

## ANALYTIC PROPERTIES OF THE $q$ -VOLKENBORN INTEGRAL ON THE RING OF $p$ -ADIC INTEGERS

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ABSTRACT. In this paper, we consider the  $q$ -Volkenborn integral of uniformly differentiable functions on the  $p$ -adic integer ring. By using this integral, we obtain the generating functions of twisted  $q$ -generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

### 1. Introduction

Let  $p$  be an odd prime.  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$  and  $\mathbb{C}_p$  will always denote, respectively, the ring of  $p$ -adic integers, the field of  $p$ -adic numbers and the completion of the algebraic closure of  $\mathbb{Q}_p$ . Let  $v_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{\infty\}$  ( $\mathbb{Q}$  the field of rational numbers) will denote the  $p$ -adic valuation of  $\mathbb{C}_p$  normalized so that  $v_p(p) = 1$ . The absolute value on  $\mathbb{C}_p$  will be denoted as  $|\cdot|_p$ , and  $|x|_p = p^{-v_p(x)}$  for  $x \in \mathbb{C}_p$ . We let  $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid 1/x \in \mathbb{Z}_p\}$ . A  $p$ -adic integer in  $\mathbb{Z}_p^\times$  is sometimes called a  $p$ -adic unit. Let  $UD(\mathbb{Z}_p, \mathbb{C}_p)$  denote the space of all uniformly (or strictly) differentiable  $\mathbb{C}_p$ -valued functions on  $\mathbb{Z}_p$ . For each integer  $N \geq 0$ ,  $C_{p^N}$  will denote the multiplicative group of the primitive  $p^N$ -th roots of unity in  $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$ . Set  $\mathbf{T}_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^N} = 1 \text{ for some } N \geq 0\} = \bigcup_{N \geq 0} C_{p^N}$ . The dual of  $\mathbb{Z}_p$ , in the sense of  $p$ -adic Pontrjagin duality, is  $\mathbf{T}_p = C_{p^\infty}$ , the direct limit (under inclusion) of cyclic groups  $C_{p^N}$  of order  $p^N$  ( $N \geq 0$ ), with the discrete topology.  $\mathbf{T}_p$  admits a natural  $\mathbb{Z}_p$ -module structure which we shall write exponentially, viz  $\omega^x$  for  $\omega \in \mathbf{T}_p$  and  $x \in \mathbb{Z}_p$ .  $\mathbf{T}_p$  can be embedded discretely in  $\mathbb{C}_p$  as the multiplicative  $p$ -torsion subgroup and we now choose, for once and all, one such embedding. If  $\omega \in \mathbf{T}_p$ , then we denote by

$$(1.1) \quad \phi_\omega : (\mathbb{Z}, +) \longrightarrow (\mathbb{C}_p^\times, \cdot)$$

for the locally constant character  $x \mapsto \omega^x$ , which is locally analytic character if  $\omega \in \{\omega \in \mathbb{C}_p \mid v_p(\omega - 1) > 0\}$ . Then  $\phi_\omega$  has continuation to a continuous group homomorphism from  $(\mathbb{Z}_p, +)$  to  $(\mathbb{C}_p^\times, \cdot)$  (see [24]).

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The indefinite sum operator  $S$  is defined by  $Sf(0) = 0$  and  $Sf(n) = \sum_{i=0}^{n-1} f(i)$  for  $n \geq 1$ . It is well known that for  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ , its Volkenborn integral is defined to be the limit of average

$$\frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = \frac{Sf(p^N) - Sf(0)}{p^N}$$

as  $N \rightarrow \infty$  (see [2], [19], [22]). We see that the uniform differentiability guarantees the existence of limits. Write down this integral as

$$(1.2) \quad I_0(f) = \int_{\mathbb{Z}_p} f(x) dx = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = (Sf)'(0) \in \mathbb{Q}_p$$

(see [7] and [19] for more details). For  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ , the  $I_0$ -Fourier transform of  $f$  is the function  $\widehat{f}: \mathbf{T}_p \rightarrow \mathbb{C}_p$  defined by

$$(1.3) \quad \widehat{f}_\omega = I_0(f\phi_\omega) \quad \text{for all } \omega \in \mathbf{T}_p.$$

This can be found in [24, Definition 3.1]. In fact we can extend  $\widehat{f}$  to  $\{\alpha \in \mathbb{C}_p \mid v_p(\alpha - 1) > 0\}$  by same formula, where it turns out to be an analytic function whose Taylor expansion has logarithmic growth (see [24, Proposition 5.3]). The analogue with the classical (complex) theory is substantially complicated by the absence of a  $p$ -adic valued Haar measure on  $\mathbb{Z}_p$ . So, C. F. Woodcock had to do various attempts to construct a satisfactory analogue of integral analysis for the spaces of functions on  $\mathbb{Z}_p$ . One, in [23] and [24], can find the fully detailed study of the  $I_0$ -Fourier transform on the space of all uniformly differentiable functions  $f: \mathbb{Z}_p \rightarrow \mathbb{C}_p$ .

The  $q$ -extension of the  $I_0$ -Fourier transform has been constructed by T. Kim [11], called the  $I_q$ -Fourier transform. The  $I_q$ -Fourier transform has the form

$$(1.4) \quad (\widehat{f}_\omega)_q = I_q(f\phi_\omega) \quad \text{for all } \omega \in \mathbf{T}_p,$$

see Section 2, Eq. (2.1).

For any integer  $n \geq 1$ , we shall use the following standard notation

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0.$$

As  $q \rightarrow 1$ ,  $[n]_q \rightarrow n$ , and this is the hallmark of a  $q$ -extension: the limit as  $q \rightarrow 1$  recovers the classical object. If  $q \in \mathbb{C}$ , one normally assume that  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , then we assume that  $|q - 1|_p < p^{-1/(p-1)}$ , so that  $q^x = \exp(x \log q)$  for  $x \in \mathbb{Z}_p$ . For  $N \geq 1$ , the  $q$ -extension  $\mu_q$  (originally introduced by T. Kim [6]) of the  $p$ -adic Haar distribution  $\mu_{\text{Haar}}$

$$(1.5) \quad \mu_q(a + p^N \mathbb{Z}_p) = \frac{q^a}{[p^N]_q}$$

is known as a distribution on  $\mathbb{Z}_p$ , where  $\mu_{\text{Haar}}(a + p^N \mathbb{Z}_p) = \frac{1}{p^N}$  and  $a + p^N \mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq p^{-N}\}$ . Note that  $\lim_{q \rightarrow 1} \mu_q = \mu_{\text{Haar}}$ . We shall write  $d\mu_q(x)$

to remind ourselves that  $x$  is the variable of integration. This distribution  $\mu_q$  yields an  $I_q$ -integral for  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$  :

$$(1.6) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x.$$

The  $I_q$ -integral for  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$  was defined by T. Kim ([6], [7], [8], [10], [11]) and basic properties were studied by many authors. Also, by (1.2) and (1.6), it is well-known that the numbers  $B_n$ ,  $\beta_n(q)$  and  $B_n(q)$  are connected with  $I_0$ -integrals and  $I_q$ -integrals as follows.

- For any  $n \geq 0$ ,  $I_0(x^n) = \int_{\mathbb{Z}_p} x^n dx = B_n$ , where  $B_n$  is the ordinary Bernoulli numbers (see [19], [22]);
- For any  $n \geq 0$ ,  $I_q([x]_q^n) = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \beta_n(q)$ , where  $\beta_n(q)$  is the Carlitz's  $q$ -Bernoulli numbers (see [6], [7], [8], [9], [11] [12]);
- For any  $n \geq 0$ ,  $I_q(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_n(q)$ , where  $B_n(q)$  is the modified Bernoulli numbers (see [3], [4]).

In this paper, we consider the  $q$ -Volkenborn integral of uniformly differentiable functions on  $\mathbb{Z}_p$ . By using this integral, we obtain the generating functions of twisted  $q$ -generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

## 2. The $q$ -extension of the $I_0$ -integral transform and related numbers and polynomials

Given  $\omega \in \mathbf{T}_p$ , we will denote by  $\phi_\omega : \mathbb{Z}_p \rightarrow \mathbb{C}_p, x \mapsto \omega^x$ , the locally constant extension of the power function from  $\mathbb{Z}$  to  $\mathbb{Z}_p$ . In [11], the  $I_q$ -Fourier transform for  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$  is the function  $I_q(f) : \mathbf{T}_p \rightarrow \mathbb{C}_p$  defined by

$$(2.1) \quad I_q(f\phi_\omega) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \phi_\omega(x) f(x) q^x.$$

This is an  $q$ -extension of the  $I_0$ -Fourier transform (1.3), that is, the  $I_0$ -Fourier transform of  $f$  is the case  $q \rightarrow 1$  of  $I_q(f\phi_\omega)$  in (2.1). In terms of integration, one can have the integral form

$$(2.2) \quad I_q(f\phi_\omega) = \int_{\mathbb{Z}_p} \phi_\omega(x) f(x) d\mu_q(x), \quad \omega \in \mathbf{T}_p.$$

Also its inverse  $I_q$ -Fourier transform is seem to be equivalent to the limit

$$(2.3) \quad f(x) q^x = \frac{\log q}{q-1} \lim_{N \rightarrow \infty} \sum_{\omega \in C_{p^N}} \phi_{\omega^{-1}}(x) I_q(f\phi_\omega)$$

for all  $x \in \mathbb{Z}_p$  (see [11]).

Note that the distribution  $\mu_q$  on  $\mathbb{Z}_p$  has the property

$$\mu_q(ap + p^{N+1}\mathbb{Z}_p) = [p]_q^{-1} \mu_{q^p}(a + p^N\mathbb{Z}_p)$$

followed trivially from the definition (1.5). Since  $\mathbb{Z}_p^\times = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ , this implies that for any  $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$  on  $\mathbb{Z}_p^\times$

$$(2.4) \quad \int_{\mathbb{Z}_p^\times} \phi_\omega(x) f(x) d\mu_q(x) = I_q(f\phi_\omega) - [p]_q^{-1} I_q(f(p\mathbb{Z}_p)\phi_{\omega^p}).$$

In the following proposition we obtain the shift versus integration of the  $I_q$ -Fourier transform for uniformly differentiable functions on  $\mathbb{Z}_p$ .

**Proposition 2.1.** *Let  $f$  be a uniformly differentiable function on  $\mathbb{Z}_p$ . For some fixed  $s \in \mathbb{Z}_p$  and  $\omega \in \mathbf{T}_p$*

$$I_q(f(x+s+1)\phi_\omega) - \frac{1}{\omega q} I_q(f(x+s)\phi_\omega) = \frac{q-1}{\omega q \log q} (f'(s) + f(s) \log q).$$

*Proof.* From the definition (2.1), it is easy to see that

$$\begin{aligned} & \frac{\omega q \log q}{q-1} \int_{\mathbb{Z}_p} \phi_\omega(x) f(x+s+1) d\mu_q(x) \\ &= \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_\omega(x) f(x+s) q^x + \lim_{N \rightarrow \infty} \frac{\phi_\omega(p^N) f(p^N+s) q^{p^N} - f(s)}{p^N}. \end{aligned}$$

It is easy to check that a uniformly differentiable function  $\phi_\omega(x) f(x+s) q^x$  on  $x \in \mathbb{Z}_p$  can be differentiated in the usual way:

$$\lim_{p^N \rightarrow 0} \frac{\phi_\omega(p^N) f(p^N+s) q^{p^N} - f(s)}{p^N} = f'(s) + f(s) \log q \quad \text{for } s \in \mathbb{Z}_p.$$

The result now follows easily.  $\square$

**Corollary 2.2.** 1. *Suppose that  $\omega \in \mathbf{T}_p$ . Let  $I_q(e^{tx}\phi_\omega)$  be a power series about the origin as follows*

$$I_q(e^{tx}\phi_\omega) = \sum_{k=0}^{\infty} \frac{B_k(q, \omega)}{k!} t^k \in \mathbb{C}_p[[t]],$$

where  $|t|_p < p^{-1/(p-1)}$  and  $t \neq 0 \in \mathbb{C}_p$ . Then the coefficients of expansion  $\{B_k(q, \omega)\}$  can be written by

$$\begin{aligned} B_k(q, \omega) &= I_q(x^k \phi_\omega) \\ &= \begin{cases} \frac{q-1}{\omega q-1} \left( H_k\left(\frac{1}{\omega q}\right) + \frac{k}{\log q} H_{k-1}\left(\frac{1}{\omega q}\right) \right), & \text{if } k \geq 1 \\ \frac{q-1}{\omega q-1}, & \text{if } k = 0. \end{cases} \end{aligned}$$

Here the generalized  $k$ -th Euler number  $H_k(u)$  attached to an algebraic  $u \neq 1$  has been defined by Frobenius (1910):  $(1-u)/(e^t - u) = \sum_{k=0}^{\infty} H_k(u) t^k / k!$ .

2. For  $\omega \in \mathbf{T}_p$  and  $k \geq 0$ ,

$$I_q(q^{kx}\phi_\omega) = \sum_{i=0}^{\infty} (q^k - 1)^i I_q\left(\binom{x}{i}\right) \phi_\omega = \frac{k+1}{[k+1]_{q, \omega}},$$

where  $[k]_{q,\omega} = (\omega q^k - 1)/(q - 1)$  and  $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$ . Moreover, the sequence  $\{q^k\}$  can be extended to a locally analytic function  $q^x$  for  $x \in \mathbb{Z}_p$ .

We define now the generating function of a new Bernoulli numbers by  $I_q$ -Fourier transform. The twisted  $q$ -extension of Bernoulli numbers is define by

$$(2.5) \quad I_q(e^{tx}\phi_\omega) = \frac{q-1}{\log q} \frac{t + \log q}{\omega q e^t - 1} = \sum_{k=0}^{\infty} B_k(q, \omega) \frac{t^k}{k!}, \quad \omega \in \mathbf{T}_p$$

for  $|t|_p < p^{-1/(p-1)}$ . If  $q \rightarrow 1$ , then from application of L'Hospital's rule the expression (2.5) is reduced to

$$\lim_{q \rightarrow 1} I_q(e^{tx}\phi_\omega) = \frac{t}{\omega e^t - 1}, \quad \omega \in \mathbf{T}_p.$$

The exciting properties of this formula were shown by T. Kim (see [5]) and C. F. Woodcock ([23, Proposition 7.1 (i)] and [24]). If  $\omega = 1$ , then

$$I_q(e^{tx}) = \frac{q-1}{\log q} \frac{t + \log q}{q e^t - 1} = \sum_{k=0}^{\infty} B_k(q, 1) \frac{t^k}{k!},$$

where  $|t|_p < p^{-1/(p-1)}$  (cf. [1], [3], [4], [9], [10], [12], [16], [20]).

**Corollary 2.3.** Let  $\omega \in \mathbf{T}_p$  with  $\omega^N = 1$ ,  $\omega \neq 1$  for  $N > 1$ . Set

$$\lim_{q \rightarrow 1} I_q(e^{tx}\phi_\omega) = \sum_{k=0}^{\infty} B_k(1, \omega) t^k / k!.$$

Then

$$B_k(1, \omega) = N^{k-1} \sum_{i=0}^{N-1} \omega^i B_k\left(\frac{i}{N}\right),$$

where  $B_k(\cdot)$  is the usual Bernoulli polynomials and  $k \geq 1$ .

**Corollary 2.4.** Let  $k \geq 0$ . Then

1.  $\frac{q-1}{\log q} q^x x^k = \sum_{\omega \in \mathbf{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) B_k(q, \omega) + B_k(q, 1)$ .
2.  $\frac{q-1}{\log q} q^{(k+1)x} = \sum_{\omega \in \mathbf{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) \frac{[k+1]_{q,\omega}}{[k+1]_{q,\omega}} + \frac{[k+1]_{q,1}}{[k+1]_{q,1}}$ .

*Proof.* From (2.3) and Corollary 2.2, the series

$$\sum_{\omega \in C_{p,N}} \phi_{\omega^{-1}}(x) I_q(x^k \phi_\omega)$$

converges uniformly to  $\frac{\log q}{q-1} q^x x^k$  as  $N \rightarrow \infty$ . So Part 1 follows directly. Part 2 follows by a similar method of Part 1.  $\square$

In Part 1 and Part 2 of Corollary 2.4, putting  $k = 0$ , we obtain the integral series expansion

$$\frac{q^x}{\log q} = \sum_{\omega \in \mathbf{T}_p} \phi_{\omega^{-1}}(x) \frac{1}{\omega q - 1},$$

whence, for  $x = 0$  in the above, we have  $\frac{1}{\log q} = \sum_{\omega \in \mathbb{T}_p} \frac{1}{\omega q - 1}$  for  $q \neq 1$ . This formula gives an explicit expression for  $\frac{1}{\log q}$  in terms of  $\frac{1}{\omega q - 1}$  (see [23, p. 692]).

Now we consider the recursion formula for the sequence of numbers  $\{B_k(q, \omega)\}$ . From Proposition 2.1 we obtain the difference formula

$$I_q(f_1 \phi_\omega) - \frac{1}{\omega q} I_q(f \phi_\omega) = \frac{q-1}{\omega q \log q} (f'(0) + f(0) \log q),$$

where  $f_1(x) = f(x+1)$ . From this expression, when  $f(x) = x^k$  for  $k \geq 0$ , we easily deduce that

$$(2.6) \quad I_q((x+1)^k \phi_\omega) - \frac{1}{\omega q} I_q(x^k \phi_\omega) = \begin{cases} \frac{q-1}{\omega q}, & k = 0, \\ \frac{q-1}{\omega q \log q}, & k = 1, \\ 0, & k \geq 2. \end{cases}$$

We expand the left-hand side of Equation (2.6) by the binomial theorem. It may be stated as

$$(2.7) \quad I_q((x+1)^k \phi_\omega) = \sum_{i=0}^k \binom{k}{i} \int_{\mathbb{Z}_p} \phi_\omega(x) x^i d\mu_q(x).$$

From (2.5), (2.7) and Part 1 of Corollary 2.2, we derive

$$(2.8) \quad \begin{aligned} & I_q((x+1)^k \phi_\omega) - \frac{1}{\omega q} I_q(x^k \phi_\omega) \\ &= \sum_{i=0}^k \binom{k}{i} B_i(q, \omega) - \frac{1}{\omega q} B_k(q, \omega) \\ &= \begin{cases} \frac{\omega q - 1}{\omega q} B_0(q, \omega), & k = 0, \\ \sum_{i=0}^{k-1} \binom{k}{i} B_i(q, \omega) + \frac{\omega q - 1}{\omega q} B_k(q, \omega), & k \geq 1. \end{cases} \end{aligned}$$

As a consequence of the above formulae (2.6) and (2.8) we deduce the recurrence relation for the sequence of numbers  $\{B_k(q, \omega)\}$  as follows

**Proposition 2.5.** *The numbers  $\{B_k(q, \omega)\}$  satisfies*

$$B_0(q, \omega) = \frac{1}{[1]_{q, \omega}}, \quad B_1(q, \omega) = \frac{1}{[1]_{q, \omega} \log q} - \frac{\omega q}{[1]_{q, \omega} (\omega q - 1)}$$

and

$$B_k(q, \omega) = \frac{\omega q}{1 - \omega q} \sum_{i=0}^{k-1} \binom{k}{i} B_i(q, \omega) \quad \text{for } k \geq 2.$$

### 3. Twisted $p$ -adic $q$ - $L$ -functions

Let  $d$  be a fixed integer and  $p$  be a fixed prime number. We set

$$(3.1) \quad X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where  $a \in \mathbb{Z}$  with  $0 \leq a < dp^N$ . Let  $D = \{q \in \mathbb{C}_p \mid |q-1|_p < 1\}$ , and let  $\overline{D} = \mathbb{C}_p \setminus D$  be the complement of the open unit disc around 1. Note that if  $q \in \overline{D}$  and  $\text{ord}_p(1-q) \neq -\infty$ , then  $\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$  is the measure (cf. [7]). Hereafter, we assume that  $q \in \overline{D}$  and  $\text{ord}_p(1-q) \neq -\infty$ .

Let  $\chi$  be a primitive Dirichlet character with conductor  $d$ . Defining the generalized numbers of  $B_k(q, \omega)$  by the formula

$$(3.2) \quad B_{k,\chi}(q, \omega) = k! \cdot \text{coefficient of } t^k \text{ in } \frac{q-1}{\log q} \sum_{a=1}^d \frac{\chi(a)\phi_\omega(a)q^a(t + \log q)e^{ta}}{\phi_\omega(d)q^d e^{dt} - 1}.$$

Thus we deduce the integral of the generalized numbers

$$(3.3) \quad B_{k,\chi}(q, \omega) = \int_X \phi_\omega(x)\chi(x)x^k d\mu_q(x) \quad \text{for } k \geq 0.$$

To see that (3.3) follows from

$$(3.4) \quad \int_X \phi_\omega(x)\chi(x)e^{tx} d\mu_q(x) = \sum_{k=0}^{\infty} \int_X \phi_\omega(x)\chi(x)x^k d\mu_q(x) \frac{t^k}{k!},$$

we note

$$\begin{aligned} & \int_X \phi_\omega(x)\chi(x)e^{tx} d\mu_q(x) \\ &= \frac{q-1}{\log q} \lim_{N \rightarrow \infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N} \chi(x)\phi_\omega(x)e^{tx} q^x \\ &= [d]_q^{-1} \sum_{a=1}^d \chi(a)\phi_\omega(a)q^a e^{ta} \frac{q^d - 1}{\log q^d} \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_{\omega^d}(x)e^{tdx} q^{dx} \\ &= [d]_q^{-1} \sum_{a=1}^d \chi(a)\phi_\omega(a)q^a e^{ta} \int_{\mathbb{Z}_p} \phi_{\omega^d}(x)e^{tdx} d\mu_{q^d}(x) \\ &= \frac{q-1}{\log q} \sum_{a=1}^d \frac{\chi(a)\phi_\omega(a)q^a(t + \log q)e^{ta}}{\phi_\omega(d)q^d e^{dt} - 1} \quad (\text{by (2.5)}). \end{aligned}$$

**Proposition 3.1.** *Let  $\chi$  be a primitive Dirichlet character with conductor  $d$  and  $x \in \mathbb{Z}_p$ . Then*

$$\frac{q-1}{\log q} q^x \chi(x) x^k = \sum_{\omega \in \mathbf{T}_p, \omega \neq 1} \phi_{\omega^{-1}}(x) B_{k, \chi}(q, \omega) + B_{k, \chi}(q, 1)$$

for  $k \geq 0$ .

Let  $p$  be odd rational prime and let  $\omega_p : X^* \rightarrow X$  be the function defined by (see [2], [15], [19], [22])

$$\omega_p(x) = \lim_{\substack{n \rightarrow \infty \\ p\text{-adically}}} x^{p^n}.$$

The function  $\omega_p$  is called the Teichmüller character, and it appears quite frequently in many different guises. For  $s \in \mathbb{Z}_p$  and  $\omega \in \mathbf{T}_p$ , we define

$$(3.5) \quad L_{p,q}(s, \chi, \omega) = \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{\substack{0 \leq x \leq dp^N - 1 \\ (p, x) = 1}} q^x \phi_\omega(x) \chi(x) \left( \frac{x}{\omega_p(x)} \right)^{1-s}.$$

For  $k \geq 0$ , we set  $\chi_k = \chi \omega_p^{-k}$ . Since  $\mu_q(pU) = [p]_q^{-1} \mu_{q^p}(U)$  for  $U \subset X$ , the value of the function  $L_{p,q}(s, \chi, \omega)$  at non-positive integers are given by

$$(3.6) \quad L_{p,q}(1-k, \chi, \omega) = B_{k, \chi_k}(q, \omega) - p^k [p]_q^{-1} \chi_k(p) B_{k, \chi_k}(q^p, \omega^p)$$

for  $k \geq 1$ . We thus obtain the following

**Theorem 3.2.** *Let  $\chi$  be a primitive Dirichlet character with conductor  $d$  and  $s \in \mathbb{Z}_p, \omega \in \mathbf{T}_p$ . Then the function  $L_{p,q}(s, \chi, \omega)$  interpolates the values  $B_{k, \chi_k}(q, \omega) - p^k [p]_q^{-1} \chi_k(p) B_{k, \chi_k}(q^p, \omega^p)$  when  $s = 1 - k$  for  $k \geq 1$ .*

For  $q \in \overline{\mathbb{D}}$ , we have

$$\left| \frac{\mu_q(a + dp^N \mathbb{Z}_p)}{1 - q} \right|_p = \left| \frac{q^a}{(1 - q)[dp^N]_q} \right|_p = \left| \frac{q^a}{1 - q^{dp^N}} \right|_p \leq 1.$$

By [13, p. 31, Eq. (3.4)], if  $k \equiv k' \pmod{(p-1)p^N}$ , then we obtain the assertion that

$$\left| x^k - x^{k'} \right|_p \leq \frac{1}{p^{N+1}} \quad \text{for } x \in X^*.$$

Using the corollary at the end of [14, Chapter II, §5] and (3.5), their integrals over the compact set  $X^*$  are also close together, and in fact, it is easy to see



that for  $k \geq 1$

$$\begin{aligned}
& (1-q)^{-1}L_{p,q}(1-k, \chi_{-k}, \omega) \\
&= \lim_{N \rightarrow \infty} \frac{1}{1-q^{dp^N}} \sum_{\substack{0 \leq x \leq dp^N - 1 \\ (p,x)=1}} q^x \phi_\omega(x) \chi \omega_p^k(x) \left( \frac{x}{\omega_p(x)} \right)^k \\
&= \int_{X^*} \chi(x) \phi_\omega(x) x^k \frac{d\mu_q(x)}{1-q} \\
&\equiv \int_{X^*} \chi(x) \phi_\omega(x) x^{k'} \frac{d\mu_q(x)}{1-q} \pmod{p^{N+1}} \\
&= (1-q)^{-1}L_{p,q}(1-k', \chi_{-k'}, \omega).
\end{aligned}$$

Hence we can prove the following congruence.

**Theorem 3.3.** *Let  $\chi$  be a primitive Dirichlet character with conductor  $d$ , and let  $k \equiv k' \pmod{(p-1)p^N}$  and  $\omega \in \mathbf{T}_p$ . Then*

$$(1-q)^{-1}L_{p,q}(1-k, \chi_{-k}, \omega) \equiv (1-q)^{-1}L_{p,q}(1-k', \chi_{-k'}, \omega) \pmod{p^{N+1}}.$$

Finally, we shall also want to consider modified twisted  $L$ -functions in the complex field  $\mathbb{C}$ . Let  $q \in \mathbb{C}$  with  $0 < |q| < 1$ , and let  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 1$ . We set

$$(3.7) \quad L_q(s, \chi, \omega) = \frac{q-1}{1-s} \sum_{n=1}^{\infty} \frac{\omega^n q^n \chi(n)}{n^{s-1}} + \frac{q-1}{\log q} \sum_{n=1}^{\infty} \frac{\omega^n q^n \chi(n)}{n^s},$$

the series being absolutely convergent (cf. [3], [4], [6], [9], [12], [17], [18], [21]). In particular, if we replace  $s$  by  $1-k$ , one then sees easily that

$$\begin{aligned}
& L_q(1-k, \chi, \omega) \\
&= \frac{q-1}{k} \sum_{n=1}^{\infty} \omega^n q^n \chi(n) n^k + \frac{q-1}{\log q} \sum_{n=1}^{\infty} \omega^n q^n \chi(n) n^{k-1} \\
&= \frac{q-1}{k} \left( \frac{d}{dt} \right)^k \left( \sum_{n=1}^{\infty} \omega^n q^n \chi(n) e^{nt} + \frac{t}{\log q} \sum_{n=1}^{\infty} \omega^n q^n \chi(n) e^{nt} \right) \Big|_{t=0}.
\end{aligned}$$

We consider the function

$$\Psi_q(t) = (1-q) \sum_{n=1}^{\infty} \omega^n q^n \chi(n) e^{nt} + \frac{(1-q)t}{\log q} \sum_{n=1}^{\infty} \omega^n q^n \chi(n) e^{nt}.$$

Since  $\chi$  is a character mod  $d$  we rearrange the terms in the series for  $\Psi_q(t)$  according to the residue classes mod  $d$ . Then we have

$$\begin{aligned}\Psi_q(t) &= \left( (1-q) + \frac{(1-q)t}{\log q} \right) \sum_{a=1}^d \sum_{b=0}^{\infty} \omega^{a+bd} q^{a+bd} \chi(a+bd) e^{(a+bd)t} \\ &= \frac{q-1}{\log q} \sum_{a=1}^d \frac{\chi(a) \omega^a q^a (t + \log q) e^{ta}}{\omega^d q^d e^{dt} - 1},\end{aligned}$$

which is equal to the formula in (3.2). We will apply the recipe above. Then we see that for  $k \geq 1$

$$(3.8) \quad L_q(1-k, \chi, \omega) = -\frac{1}{k} \left( \frac{d}{dt} \right)^k (\Psi_q(t)) \Big|_{t=0}.$$

Comparing (3.2) and (3.8), we arrive at the following

**Proposition 3.4.** *Let  $\chi$  be a primitive Dirichlet character with conductor  $d$ , and let  $q \in \mathbb{C}$  with  $0 < |q| < 1$ . Then*

$$L_q(1-k, \chi, \omega) = -\frac{1}{k} B_{k, \chi}(q, \omega) \quad \text{for } k \geq 1.$$

Let  $\overline{\mathbb{Q}}$  be an algebraic closure of  $\mathbb{Q}$ . Using (3.6) and Proposition 3.4 we have

$$-\frac{L_{p,q}(1-k, \chi, \omega)}{k} = L_q(1-k, \chi_k, \omega) - p^k [p]_q^{-1} \chi_k(p) L_{q^p}(1-k, \chi_k, \omega^p)$$

for  $k \geq 1$ . Here the right-hand side is the value of the complex  $L$ -function which the left-hand side is the values of the  $p$ -adic  $L$ -function and the value are equal in the field  $\overline{\mathbb{Q}}$  common to  $\mathbb{C}_p$  and  $\mathbb{C}$ .

**Theorem 3.5.** *Let  $\omega \in \mathbf{T}_p$  and  $q \in \overline{D}$  with  $\text{ord}_p(1-q) \neq -\infty$ . Let  $\omega_p$  be the Teichmüller character. For  $\chi$  a primitive Dirichlet character with conductor  $d$ , the function from  $\mathbb{Z}_p \setminus \{1\}$  to  $\mathbb{C}_p$*

$$\frac{L_{p,q}(s, \chi, \omega)}{s-1} = \frac{1}{s-1} \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{\substack{0 \leq x \leq dp^N - 1 \\ (p,x)=1}} q^x \phi_\omega(x) \chi(x) \left( \frac{x}{\omega(x)} \right)^{1-s}$$

*interpolates the values  $L_q(1-k, \chi_k, \omega) - p^k [p]_q^{-1} \chi_k(p) L_{q^p}(1-k, \chi_k, \omega^p)$  when  $s = 1 - k$ .*

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