) \cap E(y))$, and the (B_n) are unique and given by (1.6). Moreover, f is the sum of a function analytic on $\mathrm{Cl}\,(E(y))$ and a function analytic on $\mathrm{Cl}\,(e(x))$ and zero at infinity. The sequence (Q_n), defined by (1.3) in the bounded case or (1.4) in the unbounded case, will be called an *interpolation base* for E. If (Q_n) is *any* sequence of functions analytic on $e(x) \cap E(y)$ (resp. $\mathrm{Cl}\,(e(x) \cap E(y))$) such that every function f analytic on $e(x) \cap E(y)$ (resp. $\mathrm{Cl}\,(e(x) \cap E(y))$) can be expanded uniquely in the form (1.5), the convergence being absolute and compact-open on $e(x) \cap E(y)$ (resp. $\mathrm{Cl}\,(e(x) \cap E(y))$), then (Q_n) is called an *absolute base* for $e(x) \cap E(y)$ (resp. $\mathrm{Cl}\,(e(x) \cap E(y))$) with compact-open convergence. (Cf. [3, pp. 432, 434].)

We may summarize our previous remarks in

THEOREM 1. *Let E be a nonempty region of finite connectivity, whose boundary consists of a finite number of nonintersecting analytic Jordan curves, and let (Q_n) be any interpolation base for E. Then* $1/(w-z)=\sum_{n=0}^{\infty} Q_n(z) Q_{-n-1}(w)$; *where the convergence is absolute and uniform for z on any compact subset of the interior of* $E(x)$; *and w on any compact subset of* $e(x) \cap E(1)$ *in the bounded case or w on any compact subset of* $e(x)$ *in the unbounded case. Moreover, (Q_n) is an absolute base, with compact-open convergence, for every* $e(x) \cap E(y)$ ($0 \leqq x<y \leqq 1$ *in the bounded case or* $0 \leqq x<y$ *in the unbounded case*) *and every* $\mathrm{Cl}\,(e(x) \cap E(y))$ ($0<x<y<1$ *in the bounded case or* $0<x<y$ *in the unbounded case*). *Finally, any function analytic on* $e(x) \cap E(y)$ (*resp.* $\mathrm{Cl}\,(e(x) \cap E(y))$) *is the sum of a function analytic on* $E(y)$ (*resp.* $\mathrm{Cl}\,(E(y))$) *and a function analytic on* $e(x)$ (*resp.* $\mathrm{Cl}\,(e(x))$) *and zero at infinity.*

2. **Absolutely effective basic sets.** Let (Q_n) be any interpolation base for E. Let x and y be extended real numbers such that $0 \leqq x<y \leqq 1$ in the bounded case, or $0 \leqq x<y$ in the unbounded case. Let $(P_n, n=0, \pm 1, \pm 2, \ldots)$ be a sequence of functions analytic on $e(x) \cap E(y)$, and for each r such that $x<r<y$ let

$$M_n(r) = (\max |P_n(z)|,\ z \text{ on } L(r)), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Suppose that a matrix (G_{nk}) of complex numbers is given such that

$$Q_n = \sum_{k=-\infty}^{\infty} G_{nk} P_k, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

the convergence being compact-open on $e(x) \cap E(y)$, and suppose also that

$$R_n(r) = \sum_{k=-\infty}^{\infty} |G_{nk}| M_k(r), \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{2.2}$$

is finite for every r. Let (D_{jn}) be the (unique) matrix of complex numbers such that

$$P_j = \sum_{n=-\infty}^{\infty} D_{jn} Q_n, \qquad j = 0, \pm 1, \pm 2, \ldots.$$

In certain cases it will happen that

$$\sum_{n=-\infty}^{\infty} D_{jn} G_{nk} = \delta_{jk}. \tag{2.3}$$

If (2.1) and (2.3) hold and (2.2) is finite for every r, then (P_n) will be called a *basic set* for $e(x) \cap E(y)$. (Cf. [3, pp. 439, 443].) We define $I(r) = \limsup_{n\to\infty} (R_n(r))^{1/n}$ and $i(r) = \limsup_{n\to-\infty} (R_n(r))^{-1/n}$. Since $|Q_n(z)| \leqq R_n(r)$ for z on $L(r)$, (1.1) or (1.2) imply $1/i(r) \leqq K_1 \exp(K_2 r) \leqq I(r)$ in the bounded case, or $1/i(r) \leqq K \exp(r) \leqq I(r)$ in the unbounded case, where K_1, K_2 and K are defined as in (1.1) and (1.2). If (P_n) is a basic set for $e(x) \cap E(y)$ and any function f which is analytic on $e(x) \cap E(y)$ can be expanded in the form

$$f(z) = \sum_{k=-\infty}^{\infty} A_k P_k(z) \tag{2.4}$$

where the convergence is absolute and compact-open in $e(x) \cap E(y)$, and the (A_k) are given by

$$A_k = \sum_{n=-\infty}^{\infty} B_n G_{nk}, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{2.5}$$

the (B_n) being defined by (1.6), then (P_n) will be said to be *absolutely effective* on $e(x) \cap E(y)$ (with compact-open convergence). (Cf. [3, p. 440].) (If $0 < x < y < 1$ in the bounded case, or $0 < x < y < \infty$ in the unbounded case, we can define absolute effectivity on $\mathrm{Cl}\,(e(x) \cap E(y))$ by the condition that any function f which is analytic on $\mathrm{Cl}\,(e(x) \cap E(y))$ can be expanded in the form (2.4), the convergence being absolute and compact-open on $\mathrm{Cl}\,(e(x) \cap E(y))$, and the (A_k) being given by (2.5). Here we assume that the (P_n) are all analytic on a fixed open set containing $\mathrm{Cl}\,(e(x) \cap E(y))$.)

For any subset S of the extended plane, let $H(S)$ be the set of all functions analytic on S (and zero at infinity if the point at infinity is in S). In order to apply theorems on absolute effectivity from (3), we put topologies on $H(e(x) \cap E(y))$ and $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ which give rise to compact-open convergence. Let T be the topology of compact convergence on $H(e(x) \cap E(y))$. (Cf. [1, p. 236].) For $H(\mathrm{Cl}\,(e(x) \cap E(y)))$, let (x_n) be an increasing sequence of positive numbers approaching x and let (y_n) be a decreasing sequence approaching y. Let $H_b(x_n, y_n)$

be the set of all functions analytic on $e(x_n) \cap E(y_n)$ and continuous on $\mathrm{Cl}\,(e(x) \cap E(y))$; then $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ is the increasing union of the $(H_b(x_n, y_n))$. Let T_c be the finest locally convex topology on $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ such that all the natural injection maps from $H_b(x_n, y_n)$ (with the topology of uniform convergence on $\mathrm{Cl}\,(e(x_n) \cap E(y_n)))$ into $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ are continuous. (Cf. [1, p. 157].) $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ is isomorphic to the dual of $H(E(x)) + H(e(y))$ (with the topology of compact convergence) under the map v which associates with each f in $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ the linear functional $v(f)$ whose value at any $g+h$ in $H(E(x)) + H(e(y))$ is given by

$$v(f)(g+h) = (1/2\pi i) \int_{L(r)} f(t)g(t)\,dt - \int_{L(s)} f(t)h(t)\,dt$$

where $r < x$, $y < s$, and f is analytic on $\mathrm{Cl}\,(e(r) \cap E(s))$. (Cf. [2, p. 378].) T_c is identical with the strong topology induced on $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ by this duality [2, p. 381], so T_c contains the weak topology induced by the duality. If (f_j) is any sequence convergent for T_c, then it is bounded for T_c and therefore bounded for the weak topology, and therefore equicontinuous as a set of functions on $H(E(x)) + H(e(y))$, since $H(E(x)) + H(e(y))$ is barrelled. (Cf. [1, pp. 212, 214].) Thus for any $r > 0$, there exist M, x_n and y_n such that $|v(f_j)(g+h)| < r$ whenever $|g| < M$ on $\mathrm{Cl}\,(E(x_n))$ and $|h| < M$ on $\mathrm{Cl}\,(e(y_n))$. Thus all the (f_j) are in $H_b(x_n, y_n)$ and are uniformly bounded there. But each $f = f_j - f_k$ can be expanded in the form (1.5), with coefficients given by (1.6), where $L = L(y_n) - L(x_n)$. For each n in (1.6) the coefficients approach zero as j and k increase, since (f_j) is a weak Cauchy sequence. This, together with the uniform boundedness of (f_j), implies that (f_j) converges uniformly on $\mathrm{Cl}\,(e(x_n) \cap E(y_n))$. Thus T_c gives rise to compact-open convergence.

We may now apply various theorems on absolute effectiveness, from (3).

Theorem 2. *Let (P_n) be a basic set for $e(x) \cap E(y)$. Then (P_n) is absolutely effective on $e(x) \cap E(y)$ if and only if $1/i(r) > K_1 \exp(K_2 x)$ and $K_1 \exp(K_2 y) > I(r)$ for all r such that $x < r < y$ in the bounded case, or $1/i(r) > K \exp(x)$ and $K \exp(y) > I(r)$ for all r such that $x < r < y$ in the unbounded case. If the (P_n) are all analytic on a fixed open set containing $\mathrm{Cl}\,(e(x) \cap E(y))$ (where $0 < x < y < 1$ in the bounded case or $0 < x < y < \infty$ in the unbounded case), then (P_n) is absolutely effective on $\mathrm{Cl}\,(e(x) \cap E(y))$ if and only if $1/i(x-) = K_1 \exp(K_2 x)$ and $K_1 \exp(K_2 y) = I(y+)$ in the bounded case, or $1/i(x-) = K \exp(x)$ and $K \exp(y) = I(y+)$ in the unbounded case.*

Proof. Apply Theorem 8.2, and the analog of Corollary 2 to Theorem 7.3, from [3, p. 442].

Theorem 3. *Let (P_n) be a basic set for $e(x) \cap E(y)$ and let $f(z) = \sum_{k=-\infty}^{\infty} A_k P_k(z)$, where the convergence is absolute and compact-open on $\mathrm{Cl}\,(e(x) \cap E(y))$. Let (G_{nk}) be defined by (2.1) and suppose that for some k, $\limsup_{n\to\infty} |G_{nk}|^{1/n} \leqq K_1 \exp(K_2 y)$*

and $\limsup_{n\to-\infty} |G_{nk}|^{-1/n} \leqq 1/(K_1 \exp(K_2 x))$ *in the bounded case, or*

$$\limsup_{n\to\infty} |G_{nk}|^{1/n} \leqq K \exp(y)$$

and $\limsup_{n\to-\infty} |G_{nk}|^{-1/n} \leqq 1/(K \exp(x))$ *in the unbounded case. Then* A_k *is given by* (2.5), *where the* (B_n) *are defined by* (1.6). *In particular, if* (P_n) *is absolutely effective on* $e(x) \cap E(y)$ *or* $\mathrm{Cl}\,(e(x) \cap E(y))$, *then the basic coefficients are unique.*

Proof. Apply Theorem 9.1 of [3, p. 444], which applies to the topology T_c, and its analog for the topology T.

THEOREM 4. *Let* (P_n) *be a sequence of elements of* $H(e(x) \cap E(y))$ *such that every element of* $H(e(x) \cap E(y))$ *has a unique representation of the form* (2.4), *the convergence being compact-open on* $e(x) \cap E(y)$, *and suppose that the numbers* $R_n(r)$ *defined by* (2.2) *and* (2.1) *are finite for every* r *such that* $x<r<y$. *Then* (P_n) *is a basic set for* $e(x) \cap E(y)$, *and the representations are basic. If* (P_n) *are all in* $H(e(x_0) \cap E(y_0))$ *where* $x_0<x<y<y_0$, *if every element* f *of* $H(\mathrm{Cl}\,(e(x) \cap E(y)))$ *has a unique representation of the form* (2.4), *the convergence being compact-open on* $\mathrm{Cl}\,(e(x) \cap E(y))$, *and if* $R_n(r)$ *is finite for every* r *such that* $x_0<r<y_0$, *then* (P_n) *is a basic set for* $\mathrm{Cl}\,(e(x) \cap E(y))$, *and the representations are basic.*

Proof. Apply Theorems 9.2 and 9.3 from [3, p. 444].

Note that Theorem 4 implies, in particular, that given two interpolation bases for E, the change of coordinates is effected by matrix multiplication. In fact this is true about any two absolutely effective bases, since any absolutely effective base may be used in place of an interpolation base. Also, Theorem 4 implies that effectivity and the basic coefficients do not depend on which interpolation base is used, subject to the condition that the numbers defined by (2.2) be finite.

3. **Linear transformations.** Let (P_n) be a basic set for $e(x) \cap E(y)$ which is absolutely effective on $e(x) \cap E(y)$. Let U be a linear transformation of $H(e(x) \cap E(y))$ into itself which is continuous for the topology T. Let (U_{kn}) be the unique matrix of complex numbers such that

$$U(P_n) = \sum_{k=-\infty}^{\infty} U_{kn} P_k, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

For any r and s such that $x<r<s<y$, there exist t, u and N such that

$$\sum_{k=-\infty}^{\infty} |U_{kn}|(M_k(r)+M_k(s)) \leqq N((\max |UP_n(z)|,\ z \text{ on } L(t)) + (\max |UP_n(z)|,\ z \text{ on } L(u))).$$

(Cf. [3, p. 432].) Since U is continuous for T, there exist N_1, v and w such that

$$(\max |UP_n(z)|,\ z \text{ on } L(t)) + (\max |UP_n(z)|,\ z \text{ on } L(u)) \leqq N_1(M_n(v)+M_n(w)).$$

Thus if f is any element of $H(e(x) \cap E(y))$ and f is expanded in the form (2.4), then $\sum_{n=-\infty}^{\infty} |A_n|(M_n(v)+M_n(w))$ converges, and therefore

$$\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |A_n|\ |U_{kn}|(M_k(r)+M_k(s)) < \infty.$$

Thus U can be defined by a matrix with respect to (P_n), the convergence being absolute and compact-open on $e(x) \cap E(y)$. A double application of the same principle shows that if V is any other linear transformation continuous with respect to T, then the matrix of UV is the product of the matrices of U and V, the convergence being absolute. Similarly, if (P_n) and (p_n) are any two absolute bases then there is an invertible matrix (W_{kn}) such that $P_n = \sum_{k=-\infty}^{\infty} W_{kn} p_k$, and the matrices of U with respect to (P_n) and (p_n) are similar.

If (Q_n) is any interpolation base and (U_{kn}) is a matrix of complex numbers, then, from (1.1) in the bounded case or (1.2) in the unbounded case, it follows that a necessary and sufficient condition that (U_{kn}) be the matrix with respect to (Q_n) of a T-continuous linear transformation on $e(x) \cap E(y)$, is that, if $R = K_1 \exp(K_2 y)$ and $r = K_1 \exp(K_2 x)$ in the bounded case, or $R = K \exp(y)$ and $r = K \exp(x)$ in the unbounded case, then

$$\limsup_{k\to\infty} \left| \sum_{n=-\infty}^{-1} U_{nk}/r^n + \sum_{n=0}^{\infty} U_{nk}/R^n \right|^{1/k} \leqq 1/R$$

and

$$\limsup_{k\to-\infty} \left| \sum_{n=-\infty}^{-1} U_{nk}/r^n + \sum_{n=0}^{\infty} U_{nk}/R^n \right|^{-1/k} \leqq r$$

(and all the summations must converge absolutely). An analogous condition holds for the T_c topology.

4. **A class of examples.** Let (Q_n) be any interpolation base for E. The set $(Q_n, n=0, 1, 2, \ldots)$ is an absolute base for every $E(y)$ (resp. $\mathrm{Cl}(E(y))$) with compact-open convergence, as we saw in the remarks prior to Theorem 1. If $(P_n, n=0, 1, 2, \ldots)$ is a sequence of functions analytic on $E(y)$ we may investigate the absolute effectivity of (P_n) on $E(y)$ (resp. $\mathrm{Cl}(E(y))$), by dropping all negative indices from (2.1) to (2.5), and considering only the function I instead of I and i. All the theorems hold with these changes. In particular, we state, for convenience

THEOREM 2′. *Let $(P_n, n=0, 1, 2, \ldots)$ be a basic set for $E(y)$. Then (P_n) is absolutely effective on $E(y)$ if and only if $K_1 \exp(K_2 y) > I(r)$ for all r such that $0 < r < y$ in the bounded case, or $K \exp(y) > I(r)$ for all r such that $0 < r < y$ in the unbounded case.*

THEOREM 4′. *Let $(P_n, n=0, 1, 2, \ldots)$ be a sequence of elements of $H(E(y))$ (resp. $H(E(y_0))$ for some $y_0 > y$) such that every element of $H(E(y))$ (resp. $H(\mathrm{Cl}(E(y)))$) has a unique representation of the form* (2.4) *(with $A_k = 0$ for negative k), the convergence being absolute and compact-open on $E(y)$ (resp. $\mathrm{Cl}(E(y))$), and suppose* *that the numbers* $R_n(r)$, $n=0, 1, 2, \ldots$, *defined by* (2.2) *and* (2.1) *are finite for every* r *such that* $0 \leqq r < y$ (*resp.* $0 \leqq r < y_0$). *Then* (P_n) *is a basic set for* $E(y)$ (*resp.* $E(y_0)$) *and the representations are basic.*

THEOREM 5. *Let* (P_n) *satisfy the conditions of Theorem* 4′. *If* (P_n) *is also absolutely effective on* $\mathrm{Cl}\,(E(x))$ (*resp.* $E(x)$) *for some* x *such that* $0 \leqq x < y$ (*resp.* $0 < x < y$), *then there exist functions* P_n, $n=-1, -2, \ldots$, *analytic in* $e(x)$ (*resp.* $\mathrm{Cl}\,(e(x))$), *such that* $(P_n, n=0, \pm 1, \pm 2, \ldots)$ *is absolutely effective on* $e(x) \cap E(y)$ (*resp.* $\mathrm{Cl}\,(e(x) \cap E(y))$).

Proof. Suppose first that $(P_n, n=0, 1, 2, \ldots)$ is absolutely effective on $E(y)$ and on $\mathrm{Cl}\,(E(x))$. Let $(G_{nk}, n, k=0, 1, 2, \ldots)$ be defined by (2.1). For each $k=1, 2, \ldots$, let

$$P_{-k}(z) = \sum_{n=0}^{\infty} G_{n,k-1} Q_{-n-1}(z). \tag{4.1}$$

Since $(P_k, k=0, 1, 2, \ldots)$ is effective on $\mathrm{Cl}\,(E(x))$, the sum in (2.5) (nonnegative indices) must converge absolutely for all (B_n) such that $\limsup_{n\to\infty} |B_n|^{1/n} < 1/K_1 \exp(K_2 x)$. (We treat only the bounded case. The unbounded case is exactly similar throughout.) Thus $\limsup_{n\to\infty} |G_{n,k-1}|^{1/n} \leqq K_1 \exp(K_2 x)$ for each k, so from (1.1) the sum in (4.1) converges compact-openly in $e(x)$ for each k. Thus P_{-k} is analytic in $e(x)$. Let z be any point in $e(x) \cap E(y)$ and suppose z is on $L(u)$ for some u. Since $(P_n, n=0, 1, 2, \ldots)$ is absolutely effective on $E(y)$, Theorem 2′ says that $I(u) < K_1 \exp(K_2 y)$. By (1.1) and (2.2) this means that we can find s such that $u < s < y$ and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |G_{nk} P_k(z) Q_{-n-1}(w)| < \infty, \tag{4.2}$$

the convergence being uniform for z on $\mathrm{Cl}\,(E(u))$ and w on $\mathrm{Cl}\,(e(s))$. Similarly, let w be any point in $e(x) \cap E(y)$, and let w lie on $L(u)$. Since $(P_n, n=0, 1, 2, \ldots)$ is absolutely effective on $\mathrm{Cl}\,(E(x))$, then $I(x+) = K_1 \exp(K_2 x)$. Thus there exists r such that $x < r < u$ and (4.2) holds, the convergence being uniform for z on $\mathrm{Cl}\,(E(r))$ and w on $\mathrm{Cl}\,(e(u))$. Therefore, if z is on $\mathrm{Cl}\,(E(u))$ and w is on $\mathrm{Cl}\,(e(s))$, then

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(z) P_{-k-1}(w) &= \sum_{k=0}^{\infty} P_k(z) \sum_{n=0}^{\infty} G_{nk} Q_{-n-1}(w) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} G_{nk} P_k(z) \right) Q_{-n-1}(w) = \sum_{n=0}^{\infty} Q_n(z) Q_{-n-1}(w) \\ &= 1/(w-z)\end{aligned}$$

by Theorem 1, the convergence being absolute and uniform. Similarly, if z is on $\mathrm{Cl}\,(e(u))$ and w is on $\mathrm{Cl}\,(E(r))$ then

$$\sum_{k=0}^{\infty} P_k(w) P_{-k-1}(z) = 1/(z-w), \tag{4.3}$$

the convergence being absolute and uniform. Now let f be any function analytic on $e(x) \cap E(y)$. If z is on $L(u)$, choose r and s as above. Then

$$\begin{aligned} f(z) &= (1/2\pi i) \int_{L(s)} (f(w)/(w-z))\, dw + (1/2\pi i) \int_{L(r)} (f(w)/(z-w))\, dw \\ &= \sum_{k=0}^{\infty} P_k(z)(1/2\pi i) \int_{L(s)} f(w) P_{-k-1}(w)\, dw \\ &\quad + \sum_{k=0}^{\infty} P_{-k-1}(z)(1/2\pi i) \int_{L(r)} f(w) P_k(w)\, dw \\ &= \sum_{k=-\infty}^{\infty} A_k P_k(z), \end{aligned}$$

where

$$A_k = (1/2\pi i) \int_L f(w) P_{-k-1}(w)\, dw, \qquad k = 0, \pm 1, \pm 2, \ldots, \tag{4.4}$$

$L = L(r)$ for negative k, and $L = L(s)$ otherwise. Thus every f in $H(e(x) \cap E(y))$ can be expanded in the form (2.4), the convergence being compact-open in $e(x) \cap E(y)$. Moreover, from (4.4),

$$\begin{aligned} A_k &= (1/2\pi i) \int_L \sum_{n=-\infty}^{\infty} B_n Q_n(w) P_{-k-1}(w)\, dw \\ &= \sum_{n=-\infty}^{\infty} B_n (1/2\pi i) \int_L Q_n(w) P_{-k-1}(w)\, dw, \qquad k = 0, \pm 1, \pm 2, \ldots. \end{aligned}$$

If we define

$$G_{nk} = (1/2\pi i) \int_L Q_n(w) P_{-k-1}(w)\, dw \tag{4.5}$$

whenever n and k are not both nonnegative, then (4.4) shows that (2.1) holds. Since $(P_n, n = 0, 1, 2, \ldots)$ is absolutely effective on $E(y)$, and Q_n and P_n are analytic on $E(y)$ for nonnegative n, while P_n is given by (4.1) for negative n, the uniqueness of representation on $E(y)$ given by Theorem 4′ implies that (4.5) holds also when n and k are both nonnegative. We conclude from (4.4) that (P_n) is an absolute basis (because of the convergence in (4.3)). Moreover,

$$(1/2\pi i) \int_L P_j(z) P_{-k-1}(z)\, dz = \delta_{jk}. \tag{4.6}$$

For, the integral is zero if j and $-k-1$ are both nonnegative, since then the integrand is analytic on the region bounded by L. A similar remark holds when j and $-k-1$ are both negative. It suffices to consider the case j nonnegative, and $-k-1$ negative. In this case (4.1) implies

$$\begin{aligned} (1/2\pi i) \int_L P_j(z) P_{-k-1}(z)\, dz &= \sum_{n=0}^{\infty} G_{nk} (1/2\pi i) \int_L P_j(z) Q_{-n-1}(z)\, dz \\ &= \sum_{n=0}^{\infty} G_{nk} D_{jn} = \delta_{jk}, \end{aligned}$$

since (2.3) holds for j and k nonnegative, since $(P_n, n=0, 1, 2, \ldots)$ is a basic set on $E(y)$. (4.6) and (4.4) imply that expansion with respect to (P_n) is unique on $e(x) \cap E(y)$, so Theorem 4 implies that (P_n) is effective on $e(x) \cap E(y)$. An analogous proof holds for $\mathrm{Cl}\,(e(x) \cap E(y))$. This completes the proof of the theorem.

For a particular example, let $(P_n, n=0, 1, 2, \ldots)$ be a set of polynomials, orthogonal with respect to integration over $\mathrm{Cl}\,(E(x_0))$ or around $L(x_0)$, $0 \leqq x_0$, with any positive continuous function allowed as weight function. There exists $y > x_0$ such that if x is any number with $x_0 \leqq x < y$ (resp. $x_0 < x < y$), then (P_n) satisfies the conditions of Theorem 5. (Cf. [4, pp. 91–97, 128–130].) Thus the orthogonal polynomials can be extended to be an absolutely effective set of rational functions on $e(x) \cap E(y)$ (resp. $\mathrm{Cl}\,(e(x) \cap E(y))$).

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