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ANALYTIC GEOMETRIC DESIGN OF SPATIAL R-R ROBOT MANIPULATORS

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ABSTRACT

This paper studies the geometric design of spatial two degrees of freedom, open loop robot manipulators with revolute joints that perform tasks, which require the positioning of the end-effector in three spatial locations. This research is important in situations where a robotic manipulator or mechanism with a small number of joint degrees of freedom is designed to perform higher degree of freedom end-effector tasks. The loop-closure geometric equations provide eighteen design equations in eighteen unknowns. Polynomial Elimination techniques are used to solve these equations and obtain the manipulator Denavit and Hartenberg parameters. A sixth order polynomial is obtained in one of the design parameters. Only two of the six roots of the polynomial are real and they correspond to two different robot manipulators that can reach the desired end-effector poses.

KEYWORDS
Robot Manipulators, Geometric Design, Kinematics

1. INTRODUCTION

Designers of robotic systems are often caught in a dilemma whether to design a new mechanical system for moving a rigid body through specified locations or to use a generic, off-the-shelf multi-axis robot to perform the task. In most of the cases, the off-the-shelf robots are preferred, as they are ready to be used and do not require a prototype phase and a test phase as in the case of a newly designed mechanism. However, off-the-shelf robots are very expensive and considerably large in size. For a particular task, the cost of an off-the-shelf robot manipulator may be more than a thousand times and the system size may be more than a hundred times higher than those of a mechanism specifically designed for the task. In addition, for repetitive tasks, which is usually the case, the use of multi-axis robots is highly unjustified as several of the axes remain under-utilized because of the redundancy in degrees of freedom (Kota and Erdman, 1997). On the other hand, if the designer decides to design and build a new system, the design algorithms either do not exist or are very complicated. Therefore, there is a need to develop design methodologies for spatial, task oriented robotic systems, that have a small number of joint degrees of freedom and that are able to perform higher degree of freedom end-effector tasks (McCarthy, 1998).

In this paper, the geometric design problem of revolute-revolute (R-R) spatial manipulators is studied and solved analytically. In this problem, three spatial positions and orientations are defined and it is desired to calculate the dimensions of the geometric parameters of the R-R manipulator

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that will be able to place its end-effector at these three pre-specified locations. Tsai and Roth (1973) extending and completing previous work by Suh (1969) and Roth (1968, 1967) solved this problem first. Tsai and Roth showed that this problem has six solutions at the most. Two of these roots are real while the other four always stay in the complex domain. When the two open loop solutions are combined to form a closed loop mechanism, then a one degree of freedom 4-bar spatial Bennett mechanism is obtained that can guide its coupler link through the specified locations. Tsai and Roth used screw parameters to describe the kinematic topology of the R-R manipulator and screw displacements to obtain the design equations. However, during the last two decades Denavit and Hartenberg parameters have become the main tool to describe kinematically a robot manipulator (Denavit and Hartenberg, 1955). Also, 4x4 homogeneous matrices are used to formulate and obtain the kinematic equations. In this paper, the geometric design problem of R-R spatial manipulators is solved using the Denavit and Hartenberg parameters. The loop-closure geometric equations provide the required number of design equations. Polynomial Elimination techniques are used to solve these equations and obtain the manipulator Denavit and Hartenberg parameters. A sixth order polynomial is obtained in one of the design parameters. All six roots of this polynomial correspond to solutions of the problem but only two are real. The results by Tsai and Roth are verified.

2. BACKGROUND

The calculation of the geometric parameters of a mechanical system so that it guides a rigid body in a number of specified locations or *precision points* is referred in the literature as the *Rigid Body Guidance Problem*. In this paper, it will also be called the *Geometric Design Problem*. The precision points, i.e. spatial end-effector locations, are described by six parameters: three for position and three for orientation. This problem has been studied extensively for planar mechanisms and to a much lesser extent for spatial mechanisms. The number of precision points that may be prescribed for a given mechanism or manipulator is limited by the system type and the number of design parameters that are selected to be free choices (Suh and Radcliffe, 1978). Solution techniques for the geometric design problem may be classified into two categories: *exact synthesis* and *approximate synthesis* (Larochelle, 1994.)

Exact synthesis methods result in mechanisms and manipulators, which guide a rigid body exactly through the specified precision points. Solutions in the exact synthesis exist only if the number of equations obtained by the precision points is less than or equal to the number of design parameters. If the number of design equations is less than the number of design parameters, then the values of several of the design parameters become free choices so that a well-determined system is obtained. Obviously in this case, there is an infinite number of exact solutions, because any value can be selected for the design parameters that are free choices. For each manipulator/mechanism there is a specific number of precision points for which without selecting any free choices, there is a finite number of exact solutions to the geometric design problem. This number of precision points depends on the number of design parameters and the type of joints and can be calculated using Tsai and Roth's formula (Tsai, 1972; Roth, 1986).

In *approximate synthesis*, using an optimization algorithm, a mechanism is found that, although not guiding a rigid body exactly through the desired poses, it minimizes a distance criterion from all the desired poses. Approximate synthesis is mainly used in over-determined geometric design problems where more precision points are defined than required for exact synthesis and therefore no exact solution exists. Obviously, from a designer's point of view, it is always preferable to find exact solutions, if they exist, rather than approximate. Therefore it is important to know: a) the maximum number of precision points for which exact solutions to the geometric design problem of open and closed loop chains exist, b) the methods to calculate the exact solutions, and c) the number of distinct solutions to the problem.

The equations for the geometric design problem of mechanisms and manipulators are mathematically represented by a set of non-linear, highly coupled polynomial equations. All the solutions of these equations can be obtained by either numerical continuation methods or algebraic methods (Raghavan and Roth, 1995.) In very simple planar systems, graphical methods have also been proposed, but they become inefficient to solve the design problems in complex planar and spatial systems (Erdman and Sandor, 1997; Sandor and Erdman, 1984). Roth and Freudenstein (1963) were the first to use continuation methods to solve polynomial systems obtained in the kinematic synthesis of mechanisms. Later on, Morgan, Wampler, and Sommese (1990) described the way continuation methods can be used to obtain all solutions to systems of polynomial equations arising in kinematics. Continuation methods are very efficient in obtaining all solutions in a unified way. However, they aren't so fast and they mask the effect that each design parameter has to the solution. Algebraic methods are of interest because they give all the solutions, they are fast, and they give full insight to the solution process.

Exact synthesis of planar mechanisms for rigid body guidance, using algebraic methods, has been studied extensively by many researchers and is described in most textbooks on mechanism synthesis such as Hartenberg and Denavit (1964), Sandor and Erdman (1984) and Erdman and Sandor (1997). Very little work has been done on the exact synthesis of spatial mechanisms and manipulators using algebraic methods. Only in very few spatial manipulators and mechanisms has the geometric design problem, where no free choices are selected, been solved with algebraic methods. Most of the work in this area was performed for solving the exact synthesis of the spatial revolute-revolute (R-R) manipulators (see Section 1). Other than the R-R binary links, the geometric design problem has been solved algebraically for the following manipulators/mechanisms. Innocenti (1994) solved the geometric design problem for the sphere-sphere binary link. He showed that seven precision points are required and there are twenty distinct solutions at the most. Neilsen and Roth (1995) solved the same problem for the slider-slider sphere dyad, cylinder-cylinder binary link, revolute-slider-sphere dyad and cylinder-sphere binary link. McCarthy (1999) in a very recent and still unpublished work proposed a new method, based on screw theory, to solve the exact synthesis problem for several types of dyads. There exist, however, many types of robotic and other mechanical systems that are used very often in practical applications, such as the 3R, 4R and 5R manipulators, for which the exact synthesis of the geometric design problem, without selecting free choices, has not been solved before.

3. PROBLEM FORMULATION

In this work, the relative position of links and joints in mechanisms and manipulators is described using the variant of Denavit and Hartenberg notation (Denavit and Hartenberg, 1955), in which the parameters a_i, α_i, d_i and θ_i are defined so that: a_i is the length of link i, α_i is the twist angle between the axes of joints i and i+1, d_i is the offset along joint i and θ_i is the rotation angle about joint axis i as shown in Figure 1. When joint i is revolute, then a_i, α_i and d_i are constants and are called structural parameters, while the value for θ_i depends on the configurations and is called the joint variable.

Reference frame R_i is attached at link i and its origin O_i is the intersection point of the common perpendicular between axes i and i-1 with joint axis i. Unit vector $\mathbf{z_i}$ of frame R_i is along joint axis i; unit vector $\mathbf{x_i}$ is along the common perpendicular of joint axes i and i-1. Positive directions for $\mathbf{x_i}$ and $\mathbf{z_i}$ are arbitrarily selected. (Note: letters in bold indicate vectors and matrices.)

The homogeneous transformation matrix $\mathbf{A_i}$ that describes reference frame R_{i+1} into R_i, and its inverse matrix $\mathbf{A_i^{-1}}$ are found to be equal to:

$$\mathbf{A_i} = \begin{pmatrix} c_i & -s_i c_{\alpha_i} & s_i s_{\alpha_i} & a_i c_i \\ s_i & c_i c_{\alpha_i} & -c_i s_{\alpha_i} & a_i s_i \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1.a)$$

$$\mathbf{A_i^{-1}} = \begin{pmatrix} c_i & s_i & 0 & -a_i \\ -s_i c_{\alpha_i} & c_i c_{\alpha_i} & s_{\alpha_i} & -d_i s_{\alpha_i} \\ s_i s_{\alpha_i} & -c_i s_{\alpha_i} & c_{\alpha_i} & -d_i c_{\alpha_i} \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1.b)$$

where: $c_i=\cos(\theta_i)$, $s_i=\sin(\theta_i)$, $c_{\alpha i}=\cos(\alpha_i)$ and $s_{\alpha i}=\sin(\alpha_i)$.

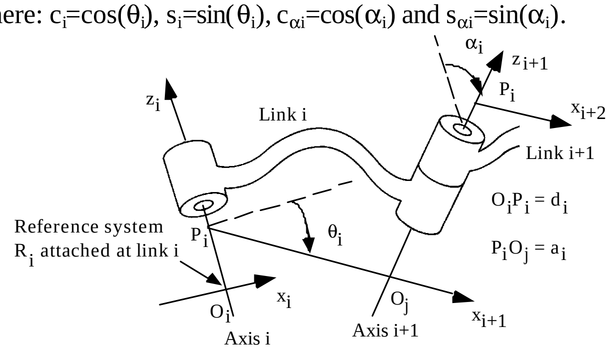

Figure 1: Denavit and Hartenberg Parameters

Consider the two-link open loop spatial chain with revolute (R) joints shown in Figure 2. Two frames are selected arbitrarily: a fixed reference frame R_0 and a moving end-effector frame R_e. Frame R_e will be defined in three distinct spatial locations. In addition to the two links of the manipulator, a stationary virtual link 0 is also assumed between axis z_0 of frame R_0 and the first revolute joint axis. Frames are defined at each link using the Denavit and Hartenberg procedure described above. Frame R_1 which is stationary is defined attached at link 0 having its z_1 axis along the first revolute joint and its x_1 axis along the common perpendicular of z_0 and z_1 (Note: axes z_0 and z_1 are not parallel in the general case.) Frame R_2 is attached at the tip of link 1, and frame R_3 is attached at the tip of link 2. The axis z_3 is coincident with the axis z_e of the end-effector frame. The axis x_3 is defined along the common perpendicular of z_2 and z_e and the origin O_3 of R_3 is the point of intersection of z_e with its common perpendicular with z_2. So frames R_3 and R_e have the same z axis.

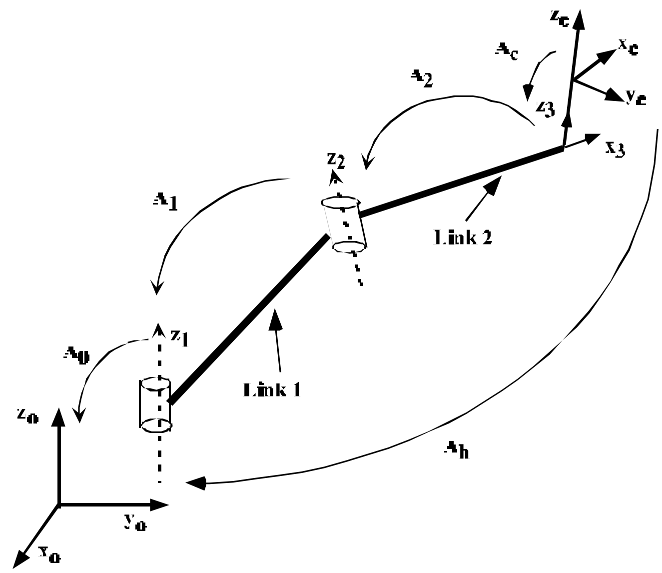

Figure 2: R-R Open loop Spatial Manipulator

The homogeneous transformation matrices $\mathbf{A_i}$, with i=0, 1, 2, describe frame R_{i+1} to R_i. The homogeneous transformation matrix $\mathbf{A_c}$ describes R_e into R_3. The relationship between these frames is a screw displacement: a rotation ψ around the z_3 axis and a translation d along the z_3 axis. Homogeneous transformation matrix $\mathbf{A_h}$ relates directly the end-effector reference frame R_e to the frame R_0. Matrices $\mathbf{A_c}$ and $\mathbf{A_h}$ are written as:

$$\mathbf{A_c} = \begin{pmatrix} c_\psi & -s_\psi & 0 & 0 \\ s_\psi & c_\psi & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A_h} = \begin{pmatrix} l_1 & m_1 & n_1 & x_d \\ l_2 & m_2 & n_2 & y_d \\ l_3 & m_3 & n_3 & z_d \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

where $\mathbf{l}=[l_1, l_2, l_3]^T$, $\mathbf{m}=[m_1, m_2, m_3]^T$, and $\mathbf{n}=[n_1, n_2, n_3]^T$, are the 3 by 1 vectors of the direction cosines of R_e in R_0 (Note: the superscript ^{T} denotes the transpose of a vector). The parameters x_d, y_d, and z_d are the coordinates of the origin of R_e in R_0.

The loop closure equation of the manipulator is used to obtain the design equations:

$$\mathbf{A_0}\,\mathbf{A_1}\,\mathbf{A_2}\,\mathbf{A_c} = \mathbf{A_h} \qquad (3)$$

Equation (3) is a 4 by 4 matrix equation that results in six scalar independent equations. The right side of Equation (3), i.e. the elements of matrix $\mathbf{A_h}$, are known since they represent the position and orientation of frame R_e at each precision point. The left side of Equation (3) contains all the unknown geometric parameters of the manipulator which are the Denavit and Hartenberg parameters a_i, α_i, d_i and θ_i for i=0, 1, 2 , and parameters ψ and d of matrix $\mathbf{A_c}$. Joint angles θ_1 and θ_2 have a different value for each precision point while all other 12 geometric parameters are constant. Thus for three precision points there are 18 unknown parameters in total, and there are 18 scalar equation that are obtained. Therefore, it is possible to solve this system of equation and obtain a finite number of solutions.

Due to the arbitrary selection of the positive direction of $\mathbf{z_i}$ there will be two values for the twist angle, i.e α_i and $\alpha_i+\pi$, that correspond to the same joint axes i and i+1. Similarly, due to the arbitrary selection of the positive direction of $\mathbf{x_i}$, there will be two values for the joint angle, i.e. θ_i and $\theta_i+\pi$, that describes the angle between $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$. The consequence is that in problems, such as the one that is studied in this paper, where angles α_i and θ_i are calculated (see Section 4), both values for each one of these parameters will appear among the set of solutions. Obviously, only one of these values will be retained because they correspond to the same set of axes.

4. ELIMINATION TECHNIQUE

The objective is to obtain one polynomial equation in one of the unknown design parameters. This polynomial will be obtained after consecutive eliminations of all other unknowns from the initial set of design equations.

Elimination of q_1 and q_2

In a first step of the elimination procedure, only the joint variables are eliminated. This will result in a new set of equations that contain only unknowns that do not change from precision point to precision point. In this way, for each new precision point that is defined, new equations are added that have exactly the same form as for the first precision point.

From Equation (1), it can be seen that the 3rd and 4th columns of matrix $\mathbf{A_i^{-1}}$ are independent of joint angle θ_i. Therefore, if Equation (3) is written as:

$$\mathbf{A_0\,A_1 = A_h\,A_c^{-1}\,A_2^{-1}} \tag{4}$$

then the scalar equations that are obtained by equating the left and right side of the third and fourth columns of matrix Equation (4) will be devoid of joint angle θ_2.

From the third column of Equation (4), three scalar equations are obtained:

$$c_0 s_1 s_{\alpha_1} + s_0 c_{\alpha_0} c_1 s_{\alpha_1} + s_0 s_{\alpha_0} c_{\alpha_1} = s_{\alpha_2} p_1 + c_{\alpha_2} n_1 \tag{5}$$

$$s_0 s_1 s_{\alpha_1} - c_0 c_{\alpha_0} c_1 s_{\alpha_1} - c_0 s_{\alpha_0} c_{\alpha_1} = s_{\alpha_2} p_2 + c_{\alpha_2} n_2 \tag{6}$$

$$-s_{\alpha_0} c_1 s_{\alpha_1} + c_{\alpha_0} c_{\alpha_1} = s_{\alpha_2} p_3 + c_{\alpha_2} n_3 \tag{7}$$

where: $p_i = l_i s_\psi + m_i c_\psi$, with i = 1, 2, 3.

From the fourth column of equation (4), another three scalar equations are obtained:

$$\begin{aligned} &a_1 c_1 c_0 - a_1 s_1 s_0 c_{\alpha_0} + d_1 s_0 s_{\alpha_0} + a_0 c_0 = \\ &-a_2 q_1 - d_2 s_{\alpha_2} p_1 - d_2 c_{\alpha_2} n_1 - n_1 d + x_d \end{aligned} \tag{8}$$

$$\begin{aligned} &a_1 c_1 s_0 + a_1 s_1 c_0 c_{\alpha_0} - d_1 c_0 s_{\alpha_0} + a_0 s_0 = \\ &-a_2 q_2 - d_2 s_{\alpha_2} p_2 - d_2 c_{\alpha_2} n_2 - n_2 d + y_d \end{aligned} \tag{9}$$

$$\begin{aligned} &a_1 s_1 s_{\alpha_0} + d_1 c_{\alpha_0} + d_0 = \\ &-a_2 q_3 - d_2 s_{\alpha_2} p_3 - d_2 c_{\alpha_2} n_3 - n_3 d + z_d \end{aligned} \tag{10}$$

where: $q_i = l_i c_\psi - m_i s_\psi$, with: i = 1, 2, 3.

By calculating c_1 from Equation (7) and s_1 from Equation (10) and then substituting them in Equations (5), (6), (8) and (9) four equations devoid of θ_1 and θ_2 are obtained.

Elimination of Constant Parameters

Equations (5), (6), (8) and (9), after the elimination of θ_1, are written for each one of the three precision points. In total, there are 12 equations and there are 12 structural parameters to calculate which are: a_0, α_0, d_0, θ_0, a_1, α_1, d_1, a_2, α_2, d_2, ψ and d. In these equations, there are terms that do not change from precision point to precision point and there are terms which have coefficients which are elements of matrix $\mathbf{A_h}$ and therefore depend on each precision point. By subtracting each one of these equations for the second and third precision points from the corresponding equation for the first precision point, eight equations are obtained where all constant terms have been eliminated. The eliminated terms contained the parameters a_0, d_0 and d_1. Also, the subtraction of the equations combined with the substitution of the expressions for s_1 and c_1 that was found from Equations (7) and (10) has as a consequence that the eight equations to depend only on the ratio of $a_1/s_{\alpha 1}$ and not on both of these variables. Therefore, the eight equations depend on eight unknown parameters which are: α_0, θ_0, $s_{\alpha 1}/a_1$, a_2, α_2, d_2, ψ and d. In these equations, two of the unknowns α_0, θ_0, (i.e. their sine and cosine functions), are suppressed in the coefficients while the other six form seven power products which are:

$$A = \frac{a_1 s_{\alpha_2} s_\psi}{s_{\alpha_1}}, \quad B = \frac{a_1 s_{\alpha_2} c_\psi}{s_{\alpha_1}}, \quad C = \frac{a_1 c_{\alpha_2}}{s_{\alpha_1}}, \quad D = 1,$$

$$E = a_2 c_\psi + d_2 s_{\alpha_2} s_\psi, \; F = a_2 s_\psi - d_2 s_{\alpha_2} c_\psi, \; G = d_2 c_{\alpha_2} + d$$

Then the eight equations are written in the following matrix form:

$$\mathbf{M_1 X_1} = \mathbf{M_1}\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{pmatrix} = 0 \tag{11}$$

where: $\mathbf{M_1}$ is an 8 by 7 matrix whose elements are functions of α_0, θ_0, and $\mathbf{X_1}$ is the 7 by 1 vector of power products. By choosing any 7 equations of the 8 of Equation (11), then a square homogeneous system is formed:

$$\mathbf{M_1^i}\ \mathbf{X_1}=\mathbf{0} \tag{12}$$

where $\mathbf{M_1^i}$ is a 7 by 7 submatrix of $\mathbf{M_1}$ and superscript i=1...8, indicates the 7 by 7 submatrix of $\mathbf{M_1}$ where row i has been deleted. This homogeneous system can not accept the trivial solution, i.e. zero value for all elements of $\mathbf{X_1}$ because one power product is equal to unity and can not take the zero value. Therefore, the determinant of each one of those eight submatrices is set to zero and eight equations in α_0 and θ_0, are obtained:

$$\det(\mathbf{M_1^i}) = f_i(\alpha_0, \theta_0) = 0 \tag{13}$$

In all Equations (13), the functions of θ_0 are suppressed in the coefficients while the functions of α_0 form a new vector of power products:

$$\mathbf{M_2}\ \mathbf{X_2}=\mathbf{0} \tag{14}$$

where $\mathbf{M_2}$ is an 8 by 8 matrix whose elements are functions of θ_0 and $\mathbf{X_2}$ is a 8 by 1 vector equal to:

$$\mathbf{X_2} = [c_{\alpha 0}{}^8, c_{\alpha 0}{}^7 s_{\alpha 0}, c_{\alpha 0}{}^6, c_{\alpha 0}{}^5 s_{\alpha 0}, c_{\alpha 0}{}^4, c_{\alpha 0}{}^3 s_{\alpha 0}, c_{\alpha 0}{}^2, c_{\alpha 0} s_{\alpha 0}, 1]^T$$

Note: In the first 4 lines of $\mathbf{M_2}$ the elements of the last column are zero, and in the last 4 lines of $\mathbf{M_2}$ the elements of the first two columns are zero.

The following trigonometric substitutions are performed for the sine and cosine of α_0:

$$c_{\alpha_0}^2 = \frac{1}{1+t_{\alpha_0}{}^2} \qquad s_{\alpha_0} c_{\alpha_0} = \frac{t_{\alpha_0}}{1+t_{\alpha_0}{}^2} \tag{15}$$

where: $t_{\alpha 0}=\tan(\alpha_0)$. It has to be noted that these substitutions for $s_{\alpha 0}$ and $c_{\alpha 0}$ are functions of the tangent of α_0 and not of the tangent of half angle of α_0 that is very often used in kinematics.

Doing these substitutions the following equation is obtained:

$$\mathbf{M_3}\ \mathbf{X_3}=\mathbf{0} \tag{16}$$

where $\mathbf{M_3}$ is an 8 by 5 matrix whose elements are functions of θ_0. The columns of $\mathbf{M_3}$ are obtained from linear combinations of the columns of $\mathbf{M_2}$ after substitutions of Equations (15) are performed. Vector $\mathbf{X_3}$ is a 5 by 1 vector equal to:

$$\mathbf{X}_3 = [\ t_{a_0}{}^4,\ \ t_{a_0}{}^3,\ \ t_{a_0}{}^2,\ \ t_{a_0},\ \ 1]^T$$

Selecting any 5 equations from the 8 of Equation (16), then a square homogeneous system is formed:

$$\mathbf{M_3}'\ \mathbf{X_3}=\mathbf{0} \tag{17}$$

where $\mathbf{M_3}'$ is any 5 by 5 submatrix of $\mathbf{M_3}$. Setting the determinant of $\mathbf{M_3}'$ to zero one equation is obtained that depends on the cosine and sine of θ_0:

$$\det(\mathbf{M_3'}) = g(c_0{}^6, c_0{}^5 s_0, c_0{}^4, c_0{}^3 s_0, c_0{}^2, s_0 c_0, 1) = 0 \tag{18}$$

The expressions of sine and cosine of θ_0 are written as a function of the tangent of θ_0:

$$c_0^2 = \frac{1}{1+t_0^2} \qquad s_0 c_0 = \frac{t_0}{1+t_0^2} \tag{19}$$

where $t_0 = \tan(\theta_0)$. After these substitutions, a polynomial of order 6 in t_0 is obtained:

$$\sum_{i=0}^{6} k_i t_0^i = 0 \tag{20}$$

The six roots of Equation (20), give six solutions for t_0 and hence for θ_0. Obviously for each value of t_0 two solutions for θ_0 are obtained: θ_0 and $\theta_0+\pi$. As it was explained in Section 3, both values correspond to the same axes arrangement and only one of them is considered.

Tsai and Roth (1973) also obtained a sixth order polynomial which was reduced to a 3rd order polynomial in the square of an intermediate variable. They have shown that the sixth order polynomial can only have two real roots. All other roots, although they satisfy the original set of equations and mathematically they are valid solutions to the problem, they always remain in the complex domain and hence they do not have any physical meaning. Although, we do not have a new proof that four roots of this polynomial always lie in the complex domain, extensive numerical examples verified Tsai and Roth's conclusion.

Back-substitution

Each value of θ_0 is substituted in the set of Equation (16). Any four of these equations can be used to form a linear system and calculate one value for t_{a0} and hence for α_0.

The values of α_0 and θ_0, are substituted in Equation (11). Seven equations out of the eight from Equation (11) can be selected and form a linear systems with the power products A, B, C, D, E, F, G as unknowns. Solving this linear system, the values of A, B, C, D, E, F, G are obtained. By dividing A with B the tangent of ψ is obtained and hence angle ψ. By dividing A or B with C and knowing ψ, the tangent of α_2 is obtained and hence angle α_2. Knowing α_2 and ψ, from anyone of A, B and C the value of $a_1/s_{\alpha 1}$ is obtained. In the expressions of E and F, only a_2 and d_2 are unknowns and they can be calculated solving a simple 2 by 2 linear system. Using all values that have been calculated so far, the parameter d can be calculated from the value of G.

The calculated values of α_0, θ_0, $s_{\alpha 1}/a_1$, a_2, α_2, d_2, ψ and d are substituted in Equations (5), (6), (8) and (9) where the terms c_1 and s_1 have been substituted with their symbolic expressions from Equations (7) and (10). In this step of the solution process, only one of the three precision points is needed. The left sides of the equations have four linear unknowns: a_0, d_0, d_1 and $\cos(\alpha_1)$. The right sides have already been calculated. Solving this 4 by 4 linear system the 4 unknowns are calculated. Finally, from the value of $\cos(\alpha_1)$ two values are obtained for angle α_1 that have opposite sign but only one of them is correct. The correct α_1 is the one that results in a positive value for a_1 when this parameter is calculated from the expression $s_{\alpha 1}/a_1$.

Thus all Denavit and Hartenberg parameters are calculated. In total for three precision points only two open loop R-R spatial chains can be found. By combining these two chains into one closed-loop mechanism, a four bar Bennett mechanism is obtained.

It has to be noted that the solution methodology developed in this section is based on the assumption that all orientations in the three precision points are distinct. It can be shown that: (a) when all the orientations are identical, the geometric design problem for a R-R manipulator reduces to a planar geometric design problem, and (b) when any two orientation are identical, and the third is distinct this problem has no solution in general.

5. NUMERICAL EXAMPLE

A numerical example for solving the geometric design problem of a spatial open loop R-R robot manipulator using the algorithm that was presented in Section 4, is shown below. All calculations were performed with 20 decimal numbers using the symbolic calculation software package Maple V, Release 5. Due to space limitations and for simplicity, only 5 decimal numbers have been included in the numerical results shown below.

Three precision points are arbitrarily selected. These precision points are defined by the position coordinates of the origin of the end-effector frame with respect to the fixed reference frame and the direction cosines of the end-effector frame with respect to the fixed reference frame. The three precision points are graphically represented in Figure (3). These three precision points that are selected give the following $\mathbf{A_{hi}}$ matrices where i=1, 2, 3 :

$$A_{h1} = \begin{pmatrix} 0.65623 & -0.11296 & 0.74605 & 162.03673 \\ -0.21885 & -0.97472 & 0.04491 & 82.18408 \\ 0.72212 & -0.19275 & -0.66437 & 48.74290 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{h2} = \begin{pmatrix} 0.14391 & 0.92897 & 0.34102 & 72.96453 \\ -0.98949 & 0.13986 & 0.03655 & 42.77492 \\ -0.01373 & -0.34270 & 0.93934 & 113.39977 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_{h3} = \begin{pmatrix} 0.76457 & 0.64084 & -0.06892 & 43.27298 \\ -0.47704 & 0.49073 & -0.72911 & 16.14093 \\ -0.43341 & 0.59034 & 0.68091 & 88.68218 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 6^{th} degree polynomial in t_0 is:

$$-43215.44589t_0^6 + 48238.55854t_0^5 - 2559.15192t_0^4 + 50727.96989t_0^3 - 114681.08559t_0^2 + 75359.73016t_0 - 16254.11701 = 0$$

The roots of the polynomial are: 0.57735, 0.816079, -0.80246 - 0.98242 I, -0.80246 + 0.98242 I, 0.66386 – 0.23534 I, 0.66386 + 0.23534 I.

Only two roots are real which give two values for t_0 and hence for θ_0. Doing back substitution of the values of θ_0, all other Denavit and Hartenberg parameters are calculated. There are two sets of solutions that are given in Table 1. The solutions have been graphically represented using the CAD package Pro-Engineer. Figure 3 shows the manipulator that corresponds to the first solution as it places its end-effector frame in all the precision points. Figure 4 shows the manipulator that corresponds to the second solution.

Table 1: Solution Sets of R-R Open loop Manipulator

Variables	Set I	Set II
θ_0 (deg)	30.00000	39.21717
α_0 (deg)	55.00000	71.57945
d_0	10.00000	32.15932
a_0	50.00000	91.98281
α_1 (deg)	35.99999	35.99999
a_1	89.99999	89.99999
d_1	12.00000	11.97707
α_2 (deg)	63.00000	52.41377
a_2	35.99999	14.31500
d_2	14.99999	34.45982
ψ (deg)	42.99999	24.99159
d	25.99999	-21.63223
$\theta_1^{(1)}$ (deg)	19.99999	-1.143387
$\theta_1^{(2)}$ (deg)	55.99999	96.85661
$\theta_1^{(3)}$ (deg)	86.00000	141.85661
$\theta_2^{(1)}$ (deg)	25.00000	51.66870
$\theta_2^{(2)}$ (deg)	122.99999	87.66870
$\theta_2^{(3)}$ (deg)	168.00000	117.66870

6. CONCLUSIONS

In this paper, the geometric design problem of a two link robot manipulator with revolute joints is solved analytically. The method that has been presented, calculates the manipulator Denavit and Hartenberg parameters given three spatial end-effector locations.

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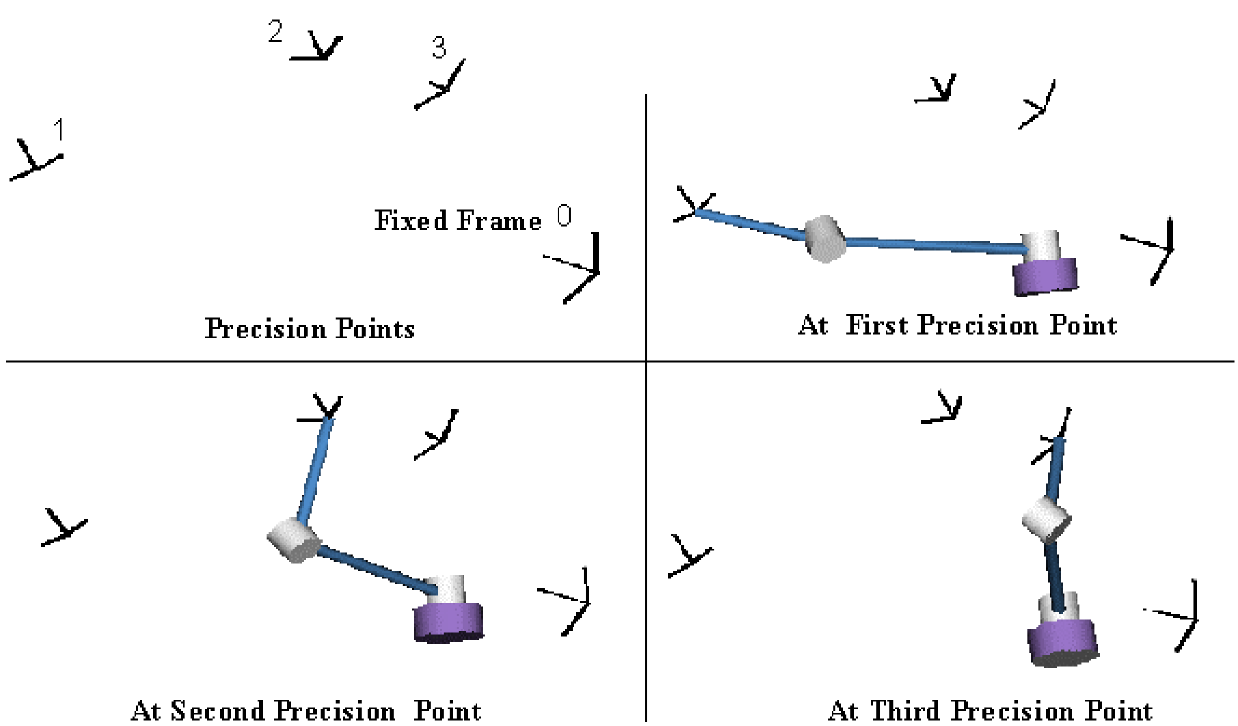

Figure 3: R-R Spatial Manipulator Corresponding to Solution Set I

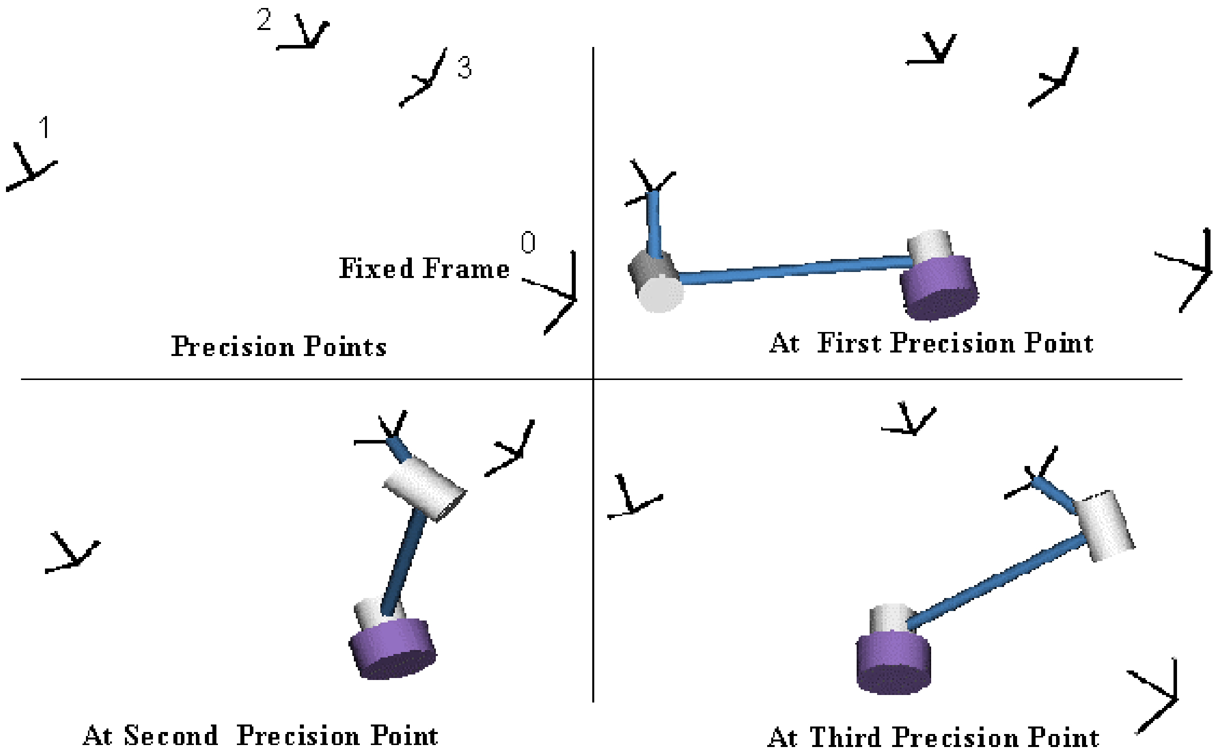

Figure 4: R-R Spatial Manipulator Corresponding to Solution Set II