in the proof of 4.1 there is a $\delta>0$ such that $\|U_n(x)\|^*<\varepsilon$ if $\|x\|^*<\delta$, $\|x\|\leqslant 1$.

Bibliography

[1] A. Alexiewicz, *On sequences of operations I*, Studia Mathematica 11 (1950), p. 1-30.

[2] A. Alexiewicz, *On sequences of operations II*, ibidem, p. 200-236.

[3] S. Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Warszawa 1932.

[4] G. Fichtenholz, *Sur les fonctionnelles linéaires, continues au sens généralisé*, Recueil Mathématique 4 (1938), p. 193-214.

[5] C. Kuratowski, *Topologie I*, 2e éd., Monografie Matematyczne, Warszawa 1948.

[6] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires*, Studia Mathematica 10 (1948), p. 184-208.

[7] W. Orlicz, *Linear operations in Saks spaces*, ibidem 11 (1950), p. 237-272.

[8] S. Saks, *On some functionals*, Transactions of the American Mathematical Society 35 (1933), p. 549-556.

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Analytic operations in real Banach spaces

by

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In the treatise Functional Analysis and Semigroups Hille [5] has developped a complete theory of analytic operations in complex Banach spaces. The fact that the spaces under consideration are complex is essential for the methods used by Hille, for they are grounded on the use of the theorem of Hartoggs. On the other hand in many applications of Functional Analysis analytic operations in real Banach spaces play an essential role. Therefore it seems worth while to transfer the theory of Hille to the case of real Banach spaces. That is the purpose of this paper. We show that *mutatis mutandis* the main theorems of the theory can be restored in real Banach spaces. In the development of the theory we follow closely the ideas of Hille: we consider firstly the series of discontinuous powers (called the *p-powers*) and then pass to the continuous powers. The paper is divided into two parts. In the first, after introductory considerations, we establish some properties of analytic operations, in the second, sections 5 and 6, we formulate the extension principle and by its use we extend locally every convergent power series to a power series in an appropriate complex Banach space, converging also locally. Although the results of the first part of the paper might be obtained from the extension principle, we preferred to exhibit them independently, and so to use the extension principle only in the cases where its use seems to be indispensable. The principal tool in tackling the real case is the use of Leja's theorems on sequences of polynomials.

1. Preliminary theorems. In this section we present the auxiliary theorems dealing with the polynomials of real and complex variables taking on values from Banach spaces. Those theorems are well known for numerically valued polynomials.

Let

$$P(u)=\sum_{k=0}^{n}a_k u^k$$

denote a polynomial of degree n of the variable u with coefficients in a Banach space Y. The space Y is supposed to be real or complex according as the variable u is so. Then the following inequalities are valid:

(1.1) INEQUALITY OF MARKOV. *Let the variable u be real. Then*

$$\max_{a\leqslant u\leqslant b}\|P'(u)\|\leqslant\frac{n^2}{b-a}\max_{a\leqslant u\leqslant b}\|P(u)\|.$$

(1.2) INEQUALITY OF BERNSTEIN. *Let the variable u be complex. Then*

$$\|P(u)\|\leqslant(a+b)^n\max_{-1\leqslant v\leqslant 1}\|P(v)\|,$$

where a and b denote the semiaxes of the ellipse with foci $+1$ and -1, passing through the point u.

Proof. Let η be any functional linear in Y. Then $\eta P(u)$ is a real valued polynomial of degree n in the variable u and

$$\frac{d}{du}\eta P(u)=\eta P'(u).$$

The inequality of Markov being valid for real valued polynomials [10] we have

$$\max_{a\leqslant u\leqslant b}|\eta P'(u)|\leqslant\frac{n^2}{b-a}\max_{a\leqslant u\leqslant b}|\eta P(u)|\leqslant\frac{n^2}{b-a}\|\eta\|\max_{a\leqslant u\leqslant b}\|P(u)\|.$$

Choosing now u_0 so that

$$\max_{a\leqslant u\leqslant b}\|P'(u)\|=\|P'(u_0)\|,$$

and then choosing η so that $\|\eta\|=1$ and $\eta P'(u_0)=\|P'(u_0)\|$ we get the inequality (1.1).

The proof of the inequality (1.2) is analogous (for the numerical case see [4]).

We shall need some theorems dealing with the behaviour of polynomials, proved in the case of numerically valued polynomials by Leja. We start with the

(1.3) THEOREM OF LEJA. *Let C denote a continuous and bounded curve in the u-plane if the variable u is complex, or an interval if the variable u is real. If for the polynomials*

$$P_n(u)=\sum_{k=0}^{n}a_{nk}u^k$$

we have

$$\sup_{n=1,2,\ldots}\|P_n(u)\|<\infty\quad \textit{for every}\quad u\in C,$$

then for $|\lambda|<1$ the sequence $\{\|\lambda^nP_n(u)\|\}$ is uniformly bounded on C.

Proof. The theorem is known if the polynomials under consideration are numerically valued (Leja [7], p. 520). Denote by η any linear functional in Y. By hypothesis

$$\sup_{n=1,2,\ldots}|\eta P_n(u)|<\infty\quad\text{for every}\quad u\in C,$$

whence

$$\sup_{n=1,2,\ldots}\sup_{u\in C}|\lambda^n\eta P_n(u)|=\sup_{\substack{n=1,2,\ldots\\ u\in C}}|\lambda^n\eta P_n(u)|<\infty$$

if $|\lambda|<1$ and λ is fixed. By a general principle of Functional Analysis [1]) there exists a constant K such that

$$\sup_{n=1,2,\ldots}\sup_{u\in C}\|\lambda^nP_n(u)\|<K<\infty.$$

We shall use the following consequence of Theorem (1.3):

(1.4) THEOREM OF LEJA. *Let the sequence*

(1.5) $$P_n(t_1,t_2)=\sum_{i+k=n}a_{ik}t_1^it_2^k$$

of homogeneous polynomials of real variables t_1,t_2 be bounded at every point (t_1,t_2) such that $|t_1|<\varepsilon_1$, $|t_2|<\varepsilon_2$. Then $0<\omega_i<\varepsilon_i$, $1<\alpha$, $\alpha\omega_i<\varepsilon_i$ implies

$$\sup_{n=1,2,\ldots}\sup_{|t_i|\leqslant\omega_i}\|\alpha^nP_n(t_1,t_2)\|<\infty.$$

Proof[2]). Let $\alpha\omega_i<\bar\varepsilon_i<\varepsilon_i$, then the polynomials

$$p_n(t_2)=\sum_{i+k=n}a_{ik}\bar\varepsilon_1^it_2^k$$

are bounded for $|t_2|\leqslant\bar\varepsilon_2$, whence by Theorem (1.3)

$$\sup_{n=1,2,\ldots}\sup_{|t_2|\leqslant\bar\varepsilon_2}\|\lambda^nP_n(\bar\varepsilon_1,t_2)\|<K<\infty.$$

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[1]) We apply the following theorem: *if $y(\omega)$ is a function from an abstract set Ω to Y and if for every linear functional η the inequality*

$$\sup_{\omega\in\Omega}|\eta y(\omega)|<\infty$$

is true, then $\sup_{\omega\in\Omega}\|y(\omega)\|<\infty$. To prove it, choose $\omega_n\in\Omega$ so that $\|y(\omega_n)\|\to\sup_{\omega\in\Omega}\|y(\omega)\|$ as $n\to\infty$. By hypothesis $\sup_n|\eta y(\omega_n)|<\infty$ for every η. Hence by a theorem of Banach ([3], p. 80) $\sup_n\|y(\omega_n)\|<\infty$. To apply the principle in our case consider $\lambda^nP_n(u)$ as a function defined in $N\times C=\Omega$, where N stands for the set of natural numbers.

[2]) Following Leja ([8], p. 3).

Similarly

$$\sup_{n=1,2,\ldots} \sup_{|t_1|\leqslant \bar{\varepsilon}_1} \|\lambda^n P_n(t_1, \bar{\varepsilon}_2)\| < L < \infty.$$

This implies

$$\sup_{n=1,2,\ldots} \sup_{|t_i|\leqslant \bar{\varepsilon}_i} \|\lambda^n P_n(t_1, t_2)\| \leqslant K + L.$$

Choosing μ so that $1 < \mu$, $a\omega_i < \mu\omega_i < \bar{\varepsilon}_i$, we get

$$\mu^n \sup_{|t_i|\leqslant \omega_i} \|\lambda^n P_n(t_1,t_2)\| = \sup_{|t_i|\leqslant \mu\omega_i} \|\lambda^n P_n(t_1,t_2)\| \leqslant \sup_{|t_i|\leqslant \bar{\varepsilon}_i} \|\lambda^n P_n(t_1,t_2)\|,$$

then choosing λ so that $a < \lambda\mu$ we obtain

$$\sup_{n=1,2,\ldots} \sup_{|t_i|\leqslant \omega_i} \|a^n P_n(t_1,t_2)\| \leqslant K + L.$$

(1.6) THEOREM OF LEJA. *Let the series*

$$\sum_{n=0}^{\infty} P_n(t_1, t_2)$$

composed of the polynomials (1.5) *converge for* $t_1 = a > 0$, $|t_2| < \beta$. *Then the series converges absolutely in the domain*[3]) D *bounded by the straight lines* $t_1 = \pm a$ *and the parabolas*

$$t_2 = \pm \frac{\beta}{2}\left(1 + \frac{t_1^2}{a^2}\right).$$

Proof. We suppose that the theorem is known for numerically valued polynomials (Leja [8], p. 2). If the polynomials are vector-valued, then for every linear functional η the series

$$\sum_{n=0}^{\infty} \eta P_n(t_1, t_2)$$

converges for $(t_1, t_2) \epsilon D$. Then for fixed $(t_1, t_2) \epsilon D$ choose $\lambda > 1$ so that $(\lambda t_1, \lambda t_2) \epsilon D$; the convergence $\eta P_n(\lambda t_1, \lambda t_2) \to 0$ for every η implies

$$\sup_{n=1,2,\ldots} \|P_n(\lambda t_1, \lambda t_2)\| < \infty,$$

whence

$$\sum_{n=0}^{\infty} \|P_n(t_1, t_2)\| < \infty.$$

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[3]) By *domain* we mean any open connected set.

(1.7) THEOREM. *Let the series*

$$\sum_{n=0}^{\infty} P_n(t_1, t_2)$$

of polynomials (1.5) *converge for* $|t_i| < \varepsilon_i$. *Then the double series*

$$\sum_{i,k=0}^{\infty} a_{ik} t_1^i t_2^k$$

converges absolutely for $(|t_1|/\varepsilon_1) + (|t_2|/\varepsilon_2) < 1/2$ [4]).

Proof. Let $0 < \bar{\varepsilon}_i < \varepsilon_i$; by Theorem (1.4) there is a constant K such that

$$\sup_{|t_i|<\bar{\varepsilon}_i} |P_n(t_1, t_2)| \leqslant K.$$

By a Theorem of Ree[5]) ([13], p. 575)

$$\|a_{ik}\| \leqslant \frac{2^n K}{\bar{\varepsilon}_1^i \bar{\varepsilon}_2^k}.$$

Hence

$$\sum_{i+k=n} \|a_{ik}\|\, |t_1|^i |t_2|^k \leqslant 2^n K \sum_{i+k=n} \left|\frac{t_1}{\bar{\varepsilon}_1}\right|^i \left|\frac{t_2}{\bar{\varepsilon}_2}\right|^k \leqslant 2^n K \left(\frac{|t_1|}{\bar{\varepsilon}_1} + \frac{|t_2|}{\bar{\varepsilon}_2}\right)^n,$$

and this implies the convergence of the double series for $(|t_1|/\varepsilon_1) + (|t_2|/\varepsilon_2) < 1/2$.

2. The space $X + iX$. Every real Banach space X may be imbedded isometrically into a complex Banach space (Taylor [14], p. 312). This space may be constructed as follows: the elements of it are the couples (x_1, x_2) of elements of X with the following definition of addition and multiplication by complex scalars:

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2),$$

$$(\lambda + i\mu)(x_1, x_2) = (\lambda x_1 - \mu x_2, \lambda x_2 + \mu x_1).$$

The norm is defined as[6])

$$\|(x_1, x_2)\| = \sup_{\|\xi\|\leqslant 1} \sqrt{\xi^2(x_1) + \xi^2(x_2)},$$

ξ denoting linear functionals over X.

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[4]) Using a deeper result of Leja ([9], p. 22), we might replace the constant 1/2 in this inequality by 1.

[5]) Ree has proved the theorem if a_{ik} are real; the extension of this result by use of linear functionals is similar to the proof of Theorem (1.1).

[6]) This is due to A. E. Taylor ([15], p. 665).

An isomorphical norm is

$$\|(x_1,x_2)\|'=\sup_{\vartheta}\left(\|x_1\cos\vartheta-x_2\sin\vartheta\|+\|x_1\sin\vartheta+x_2\cos\vartheta\|\right).$$

The norm $\|(x_1,x_2)\|$ is such that $\|(x_1,x_2)-(x_1^0,x_2^0)\|\to 0$ is equivalent to $x_1\to x_1^0, x_2\to x_2^0$.

The couple (x_1,x_2) will be written x_1+ix_2, and the space of these couples will be denoted by $X+iX$. The mapping $F(x)=x+i0$ imbeds isometrically X into $X+iX$. Every linear functional in X may be extended to a (complex) linear functional in $X+iX$ without increase of the norm; this is done by the formula $\xi(x+iy)=\xi(x)+i\xi(y)$.

3. *p*-analytic operations. Let X be a linear space, Y a Banach space, both real.

An operation $P(x_1,\dots,x_n)$ from $X\times\dots\times X$ (n-times) to Y is said to be *n-distributive* if it is distributive [7]) with respect to every variable separately. It is called *symmetric* if it does not change its value under an permutation of the variables. If $P(x_1,\dots,x_n)$ is an n-distributive operation, then the operation $P(x,\dots,x)$ is called the *p-power* (or *homogeneous operation*) *of degree n*. A *power* of degree n is a continuous p-power of degree n. For every p-power $P(x)$ of degree n there is a symmetric (uniquely determined) n-distributive operation $P(x_1,\dots,x_n)$ such that $P(x)=P(x,\dots,x)$; this operation is called the *generating* or *polar* operation. It is expressible by the formula

$$P(x_1,\dots,x_n)=\sum_{\varepsilon_1,\dots,\varepsilon_n=0}^{1}(-1)^{n-(\varepsilon_1+\dots+\varepsilon_n)}P(x_0+\varepsilon_1x_1+\dots+\varepsilon_nx_n),$$

with arbitrary x_0 (Mazur and Orlicz [11], p. 52-54).

The operation $P(x)$ is a p-power of degree n if $P(tx)=t^nP(x)$ for every t and if for every $x,h\in X$

$$P(x+th)=\sum_{k=0}^{n}a_k(x,h)t^k,$$

where $a_n(x,h)\in Y$ (Mazur and Orlicz [11], p. 56).

A set $D\subset X$ is said to be *finitely open* (Hille [5], p. 71) if the intersection of D with every finitely dimensional subspace X_1 of X is open relatively to X_1.

An operation $F(x)$ from a set $A\subset X$ to the space Y is said to have the *p-differential of order n at x* if

(i) the set A contains a finitely open set D containing x;

[7]) An operation U is *distributive* if it satisfies the equation $U(t_1x_1+t_2x_2)=t_1U(x_1)+t_2U(x_2)$. Since there are differences in the usual terminology (additive and homogeneous in the language of the Polish school, linear — adopted by the others), we have preferred to introduce a third term.

(ii) for every $h\in X$ there exists

$$\left[\frac{d^n}{dt^n}F(x+th)\right]_{t=0}=\delta^nF(x,h);$$

(iii) $\delta^nF(x,h)$ is a p-power of degree n in the variable h.

The operation $\delta^nF(x,h)$ is called the *pseudodifferential*, or shortly the *p-differential*, of order n of F with increment h. For $n=1$ the definition gives

$$\delta F(x,h)=\lim_{t\to 0}\frac{F(x+th)-F(x)}{t},$$

under the supplementary hypothesis that $\delta F(x,h)$ is distributive with respect to h. The derivative (ii) is always homogeneous of degree n in the increment, however it is not necessarily a p-power. This explains the hypothesis (iii).

We shall introduce also the *mixed p-differential* of order n. It is defined as

$$\left[\frac{\partial^n}{\partial t_n\dots\partial t_1}F(x+t_1h_1+\dots+t_nh_n)\right]_{t_i=0}=\delta^nF(x,h_1,\dots,h_n)$$

under the hypothesis that (i) is satisfied and

(ii′) the above derivative exists for every $h_1,\dots,h_n\in X$;

(iii′) $\delta^nF(x,h_1,\dots,h_n)$ is an n-distributive operation.

An operation $F(x)$ from a set $D\subset X$ to Y will be called *p-analytic* in the set D if

1° $F(x)$ is defined in D and D is finitely open;

2° $\delta^nF(x,h)$ exists for every $x\in D$;

3° for every $x\in D$ there exists a finitely open set H containing 0 such that $h\in H$ implies

$$F(x+h)=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^nF(x,h).$$

(3.1) Theorem. *Every p-analytic operation in D has the mixed p-derivatives of all orders and*

$$\delta^kF(x,h_1,\dots,h_k)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{\partial}{\partial t_k\dots\partial t_1}\delta^nF(x,t_1h_1+\dots+t_kh_k)\right]_{t_i=0}.$$

The series converges for every $h_1,\dots,h_k\in X$.

Proof. We shall prove the Theorem for $n=2$. There exist $\varepsilon_1>0$, $\varepsilon_2>0$ such that the series

$$\sum_{n=0}^{\infty}\frac{1}{n!}\delta^n F(x_0,t_1h_1+t_2h_2)=F(x_0+t_1h_1+t_2h_2) \tag{3.2}$$

converges for $|t_1|\leqslant\varepsilon_1$, $|t_2|\leqslant\varepsilon_2$. Set

$$P_n(t_1,t_2)=\frac{1}{n!}\delta^n F(x_0,t_1h_1+t_2h_2);$$

these are homogeneous polynomials of degree n in t_1,t_2 and the series

$$\sum_{n=0}^{\infty}P_n(t_1,t_2)$$

converges for $|t_1|<\varepsilon_1$, $|t_2|<\varepsilon_2$. Let $0<\omega_i<\varepsilon_i$,

$$M_n=\sup_{|t_i|\leqslant\omega_i}\|P_n(t_1,t_2)\|.$$

By Theorem (1.4) of Leja there exist an σ and K such that $0<\sigma<1$ and $M_n<K\sigma^n$, whence by the inequality of Markov (1.1)

$$\left\|\frac{\partial}{\partial t_1}P(t_1,t_2)\right\|\leqslant\frac{n^2}{2\omega_1}K\sigma^n \quad \text{for} \quad |t_i|\leqslant\omega_i.$$

Thus the series

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial t_1}P_n(t_1,t_2)$$

converges normally[8]) for $|t_i|\leqslant\omega_i$. Applying the inequality (1.1) again we get

$$\left\|\frac{\partial^2}{\partial t_2\,\partial t_1}P(t_1,t_2)\right\|\leqslant\frac{n^4}{4\omega_1\omega_2}K\sigma^n;$$

hence we may differentiate twice the series (3.2):

$$\frac{\partial^2}{\partial t_1\,\partial t_2}F(x_0,t_1h_1+t_2h_2)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^2}{\partial t_2\,\partial t_1}\delta^n F(x_0,t_1h_1+t_2h_2)$$

for $|t_i|\leqslant\omega_i$.

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[8]) The series $\sum_{n=0}^{\infty}\varphi_n(u)$ *converges normally in* A (or for $u\in A$) if

$$\sum_{n=0}^{\infty}\sup_{u\in A}\|\varphi_n(u)\|<\infty.$$

(3.3) THEOREM. *If the operation $F(x)$ is p-analytic in D, then for $x_0\in D$, $h_2\in X$ and h_1 belonging to a finitely open set H such that $0\in H$ we have*

$$\delta F(x_0+h_1,h_2)=\sum_{n=0}^{\infty}\frac{1}{n}\delta^{n+1}F(x_0,\underbrace{h_1,\ldots,h_1}_{n\text{ times}},h_2).$$

Proof. There exists a finitely open set H such that $0\in H$ and $x_0+th\in D$ if $h\in H$, $|t|\leqslant3$. Let $h_1\in H$. Using the same notation as in the proof of Theorem (3.1) we see that $\varepsilon_1\geqslant3$ and the series (3.2) converges for $|t_1|<\varepsilon_1$, $|t_2|<\varepsilon_2$. Write

$$P_n(t_1,t_2)=\sum_{i+k=n}^{\infty}a_{ik}t_1^it_2^k;$$

by Theorem (1.7) the double series

$$\sum_{i,k=0}^{\infty}\|a_{ik}\|t_1^it_2^k$$

converges for $|t_i|<\varepsilon_i/2$, hence we may rearrange its terms arbitrarily and write

$$F(x_0+t_1h_1+t_2h_2)=\sum_{i,k=0}^{\infty}a_{ik}t_1^it_2^k \tag{3.4}$$

for $|t_i|<\varepsilon_i/2$. We may also differentiate this series an arbitrary number of times term by term. Rewriting the series (3.4) as

$$F(x_0+t_1h_1+t_2h_2)=\sum_{i=0}^{\infty}\left(\sum_{k=0}^{\infty}a_{ik}t_2^k\right)t_1^i=\sum_{k=0}^{\infty}\left(\sum_{i=0}^{\infty}a_{ik}t_1^i\right)t_2^k$$

we get

$$\delta^n F(x_0+t_2h_2,\underbrace{h_1,\ldots,h_1}_{n})=n!\sum_{k=0}^{\infty}a_{nk}t_2^k,$$

$$\delta^{n+1}F(x_0,\underbrace{h_1,\ldots,h_1}_{n},h_2)=n!\,a_{n1},$$

whence for $t_1=1$

$$\delta F(x_0+h_1,h_2)=\left[\frac{\partial}{\partial t_2}F(x_0+h_1+t_2h_2)\right]_{t_2=0}=\sum_{i=0}^{\infty}a_{i1}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^{n+1}F(x_0,\underbrace{h_1,\ldots,h_1}_{n},h_2).$$

The formula in Theorem (3.3) may easily be generalized to read

$$\delta^k F(x_0+h,h_1,\ldots,h_k)=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^{n+k}F(x_0,\underbrace{h,\ldots,h}_{n},h_1,\ldots,h_k).$$

4. *p*-power series. A series

$$(4.1) \qquad \sum_{n=0}^{\infty} P_n(x)$$

where $P_n(x)$ is a p-power of degree n will be called the *p-power series*.

If the set H is such that $h \in H$, $|t|<1$ implies $th \in H$, then the set x_0+H [9]) is called a *star with centre* x_0 or simply a *star* (Hille [5], p. 71).

It is obvious that if x_0 is an element of a finitely open set D, then there exists a finitely open star $x_0+H \subset D$.

(4.2) THEOREM. *If the p-power series* (4.1) *converges in the star* x_0+H, *then it converges in every star*

$$tx_0+(1+t^2)/2H, \qquad -1<t<1.$$

Proof. Let $h \in H$ and write

$$Q_n(t_1,t_2)=P_n(t_1x_0+t_2h).$$

These are homogeneous polynomials of degree n in t_1,t_2 and the series

$$\sum_{n=0}^{\infty} Q_n(t_1,t_2)$$

converges for $t_1=1$, $|t_2|<1$. Hence by Theorem (1.6) of Leja, if $-1<t<1$, it converges for $t_1=t$, $|t_2|\leqslant(1+t^2)/2$, which implies the conclusion.

(4.3) COROLLARY. *If the p-power series* (4.1) *converges in a finitely open* [*open*] *set, it converges in a finitely open* [*open*] *set containing the element* 0.

A set A will be called a *set of Leja* or shortly an *L^*-set* if

(i) it is finitely open;

(ii) for every H, if the star $x_0+H \subset A$, then for every $|t|<1$ the set $tx_0+(1+t^2)/2H \subset A$.

An *L-set* is an open L^*-set. Every L^*-set is a star with centre 0. Since the union of every family of finitely open [open] sets is finitely open [open], for every p-power series there exists the greatest finitely open [open] set D in which the series converges. By Theorem (4.2) this set is an L^*-or L-set respectively.

Notice that the finite interior [10]) of the intersection of every family of L^*-sets is an L^*-set.

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[9]) We use the following notation: $A \cup B$ and $A \cap B$ denote set-theoretic union and intersection of sets. $A+B$ and AB denote the algebraic sum and product of the sets A and B, *i. e.* the set of the elements $a+b$ and ab respectively, with $a \in A$, $b \in B$. The set $\{x\}$ composed of the element x will be denoted simply by x. Hence $x+H$ is to be understood as $\{x\}+H$.

[10]) The finite interior of the set A is the interior of the set A in the topology in which the finitely open sets are called *open*.

(4.4) THEOREM. *If the p-power series* (4.1) *converges in a finitely open set* D, *its sum is a p-analytic operation in* D.

Proof. Denote by $F(x)$ the sum of the series (4.1). Let $x_0 \in D$. Finite opennes of the set D implies the existence of an $a>1$ such that $ax_0 \in D$, whence there exists a finitely open star $ax_0+H \subset D$. The set $a^{-1}H=H_1$ is a finitely open star with centre 0 and $a(x_0+H_1)\subset D$. Let h be an arbitrary element of H_1 and write

$$Q_n(t)=P_n(a(x_0+th));$$

these are polynomials of degree n of the variable t and

$$\lim_{n\to\infty} Q_n(t)=0 \quad \text{for} \quad |t|\leqslant 1,$$

whence by Theorem (1.3) of Leja, for every $|\mu|<1$,

$$\sup_{n=1,2,\ldots}\ \sup_{|t|\leqslant 1} \|\mu^n Q_n(t)\| \leqslant K(\mu)<\infty.$$

Choose μ_0 so that $0<\mu_0<1$, $\mu a_0=\lambda>1$; then

$$\sup_{n=1,2,\ldots}\ \sup_{|t|\leqslant 1} \|\lambda^n P_n(x_0+th)\| \leqslant K(\mu_0)=K,$$

i. e.

$$\|P_n(x_0+th)\|\leqslant K/\lambda^n$$

for $|t|\leqslant 1$, $n=1,2,\ldots$ The polynomial $P_n(x_0+th)$ may be written in the form

$$P_n(x_0+th)=\sum_{i+k=n} b_{ik}t^k.$$

Now consider polynomials of the complex variable ζ with values in the space $X+iX$ defined as

$$R_n(\zeta)=\sum_{i+k=n} b_{ik}\zeta^k.$$

The degree of R_n is $\leqslant n$, and $R_n(\zeta)=P_n(x_0+\zeta h)$ for real ζ. Now, $\|R_n(\zeta)\|\leqslant \leqslant K\lambda^{-n}$ for $\zeta \in(-1,+1)$, thus the inequality of Bernstein (1.2) yields for every complex ζ

$$\|R_n(\zeta)\|\leqslant K\left(\frac{a+b}{\lambda}\right)^n,$$

where a and b denote the semiaxes of the ellipse with foci ± 1, passing through the point ζ. We easily see that $a+b\leqslant \varrho+\sqrt{1+\varrho^2}<1+2\varrho$ if $|\zeta|\leqslant\varrho$. Then $0<\varrho\leqslant\varrho_0=(\lambda-1)/2$ implies $0<\sigma=(a+b)\lambda<1$, whence

$$\|R_n(\zeta)\|\leqslant K\sigma^n,$$

and it follows that the series

$$\sum_{n=0}^{\infty} R_n(\zeta)$$

converges uniformly and normally for $|\zeta|\leqslant\varrho_0$; the sum, $R(\zeta)$, of this series is a holomorphic function of ζ, with values in $X+iX$, moreover $R(\zeta)=F(x_0+\zeta h)$ for real ζ. Writing

$$R(\zeta)=\sum_{n=0}^{\infty} C_n\zeta^n$$

we have

$$C_n=\frac{1}{n!}\left\{\frac{d^n}{dt^n}R(\zeta)\right\}_{\zeta=0}=\frac{1}{n!}\left\{\frac{d^n}{dt^n}F(x_0+th)\right\}_{t=0}=\frac{1}{n!}\tilde{\delta}^n F(x_0,h).$$

Let us put off for a moment the proof that $\tilde{\delta}^n F(x_0,h)$ are p-differentials. We have

$$F(x_0+th)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\tilde{\delta}^n F(x_0,h) \quad \text{for} \quad |t|\leqslant\varrho_0,$$

and this gives

$$F(x_0+h)=\sum_{n=0}^{\infty}\frac{1}{n!}\tilde{\delta}^n F(x_0,h)$$

for $h\epsilon\varrho_0 H_1$. Now we prove that $\tilde{\delta}^n F(x_0,h)$ are p-differentials. Since for every $h\epsilon X$ there exists a $\lambda\neq 0$ such that $\lambda h\epsilon H_1$, the derivative

$$\left\{\frac{d^n}{dt^n}F(x_0+th)\right\}_{t=0}$$

exists for every $h\epsilon X$. Now write for $h_1,h_2\epsilon X$

$$W_n(t_1,t_2)=P_n(x_0+t_1h_1+t_2h_2);$$

there exist $\varepsilon_i>0$ such that $|t|\leqslant\varepsilon$ implies $t_1h_1+t_2h_2\epsilon\varrho_0 H_1$. Since the series

$$\sum_{n=0}^{\infty} W_n(t_1,t_2)=F(x_0+t_1h_1+t_2h_2)$$

converges for $|t_i|\leqslant\varepsilon_i$, we see by Theorem (1.4) of Leja that for $0<\bar{\varepsilon}_i<\varepsilon_i$ there is a γ such that $0<\gamma<1$ and

$$\|W_n(t_1,t_2)\|\leqslant L\gamma^n \quad \text{for} \quad |t_i|\leqslant\bar{\varepsilon}_i.$$

Now write

$$W_n(t_1,t_2)=\sum_{i+k=n} d_{ik}t_1^it_2^k.$$

By Theorem (1.7) the double series

$$\sum_{i,k=0}^{\infty} d_{ik}t_1^it_2^k=F(x_0+t_1h_1+t_2h_2)$$

converges absolutely for $|t_i|\leqslant\bar{\varepsilon}_i/4$. For these values of t_1,t_2 the function has continuous partial derivatives of all orders. Thus for $|t|<1$, $|u|<1$

$$\left\{\frac{d^n}{dt^n}P\big(x_0+t(t_1h_1+ut_2h_2)\big)\right\}_{t=0}$$
$$=\sum_{k=0}^{n}\binom{n}{k}\left\{\frac{\partial^n}{\partial t_2^k\partial t_1^{n-k}}P\big(x_0+t(t_1h_1+ut_2h_2)\big)\right\}_{t=0}u^k=\sum_{k=0}^{n}g_k(t_1,t_2)u^k,$$

hence for $t_1=t_2=\vartheta\neq 0$

$$\tilde{\delta}^n F(x_0,h_1+uh_2)=\vartheta^{-n}\sum_{k=0}^{n}g_k(\vartheta,\vartheta)u^k,$$

and this shows that $\tilde{\delta}_n F(x_0,h)$ is a p-power of degree n in the increment.

A set Γ of linear functionals in Y is said to be *fundamental* if there are two constants $C>0$ and $c>0$ such that for every $y\epsilon Y$

$$\sup_{\|\eta\|\leqslant C,\eta\epsilon\Gamma}\eta y\geqslant c\|y\|.$$

A set Γ of linear functionals η satisfying the condition

(n) if $y_n\epsilon Y$ and $\sup_{n=1,2,\dots}|\eta y_n|<\infty$ for $\eta\epsilon\Gamma$, then $\sup_{n=1,2,\dots}\|y_n\|<\infty$,

is obviously fundamental; such a set will be called *strictly fundamental*.

(4.5) THEOREM. *If the p-power series (4.1) is such that for every functional η belonging to a strictly fundamental set Γ the numerical series*

$$\sum_{n=0}^{\infty}\eta P_n(x)$$

converges in a finitely open set D, then the series (4.1) converges (strongly) in D.

Proof. Let $x\epsilon D$, then there exists a $\lambda>1$ such that $\lambda x\epsilon D$. Since

$$\sup_{n=1,2,\dots}|\eta P_n(\lambda h)|<\infty \quad \text{for every} \quad \eta\epsilon\Gamma,$$

we have

$$\sup_{n=1,2,\dots}\|P_n(\lambda x)\|<\infty,$$

whence the series

$$\sum_{n=0}^{\infty}P_n(tx)=\sum_{n=0}^{\infty}t^nP_n(x)$$

converges for $|t|<\lambda$.

The Theorem (4.5) may be strengthened as follows:

(4.6) THEOREM. *Let $F(x)$ be an operation defined in a finitely open set D. If for every linear functional belonging to a strictly fundamental set Γ the functional $\eta F(x)$ may be developed in a p-power series converging in D, then there are p-powers of degree n $P_n(x)$ such that*

(4.7) $$F(x)=\sum_{n=0}^{\infty}P_n(x),$$

the series being convergent in D.

Proof. For every $\eta\epsilon\Gamma$ denote by D_η the set in which the functional $\eta F(x)$ may be developed into a convergent p-power series. Then the finite interior A of the intersection $\bigcap_\eta D_\eta$ is finitely open and non vacuous because it contains the set D. We shall prove that $F(x)$ may be represented in A in the form (4.7). Let $x_0\epsilon A$, $h\epsilon X$, then $x_0+th\epsilon A$ for $|t|<\delta$; hence there exists

$$\frac{d^n}{dt^n}\eta F(x_0+th)\quad \text{for}\quad |t|<\delta,\ \ \eta\epsilon\Gamma,$$

and is a continuous functional of t by Theorem (4.4). By a theorem of the authors ([1], p. 109)

$$\frac{d^n}{dt^n}F(x_0+th)$$

exist for $|t|<\delta$. Hence $\delta^n(x_0,h)$ exists for every $x\epsilon A$, $h\epsilon X$, moreover

$$\eta\delta^n F(x_0,h)=\delta^n\eta F(x_0,h).$$

Now write

$$P_n(x)=\frac{1}{n!}\,\delta^n F(0,h),$$

then $\eta\epsilon\Gamma$, $x\epsilon A$ implies

$$\eta F(x)=\sum_{n=0}^{\infty}\delta^n\eta F(0,x)=\eta\sum_{n=0}^{\infty}\delta^n F(0,x)=\eta\sum_{n=0}^{\infty}P_n(x).$$

We now apply Theorem (4.5).

5. The extension principle. A. E. Taylor ([14], p. 313) has shown that every p-power $P(x)$ of degree n from X to Y may be extended to a p-power $P^*(x_1+ix_2)$ of degree n from $X+iX$ to $Y+iY$ in such a manner that $P^*(x_1+i0)=P(x_1)$. This may be done only in one fashion: denote by $P(x_1,\ldots,x_n)$ the generating operation of P and set

$$P^*(x_1+ix_2)=\sum_{k=0}^{\infty}i^{n-k}\binom{n}{k}P(\underbrace{x_1,\ldots,x_1}_{k},\underbrace{x_2,\ldots,x_2}_{n-k})$$
$$=\sum_{k=0}^{En/2}(-1)^k\binom{n}{2k}P(x_1,\ldots,x_1,\underbrace{x_2,\ldots,x_2}_{2k})$$
$$+i\sum_{k=1}^{E(n-1)/2}(-1)^k\binom{n}{2k-1}P(x_1,\ldots,x_1,\underbrace{x_2,\ldots,x_2}_{2k-1});$$

it is easy to show that this operation has the desired properties. Taylor ([14], p. 313) has given an estimation of $\|P^*(x_1+ix_2)\|$ — valid for powers (not p-powers) under the supplementary hypothesis that X and Y are (abstract) Hilbert spaces. We shall need other estimations valid for the p-powers.

The function $f(\zeta)=P^*(x_1+\zeta x_2)$ is a polynomial of degree n in ζ and $P^*(x_1+tx_2)=P(x_1+tx_2)$ for real t. Hence, by Bernstein's inequality (1.2),

$$\|P^*(x_1+\zeta x_2)\|\leqslant(a+b)^n\sup_{|t|\leqslant 1}\|P(x_1+tx_2)\|.$$

For $\zeta=i\varrho$, $\varrho<0$, this gives

(5.1) $$\|P^*(x_1+i\varrho x_2)\|\leqslant(\varrho+\sqrt{1+\varrho^2})^n\sup_{|t|\leqslant 1}\|P(x_1+tx_2)\|,$$

in particular for $\varrho=1$

$$\|P^*(x_1+ix_2)\|\leqslant(1+\sqrt{2})^n\sup_{|t|\leqslant 1}\|P(x_1+tx_2)\|.$$

If X is an (abstract) Hilbert space and P is a power of degree n, a more precise estimation than (5.1) is possible. By a theorem of Banach ([3], p. 42)[11])

$$\sup_{\|x_i\|\leqslant 1}\|P(x_1,\ldots,x_n)\|=\sup_{\|x\|\leqslant 1}\|P(x)\|=\|P\|,$$

whence

$$\|P(\underbrace{x_1,\ldots,x_1}_{k},x_2,\ldots,x_2)\|\leqslant\|P\|\,\|x_1\|^k\|x_2\|^{n-k},$$

$$\|P^*(x_1+ix_2)\|\leqslant\sum_{k=0}^{n}\|P\|\binom{n}{k}\|x_1\|^k\|x_2\|^{n-k}=(\|x_1\|+\|x_2\|)^n\|P\|.$$

[11]) Banach supposes that X and Y are the Hilbert spaces L^2. An inspection of the proof shows immediately that it suffices to suppose that X is an abstract Hilbert space.

The inequality (5.1) gives in this case only

$$\|P^*(x_1+ix_2)\| \leqslant (1+\sqrt{2})^n(\|x_1\|+\|x_2\|)^n\|P\|.$$

Before going further we need some theorems on the formation of finitely open sets in the space $X+iX$. Let A be any set in X. Denote by $\mathfrak{Z}_a(A)$ the set of the elements x_1+ix_2 of $X+iX$ satisfying the following condition:

$$\text{if } |\vartheta|\leqslant a, \text{ then } x_1+\vartheta x_2 \in A.$$

(5.2) THEOREM. *If the set A is finitely open, then so is the set $\mathfrak{Z}_a(A)$ in $X+iX$, and if $A\neq 0$, then $\mathfrak{Z}_a(A)\neq 0$.*

Proof. Let $z_0=x_1^0+ix_2^0\in\mathfrak{Z}_a(A)$, $h_k=p_k+iq_k\in X+iX$, $\zeta_k=a_k+i\beta_k$. Then

$$z_0+\zeta_1h_1+\ldots+\zeta_nh_n = x_1^0+\sum_{k=1}^n(a_kp_k-\beta_kq_k)+i[x_2^0+\sum_{k=1}^n(a_kq_k+\beta_kp_k)].$$

By the definition of the set $\mathfrak{Z}_a(A)$, $|\vartheta|\leqslant a$ implies $x_1^0+\vartheta x_2^0\in A$ and finite openness of the set A implies further that for every ϑ satisfying the above inequality there is an $r(\vartheta)>0$ such that

$$x_1^0+\vartheta x_2^0+\alpha x_1^0+\beta x_2^0+\sum_{k=1}^n[(a_kp_k-\beta_kq_k)+(a_k'q_k+\beta_k'p_k)]\in A$$

if $|\alpha|,|\beta|,|a_k|,|\beta_k|,|a_k'|,|\beta_k'|$ are not greater that $r(\vartheta)$; if moreover

$$|\vartheta-\vartheta'|<r(\vartheta),$$

then

$$x_1^0+\vartheta' x_2^0+\sum_{k=1}^n[(a_kp_k-\beta_kq_k)+(a_k'q_k+\beta_k'p_k)]\in A.$$

We cover every $\vartheta\in\langle -a,a\rangle$ by the interval $(\vartheta-r(\vartheta),\vartheta+r(\vartheta))=I(\vartheta)$. By the Heine-Borel theorem there exist $\vartheta_1,\ldots,\vartheta_m\in\langle -1,1\rangle$ such that

$$\langle -1,1\rangle\subset\bigcup_{j=1}^m I(\vartheta_j).$$

Set

$$r=\min_{j=1,2,\ldots,m} 2^{-1}r(\vartheta_j);$$

it is easily seen that $|\vartheta|\leqslant a$, $|a_k|,|a_k'|,|\beta_k|,|\beta_k'|<r$ implies

$$x_1^0+\vartheta x_2^0+\sum_{k=1}^n[(a_kp_k-\beta_kq_k)+(a_k'q_k+\beta_k'p_k)]\in A,$$

and, *a fortiori*, if $|a_k+i\beta_k|<r[\max(1,a)]^{-1}=r_1$, then

$$x_1^0+\sum_{k=1}^n(a_kp_k-\beta_kq_k)+\vartheta[x_2^0+\sum_{k=1}^n(a_kq_k+\beta_kp_k)]\in A,$$

whence $|\zeta_k|<r_1$ implies

$$z_0+\zeta_1h_1+\ldots+\zeta_nh_n\in\mathfrak{Z}_a(A).$$

Thus the set $\mathfrak{Z}_a(A)$ is finitely open. The second statement of the theorem is obvious.

(5.3) REMARKS. A simple proof shows that *if the set D is open, then the set $\mathfrak{Z}_a(D)$ is also open, and if $D\neq 0$ — non empty.*

If the set D is open [finitely open] in $X+iX$, then the set $X\cap D$ is so in X.

(5.4) THEOREM. *Let the p-power series (4.1) converge in a finitely open set D and let $x_0\in D$. Then there exists a finitely open star H with centre 0 and an $a>0$ such that the extended series*

$$\sum_{n=0}^\infty P_n^*(z) \qquad (z=x_1+ix_2) \tag{5.5}$$

converges in the set $x_0+\mathfrak{Z}_a(H)$.

Proof. Finite openness of the set D implies the existence of a $\lambda>1$ such that $\lambda x_0\in D$; then there exists a finitely open star H_1 with centre 0 such that $\lambda x_0+H_1\subset D$. Obviously

$$\lambda(x_0+\lambda^{-1}H_1)\subset D,$$

and the set $H=\lambda^{-1}H_1$ is a finitely open star with centre 0. Choose first μ and then a so that

$$0<\mu<1,\qquad \mu\lambda>1,\qquad 0<a,\qquad \frac{1+2a}{\mu\lambda}=b<1.$$

Let $z\in\mathfrak{Z}_a(H)$, then $z=x_1+iax_2$, and x_2 is such that $|\vartheta|\leqslant 1$ implies $x_1+\vartheta x_2\in H$, and $\lambda(x_0+x_1+\vartheta x_2)\in\lambda x_0+\lambda H=\lambda x_0+H_1$. Write

$$Q_n(\vartheta)=P_n(\lambda(x_0+x_1+\vartheta x_2))=\lambda^nP_n(x_0+x_1+\vartheta x_2);$$

these are polynomials of degree $\leqslant n$ in ϑ, and $\lambda x_0+H_1\subset H$ implies that

$$\lim_{n\to\infty}Q_n(\vartheta)=0 \quad\text{ for } |\vartheta|\leqslant 1.$$

Hence by Theorem (1.3) of Leja there is an A such that

$$\sup_{|\vartheta|\leqslant 1}\|Q_n(\vartheta)\|\leqslant\frac{A}{\mu^n},$$

i. e.

$$\sup_{|\vartheta|\leqslant 1}\|P_n(x_0+x_1+\vartheta x_2)\|\leqslant\frac{A}{(\mu\lambda)^n}.$$

By the inequality (5.2)

$$\|P_n^*(z)\| = \|P_n^*(x_0+x_1+ix_2)\| \leqslant (a+\sqrt{1+a^2})^n \frac{A}{(\mu\lambda)^n} < \left(\frac{1+2a}{\mu\lambda}\right)^n A = Ab^n,$$

and this implies

$$\sum_{n=0}^{\infty} \|P_n^*(z)\| < \infty.$$

(5.6) REMARK. If the set D in Theorem (5.5) is open, the set H may be supposed to be open too. Thus under the hypotheses of Theorem (5.5) *if the set D is open, there exists an open set $H\epsilon 0$ such that the series converges in the open set $x_0+\mathfrak{Z}_a(H)$.*

If the set D is finitely open, so is the set $x_0+\mathfrak{Z}_a(H)$ — if D is open, the set $x_0+\mathfrak{Z}_a(H)$ is open. By Theorem (5.5) for every x there is an a_x and H_x such that the extended series converges in $x+\mathfrak{Z}_{a_x}(H_x)$. Hence the extended series converges in the set

$$\bigcup_{x\epsilon D} [x+\mathfrak{Z}_{a_x}(H_x)],$$

which is finitely open or open in $X+iX$ according to the set D. Hence

(5.7) THEOREM. *Let the p-power series converge in a finitely open [open] set D, then there exists a finitely open [open] set Z in $X+iX$ such that $D\subset Z$, and the extended series* (5.5) *converges in Z.*

Theorem (5.7) implies that if the operation $F(x)$ is p-analytic in the finitely open set D, then for every $x\epsilon D$ there exists a p-analytic operation $F^*(z)$ defined in a finitely open set in $X+iX$ containing the point x such that $F^*(x+i0)=F(x)$.

We are now able to transfer the theorems dealing with analytic operations in complex spaces to the case of real ones. Applying a theorem of Hille ([5], p. 75) we get the

(5.8) THEOREM. *Let the p-power series* (4.1) *converge in a finitely open set D. Then the series converges locally normally in D, i. e. for every $x\epsilon D$ there is a finitely open set $H\subset D$ such that $x\epsilon H$ and*

$$\sum_{n=0}^{\infty} \sup_{x\epsilon H} \|P_n(x)\| < \infty .$$

(5.9) REMARK. Theorem (5.8) implies: *if the operation $F(x)$ is p-analytic in D, then for every $x\epsilon D$ there exists a star H with centre 0 such that*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sup_{h\epsilon H} \|\delta^n F(x,h)\| < \infty.$$

By inspection of the proof of Theorem (5.5) we obtain the following result. If the p-power series converges in the sphere $S=S(0,r)$ then $S(0,r/2)\subset \mathfrak{Z}_1(S)$. Thus for $x_1+ix_2\epsilon [2(1+\sqrt{2})]^{-1} S(0,r/2)$

$$\|P_n^*(x_1+ix_2)\| \leqslant \left(\frac{1}{2(1+\sqrt{2})}\right)^n \sup_{|\vartheta|\leqslant 1} \|P_n(x_1+\vartheta x_2)\|.$$

Hence *if the p-power series converges for $\|x\|\leqslant r$, the extended series converges for $\|x_1+ix_2\|\leqslant r[2(1+\sqrt{2})]^{-1}$.*

6. Analytic operations. We now suppose that X is a Banach space. The operation $F(x)$ is said to be (Fréchet-) *differentiable at* x_0 if it is p-differentiable at x_0, the p-differential $\delta F(x_0,h)$ is continuous with respect to h, and

$$\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)-\delta F(x_0,h)}{\|h\|} = 0;$$

the existence of the last limit is equivalent to the existence of the limit

$$\lim_{t\to 0} \frac{F(x_0+th)-F(x_0)}{t} = \delta F(x_0,h)$$

uniformly in the sphere $\|h\|\leqslant 1$. In this case $\delta F(x_0,h)$ is called the *Fréchet-differential* or shortly the *differential* and will be writen $dF(x_0,h)$. Similarly we define the differentials of n-th order $d^nF(x_0,h)$ and the mixed differentials $d^nF(x_0,h_1,\ldots,h_n)$. Every differential $d^nF(x,h)$ is a power of degree n in the increment h. Every operation which is differentiable at x_0 is continuous there.

An operation $F(x)$ is said to be *analytic in* D if

1° $F(x)$ is defined in D and the set D is open;

2° the differentials $d^nF(x,h)$ exist at every point of D;

3° for every $x\epsilon D$ there is a $\delta>0$ such that

$$F(x+h) = \sum_{n=0}^{\infty} \frac{1}{n!} d^nF(x,h) \qquad \text{for} \quad \|h\|<\delta.$$

A *power series* is a series

$$\sum_{n=0}^{\infty} P_n(x), \tag{6.1}$$

where P_n is a power of degree n.

(6.2) THEOREM. *If the power series* (6.1) *converges in an open set D, its sum is an analytic operation in D and for every $x_0\epsilon D$ there is an open set H such that $x_0\epsilon H\subset D$ and*

$$\sum_{n=0}^{\infty} \sup_{x\epsilon H} \|P_n(x)\| < \infty.$$

Proof. By Theorem (5.7) the extended series converges in an open set Z in $X+iX$, such that $D\subset Z$. Hence by a theorem of Hille ([5], p. 87) for $x_0 \epsilon D$ there is an open set K such that $x_0 \epsilon K$ and

$$\sum_{n=0}^{\infty} \sup_{z\epsilon K} \|P_n^*(z)\| < \infty;$$

this involves the second part of the Theorem, if we set $H=K\cap D$. To prove the first part it suffices to notice that for every $x\epsilon D$ the extended series converges in a neighbourhood of x and we may apply the theorem dealing with the complex case (Hille [5], p. 87).

(6.3) REMARK. Theorem (6.2) implies that *if a power series converges in an open set D, then every compact set $C\subset D$ may be covered by an open set H on which the series converges normally* (and hence uniformly).

(6.4) THEOREM. *Every analytic operation in D has all mixed derivatives in D.*

Proof. The Theorem is valid in the complex case (Hille [5], p. 82).

Now we shall prove some sufficient (and trivially also necessary) conditions that an operation be analytic. Theorem (4.6) easily implies

(6.5) THEOREM. *Let the operation $F(x)$ defined in an open set D be such that for every functional η belonging to a strictly fundamental set the functional $\eta F(x)$ is expressible as the sum of a power series convergent in D; then there are powers P_n such that*

$$F(x)=\sum_{n=0}^{\infty} P_n(x),$$

and the series converges in D.

An operation $F(x)$ is said to be *weakly continuous* with respect to the set Γ of functionals if for every $\eta\epsilon\Gamma$ the functional $\eta F(x)$ is continuous.

(6.6) THEOREM. *Every operation $F(x)$ which is p-analytic in an open set, and weakly continuous with respect to a strictly fundamental set of functionals Γ, is analytic in D.*

Proof. Let $\eta\epsilon\Gamma$, then if $t_p\to 0$

(6.7) $$\eta\delta^n F(x,h)=\delta^n \eta F(x,h)=\lim_{p\to\infty}\frac{1}{t_p}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\eta F(x+kt_p h),$$

hence $\eta\delta^n F(x,h)$ as a functional of h is of Baire's first class, hence bounded for $\|h-h_0\|<r>0$. By a theorem of Mazur and Orlicz ([12], p. 182) $\eta\delta^n F(x,h)$ is continuous in h, hence a power of degree n. Since $\eta\epsilon\Gamma$ may be chosen arbitrarily, $\delta^n F(x,h)$ must be bounded in a sphere — and this implies again, by the theorem of Mazur and Orlicz, that $\delta^n F(x,h)$ is a power. By hypothesis

$$F(x+h)=\sum_{n=0}^{\infty}\frac{1}{n!}\delta^n F(x,h),$$

and by Theorem (5.8) for every point x there is a finitely open set K such that

$$\sum_{n=0}^{\infty}\sup_{h\epsilon K}\frac{1}{n!}\|\delta^n F(x,h)\|<\infty;$$

continuity in h of $\delta^n F(x,h)$ implies

$$\sum_{n=0}^{\infty}\sup_{h\epsilon \overline{K}}\frac{1}{n!}\|\delta^n F(x,h)\|<\infty$$ [12].

Since every finitely open set is of the second category, $\overline{K}$ contains a sphere

$$\overline{K}\supset x_1+S,$$

where $S=S(0,r)$.

Then by Theorem (4.2) $S(0,r/2)\subset K$, hence the series converges for h in a sphere with centre 0.

The next theorem presents a slight generalization of a theorem proved by Zorn [16] in the complex case. An operation $F(x)$ *satisfies the condition of Baire* in the set G if the set G is open and $F(x)$ is continuous in the set $G-N$ where N is of the first category.

(6.8) THEOREM. *Let the operation $F(x)$ be p-analytic in an open set D and suppose that for every linear functional η belonging to a strictly fundamental set Γ the functional $\eta F(x)$ satisfies the condition of Baire in D. Then the operation is analytic in D.*

Proof. Let $\eta\epsilon\Gamma$. Formula (6.7) shows that $\eta\delta^n F(x,h)$ is the limit of functionals satisfying the condition of Baire, hence it satisfies this condition too (Kuratowski [6], p. 308). By a theorem of Mazur and Orlicz ([12], p. 182) $\eta\delta^n F(x,h)$ is continuous in h. We can now repeat the proof of Theorem (6.6).

References

[1] A. Alexiewicz and W. Orlicz, *On analytic vector valued functions of a real variable,* Studia Mathematica 12 (1951), p. 108-111.

[2] S. Banach, *Théorie des opérations linéaires,* Monografie Matematyczne, Warszawa 1932.

[3] — *Über homogene Polynome in (L^2),* Studia Mathematica 7 (1937), p. 36-44.

[12]) In the formula $\overline{K}$ denotes the closure of the set K.

[4] S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes*, Mémoires de l'Académie Royale de Belgique, Classe des Sciences 4 (1912), p. 13-15.

[5] E. Hille, *Functional Analysis and Semigroups*, American Mathematical Society Colloquium Publications 1948.

[6] K. Kuratowski, *Topologie I*, II^e éd., Monografie Matematyczne, Warszawa 1948.

[7] F. Leja, *Sur les suites des polynômes bornées presque partout sur la frontière d'un domaine*, Mathematische Annalen 108 (1933), p. 517-524.

[8] — *Sur les séries des polynômes homogènes bornées sur un segment rectiligne*, Rendiconti del Circolo Matematico di Palermo 58 (1934), p. 1-7.

[9] — *Remarques sur les séries entières doubles*, Annales de la Société Polonaise de Mathématique 24 (1951), p. 19-24.

[10] A. A. Markov, *Об одном вопросе Д. М. Менделеева*, Известия Академии Наук 62 (1889), p. 1-24.

[11] S. Mazur und W. Orlicz, *Grundlegende Eigenschaften der polynomischen Operationem, Erste Mitteilung*, Studia Mathematica 5 (1935), p. 50-68.

[12] — *Grundlegende Eigenschaften der polynomischen Operationem, Zweite Mitteilung*, ibidem, p. 179-189.

[13] R. Ree, *On a problem of Max Zorn*, Bulletin of the American Mathematical Society 55 (1949), p. 575-576.

[14] A. E. Taylor, *Additions to the theory of polynomials in normed linear spaces*, Tohôku Mathematical Journal 44 (1938), p. 302-318.

[15] — *Analysis in complex Banach spaces*, Bulletin of the American Mathematical Society 49 (1943), p. 652-669.

[16] M. Zorn, *Gateaux differentiability and essential boundedness*, Duke Mathematical Journal 12 (1945), p. 579-583.

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On a theorem of Saks for abstract polynomials

by

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The purpose of this Note[1]) is to generalize a theorem due to A. Alexiewicz ([1], Theorem 1) dealing with the structure of linear operations depending on a parameter. Because of the close connection of this Note with paper [1] I use throughout the definitions adopted there.

In particular $(T, \mathfrak{E}, \mu)$ denotes a measure space, on which μ is σ-additive and $\mu(T)<\infty$. $U(x,t)$ will stand for an operation from $X\times T$ to an F-space Y, with the following properties:

1. $U(x,t)$ is Bochner measurable for fixed x;
2. $U(x,t)$ is a polynomial of degree m in x for fixed t;
3. $x\to x_0$ implies $U(x,t)\overset{as}{\to} U(x_0,t)$

(unlike [1]).

For a set $R\subset Y$ we shall denote by $\Theta(x)$ the set of those elements t for which $U(x,t)\in R$.

It is easily seen that Lemmas 1 and 2 of [1] still hold in the case considered now.

The following Lemma will be proved in place of Lemma 3:

LEMMA. *Let the set $R\subset Y$ be linear and measurable (B). If the set*

$$W = \mathop{E}_{x}\{\mu(H-\Theta(x))<\varepsilon\}$$

is of the second category, then the set

$$V = \mathop{E}_{x}\{\mu(H-\Theta(x))<(m+1)\varepsilon\}$$

is residual.

Proof. It is known ([2], p. 50-51) that we can represent the polynomial operation $U(x,t)$ in the canonical form

$$U(x,t)=U_0(x,t)+U_1(x,t)+\ldots+U_m(x,t), \tag{1}$$

[1]) I am indebted to Professor A. Alexiewicz for his help in the preparation of this Note.