Analytic Hierarchy Process-Fuzzy Sorting: an Analytic Hierarchy Process based method for fuzzy classification in sorting problems

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Abstract: Analytic Hierarchy Process (AHP) is a well-founded and popular method in the Multi-Criteria Decision Analysis (MCDA) field. Recently, AHPSort, a sorting variant, uses crisp class-assignment of alternatives. This can sometimes be misleading, especially for alternatives near the border of two classes. This paper aims at making the class assignment process in AHPSort more flexible by using fuzzy sets theory, which facilitates soft transitions between classes and provides additional information about the membership of alternatives in each class that can be used to fine tune actions beyond the crisp sorting process. This essentially complements the ordinal information of its crisp variant with cardinal information as to the degree of membership of an alternative to each class. The applicability of the proposed approach is illustrated in a case study that regards the classification of London boroughs according to their safety levels.

Keywords: Multi-criteria decision analysis, AHP, AHPSort, Fuzzy Sets, Sorting Method, Fuzzy Linguistic Approach.

1. Introduction

Multi-Criteria Decision Analysis (MCDA) relates to the process of making decisions in situations where there are multiple and conflicting criteria. Different types of decision problems can be formulated within the context of MCDA (Roy 1981) from *choice, sorting, ranking and description* problems, to *elimination* (Bana e Costa 1996) and *design* (Keeney 1992) ones. Most of the problems studied in the literature revolve around choice and ranking problems, thus many approaches have been developed and applied accordingly in real-world problems. Analytic Hierarchy Process (AHP) (Saaty 2003, Ishizaka and Labib 2011) is one of the most widespread and useful MCDA methods (Wallenius, Dyer et al. 2008). Nonetheless, it is only in 2012 that a variant of AHP for sorting problems has been proposed with AHPSort (Ishizaka, Nemery et al. 2012). Shortly after that, several other variants have been developed. In particular, AHPSort II has been proposed for problems having a large

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number of alternatives (Miccoli and Ishizaka 2017). It compares only a small number of representative points on each criterion and then the scores are interpolated to build the preference function. This permits to drastically reduce the number of pairwise comparisons. Moving forward, AHP-K and AHP-K-veto have been developed when limiting profiles cannot be defined, for example when an expert is not available or the problem is totally new (Lolli, Ishizaka et al. 2014). In this method, alternatives are clustered automatically according to the number of ordered classes. For problems involving group decisions, an AHP-K version has been described in Lolli, Ishizaka et al. (2017). GAHPSort has been presented later when several of decision-makers are involved (López and Ishizaka 2017). Furthermore, "Cost-Benefit AHPSort" (Ishizaka and López 2018) has been used when a problem contains several conflicting criteria having different optimization direction, i.e. some need to be maximized and some to be minimized. In this case a cost and benefit hierarchy is used. Finally, Krejčí and Ishizaka (2018) recently proposed Fuzzy-AHPSort (FAHPSort), where the uncertainty in the pairwise comparison is taken into account by using fuzzy numbers. It is to note that using fuzzy numbers in AHP is controversial as (Saaty and Tran 2007) claim that uncertainty is already captured in the fundamental scale. AHP would perform better than Fuzzy AHP, which may even give wrong results (Wang, Luo et al. 2008). Therefore, in the proposed Analytic Hierarchy Process-Fuzzy Sorting, we do not use fuzzy numbers for the evaluations, as our focus will be on the limiting profiles alone. In all the previously developed AHP sorting methods, a strict boundary is assumed: i.e. above a boundary an alternative belongs to class A, below that it belogs to class B. This strict boundary has two major problems though:

- There is the necessity of *fine-tuning* processes to avoid ambiguous or doubtful class assignments for alternatives that are close to the boundary (Miccoli and Ishizaka 2017).

- Insignificant differences in the priorities obtained by the alternatives can result in significant differences in the class assignment in the sorting MCDA approach.

One can think of this issue as follows. Some alternatives that are close to the boundary could present some characteristics of class A and some characteristics of class B. Yet, in previous crisp sorting approaches the classification was strict. Moreover, the output was only ordinal in the sense that an alternative belongs to a given class no matter how closely relates to the characteristics of its neighboring class. In this paper, we would like to provide a precise information on the membership of an alternative to class A and on the membership to class B. This gives cardinal information that reduces the ambiguity that is inherent in the crisp classification. For this purpose and with the objective to remove the existing shortcoming of AHPSort, Fuzzy Sets theory (Zadeh 1965) will be applied to the limiting profiles only. Fuzzy Sets theory has already been used successfully to deal with

the flexibility or the lack of boundaries for classification purposes (Baraldi and Blonda 1999, Pedrycz 2005). Moreover, it has enhanced the reliability and flexibility of classic decision models (Martínez, Ruan et al. 2009).

Due to the fact that the AHPSort approaches make use of ordered classes that are usually linguistically tagged, this paper proposes an Analytic Hierarchy Process-Fuzzy Sorting (AHP-FuzzySort) model that uses fuzzy sets theory and the fuzzy linguistic approach (Zadeh 1975, Martínez, Ruan et al. 2009) to improve the assignment of alternatives to classes such that the *fine-tuning* process will no more be necessary. Instead, a degree of "membership within a linguistic approach will be provided for a better understanding of the assignment of alternatives to the respective classes.

Finally, a case study on the safety of London boroughs is developed with AHP-FuzzySort, highlighting the advantages of using flexible transitions in the definition of classes.

The remaining of this paper is set up as follows: Section 2 presents a literature review on sorting techniques. Section 3 provides the necessary preliminaries about AHPSort method, Fuzzy Sets and Fuzzy Linguistic approach to understand the two different strands of literature on which our proposal is based on. Section 4 presents the novel proposal "AHP-FuzzySort" for MCDA sorting problems. Section 5 develops a case study with the aid of AHP-FuzzySort, showing its advantages over the crisp sorting variant AHPSort. Section 6 concludes the paper.

2. Literature review

Several MCDA methods have been developed with the purpose of solving ranking and choice problems (Ishizaka and Nemery 2013, Figueira, Greco et al. 2016). ELECTRE was the first with the sorting variant ELECTRE-Tri (Yu 1992). Many variants appeared afterwards, e.g. Electre Tri-C (Almeida-Dias, Figueira et al. 2010), ELECTRE-SORT (Ishizaka and Nemery 2014), ELECTRE Tri-nC (Almeida-Dias, Figueira et al. 2012), ELECTRE Tri-nB (Fernández, Figueira et al. 2017). Of course, soon all MCDA were modified to solve sorting problems as well. For instance, UTADIS (Jacquet-Lagrèze and Siskos 1982) is the sorting variant of UTA. TOPSIS-Sort (Sabokbar, Hosseini et al. 2016) supports sorting problems with TOPSIS. VIKORSORT (Demir, Akpınar et al. 2018) is the sorting variant of VIKOR. FlowSort (Nemery and Lamboray 2008) and its complementary visual method GAIASort (Nemery, Ishizaka et al. 2012) have been developed for PROMETHEE and GAIA respectively. MACBETHSort (Ishizaka and Gordon 2017)is the sorting extention of MACBETH and DEASORT (Ishizaka, Lolli et al. 2018) is the sorting variant of DEA.

However, uncertainty can arise at several levels: performance on each criteria, preferences given by the decision-maker (e.g. pairwise comparisons) and limiting/central profiles. In the literature, there are broadly two approaches to deal with uncertain data in sorting problems:

- *Probabilistic sorting:* The idea is to use a Monte Carlo simulation to explore the profiles, the cutting levels and/or the weights space. These spaces are defined by a density function, very often a Gaussian or a uniform distribution, in a certain interval. The outcome is a class acceptability index, which express the percentage that an alternative belong to each class. This technique has been used in SMAA-Tri (Tervonen, Figueira et al. 2009) and in CPP-Tri (Sant'Anna, Costa et al. 2015).

- *Fuzzy sorting:* Campos, Mareschal et al. (2015) observed that performances on criteria cannot always be expressed precisely or exact values can be inadequate for modelling the real life. Therefore, they proposed to use the fuzzy theory to deal with these uncertain data. The developed F-FlowSort considers only fuzzy numbers for the performance on criteria. The parameters of the model including indifference and preference thresholds, reference profiles and criteria weights are crisp numbers. Soon, Govindan and Jepsen (2016) combined fuzzy numbers and ELECTRE TRI-C. Surprisingly, the authors asked for crisp data, claiming that it creates less strain to the decision-maker but ended up creating fuzzy data randomly generated around the original crisp data. The same process is used for the central profiles and no explanation is given on how the thresholds are set actually. Recently, FAHPSort has also been proposed (Krejčí and Ishizaka 2018). In this case, the pairwise comparisons are given verbally by the decision-maker and then converted to fuzzy numbers. The limiting profiles are crisp numbers.

The main difference between probability and fuzzy sorting is that fuzzy sorting gives a precise classification on the amount an alternative belong to each class. (an alternative belongs to only one class)In probability sorting we have a percentage of chances that an alternative belongs to each class, but it is the entire alternative belongs tobelongs to one class only with a percentage of chances. This can be problematic when an alternative is close to the boundary as it may present some characteristics of both adjacent classes. In our new sorting method, AHP-FuzzySort is able to identify how much and which characteristic belong to each adjacent class.

3. Preliminaries

In this section, we review the main necessary concepts to understand our new method. We begin with the AHPSort approaches, followed by a brief revision of the main concepts of Fuzzy Sets and Fuzzy Linguistic Approaches.

3.1. AHPSort

It is an adaptation of the AHP method (Saaty 1999) for sorting MCDA problems, although it could be used for ranking proposals (even in MCDA problems with sets of medium/large sized alternatives). The AHPSort (Ishizaka, Nemery et al. 2012) aims at sorting alternatives into classes that are ordered from *most* to *least* preferred, according to the scheme depicted in Fig.1. Such a scheme is composed of eight steps, carried our in three phases (Ishizaka, Nemery et al. 2012):

A) *Phase 1*: **Problem definition**

1) The criteria c_j, $j = 1,\ldots,m$, the alternatives a_k, $k = 1,\ldots, l$ and the goal of the problem are established.

2) The classes $C_i, i=1,\ldots,n$ are defined in a way that they are ordered and may have a linguistic descriptor (e.g. *excellent, good, medium, bad, poor*)

Fig. 1. Sorting with limiting and central profiles
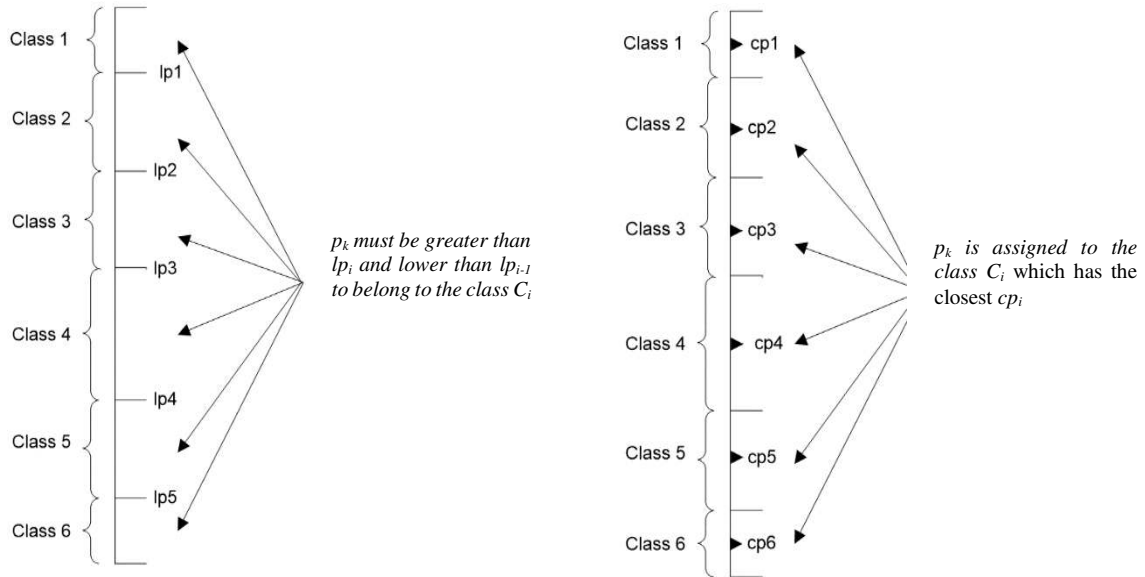
3) The ***profiles*** of each class, C_i, are defined by either local limiting profiles lp_{ij} (minimum performance that a criterion c_j should obtain to belong to the class C_i), or local central profiles cp_{ij} (characteristic example of an element in the class C_i on criterion c_j).

B) *Phase* 2: **Evaluations**

4) First, the priority for the importance of each criterion, c_j, is given by the expert, obtaining their ***weights***, w_j, by employing the AHP eigenvalue method.

$$\mathbf{A} \cdot \mathbf{p} = \lambda \cdot \mathbf{p} \qquad\qquad (1)$$

where

        A is the comparison matrix

        p is the priorities/weight vector

        λ is the maximal eigenvalue

As in AHP, a consistency index can be calculated:

$$CI = \frac{\lambda_{\max} - n}{n - 1},$$ (2)

where n = dimension of the matrix
λ_{max} = maximal eigenvalue

If the CR, ratio of CI and RI (the average CI of 500 randomly filled matrices), is less than 10%, then the matrix can be considered as having an acceptable consistency.

$$CR = CI/RI,$$ (3)

where CR is the consistency ratio
RI is the random index

Saaty (1977) calculated the following random indices:

Table 1: Random indices

n	3	4	5	6	7	8	9	10
RI	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

5) Each alternative, a_k, is pairwise compared with the limiting (lp_{ij}) or central profiles (cp_{ij}) for each criterion, c_j, For each alternative, the evaluations are gathered in a pairwise comparison matrix as regards to each criterion.

6) From each pairwise comparison matrix, the local priority for each alternative a_k (p_{kj}), and for each limiting, or central profile lp_{ij}, cp_{ij} (p_{ij}) is computed with Eq. (1).

C) *Phase 3:* **Assignment to classes**

7) Then, the global priorities are computed for every alternative a_k (p_k), and every limiting or central profile (lp_i or cp_i accordingly), by aggregating the weighted local priorities.

$$p_k = \sum_{j=1}^{m} p_{kj} \cdot w_j$$ (4)

$$lp_i \text{ or } cp_i = \sum_{j=1}^{m} p_{ij} \cdot w_j$$ (5)

The assignment of an alternative a_k to a class C_i is accomplished by the comparison of p_k with lp_i or cp_i (See Fig. 1).

8) Steps (5) to (8) are repeated for each alternative to be classified.

In (Miccoli and Ishizaka 2017), the AHPSort approach was updated (AHPSort II) by modifying the previous algorithm from step 4) onwards, as follows:

5) For all criteria c_j, a few representative points s_{oj}, $o=1,...,rp_j$ well-distributed across the scale of each criterion are selected.

6) Either the limiting, or the central profiles are pairwise compared with the set of the representative points chosen in step (5). By using these computed comparison matrices, the local priority, p_{oj}, is obtained for the representative points and p_{ij}, for the local priority of the limiting or central profiles with Eq. (1).

7) If an alternative $a_k \in [s_{oj}, s_{o+1j}]$, the local priority p_{kj} is calculated with:

$$\boldsymbol{p_{kj}} = \boldsymbol{p_{oj}} + \frac{p_{o+1j} - p_{oj}}{s_{o+1j} - s_{oj}} \cdot (\boldsymbol{g_j(a_k)} - \boldsymbol{s_{oj}}) \qquad\qquad p_{kj} = p_{oj} + \frac{p_{o+1j} - p_{oj}}{s_{o+1j} - s_{oj}} \cdot (g_j(a_k) - s_{oj}) \quad (6)$$

In which:

- s_{oj} and s_{o+1j} are two consecutive representative points on criterion c_j
- p_{oj} and p_{o+1j} are their corresponding local priorities
- $g_j(a_k)$ is the score of alternative a_k on criterion c_j
- p_{kj} is the local priority of a_k

For example, consider 10 representative points on the axis of abscissa (i.e. 100, 200,..., 1000; depicted in figure 2), whose local priorities are calculated on the axis of ordinates. By connecting these points, the piecewise linear priority function is then obtained. If an alternative is evaluated between 800 and 900, its local priority is found in the graph created.

8) To obtain the global priority p_k for alternative a_k, the local priorities are then aggregated (Eq. (4)) and the global priority lp_i or cp_i for the limiting or central profiles are obtained by (Eq. (5)).

9) Steps (5) to (9) are repeated for each alternative to be classified.

Fig. 2. Computing the local priority of alternative a_k by Eq. (6)
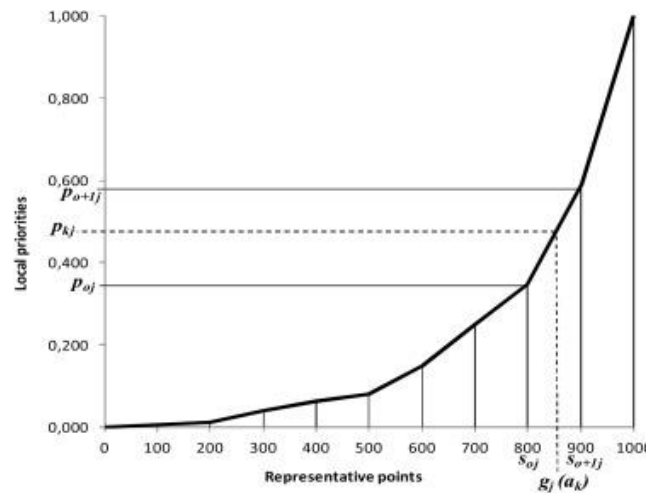
Due to the linear approximation used in Eq. (6) in AHPSort II, a *Fine-Tuning* process is recommended for the class assignment:

10) Alternatives that are found -according to AHPSort II- to be very close to the limiting profiles; they are cross-checked, in order to be exactly classified. More specifically, if their classification matches that of the AHPSort, then the classification is correct and over. Otherwise, closest alternatives above or below

the previous ones need to be also classified according to AHPSort, to check that both approaches' classifications match.

Both algorithms also provide a significant reduction of pairwise comparisons compared to the AHP method whenever the number of classes is reduced (see (Ishizaka, Nemery et al. 2012, Miccoli and Ishizaka 2017)). However, the increase of classes for sorting alternatives may result in an important increasing of pairwise comparisons

Remark 1. Both algorithms make a crisp class assignment process, in which very small variations in the global priority of alternatives may result in different class assignments.

3.2. *Fuzzy Sets, Fuzzy Linguistic Approach and Modelling*

The concept of fuzzy sets is very intuitive and captures the principle that some real world concepts cannot be represented in a precise way (Pedrycz, Ekel et al. 2011), because there exist categories of elements whose membership is a matter of degree. According to this view, a fuzzy set defined on a universe of discourse extends the notion of a set by using a degree of membership of its elements in (Zadeh 1975):

$$\mu_{\tilde{A}} : X \rightarrow [0,1] \tag{7}$$

Based on this membership function, a fuzzy set $\tilde{A}$ defined over the domain X is represented by the set of pairs of the element x and its membership:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,/\, x \in X, \mu_{\tilde{A}}(x) \in [0,1]\} \tag{8}$$

Within the fuzzy sets theory arises the fuzzy linguistic approach based on the concept of linguistic variable (Zadeh 1975) that plays a key role in many fuzzy applications for modelling uncertainty and lack of boundaries.

Definition 1 (Zadeh 1975)**.** *A linguistic variable is characterized by a quintuple (H,T(H),U,G,M) in which H is the name of the variable; T(H) (or simply T) denotes the term set of H, i.e., the set of names of linguistic values of H, with each value being a fuzzy variable denoted generically by X and ranging across a universe of discourse U which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for generating the names of values of H; and M is a semantic rule for associating its meaning with each H, M(X), which is a fuzzy subset of U.*

The fuzzy linguistic variables have been used in decision making for several purposes (Martínez, Ruan et al. 2009) to model uncertainty and compute with words (Liu, Martinez et al. 2010, Martínez and Herrera 2012, Rodríguez, Martínez et al. 2012, Rodriguez and Martinez 2013, Rodríguez, Labella et al. 2016). It is required to select the suitable linguistic descriptors for the term set and their semantics. One usual way of creating the linguistic term set is to directly supply the set, by considering a total order and all terms distributed on a scale (Yager 1995, Martinez 2007). An example of a five-term set S, might be:

$$S = \{s_0: \text{Poor}; s_1: \text{Low}; s_2: \text{Average}; s_3: \text{High}; s_4: \text{Good}\}$$

In such cases, it is required that there exists (Yager 1995, Martinez 2007):

- A negation operator $\text{Neg}(s_i) = s_j$ such that $j = g-i$ ($g+1$ stands for the number of linguistic terms of the term set)

- Minimum and maximum operators for the linguistic term set, S: $s_i \leq s_j \Leftrightarrow i \leq j$

The semantics for the terms in a linguistic term set are provided by fuzzy numbers defined in the interval [0,1], which are depicted by membership functions that can be characterized by types of functions. In this paper, we consider parametric membership functions (Delgado, Verdegay et al. 1992), as they are capable to capture the approximation of the information represented in many situations (Delgado, Vila et al. 1998). A common parametric representation is the trapezoidal one, that may be represented by (a, b, d, c), in which *b* and *d* define the interval in which the height of the fuzzy set is 1; meanwhile, *a* and *c* fix the left and right limits of the trapezoidal membership function. A specific case of this type of representation is the triangular membership functions, i.e., $b = d$, that is represented by a triplet (a; b; c) (see a graphical example of triangular membership functions in Fig. 3).

Fig. 3. A linguistic term sets with five labels

POOR   LOW   AVERAGE   HIGH   GOOD
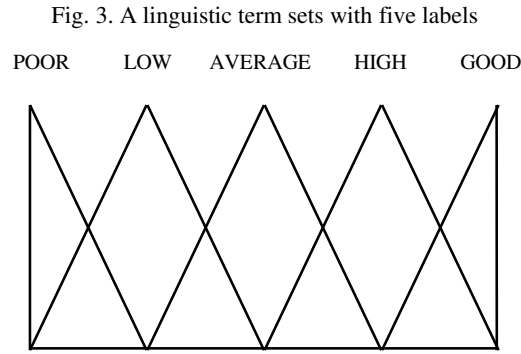
Due to the fact that AHPSort deals with ordered classes that can be linguistically described, and our aim is to deal with soft transitions between classes; among the different linguistic modelling proposals existing in the literature, we adapt the fuzzy-based linguistic representation introduced by (Wang and Hao 2006). It represents the linguistic information by means of proportional two-tuple, such as (0.2 *Average*, 0.8 *High*) that grades the membership to each linguistic label. The authors pointed out that if only the *High* label were used as an approximate grade of the information, then some information would be lost. This proportional two-tuple model is based on the concept of symbolic proportion that in our proposal is interpreted as the degree of membership of an element to each linguistic label to follow a fuzzy representation, such as it was pointed out in (Rodriguez and Martinez 2013).

Definition 2 (Wang and Hao 2006). *Let S = {s₀,s₁,...,sₘ} be an ordinal term set, I =[0,1] and*

IS ≡ I × S = {(α,sᵢ) : α ∈ [0,1] and i = 0,1,...,g}   (9)

Where S is the ordered set of g + 1 ordinal terms {s₀, s₁,...,sₘ }. Given a pair (sᵢ,sᵢ₊₁) of two successive ordinal terms of S, any two elements (α,sᵢ), (β,sᵢ₊₁) of IS is so-called a symbolic proportion pair and α,β are a pair of symbolic proportions of the pair (sᵢ,sᵢ₊₁) if α + β = 1.

Remark 2. The fact that $\alpha + \beta = 1$ means that the linguistic term set S is a fuzzy partition in the sense of Ruspini (1969).

A symbolic proportion pair (α, s_i), $(1-\alpha, s_{i+1})$ is denoted by $(\alpha \cdot s_i, (1-\alpha) s_{i+1})$ and the set of all the symbolic proportion pairs is denoted by $\bar{S}$, i.e., $\bar{S}=\{(\alpha \cdot s_i, (1-\alpha) s_{i+1}): \alpha \in [0,1]$ and $i = \{0, 1, ..., g-1\}$.. $\bar{S}$ is called the ordinal proportional two-tuple set generated by S and the members of $\bar{S}$, ordinal proportional two-tuple, which are used to represent the ordinal information for CW processes when it is necessary.

4. AHP-FuzzySort: Fuzzy sorting process in AHPSort

This section aims at developing a new sorting method for the AHPSort approach, namely *AHP-FuzzySort*, based on a fuzzy process that will use the proportional linguistic two-tuples to carry out the sorting process. It will not only facilitate the soft transition from one class to another by designing soft boundaries between the classes, but it will also provide additional information about the membership of the alternatives to the corresponding classes, allowing the gradation of membership to a class *without increasing the number of pairwise comparisons*. This can be also used for a finer classification and a better visualization/understanding of the results.

The novel sorting method consists of the following two additional steps:

1. *A Fuzzy Linguistic Representation of the classes:* To build a fuzzy linguistic scale with its corresponding fuzzy membership functions that represents the classes in which the alternatives will be sorted.
2. *Assignment to classes by using fuzzy membership degrees:* To redefine the *assignment to classes* phase of the AHPSort to assign the classes according to the new representation of the classes, in which the proportional two tuple representation (Wang and Hao 2006) will play a key role.

These steps are further detailed in the following subsections.

4.1. Fuzzy Linguistic Representation of the classes

As mentioned in *Section 2.1,* AHPSort deals with ordered classes when sorting the different alternatives of the MCDA problem, and commonly, these classes are linguistically tagged. This proposal assumes that linguistic information models the uncertainty regarding the definition of the classes and may capture it by using a fuzzy representation.

The limiting and central profiles (lp_i and cp_i respectively), are used to define the classes in which alternatives of the MCDA problem are sorted. Therefore, from such profiles, a fuzzy linguistic representation of the classes will be built that will then be used to compute proportional two tuples to carry out the assignment of alternatives to classes.

The fuzzy linguistic scale must fulfill the following requirements:
- The linguistic term set S that contains the labels describing the classes should be ordered. So, the generating process described in Section 2.2 from (Yager 1995, Martinez 2007) is hereby assumed.

- For the sake of simplicity and without loss of generality, the fuzzy membership functions that define the semantics of the linguistic labels will be parametric functions.

- For fuzzy-based assignment of classes and for dealing with proportional two tuples in a proper way, the fuzzy membership functions of the linguistic labels in *S* should form a fuzzy partition (Ruspini 1969, Wang and Hao 2006).

The building process of the linguistic scale will consist of the following steps: (i) Softening classes transitions; (ii) Selecting values for the parametric functions; (iii) Building the fuzzy membership functions. These steps are further discussed in the following subsections.

4.1.1. Softening classes transitions

In the AHPSort approaches, the sorting classes are ordered from the *most* to the *least* preferred according to Fig. 1. The classes are defined by a strict crisp interval (*sci*) based on the limiting profiles lp_i as it is indicated by the curly brackets in Fig. 4 and formally defined below:

Fig. 4. Strict crisp intervals of the classes
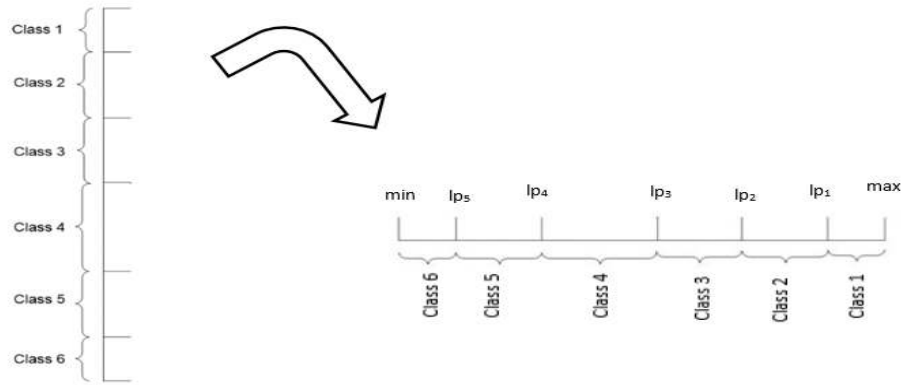
i. *Least preferred class* (*Class_n*). Let us assume *n* classes, hence the least preferred class is the Class *n,* whose *sci* is defined by the interval between the minimum value of the universe of the discourse and the limiting profile, lp_{n-1}:

$$sci(Class\ n) = [min, lp_{n-1}] \tag{10}$$

ii. *Most preferred class*, (*Class_1*). This class is defined in the following *sci:*

$$sci(Class\ 1) = [lp_1, max] \tag{11}$$

iii. *Remaining classes*, (*Class_i*, $1 < i < n$). These classes are established within their *sci* as:

$$sci(Class\ i) = [lp_i, lp_{i-1}] \tag{12}$$

However, our aim is to avoid these strict borders between classes, because small differences $lp_i \pm \varepsilon$ may imply the assignment to different classes which is not representative of many real-world situations. Therefore, to soften these borders and transitions between classes, fuzzy membership functions are defined.

4.1.2. Selecting values for the parametric membership functions

In order to increase the flexibility of the sorting process, it is necessary to define the support of the membership functions that define these classes beyond the *sci*. Our proposal deals with parametric (trapezoidal/triangular) membership functions.

As mentioned in Section 2.2, the parametric functions can be either trapezoidal ones that are defined by (a, b, d, c) or triangular ones, i.e., $b = d$, defined by a triplet (a, b, c). Therefore, the current values that define the *sci* of the classes will serve as a reference to define the membership functions according to the type of function chosen (triangular/trapezoidal):

A. *Triangular Membership Function*

In this case, it is necessary to obtain three values for the membership function that will define the corresponding classes. The use of central profiles cp_i will be enough to form such functions. Based on AHPSort approaches, the central profiles define the function according to the different classes:

i. *Least preferred class* (*Class$_n$*). The membership function for this class must be defined as a *trapezoidal one* for the definition to make sense, because values less than cp_n belong to this class, so it will be defined by:

$$(a_n = min, b_n = min, d_n = cp_n, c_n = cp_{n-1})$$ (13)

ii. *Most preferred class* (*Class$_1$*). Analogously to the previous membership function, the most preferred class must be also defined as a trapezoidal function. Albeit, in this case the function is defined as:

$$(a_1 = cp_2, b_1 = cp_1, d_1 = max, c_1 = max)$$ (14)

iii. *Remaining classes*, (*Class$_i$*, $1 < i < n$)*:* in these cases, the membership functions will be triangular ones and the parameters will be:

$$(a_i = cp_{i+1}, b_i = cp_i, c_i = cp_{i-1})$$ (15)

Remark 3. The least and most preferred classes are defined as right-angled trapezoidal membership functions in spite of the fact that the remaining n-2 classes are triangular ones. Such trapezoidal functions are necessary to make sense with the interpretation of the membership degree because below b_n of *Class$_n$* the membership must be total, i.e. 1, due to there is not a worse class analogously for *Class$_1$*.

The use of triangular membership functions is straightforward and easy to compute, but it reduces the full membership of an element to this class only to elements that match perfectly with the typical example of the class. In some cases, it could be more reasonable to use trapezoidal membership functions, in which a wider range of elements have a full membership in the class.

B. *Trapezoidal Membership Function*

In this case, it is necessary to obtain four values to define the parameters to compute the membership function that will define the corresponding classes. Our idea is that the limiting profile lp_i will be the

equilibrium point in which the degree of membership between *Classes i* and *i+1* is equal. Therefore, an interval area will be added to the central profile to fix the full membership values for *Class i* of long, δ_i, and the soft transition between classes will have a support defined by $\varphi_i = lp_{i-1} - (cp_i + \delta_i)$. Hence, a fuzzy partition is built. Similar to the triangular membership function, in this case such values will be chosen according to the different classes, and they will take into account the limiting profiles:

    i.   *Least preferred class* (*Class$_n$*). The membership function will be defined by:

$$(a_n = min, b_n = min, d_n = lp_{n-1} - \varphi_n, c_n = lp_{n-1} + \varphi_n) \tag{16}$$

    ii. *Most preferred class* (*Class$_1$*). In this case it will be:

$$(a_1 = lp_1 - \varphi_2, b_1 = lp_1 + \varphi_2, d_1 = max, c_1 = max) \tag{17}$$

    iii. *Remaining classes*, (*Class$_i$*,$1<i<n$)*:* the parameters will be:

$$(a_i = lp_i - \varphi_{i+1}, b_i = lp_i + \varphi_{i+1}, d_i = lp_{i-1} - \varphi_i, c_i = lp_{i-1} + \varphi_i) \tag{18}$$

Once the parameters are known, the membership function can be defined.

Remark 4. Without loss of generality, it is assumed that the definition of the membership functions is ordered from the class less preferred to the one most preferred. Therefore, the parameter a_i for *classes*, (*Class$_i$*) is fixed from φ_{i+1} for the sake of clarity.

4.1.3. *Defining the fuzzy membership functions*

In this subsection, we provide the triangular and trapezoidal definitions for the membership functions that will outline each class in the AHP-FuzzySort approach. It is necessary to remember that the functions must form a fuzzy partition (Ruspini 1969).

  A. *Triangular Membership Function*

    The membership function for each class is defined as:

      i. *Least preferred class* (*Class$_n$*). Taking into account that this is a special function that is defined by a trapezoidal one; the function is defined as:

$$\mu_{Class_n}(x) = \begin{cases} 0 & if \quad a_n > x > c_n \\ \frac{cp_{n-1}-x}{cp_{n-1}-cp_n} & if \quad d_n < x \le c_n \\ 1 & if \quad a_n \le x \le c_n \end{cases} \tag{19}$$

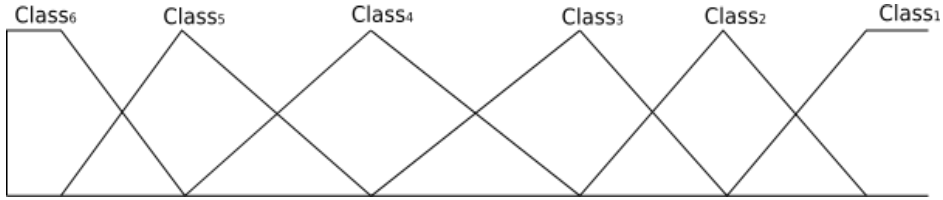   ii. Most preferred class(*Class$_1$*). Analogously, this function is defined as:

$$\mu_{Class_1}(x) = \begin{cases} 0 & if \quad a_1 > x > c_1 \\ \frac{x-cp_2}{cp_1-cp_2} & if \quad a_1 < x \le b_1 \\ 1 & if \quad b_1 \le x \le d_1 \end{cases} \tag{20}$$

      iii. Remaining classes, (*Class$_i$*,$1<i<n$)*:* these functions are triangular:

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$$\mu_{Class_i}(x) = \begin{cases} 0 & if \ a_i > x > c_i \\ \frac{x-cp_{i+1}}{cp_i-cp_{i+1}} & if \ a_i \le x < b_i \\ \frac{cp_{i-1}-x}{cp_{i-1}-cp_i} & if \ b_i < x \le c_i \\ 1 & if \ \ x = b_i \end{cases} \tag{21}$$

A graphical representation of these membership functions is shown in Fig. 5.

Fig. 5. Fuzzy triangular membership functions for classes.

B. *Trapezoidal Membership Function*

In this case, all functions are trapezoidal, and they are defined as:

i. *Least preferred class* (*Class$_n$*). The membership function is defined as:

$$\mu_{Class_n}(x) = \begin{cases} 0 & if \ \ a_n > x > c_n \\ \frac{(lp_{n-1}-\varphi_n)-x}{2\varphi_n} & if \ \ d_n < x \le c_n \\ 1 & if \ \ a_n \le x \le d_n \end{cases} \tag{22}$$

ii. *Most preferred class* (*Class$_1$*). In this case it is defined as:

$$\mu_{Class_1}(x) = \begin{cases} 0 & if \ \ a_1 > x > c_1 \\ \frac{x-(lp_1-\varphi_2)}{2\varphi_2} & if \ \ a_1 < x \le b_1 \\ 1 & if \ \ b_1 \le x \le c_1 \end{cases} \tag{23}$$

iii. *Remaining classes*, (*Class$_i$*, $1 < i < n$): the functions are defined as:

$$\mu_{Class_i}(x) = \begin{cases} 0 & if \ \ a_i > x > c_i \\ \frac{x-(lp_i-\varphi_{i+1})}{2\varphi_{i+1}} & if \ \ a_i \le x < b_i \\ \frac{(lp_{i-1}+\varphi_i)-x}{2\varphi_i} & if \ \ d_i < x \le c_i \\ 1 & if \ \ b_i \le x \le d_i \end{cases} \tag{24}$$

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These functions are graphically shown in Fig. 6.

Fig. 6. Fuzzy trapezoidal membership functions for classes
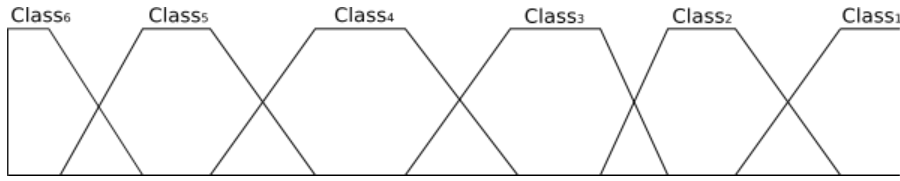
It is noteworthy to clarify that any membership function could be used, but due to the fact that the set of functions is a fuzzy partition in the sense of Ruspini, i.e. that any value of the universe of the discourse will have a total degree of membership equal to one either belonging to just one term or to two consecutive terms.

4.2. *Assignment to classes by using fuzzy membership degrees*

After obtaining the fuzzy membership functions, we can redefine the AHPSort *assignment to classes* (Phase 3, step 7 in Section 2.1) by using the proportional two-tuple (Wang and Hao 2006).

Even though the assignment of alternatives to the considered classes is carried out in AHPSort I and II in different ways, the assignment is similar in both approaches. The global priorities p_k, lp_i or cp_i for alternative a_k, limiting or central profiles are computed by aggregating local priorities according to Eqs. (4) and (5) and Fig. 1 respectively. So, our proposal can be applied to both approaches leading to the final assignment of alternatives to the considered classes.

For sake of clarity, the explanation of the proposal will be developed by using fuzzy trapezoidal membership functions though it is analogous for triangular ones. The assignment process of an alternative a_k, to a class based on fuzzy proportional two-tuples consists of the following steps:

1. Computation of the global priorities p_k, lp_i or cp_i.
2. Obtaining a proportional two-tuple for p_k.
3. Applying an assignment process based on the proportional two-tuple.

The first step is not explained here because it has been developed and extensively described in (Ishizaka, Nemery et al. 2012, Miccoli and Ishizaka 2017) and briefly summarised in Section 2.1. However, the second and third steps are further detailed below.

4.2.1. *Obtaining a proportional two-tuple*

Once the limiting, lp_i and central, cp_i profiles are obtained, the membership functions are defined and represent the classes with their respective semantics and syntax that show an order (see Fig 7):

Fig. 7: Classes with labels
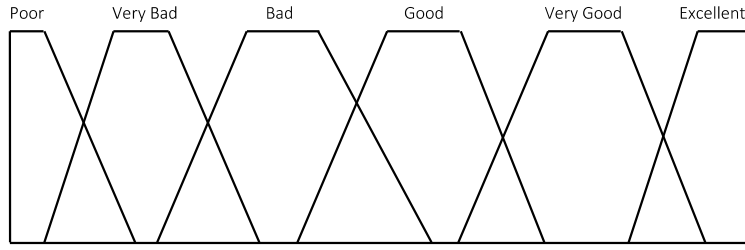
After computing the global priority p_k, for each alternative a_k, the comparison process carried out in previous AHPSort approaches is replaced by the computation of a proportional two-tuple global priority, $\overline{p_k} = h(p_k) = (\alpha \cdot s_i, (1-\alpha)\, s_{i+1})$, in the ordinal proportional 2-tuple set generated by the fuzzy linguistic class scale:

$$h: [min, max] \rightarrow \overline{S}$$

(25)

$$\gamma = \max \mu_{S_j}(p_k) /\ s_j \in S \text{ and } s_l = \arg max_{S_j}(\mu_{S_j}(p_k))$$

$$If\ \mu_{S_{l+1}}(p_k) > 0 \Rightarrow s_i = s_l\ and\ \alpha = \gamma$$

$$Otherwise\ s_i = s_{l-1}\ and\ \alpha = 1 - \gamma$$

Therefore,

$$\overline{p_k} = h(p_k) = (\alpha s_i, (1-\alpha)s_{i+1})_k$$

Due to the fact that $\bar{S}$ is a fuzzy partition (see the line from p_k in Fig. 8) just the membership degrees obtained fulfill the requirements of this proportional two-tuple representation:

Fig. 8. Computing proportional two tuple from p_k
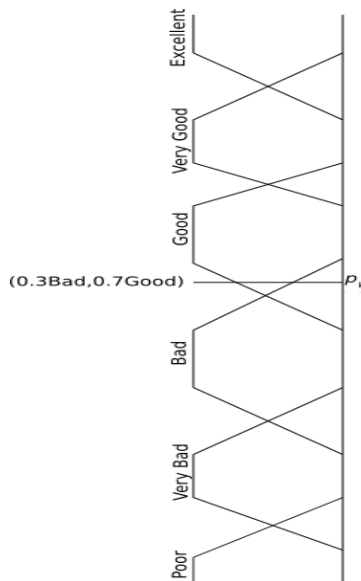
From this proportional two-tuple linguistic value, $\overline{p_k}$:

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$$\overline{p_k} \rightarrow (0.3 \text{ Bad}, 0.7 \text{ Good}),$$

that represents the global priority of the alternative a_k, the assignment process to a class is carried out, further described in the following subsection.

4.2.2. Assignment process

The input value for the assignment of alternative a_k to a class is the proportional two-tuple, $\overline{p_k} = (\alpha \cdot s_i, (1-\alpha) s_{i+1})_k$, obtained from Eq. (25) and the process is implemented as follows:

1. **If $\alpha > (1-\alpha)$**

   Then alternative a_k is assigned to class s_i.

2. **If $\alpha < (1-\alpha)$**

   Then alternative a_k is assigned to class s_{i+1}.

3. **If $\alpha = (1-\alpha)$**

   Then there are two options to assign a_k to a class

      a. *Optimistic view: a_k is assigned to s_i.*

      b. *Pessimistic view: a_k is assigned to s_{i+1}*

Fig. 9 illustrates the class-assignment process, which is further explained below with two examples:

Fig. 9. Assignment to class based on proportional two tuple

1. $p_k = (0.3 \text{ }Bad, 0.7 \text{ }Good)$

   Condition 2. is fulfilled because:

   $\alpha = 0.3 < 0.7$

   Then   Class(p_k)=$Good$

2. $p_l = (0.7 \text{ }VeryBad, 0.3 \text{ }Bad)$

   Condition 1. is fulfilled because:

   $\alpha = 0.7 > 0.3$

   Then   Class(p_l)=$VeryBad$
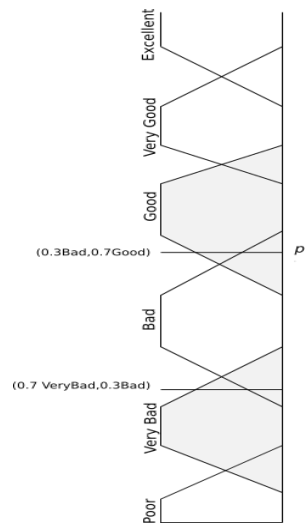
Furthermore, for all cases, despite the class assignment, we keep additional information regarding the strength of the membership (i.e. the strength according to which alternative, a_k, belongs to the assigned class, s_j). This information can be useful for proposals beyond the classification in the initial classes, as well as in the visualization and the understanding of the sorting process as it is shown in the illustrative example following.

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5. Case Study: *Safety of London boroughs*

Exhibiting the applicability of the AHP-FuzzySort approach, we will hereby apply our proposed method to classify London's boroughs according to their safety levels, evaluated from several crime-related criteria. To illustrate further the advantages of AHP-FuzzySort over its crisp variants, a comparison of our results to those of the AHPSort is offered. In the following, Section 5.1 consists a brief background discussion of the importance of crime in general and the relevance of the case study we present. Section 5.2 presents the data and and the procedure followed to obtain them. Section 5.3 contains a discussion of the results and Section 5.4 validates those by presenting a sensitivity analysis.

5.1 Introduction

Crime is inherently a difficult and complex phenomenon to be explained and measured (Osgood, McMorris et al. 2002). Several decades of dedicated research in this field have provided many ways to approach this (see e.g., (Sullivan and McGloin 2014), explaining the drivers of crime itself (Hirschi 2002, Andrews and Bonta 2010) and finding ways to prevent it (Crawford and Evans 2017). Understandably, criminality is a very important factor, directly or indirectly affecting every aspect of a community. Thus, its awareness and consideration are of utmost importance not only for citizens and businesses within a community, but also for the planners and policymaking designers' decision-making process (Paulsen 2012).

In this case study, we are considering the capital of the United Kingdom, London, evaluating its boroughs' safety levels according to some crime measures. London was among a number of cities in the United Kingdom experiencing rapidly increased levels of street crime in the late 90s; a trend that had begun falling by early to mid-00s (Curran, Dale et al. 2005). After reaching a two decades' low in 2014, the number of recorded total offences in England and Wales presented an increasing trajectory that, according to the recent ONS bulletin (ONS 2017), skyrocketed to a yearly increase of 13% over the past year; the biggest in a decade (Travis 2017). We hereby take advantage of this increasing interest in media and the respective authorities to explore the overall safety of London at the borough level.

5.2 Data & Results

Taking all 33 boroughs of London into consideration, we intend to classify them into three distinct categories that will accordingly exhibit their safety levels; namely, 'high', 'moderate' and 'low'. In doing so, we gather the latest available data[1] on the number of different types of crime recorded by the police from 'UK Crime Stats' (http://www.ukcrimestats.com), an open data platform that collects recorded crime data from the official repository of the UK's police force and supplies them in ready to download data sets. To classify the boroughs according to their overall safety levels, seven criteria were used, which illustrate the different types of recorded crime. These are described in Table 1. Since all criteria were originally obtained as raw numbers of recorded incidents, they were scaled to reflect the type of crime per 1,000 inhabitants, in order to make the comparison

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[1] At the time of writing this paper, the latest available data concern the period April, 2016 – April, 2017.

between boroughs feasible (given the noticeable difference of boroughs' population). These are reported for every borough in Table 2.

Table 1. Measuring criminality – Types of recorded crime

Criteria (Types of crime)	Description
Robbery	Includes offences where a person uses force, or threat to steal from another person, either outside of, or within one's premises.
Vehicle-related	Includes theft from, or of a vehicle, or interference with a vehicle.
Violent	Includes offences against a person, such as common assaults, grievous bodily harm and sexual offences
Misbehaviour	Includes personal, environmental and nuisance anti-social behaviour, offences that cause deliberate damage to buildings and vehicles, or any other offences causing fear, alarm, distress or a possession of a weapon such as a firearm.
Crimes without the use of force	Includes crimes that involve theft directly from the victim (including handbag, wallet, cash, mobile phones, or bicycles), but without the use or threat of physical force.
Drug-related	Includes offences related to possession, supply and production of drugs.
Weapon-related	Includes offences related to the possession of a weapon, such as a firearm or knife.

Table 2. – Criteria (Types of crime): Recorded incidents per 1,000 inhabitants

Borough	Population	Robbery	Vehicle related	Violent	Misbehaviour	Crimes without the use of force	Drug related	Weapon related
Barking and Dagenham	185,621	4.82	6.67	14.32	17.70	1.21	1.88	0.27
Barnet	355,497	6.32	6.15	9.69	14.63	0.97	1.04	0.15
Bexley	231,507	4.16	5.53	10.95	16.90	0.79	1.29	0.20
Brent	310,137	5.70	5.57	11.04	16.72	1.40	2.22	0.25
Bromley	308,883	5.90	5.76	11.14	15.14	0.70	1.01	0.15
Camden	220,121	8.03	6.93	12.11	24.46	7.57	2.14	0.30
City of London Corporation	7,288	25.80	15.37	83.56	81.92	87.95	17.56	2.88
Croydon	362,520	5.83	5.80	12.77	15.14	0.82	1.62	0.33

Ealing	338,548	4.14	4.98	8.60	11.70	1.18	1.10	0.15
Enfield	312,005	6.46	6.59	11.44	15.68	1.02	1.24	0.23
Greenwich	252,869	5.20	6.59	15.34	17.65	1.65	1.65	0.24
Hackney	245,997	7.78	5.83	13.44	20.46	8.43	1.81	0.40
Hammersmith and Fulham	181,551	5.94	10.28	12.93	17.77	3.83	2.57	0.21
Haringey	255,182	7.16	7.62	13.52	17.60	3.74	2.06	0.33
Harrow	238,060	4.64	3.55	6.83	9.38	0.57	0.65	0.07
Havering	237,104	5.82	5.74	12.29	15.63	0.64	0.98	0.13
Hillingdon	273,216	3.59	3.92	6.63	8.28	0.56	0.57	0.07
Hounslow	253,116	3.52	5.46	7.94	9.90	1.15	0.93	0.13
Islington	205,690	7.20	6.25	15.24	20.17	11.46	3.22	0.44
Kensington and Chelsea	158,959	6.76	11.07	11.49	18.77	4.21	4.49	0.41
Kingston upon Thames	159,760	4.42	3.67	8.34	13.54	1.14	1.80	0.14
Lambeth	301,049	5.88	5.67	10.65	15.43	2.74	2.42	0.44
Lewisham	275,883	5.03	5.07	11.66	13.94	1.23	1.87	0.34
Merton	199,608	5.64	6.29	10.80	14.74	1.52	1.11	0.23
Newham	307,094	5.93	8.34	14.20	19.90	1.83	2.47	0.31
Redbridge	278,759	6.60	8.12	12.13	15.99	1.02	1.41	0.19
Richmond upon Thames	186,959	5.33	5.94	7.39	13.58	2.66	0.80	0.08
Southwark	286,484	7.65	6.27	13.08	17.26	4.81	2.69	0.46
Sutton	190,191	4.96	4.93	10.71	12.39	0.81	1.12	0.18
Tower Hamlets	251,338	8.75	7.01	16.25	37.85	6.15	2.47	0.47
Waltham Forest	258,299	5.59	6.05	12.85	16.78	1.48	1.58	0.30
Wandsworth	306,599	5.33	8.50	8.66	11.96	2.25	1.05	0.17
Westminster City Council	218,318	11.52	10.99	18.07	35.44	12.94	3.45	0.46

Data obtained from UK Crime Stats' (http://www.ukcrimestats.com)

The first step is to define the problem at hand (see Section 3.1, Phase 1). That said, the objective of the case study is to classify each borough in one of the three considered classes ('low', 'moderate' and 'high'-safety), with respect to the seven considered criteria that reflect the type of reported crime. To do so, the importance of the considered criteria must be elicited, along with the profiles that characterise each class (i.e. central profiles in this case), so that the evaluation, and classification process can proceed accordingly as discussed in the steps

provided in Section 3.1, and in particularly, in Phases 2 & 3. At this stage, the input of a decision-maker is necessary to decide upon these required parameters based on his/her expertise. Thus, we consulted with an expert having several years of professional experience and academic background on crime to obtain the required information. The expert originally filled in a questionnaire containing the key parameters characterizing each safety class and a pairwise comparison among the criteria to elicit their weights. The output is given as follows: Table 3 outlines the expert's view on the profiles of each level of safety per criterion and Table 4 contains the pairwise comparisons of the criteria and the weights calculated with (1). Seemingly, 'violent crimes', followed by 'weapons' and 'drug-related' crimes are by far the most weighted crimes. This seems reasonable, being the most dangerous crimes for the public. The inconsistency ratio, computed as in (2), is 0.06, below the 0.1 threshold we had set, and thus we proceeded with the subsequent steps of the analysis.

Table 3. Types of recorded crime: Range & central profiles of safety levels

		Safety Levels (*Central Profiles*)		
Type of Crime	**Range**	**High** (*CP 1*)	**Moderate** (*CP 2*)	**Low** (*CP 3*)
Robbery	[3.52 - 25.80]	2	10	18
Vehicle-related	[3.55 - 15.37]	3	7	10
Violent	[6.63 - 83.56]	5	20	50
Misbehaviour	[8.28 - 81.92]	6	40	60
Crimes without the use of force	[0.56 - 87.95]	0.5	30	50
Drug-related	[0.57 - 17.56]	0.25	6.5	9
Weapon-related	[0.07 - 2.88]	0.05	1	1.7

Table 4. Importance of criteria – Pairwise comparisons

	Robbery	Vehicle-related	Misbehaviour	Violent	Crimes without the use of force	Drug-related	Weapon-related	Weight
Robbery	1	1	5	1/7	5	1/6	1/9	**0.060**
Vehicle-related	1	1	4	1/9	4	1/3	1/9	**0.057**
Misbehaviour	1/5	1/4	1	1/9	1	1/9	1/9	**0.022**
Violent	7	9	9	1	9	2	2	**0.351**
Crimes without the use of force	1/5	1/4	1	1/9	1	1/9	1/9	**0.022**
Drug-related	6	3	9	1/2	9	1	1	**0.210**
Weapon-related	9	9	9	1/2	9	1	1	**0.279**

Inconsistency Ratio: 0.06.

In particular, the next step required the decision-maker to pairwise compare the alternatives (i.e. 33 London boroughs) against the central profiles on each criterion. Normally, this would require a large amount of time, and a fair amount of cognitive stress attributed to the decision-maker due to the pairwise comparisons required. For this reason, reference profiles were used instead of the alternatives to be pairwise compared to the central profiles, as suggested in Miccoli and Ishizaka (2017). Thus, we manually created six reference profiles (RP) that were equally spread within the range of the considered criteria, with the first reference profile (RP1) being always equal to zero, and the last one (RP6) being always equal to the max value of each criterion. These are reported in Table 5.

Table 5. Reference profiles of the considered criteria

Reference Profiles	Robbery	Vehicle related	Violent	Misbehaviour	Crimes without the use of force	Drug related	Weapon related
RP 1	0.00	0.00	0.00	0.00	0.00	0.00	0.00
RP 2	5.16	3.07	16.71	16.38	17.59	3.51	0.58
RP 3	10.32	6.15	33.42	32.77	35.18	7.03	1.15
RP 4	15.48	9.22	50.14	49.15	52.77	10.54	1.73
RP 5	20.64	12.29	66.85	65.53	70.36	14.05	2.31
RP 6	25.80	15.37	83.56	81.92	87.95	17.56	2.88

Then, the performance of the six reference profiles is pairwise benchmarked against the three central profiles set by the crime expert as mentioned in Phase B, step (5). In this way, we elicit the local priorities of the reference profiles as mentioned in Phase 3, step (6), which will be used to obtain the local priorities of the boroughs through linear interpolation as in Phase 3, step (7). To illustrate this procedure, we hereby give an example of a single borough and a single criterion (e.g. Hackney Borough Council in 'Robbery' criterion accordingly). Over the period April 2016 to April 2017, the recorded number of robberies in Hackney Borough Council was 7.78 per each 1,000 inhabitants. To calculate its local priority, we need to search two consecutive reference or central profiles prior to and after the considered alternative (see Fig. 10). Looking at Table 6, these are RP2 and CP2, with points and local priorities (5.16, 0.139) and (10, 0.075) accordingly.

Table 6. Local Priorities for 'Robbery' criterion

Robbery		
	Reference profiles	Local Priorities
RP 1	0.00	0.32
CP 1	2	0.261
RP 2	5.16	0.139
CP 2	10	0.075
RP 3	10.32	0.074
RP 4	15.48	0.047
CP3	18	0.032
RP5	20.64	0.031
RP6	25.80	0.022

Therefore, calculating the local priority for Hackney on the 'Robbery' criterion with (6) is straightforward and equal to:

$$p\big(Hackney_{Robbery}\big) = 0.139 + \frac{0.075 - 0.139}{10 - 5.16} \cdot (7.78 - 5.16) = 0.104$$

The priority can also be read graphically in Fig. 10.

Fig. 10. Graph of local priorities of the reference and central profiles. The dotted line represents Hackney
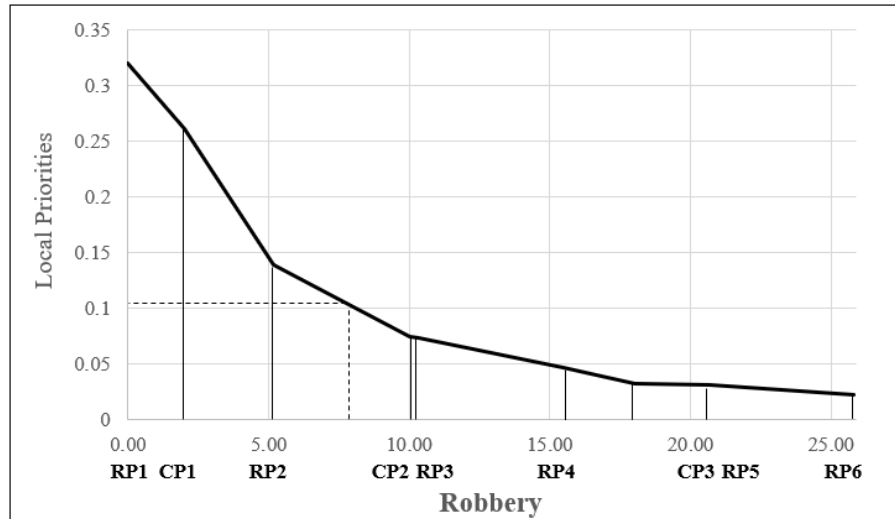
Understandably, this procedure, i.e. computing the local priority according to (6), is repeated for every borough and every criterion considered. Then, global priorities are calculated for every borough according to (4), and for every central profile according to (5). Results are reported in Tables 7 and 8 respectively. Global priorities are needed to classify the alternatives into the predefined classes. In the classic AHPSort, this is done by comparing the global priorities of the alternatives to those of the limit, or in this case, the central profiles;

whereby, an alternative belongs to a specific class according to the cutoffs specified by the global priorities of the two consecutive classes surrounding the alternative. For instance, Hackney has a global priority equal to 1.052. This is between the global priority of Class 1 (1.754) and the global priority of Class 2 (0.516), though closer to the latter. Thereby, one would classify Hackney to the second class. In the proposed approach though, in addition to the crisp classification, a degree of membership to a respective class is computed accordingly (as discussed in 4.2.2). This further facilitates the decision-maker to get a better understanding of the relative position of an alternative within the class, but also provides room for a better visualization. We report the classification results according to the classic AHPSort and the proposed, AHP-FuzzySort in Table 9, while their respective visualized heatmaps are given in Figures 11 and 12. Red (R) implies low safety, blue (B) represents moderate safety and green (G) is for a high safety.

What is noteworthy, visualizing the classification according to AHP-FuzzySort (Figure 12) provides additional comparative insight in contrast to the classic AHPSort (Figure 11). For instance, Figure 11 shows that the City of London proves to be the relatively less safe borough, with a few boroughs surrounding it being classified as moderately safe, and the rest being classified as highly safe. However, when the boroughs' membership in these classes are visualized in the form of RGB gradients of these classes (see Figure 12), one might get a better understanding of the boroughs' partitioning within the same class. For instance, the boroughs located in the west part are overall safer, while the boroughs located in the north are riskier. Understandably, these comparative insights cannot be inferred from the visualization of the classic AHPSort (Figure 11).

Table 7 – Local and global priorities of boroughs.

Borough	Population	Robbery	Vehicle related	Violent	Misbehaviour	Crimes without the use of force	Drug related	Global
Barking and Dagenham	0.152	0.065	0.162	0.143	0.285	0.187	0.203	**1.196**
Barnet	0.124	0.069	0.219	0.168	0.287	0.226	0.236	**1.328**
Bexley	0.177	0.080	0.203	0.147	0.289	0.214	0.223	**1.333**
Brent	0.132	0.079	0.202	0.147	0.283	0.171	0.209	**1.224**
Bromley	0.129	0.076	0.201	0.162	0.290	0.227	0.238	**1.324**
Camden	0.101	0.063	0.189	0.111	0.221	0.175	0.197	**1.058**
City of London Corporation	0.022	0.026	0.023	0.023	0.021	0.018	0.020	**0.153**
Croydon	0.130	0.075	0.181	0.162	0.289	0.199	0.187	**1.223**
Ealing	0.178	0.090	0.232	0.199	0.285	0.223	0.236	**1.444**
Enfield	0.122	0.066	0.197	0.156	0.287	0.216	0.214	**1.259**
Greenwich	0.138	0.066	0.149	0.143	0.280	0.198	0.212	**1.186**
Hackney	0.104	0.075	0.172	0.130	0.213	0.190	0.168	**1.052**
Hammersmith and Fulham	0.129	0.039	0.179	0.142	0.259	0.155	0.219	**1.122**

Haringey	0.113	0.059	0.172	0.143	0.260	0.179	0.189	**1.113**
Harrow	0.159	0.116	0.254	0.223	0.291	0.244	0.259	**1.546**
Havering	0.130	0.076	0.187	0.157	0.291	0.229	0.241	**1.311**
Hillingdon	0.199	0.109	0.257	0.235	0.291	0.247	0.259	**1.597**
Hounslow	0.202	0.081	0.241	0.218	0.285	0.231	0.244	**1.502**
Islington	0.112	0.068	0.150	0.131	0.182	0.125	0.156	**0.925**
Kensington and Chelsea	0.118	0.037	0.197	0.138	0.255	0.098	0.165	**1.008**
Kingston upon Thames	0.168	0.113	0.236	0.179	0.286	0.191	0.241	**1.413**
Lambeth	0.130	0.078	0.207	0.159	0.270	0.162	0.156	**1.161**
Lewisham	0.144	0.088	0.194	0.175	0.285	0.188	0.183	**1.257**
Merton	0.133	0.068	0.205	0.166	0.282	0.223	0.215	**1.292**
Newham	0.129	0.054	0.163	0.132	0.279	0.160	0.192	**1.109**
Redbridge	0.120	0.055	0.189	0.153	0.287	0.209	0.226	**1.239**
Richmond upon Thames	0.137	0.073	0.247	0.179	0.270	0.237	0.257	**1.399**
Southwark	0.106	0.068	0.177	0.145	0.249	0.150	0.152	**1.047**
Sutton	0.147	0.091	0.206	0.191	0.289	0.222	0.228	**1.374**
Tower Hamlets	0.092	0.063	0.138	0.067	0.235	0.160	0.148	**0.902**
Waltham Forest	0.133	0.071	0.180	0.147	0.282	0.201	0.196	**1.210**
Wandsworth	0.137	0.053	0.232	0.196	0.274	0.225	0.233	**1.350**
Westminster City Council	0.068	0.037	0.113	0.069	0.168	0.115	0.151	**0.720**

Table 8. Local and global priorities for the central profiles.

Safety Class	Population	Robbery	Vehicle related	Violent	Misbehaviour	Crimes without the use of force	Drug related	Global
'High' (*CP 1*)	0.261	0.138	0.277	0.259	0.292	0.262	0.265	**1.754**
'Moderate' (*CP 2*)	0.075	0.063	0.086	0.066	0.0752	0.07	0.081	**0.516**
'Low' (*CP 3*)	0.032	0.04	0.032	0.034	0.043	0.035	0.045	**0.261**

Table 9. Results: Boroughs' class membership according to AHPSort and AHP-FuzzySort

Borough	Global Priority	AHPSort Safety Class	AHP-FuzzySort % of belonging to a given safety class		
			High	Moderate	Low
Barking and Dagenham	1.196	High	54.95	45.05	0.00
Barnet	1.328	High	65.61	34.39	0.00
Bexley	1.333	High	66.01	33.99	0.00
Brent	1.224	High	57.20	42.80	0.00
Bromley	1.324	High	65.25	34.75	0.00
Camden	1.058	Moderate	43.75	56.25	0.00
City of London Corporation	0.153	Low	0.00	0.00	100
Croydon	1.223	High	57.09	42.91	0.00
Ealing	1.444	High	74.94	25.06	0.00
Enfield	1.259	High	59.98	40.02	0.00
Greenwich	1.186	High	54.15	45.85	0.00
Hackney	1.052	Moderate	43.32	56.68	0.00
Hammersmith and Fulham	1.122	Moderate	48.98	51.02	0.00
Haringey	1.113	Moderate	48.23	51.77	0.00
Harrow	1.546	High	83.23	16.77	0.00
Havering	1.311	High	64.21	35.79	0.00
Hillingdon	1.597	High	87.35	12.65	0.00
Hounslow	1.502	High	79.66	20.34	0.00
Islington	0.925	Moderate	33.06	66.94	0.00
Kensington and Chelsea	1.008	Moderate	39.73	60.27	0.00
Kingston upon Thames	1.413	High	72.42	27.58	0.00
Lambeth	1.161	High	52.12	47.88	0.00
Lewisham	1.257	High	59.89	40.11	0.00
Merton	1.292	High	62.65	37.35	0.00
Newham	1.109	Moderate	47.87	52.13	0.00
Redbridge	1.239	High	58.40	41.60	0.00
Richmond upon Thames	1.399	High	71.35	28.65	0.00
Southwark	1.047	Moderate	42.86	57.14	0.00
Sutton	1.374	High	69.29	30.71	0.00
Tower Hamlets	0.902	Moderate	31.21	68.79	0.00
Waltham Forest	1.210	High	56.06	43.94	0.00
Wandsworth	1.350	High	67.33	32.67	0.00

Westminster City Council	0.720	Moderate	16.51	83.49	0.00

5.3. Discussion

The objective of the present case study is twofold. On the one hand, it serves as an illustrative example of the proposed methodology, while at the same time it takes advantage of the recent interest in London's safety, as expressed in the press and the recent reports of the Office for National Statistics. In evaluating the safety of London at the borough level, we considered different types of crime reported over the course of the past year. The boroughs' safety levels, which consist of an overall score attributed to 7 criteria, are classified in three categories -namely, high, moderate and low- with the aid of an expert with a background on crime. In what follows, we discuss our results on a brief, but concise note. To illustrate the difference among the existing AHPSort II and the new AHP-FuzzySort, we contrast and compare their results highlighting the advantages of the latter forthwith.

The obtained results reveal some interesting points. First, almost all of London's boroughs can be classified as being of high or moderate safety. This, however, exhibits the strength of the proposed method over its non-fuzzy variant. For instance, Table 9 shows how each borough is classified in the case of AHPSort and AHP-FuzzySort. AHPSort always assigns an alternative to a given class, even though a specific assignment could hide some important information and even be misleading. For instance, Hammersmith and Fulham and Westminster boroughs are both classified as moderately safe. However, the first is almost between the high and moderate safety class (49% to 51% accordingly), while the latter is towards the verge of the moderate safety class (16.5% and 83.5% accordingly). This example highlights how AHP-FuzzySort provides a clearer picture about the safety classification of the boroughs. This, of course, boils down to the difference between the ordinal and cardinal information that is provided in the crisp and fuzzy (proposed here) variants accordingly. The former always gives a single number of a class, whereas this perception can sometimes be off the mark, with an alternative being in closer resemblance to its consecutive class, rather than the one it was assigned on. One could think of AHP-FuzzySort as a procedure that aims in sorting an alternative in a predefined class, with that alternative sharing traits of both that class and its consecutive one. This approach shows that alternative's classification with a clarification of the ambiguity revolved around the crisp classification.

On a more general note, the membership of an alternative to a respective class contains an ambiguity that could be better delineated with the aid of fuzzy theory. Obviously, the graphical representation could significantly add in this respect, by highlighting an alternative's membership to a specific class either through a graphic of the membership function or, in this particular case that we examine, an interactive map of London's boroughs. To compare AHPSort and AHP-FuzzySort, we use the geographical information system from Ordnance Survey OpenData in Tableau Software in producing the two figures. Figure 11 illustrates the safety levels of London's boroughs according to AHPSort and figure 12 depicts these safety levels according to the AHP-FuzzySort. In the latter, the degree of membership in each of these classes is computed for each borough (see Table 9). These are visually illustrated in the figure with gradients of Green, Blue and Red colors respectively. For instance, a borough that participates with 80% in the first class ("High") and with 20% in the

second class ("Moderate") is colored with these exact percentages of gradient among pure Green (255,0,0 in the RGB scale) and pure Blue (0,0,255 in the RGB scale).

It quickly becomes clear from both figures that boroughs that are remote to the centre of London are overall deemed as safer. However, more noticeable differences can be observed in Figure 12, showing the membership according to the AHP-FuzzySort results though an RGB gradient color scheme. In fact, one might get a better understanding of the sorting evaluation, as more information is conveyed due to the gradient coloring rather than the discrete red, blue and green tri-colored scheme of the crisp variant. Figure 11 shows the ordinal information, whereas Figure 12 shows the magnitude and variation within each fixed class. Finally, according to the results, there is only one borough that is assigned to the 'low'-safety class: the City of London. One could argue that it is sensible, given that this borough has a very low number of inhabitants, while at the same time attracts a high number of visitors and professionals on a daily basis.

Fig. 11. Map of London's boroughs according to their safety levels (AHPSort)
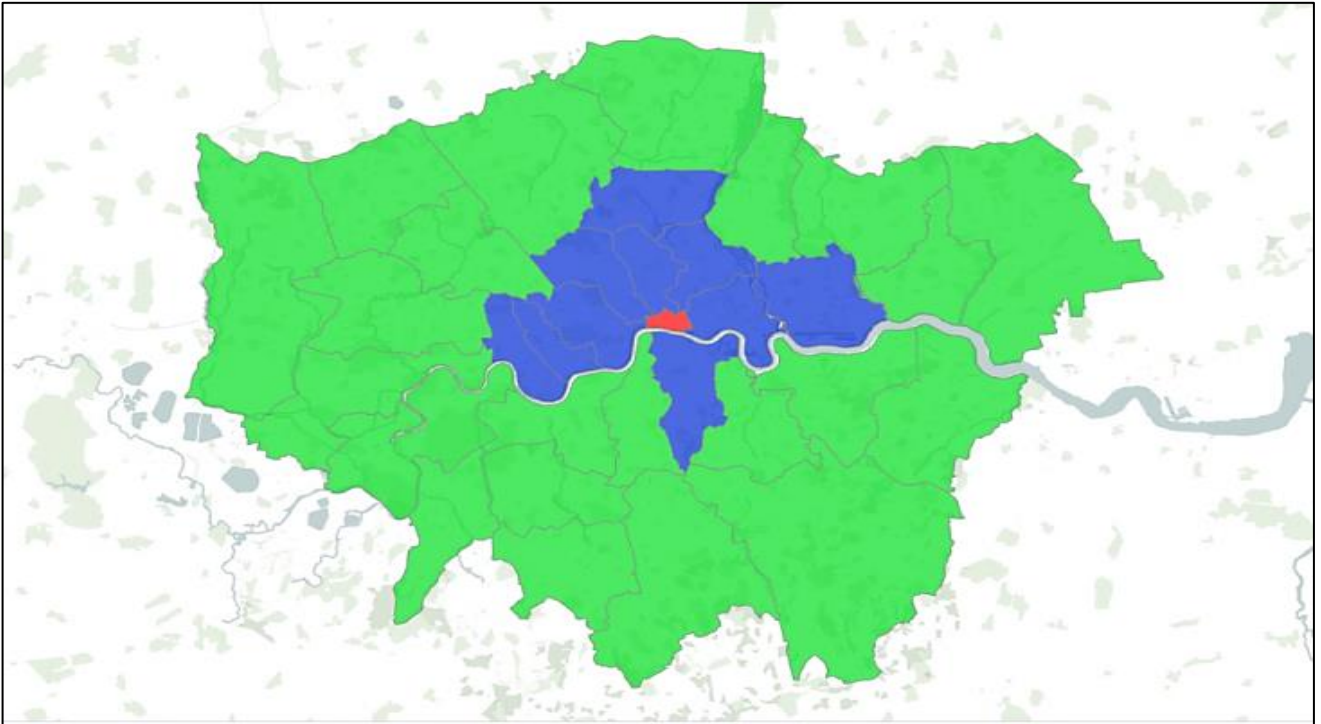
License Details: Contains National Statistics data © Crown copyright and database right [2012] and
Ordnance Survey data © Crown copyright and database right [2012].

Fig. 12. Map of London's boroughs according to their safety levels (AHP-FuzzySort)
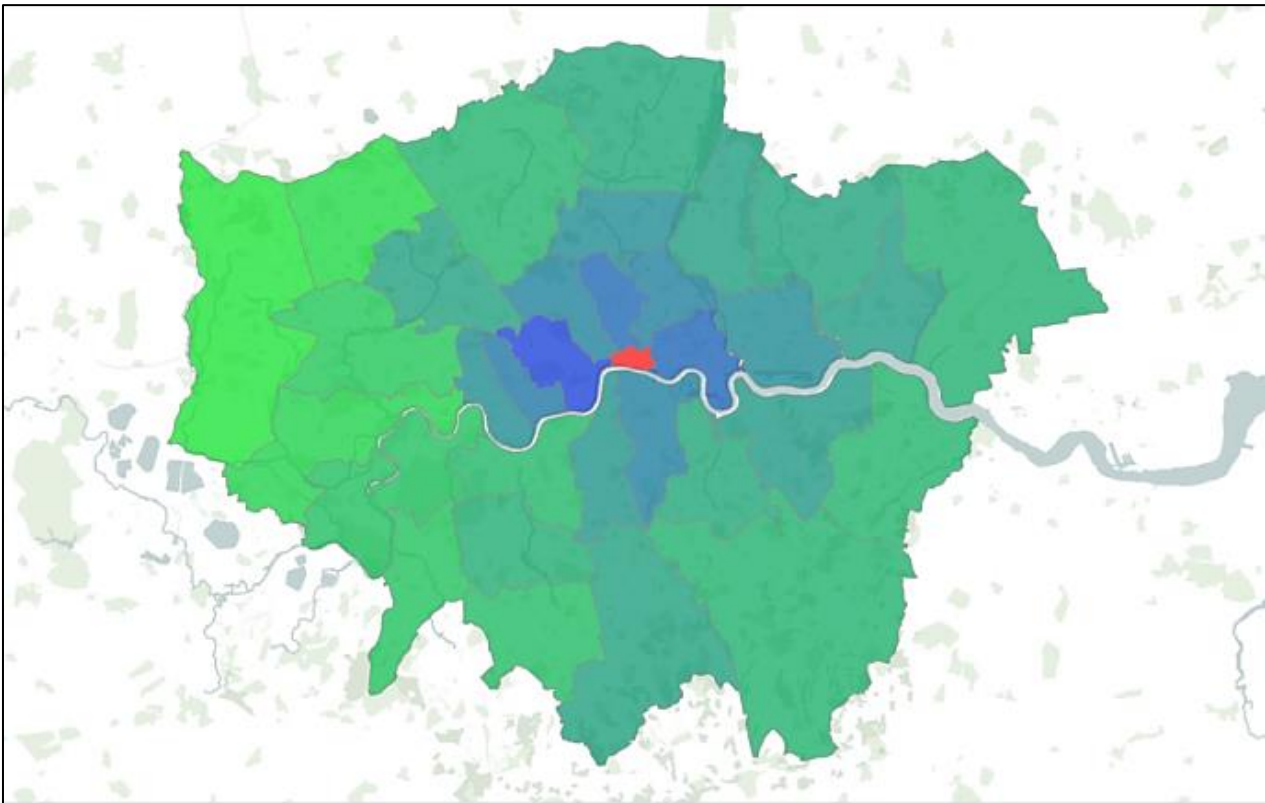
5.4. Sensitivity Analysis

A crucial issue at this point regards the robustness of the obtained results that were discussed above, particularly when it comes to the setting of limiting/central profiles and the parameters of the membership functions overall. The reason is that these could affect the sorting procedure and their change could result in sorting reversals. Indeed, while the decision-maker is an experienced individual working with crime figures, it can sometimes be hard determining said parameters, particularly the central profiles. Towards the easing of this issue, we proceed with a sensitivity analysis that we describe as follows.

We tweak the central profiles given by the decision-maker (as these are reported in Table 3) by increasing/decreasing their values by 10% in order to see whether a sorting reversal takes place. We do that for each central profile (i.e. 'High', 'Mod', 'Low') and for every criterion one at a time. That procedure essentially contains 42 replications of the procedure (7 criteria, 3 central profiles, 2 changes – i.e. increasing/decreasing by 10%). Regarding the crisp classification process (i.e. 'High', 'Mod' and 'Low' safety classes) we find no difference by means of sorting reversals for all 33 boroughs. Understandably, the membership function would fluctuate (although not by huge margins given that no crisp sorting reversal is found), the change of which is delineated in Figure 13.

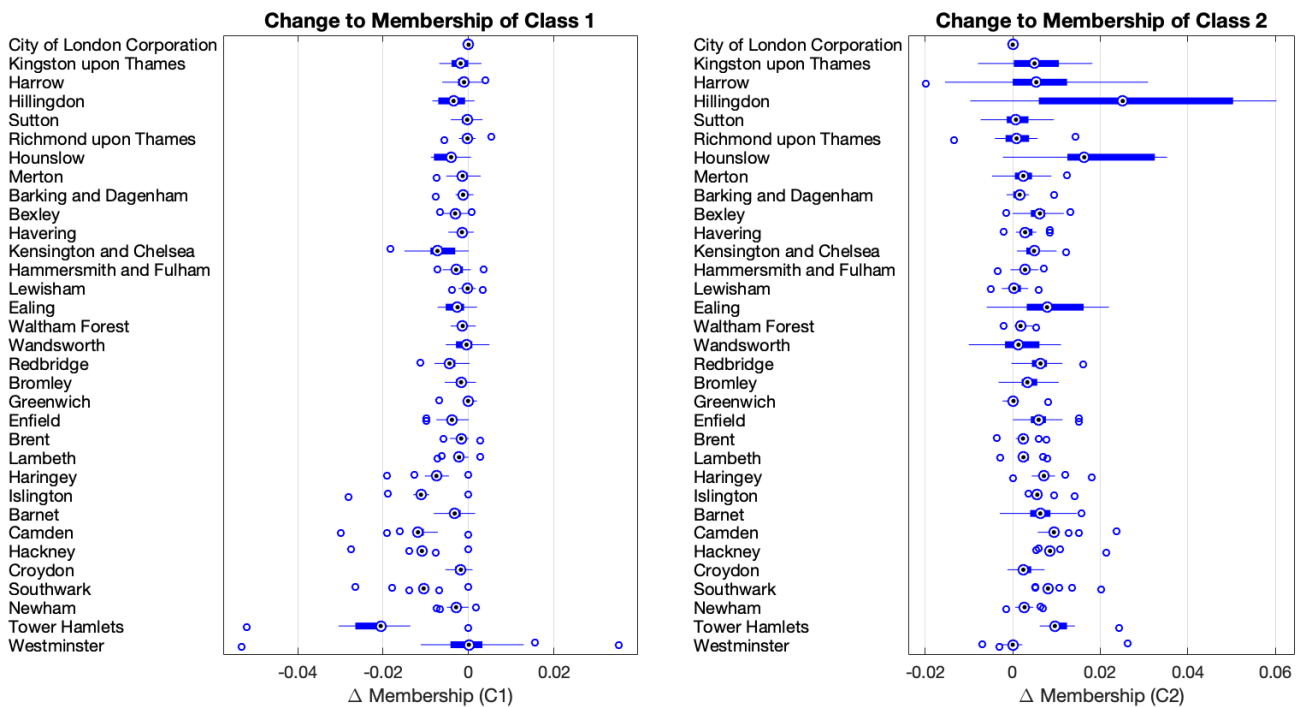Fig. 13. Sensitivity analysis: Changes observed to the class memberships tweaking the CPs by ±10%.

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Figure 13 essentially contains boxplots (one for each borrow) showing how the membership of each class[2] is affected. These are reported in a decimal format. No membership to a class is changed more than 6% (Hillingdon borough's membersip to Class 2) in the 41 changes, while the vast majority of boroughs' membership changes to these two classes revolve around the marginal ±2% range. Understandably, these membership changes were not enough to result in a sorting reversal of any borough.

6. Conclusion

AHP is a well-founded, traditional method in the field of MCDA. Its AHPSort variant came into existence to approach objectives involving the sorting of alternatives in predefined classes. An important drawback, namely the high number of pairwise comparisons required, was solved in AHPSort II, the second version of this variant. More specifically, with this approach, the number of pairwise comparisons needed to be executed by the decision-maker was significantly reduced, thus addressing a significant drawback inherent in AHP. However successful, arguably, this approach's weak point relates to the correct classification of the alternatives. More specifically, the proliferation of classes and the use of strict borders between them could highly increase the difficulty, or the uncertainty regarding the assignment of an alternative to the correct class.

In this paper, we provide a way to make the transition between classes more flexible, and a finer assignment of the alternatives into the classes without increasing the number of comparisons. We make this possible by introducing a fuzzy sorting process within the AHPSort approach, which results in smooth transitions between classes, and facilitates a fine classification of different alternatives into the respective classes. Additionally, to exhibit its applicability, a case study has been developed that regards the classification of London boroughs into three safety classes, according to a handful crime-related criteria. We compare and contrast the results of our proposal with the previous crisp approaches, showing the advantage of our method in terms of accuracy in the sorting process, but also in terms of visualization of the results in a more clear and understandable way that delineates the strength of membership of an alternative to the assigned class. By performing a sensitivity analysis on the membership parameters, we find that our results are robust. Last, but not least, and regarding its future use, as AHP-FuzzySort is a generic method we intend to apply it to solve other sorting problems.

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[2] We only delineate changes in membership of Class 1 ('High') and Class 2 ('Mod'), as no borough presented a change whatsoever as to its membership in Class 3 ('Low').

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