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ANALYTIC RENORMALIZATION OF DUAL ONE-LOOP AMPLITUDES

Peter Goddard *)
CERN - Geneva

A B S T R A C T

Off-mass shell orientable non-planar loops are used to suggest a specific way of regularizing one-loop amplitudes. The corresponding counterterms are complex because the Pomeranchukon-like singularity, present in orientable non-planar loops, is a cut rather than a pole.

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*) On leave of absence from Trinity College, Cambridge, England.

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1. INTRODUCTION

At present the method for renormalizing the exponential divergence of dual single loop amplitudes [1),2),3)] has the disadvantage that there is an infinite amount of arbitrariness in the counterterm [3),4)]. In this paper we show that considerations of zero momentum limits of off-mass shell dual loops suggest a specific way of regularizing this divergence. This way is equivalent to a particular choice of counterterm and, when the intercept $a_0 = 1$, this lies within the scheme of Refs. 3),4). However, the counterterm has the serious disadvantage that, while it has the correct singularity structure of double poles, it is complex. This is a direct result of the Pomeranchukon-like singularity [3),5)], which occurs in some non-planar loops, being a cut. It would become real if the theory could be modified so that this cut became a pole. The regularization procedure presented here is equivalent to changing the integration contour for part of the integrand in a simple way. Very similar changes in integration contour occur in the integral representation for the four-point Born term, and throughout the theory, when analytic continuations are performed.

The basis of the approach adopted here is the convergence of an orientable non-planar loop [3),5)] when a certain momentum invariant $t < -\frac{4}{3}$ (t is the square of the total momentum attached to one boundary of the analogue Riemann surface [6)]; the branch points at $t = -\frac{4}{3}+4n$, $n = 0,1,2,\ldots$, are those which have been associated with the Pomeranchukon). A planar loop can be obtained from an orientable non-planar loop by removing all the external lines on one boundary of the analogue surface. Thus, a possible way of attempting to find a finite expression for the planar loop is to continue an off-mass shell expression for an orientable non-planar loop to the point where all the momenta on one boundary are zero. It is natural to perform this continuation through the upper half complex t plane but it is apparent that the answer will not be a real analytic function because of the cut starting at $t = -\frac{4}{3}$. If there were only a pole at $t = -\frac{4}{3}$ the answer would necessarily be real analytic.

The cut necessarily creates similar problems for the orientable planar loops with only one line on one of the boundaries of the analogue surface when $a_o < \frac{4}{3}$. Such a loop is not convergent a priori but a convergent answer may be deduced from the continued expression for the loop with two lines on the given boundary by using duality and factorization; the result will be complex. This difficulty exists in the scheme of Refs. 3),4), if the cut at $t = -\frac{4}{3}$ is retained. It may be avoided by changing the integrand to remove this lowest Pomeranchukon-like cut but this introduces the sort of arbitrariness we are seek to avoid with the present discussion. However, if this method is used to remove the cut by introduction of a counterterm, adaptations of the calculations given here would suggest a corresponding counterterm for planar loops.

When the zero momentum limit just described is taken we obtain an expression of the form

$$\left(\int [\varphi - \varphi^1] + \int_{(b)} \varphi^1 \right) g_n(w)\, dw\, d\theta_i \qquad (1.1)$$

Here $\varphi = \varphi(w,\theta_i,p_i)$ is the usual integrand for planar loop, φ^1 the part of it which causes the divergence near $w=1$; w is the variable associated with the size of the hole in the analogue model and θ_i are the usual angular variables associated with the analogue positions of the external momenta p_i. The suffix (b) indicates that φ^1 is integrated along an alternative countour starting at $w=0$ but approaching $w=1$ from above, so that the integral converges. The factor $g_n(w)$ is to be expected from the point of view of the analogue model, where the measure in w is not determined. Its structure can easily be understood in terms of propagators as we explain in Section 4. If the planar loop amplitude is to be of the form of (1.1) for some $g_o(w)$ it must be $g_o(w) \equiv 1$. We show that this can be further made plausible by noting that Born terms are related in zero momentum limits if functions of very similar structure are removed from the integrand.

It is impossible to obtain non-orientable loops from orientable ones by removing external lines. However, the prescription of integrating a well-defined divergent part of the integrand along an alternative contour extends immediately to non-orientable loops. Thus, a unified scheme is obtained both for regularizing one-loop amplitudes and writing integral expressions for an orientable non-planar loop above the Pomeranchukon-like cuts.

Section 2 is solely concerned with calculating expressions for off-mass shell loop amplitudes. In Section 3, explicit expressions for analytic continuation in t of orientable non-planar loops are obtained. Then these are used to discuss renormalization of planar one-loop amplitudes in Section 4. Section 5 contains comments and conclusions. Finally, an Appendix discusses the problem of twisting for off-mass shell amplitudes incidentally encountered in Section 2.

2. OFF-MASS SHELL ONE-LOOP AMPLITUDES

A specific form of off-mass shell amplitude for scalar particles is suggested by the N Reggeon vertex [7], because the Reggeon momenta are not confined to the mass shell. For the $N+2$ point function with external momenta p_i, $0 \leq i \leq N+1$ this gives

$$\mathcal{B}_{N+2}(p_i) = \int dV \prod_{\substack{0 \leq i < j-1 \\ j \leq \min(i,2)+N}} \left[\frac{(\sigma_i - \sigma_{j+1})(\sigma_{i+1} - \sigma_j)}{(\sigma_i - \sigma_j)(\sigma_{i+1} - \sigma_{j+1})} \right]^{-\alpha(s_{ij})-1} \tag{2.1}$$

where $\sigma_{N+2+i} = \sigma_i$, $s_{ij} = -(p_{i+1}+p_{i+2}+\dots+p_j)^2$ and dV is the projectively invariant measure:

$$dV = \prod_{i=0}^{N+1} \frac{d\sigma_i}{(\sigma_i - \sigma_{i+2})} \cdot \frac{(\sigma_a - \sigma_b)(\sigma_b - \sigma_c)(\sigma_c - \sigma_a)}{d\sigma_a \, d\sigma_b \, d\sigma_c} \tag{2.2}$$

Introducing variables corresponding to a multiperipheral configuration $x_{ij} = \sigma_j / \sigma_i = x_i x_{i+1} \cdots x_{j-1}$ we can write Eq. (2.1) in the usual operator notation, Refs. 8),9);

$$B_{N+2}(p_i) = \int_0^1 \prod_{i=1}^{N-1} \left[d\mu(x_i)(1-x_i)^{-\alpha_i-\alpha_{i+1}} \right] \prod_{i=1}^{N-2} (1-x_i x_{i+1})^{\alpha_{i+1}}$$

$$\langle 0 | V_1 x_1^{L_0} V_2 x_2^{L_0} \cdots x_{N-1}^{L_0} V_N | 0 \rangle \tag{2.3}$$

where $d\mu(x_i) = x_i^{-a_0-1}(1-x_i)^{a_0-1}$, $\alpha_i = \alpha(-p_i^2)$,

$$V_i \equiv V(p_i) = \exp\left[p_i \cdot \Sigma \frac{a^{(n)+}}{\sqrt{n}}\right] \exp\left(-p_i \cdot \Sigma \frac{a^{(n)}}{\sqrt{n}}\right)$$

and L_0 is a generator of the Gliozzi group [9),10)].

We can form an off-mass shell planar loop by sewing in the usual way [9)].

$$M(p_i) = \int d^4k \int_0^1 \prod_{i=1}^{N} \left[d\mu(x_i)(1-x_i)^{-\alpha_i-\alpha_{i+1}}(1-x_i x_{i+1})^{\alpha_{i+1}} \right]$$

$$\cdot \operatorname{Tr}\left\{ V_1 x_1^{L_0} V_2 x_2^{L_0} \cdots x_{N-1}^{L_0} V_N x_N^{L_0} \right\}, \tag{2.4}$$

where i is replaced by $i-N$ if it is greater than N.

To form non-planar loops it is first necessary to have a twisting operator which works for off-mass shell tree diagrams. In the Appendix we show that the twisting operator $\theta(x)$ introduced by Amati, Le Bellac and Olive [9),11),12)] maintains its duality and twisting properties off-mass shell. Thus, to twist any line in Eq. (2.3) we replace the corresponding $x_i^{L_0}$ by $x_i^{L_0}\theta(x_i)$, and sewing as before we obtain for the loop amplitude of Fig. 1

$$M(p_i) = \int d^4k \int_0^1 \prod_{i=1}^{N} \left[d\mu(x_i)(1-x_i)^{-\alpha_i-\alpha_{i-1}}(1-x_i x_{i+1})^{\alpha_i} \right]$$

$$\operatorname{Tr}\left\{ x_1^{L_0}\theta(x_1) V_1 x_2^{L_0}\theta(x_2) \cdots x_M^{L_0}\theta(x_M) V_M x_{M+1}^{L_0} V_{M+1} \cdots x_N^{L_0} V_N \right\} \tag{2.5}$$

The operator duality relations guarantee the independence of this expression of the particular method of sewing. We can now perform the trace calculations following Refs. 9),12). Firstly, we eliminate the $(1-x_i)^W$ in the $\theta(x_i) \equiv \Omega(1-x_i)^W$ (using the notation of those references), by changing to variables y_i in the same way. Throughout this care must be taken not to use the usual mass shell conditions $\alpha_i = 0$. Thus, we use the relation

$$(1-z)^W V_i = V_i (1-z)^{W+\alpha_i - a_0} \tag{2.6}$$

and, after some algebra, obtain the expression

$$\begin{aligned} M(p_i) = & \int d^4k \int_0^1 \prod_{i=1}^{N} d\mu(y_i) \cdot \prod_{i=1}^{M-1} (1-y_i)^{-\alpha_{i-1}-\alpha_{i+1}} \\ & \prod_{i=2}^{M} (1-y_{i-1}+y_{i-1}y_i - y_{i-1}y_i y_{i+1})^{\alpha_i} \cdot (1-y_M)^{-\alpha_{M-1}-\alpha_1} \cdot \prod_{i=M+1}^{N} (1-y_i)^{-\alpha_{i-1}-\alpha_i} \\ & \prod_{i=M+1}^{N} (1-y_i y_{i-1})^{\alpha_i} \cdot (1-y_M + y_1 y_M y_{M+1} \cdots y_N (1-y_2))^{\alpha_1} \ (1-(-)^M w) \\ & \mathrm{Tr}\, \{ y_1^{L_0} \Omega V_1 y_2^{L_0} \cdots y_M^{L_0} \Omega V_M y_M^{L_0} V_{M+1} \cdots y_N^{L_0} V_N \}. \end{aligned} \tag{2.7}$$

where $w = y_1 y_2 \cdots y_N$.

Still following Refs. 9),12), we use the techniques of Ref. 3) to remove the Ω's, and, after more algebra, obtain

$$\begin{aligned} M(p_i) = & \int d^4k \int \prod_{i=1}^{M} du_i \, u_i^{-\alpha(s_i)-1} \prod_{i=M+1}^{N} v_i^{-\alpha(s_i)-1} dv_i \\ & \prod_{i=1}^{M-1} (1-u_i u_{i+1})^{a_0-1-\alpha_{i-1}-\alpha_{i+1}} \ (1-u_M v_{M+1} \cdots v_N u_1)^{a_0-1-\alpha_{M-1}-\alpha_1} \\ & \prod_{i=M+1}^{N} (1-v_i)^{a_0-1-\alpha_{i-1}-\alpha_i} \, \mathrm{Tr}\, \{ (-u_1)^H V_1 (-u_2)^H \cdots (-u_M)^H V_M v_{M+1}^H \cdots v_N^H V_N \} \\ & \prod_{i=2}^{M-2} (1-u_{i-1}u_i \cdots u_{i+2})^{\alpha_i} \ (1-u_{M-2}u_{M-1}u_M v_{M+1} \cdots v_N u_1)^{\alpha_{M-1}} (1-u_M v_{M+1} \cdots v_N u_1 u_2 u_3)^{\alpha_1} \\ & \prod_{i=M+1}^{N-1} (1-v_i v_{i+1})^{\alpha_i} (1-v_N u_1 u_2)^{\alpha_N} (1-u_{M-1}u_M)^{\alpha_M} (1-(-)^M w)^2. \end{aligned} \tag{2.8}$$

This makes it possible to give a simple extension to the off-mass shell case of the rules for one-loop amplitudes given by Gross, Neveu, Scherk and Schwarz [5),9)]. The factors of the form $u_i^{-\alpha(s_i)-1}$, $v_i^{-\alpha(s_i)-1}$, $(1-(-)^M w)^2$, and the trace factor remain exactly the same but there are two differences:

1) previously there was a factor $(1-c)^{a_0-1}$ for each pair of adjacent external lines on the (same) edge of the analogue surface, c being the product of u's and v's between them; this now becomes $(1-c)^{a_0-1-\alpha_i-\alpha_j}$ where i, j label the lines;

2) for each line, i, there is a factor $(1-c')^{\alpha_i}$ where c' is the produce of u's and v's between the two lines on each side of i on the (same) edge of the analogue surface; if there are only two lines on one edge i, j, say this factor is $(1-w)^{\alpha_i+\alpha_j}$, and if only one, i say, it is $(1-w^2)^{\alpha_i}$. These extensions are exactly analogous to those for the tree amplitudes for Eqs. (2.3) and (2.5). The integration region is just as before, of course, and is restricted to avoid multiple counting.

To bring Eq. (2.8) into a form in which we can study its renormalization we need to perform the loop integration and express the the trace in terms of Jacobi Θ functions. This can be done exactly as in Ref. 3) with identical results once it is noticed that the use of mass shell conditions there is unnecessary.

Thus, we finally obtain

$$M(p_i) = 4\pi^2 \int \prod_{i=1}^{M} du_i \prod_{i=M+1}^{N} dv_i \, (1-(-)^M w)^2 \, \frac{w^{-a_0-1}}{\ln^2 w} \left[f((-)^M w)\right]^{-4}$$
$$\nu(u_i, v_i) \prod_{i<j} \left[\psi_x(C_{ji})\right]^{p_i \cdot p_j} \tag{2.9}$$

where $\nu(u_i, v_j)$ is the part of the integrand of (2.8) consisting of factors raised to powers $a_0-1-\alpha_i-\alpha_j$ or α_i, $f(w) = \prod_{n=1}^{\infty}(1-w^n)$, $\psi_x(C_{ji})$ is one of the four functions ψ, ψ_T, ψ_N and ψ_{NT} related to the Jacobi Θ functions, defined in Ref. 3), and $C_{ji} = \xi_{i+1}\xi_{i+2}\cdots\xi_j$, $\xi_i = u_i$ if $i \le M$, $\xi_i = v_i$ if $i > M$.

[If $M=0$, and we do not eliminate spurious states, the factor $(1-w)^2$ has to be omitted.]

3. ANALYTIC CONTINUATION OF ORIENTABLE NON-PLANAR LOOPS

In this Section we consider the orientable non-planar loop of Fig. 2. The corresponding amplitude is

$$M_N(t_i) = 4\pi^2 \int \prod_{i=1}^{n+1} du_i \prod_{i=n+2}^{N} dv_i \, \nu_{N,n}(u_i;v_i)\,(1-w)^2\, w^{-a_0-1}$$

$$[f(w)]^{-4} (\ln w)^{-2} \prod_{i<j} \left[\psi_x(c_{ji})\right]^{p_i \cdot p_j} \tag{3.1}$$

where $\nu_{N,n}$ is the product of the factors given by rules 1) and 2) of Section 2, and $\psi_x \equiv \psi$ unless both $i \leq n$ and $j \geq n+1$, in which case $\psi_x \equiv \psi_T$. The behaviour of ψ, ψ_T and f as $w \to 1$ is [3]

$$\psi(x) = -\frac{1}{\pi} \ln w \sin\left[\pi \frac{\ln x}{\ln w}\right] \left(1 + O(q^2)\right) \tag{3.2}$$

$$\psi_T(x) = -\frac{1}{2\pi} \ln w \, q^{-\frac{1}{4}} \left(1 + O(q)\right) \tag{3.3}$$

$$\frac{4\pi^2}{(\ln w)^2} [f(w)]^{-4} = w^{\frac{1}{6}} q^{-\frac{1}{3}} \left(1 + O(q)\right) \tag{3.4}$$

where $q = \exp\left[2\pi^2/\ln w\right]$. Thus, if we introduce the usual angular variables $\theta_i = 2\pi \ln c_{i,N}/\ln w$ the leading behaviour of the integrand, $\varphi_{N,n}$ is given by

$$\varphi_{N,n} = \varphi^1_{N,n} \left(1 + O(q)\right) \tag{3.5}$$

$$\varphi^1_{N,n} = (1-w)^2 w^{-a_0-5/6} \nu_{N,n} \left[-\ln w\right]^{(A-Na_0)} q^{-\frac{1}{3}-\frac{1}{4}t} \lambda_1(\theta) \tag{3.6}$$

$t = -\left(\sum_{i=1}^{n} p_i\right)^2$, $A = \sum_{i=1}^{N} \alpha_i$ and λ_1 is a function of the θ_i only.

Thus, there is no convergence problem at $w = 1$ for $t < -\frac{4}{3}$. If we first sought to evaluate (3.1) when $-\frac{4}{3} < t < \frac{8}{3}$, (3.6) would seem to give a natural choice of counterterm, but the expression obtained by integrating $\varphi_{N,n} - \varphi^1_{N,n}$ would not be the expression obtained by evaluating (3.1) directly for $t < -\frac{4}{3}$ and continuing it through the upper complex t plane. To evaluate this continuation write

$$\varphi_{N,n} = \varphi^{o}_{N,n} + \varphi^{1}_{N,n} \tag{3.7}$$

Then, the integral of $\varphi^{o}_{N,n}$ is analytic at $t = -\frac{4}{3}$ and only the integral of $\varphi^{1}_{N,n}$ need be continued. At this point it should be stressed that no arbitrariness is introduced by the division of $\varphi_{N,n}$ into $\varphi^{o}_{N,n}$ and $\varphi^{1}_{N,n}$ since analytic continuations are uniquely specified. Any other division must lead to the same result. In particular, adding higher powers of q to $\varphi^{1}_{N,n}$ will merely add terms whose continuation is trivial at the lowest branch point and the result is the same.

$$\int \varphi^{1}_{N,n}\, du_i\, dv_j = \int d\theta_i \int_0^1 dw (1-w)^2 w^{-\alpha_0 - 5/6} [-\ln w]^{A-N\alpha_0+N-1} \nu_{N,n}\, q^{-\frac{1}{3}-\frac{1}{4}t}\, \lambda_2(\theta) \tag{3.8}$$

To study the analytic continuation it is convenient to introduce variables $z = -(2\pi^2)/(\ln w)$, so that $q = e^{-z}$ and $\tau = \frac{1}{3} + \frac{1}{4}t$, so that we wish to continue from $\tau < 0$ to $\tau > 0$. Then the right-hand side of (3.8) takes the form

$$\int d\theta \int_0^\infty dz\, \lambda_3(z,\theta)\, e^{\tau z}\, dz \tag{3.9}$$

where

$$\lambda_3 = (1-w)^2 w^{-\alpha_0+\frac{1}{6}} [-\ln w]^{A-N\alpha_0+N+1} (2\pi^2)^{-1} \nu_{N,n} \lambda_2$$

and is not significant in determining the convergence. The situation in the z plane is illustrated in Fig. 3. The first sheet of the w plane $-\pi \leq \arg w \leq \pi$ is mapped onto the exterior of the two circles of radius π centred at $z = \pm\pi$. The heavy lines inside the circles indicate cuts of $\nu_{N,n}$ on other sheets. If we put $\zeta = -\tau z$ we obtain

$$-\frac{1}{\tau} \int d\theta \int_0^\infty d\zeta\, \lambda_3(-\zeta/\tau, \theta)\, e^{-\zeta}\, d\zeta \tag{3.10}$$

The images in the ζ plane of the circles in the z plane rotate clockwise as we continue from $\tau < 0$ to $\tau > 0$ through the upper half τ plane. Convergence of (3.10) is maintained if the contour always approaches the origin along the common tangent to the two circles in the ζ plane. Thus, if when $\tau > 0$, we map the ζ plane back onto the z plane by $\zeta = \tau z$, we obtain the contour (a). If we still use $\zeta = -\tau z$ we obtain the contour (b). The corresponding contours in the w plane are shown in Fig. 4. The continuation of (3.8) to positive τ is given by altering the real w contour to the contour (b). If we use the contour (a) the integrand has to be modified by sending w to w^{-1}, q to q^{-1}, u_i to u_i^{-1} and v_i to v_i^{-1} and we obtain

$$\int d\theta_i \int_{\infty \, (a)}^{1} dw\, (1-w)^2\, w^{-a_0-\frac{1}{6}} \left[-\ln w\right]^{A-Na_0+N-1} \nu_{N,n}\, q^{\frac{1}{3}+\frac{1}{4}t}\, \lambda_2(\theta) \tag{3.11}$$

Using contour (b) the effect of continuation has been to integrate the divergent part of the integrand along another contour from 0 to 1 where it converges. The answer may be regarded as the divergent integral (3.1) minus the infinite counterterm formed by integrating $\varphi^{1}_{N,n}$ around a closed loop [(c) in Fig. 4], in the upper half w plane tangent to the real axis at $w = 1$. The use of complex contours complicates the evaluation of integrals but if $a_0 < -\frac{1}{6}$ we may distort the contour (b) to infinity in the w plane to leave integrals from 1 to ∞ and $-\infty$ to 0.

Similar procedures may be used to continue above the higher Pomeranchukon-like thresholds. If we had continued below the threshold the various integration contours would be reflected in the real axis.

4. REGULARIZATION OF PLANAR ONE-LOOP AMPLITUDES

In the last Section we saw how the orientable non-planar loop M_N could be continued above the Pomeranchukon-like thresholds by integrating part of the integrand along a different contour.

Now, we consider the limit $p_i \to 0$, $1 \le i \le n$ which removes all the momenta on one boundary of the analogue surface, leaving an amplitude related to the planar loop with $N-n$, $=m$ say, external particles. Let $v_o = u_1 u_2 \dots u_{n+1}$ and use $v_o, v_{n+2}, \dots, v_N, u_2, \dots, u_n$ as integration variables:

$$\prod_{i=1}^{n+1} du_i = dv_o \prod_{i=1}^{n} \frac{du_i}{u_i} \tag{4.1}$$

When $p_i = 0$, $1 \le i \le n$, and this change of variables has been made, the only place where u_i, $1 \le i \le n$, occur in the integrand is in $\nu_{N,n}(u_i, v_i)$

$$\nu_{N,n}(u_i, v_i) = \sigma_n(w, u_i)\, \nu_m(v_i) \tag{4.2}$$

where $\nu_m(v_o, v_{n+2}, \dots, v_N)$ is the appropriate part of the measure for an m particle planar loop. The limit of the continuation of (3.1) is thus

$$\left(\int [\Phi_m(v) - \Phi_m^1(v)] - \int_{(b)} \Phi_m^1(v) \right) g_n(w)\, dv_o\, dv_{n+2} \dots dv_N \tag{4.3}$$

$\Phi_m(v)$ is the usual planar loop integrand; $\Phi_m^1(v)$ is its leading behaviour near $w=1$ [defined as in Eq. (3.6)]. If we set $g_n(w) = 1$ in (4.3) we obtain a specific regularization of the planar one-loop amplitude, and, therefore, it only remains to explain the presence of this function. For $n \ge 3$

$$g_n(w) = (1-w)^2 \int_w^1 \frac{du_1}{u_1} \int_w^1 \frac{du_2}{u_2} \int_{w/u_2}^1 \frac{du_3}{u_3} \dots \int_{w/(u_2 \dots u_{n-1})}^1 \frac{du_n}{u_n} \prod_{i=2}^{n} (1-u_i)^{-a_0-1}$$

$$(1 - w/u_2 \dots u_n)^{-a_0-1} \prod_{i=2}^{n-1} (1-u_i u_{i+1})^{a_0} (1 - w/u_3 \dots u_n)^{a_0} (1 - w/u_2 \dots u_{n-1})^{a_0} \tag{4.4}$$

$$g_1(w) = (1-w^2)^{a_0}(1-w) \int_w^1 \frac{du_1}{u_1} = -(1-w^2)^{a_0}(1-w) \ln w \tag{4.5}$$

$$g_2(w) = -(1-w)^{2a_0+1} \ln w \int_w^1 du_2 u_2^{a_0}(1-u_2)(u_2-w)$$

$$= \frac{w^{a_0}}{1-w} \ln w \frac{[\Gamma(-a_0)]^2}{\Gamma(-2a_0)} {}_2F_1\left(-a_0,-a_0;-2a_0;-\frac{1-w}{w}\right) \tag{4.6}$$

$$\sim (\ln w)^2 \quad \text{as } w \to 0 \tag{4.7}$$

where we have used formulae for the hypergeometric function which can be found in the Bateman Manuscript Project [13]. In general

$$g_n(w) \sim \text{constant}. (\ln w)^n \quad \text{as } w \to 0 \tag{4.8}$$

The functions g_n have the property of introducing multiple poles on the propagators because

$$\int_0^1 (-\ln x)^r x^\alpha dx = \Gamma(r+1)(\alpha+1)^{-r-1} \tag{4.9}$$

This is to be expected because when the $p_i \to 0$, $1 \le i \le n$, there are still $(n+1)$ propagators between p_N and p_{n+1}. The term $(\ln v_0)^{n+1}$ contained in (4.8) is responsible for the corresponding multiple poles which result from using duality to alter the positions of the twists and then letting the momenta tend to zero.

While we have not advanced a priori a criterion for regularization leading to (4.3) with $g_0(w) \equiv 1$, this argument makes it a plausible choice. If the renormalized amplitude is to be of the form (4.3) for some $g_n(w)$ it is clear that we must make this choice.

Functions similar to $g_n(w)$ are encountered when taking zero momentum limits of Born terms. For example, consider letting $p_{r+i} \to 0$, $1 \le i \le n$ $(r \ge 1,\ r+n < N)$ in Eq. (2.3) while the remaining p_i's are off-mass shell. If we set $x = x_r x_{r+1}, \ldots, x_{n+r}$ the integrand becomes that for an off-mass shell $N-n+2$ point function, with integration variables $x_1,\ldots,x_{r-1},x,x_{n+r-1},\ldots,x_{N-1}$, multiplied by a function which, after the $x_r,\ldots,x_{n+r+1}$ integrations have been performed, is just a function $h_n(x_{r-1},x,x_{n+r+1})$. This function has a structure similar to g_n

$$h_n = (1-x_{r-1}x)^{-\alpha_r}(1-x x_{n+r+1})^{-\alpha_{n+r+1}} \int_x^1 \frac{dx_r}{x_r} \int_{x/x_r}^1 \frac{dx_{r+1}}{x_{r+1}} \cdots$$

$$\cdots \int_{x/x_r\ldots x_{n-r+1}} \frac{dx_{n+r-1}}{x_{n+r-1}} \prod_{i=r}^{n+r} (1-x_i)^{\alpha_0-1-\alpha_i-\alpha_{i+1}} \prod_{i=r}^{n+r-1} (1-x_i x_{i+1})^{\alpha_{i+1}}$$

(4.10)

h_n behaves like $(\log x)^n$ when $x \to 0$ and if h_n is put equal to one we obtain the $N-n+2$ point function. (When the remaining momenta go onto the mass shell these factors disappear and h_n just produces an n fold pole whose residue is exactly the $N-n+2$ point function. This must happen because the external legs whose momenta have gone to zero could have been attached to p_r or p_{n+r+1} by duality.)

5. CONCLUSIONS

The regularization of the planar loop obtained in Section 4 consists of integrating the convergent part φ_m^0 of the usual integrand φ_m, along the usual contour $0<w<1$ and subtracting the integrals of the remaining divergent part φ_m^1 from $-\infty$ to 0 and from 1 to ∞. The continuation of φ_m^1 used is that taken through the upper half plane. Stated like this we have a unified prescription for regularizing planar loops and evaluating orientable non-planar loops above the Pomeranchukon-like cuts, and this prescription can be extended immediately to non-orientable loops. [In the latter case the divergent part φ_m^1 is defined by an equation like (3.6) taking the leading power behaviour of φ_m in $q_N = \exp(\pi^2)/(2\ln w)$.] In each case this procedure is equivalent to adding a counter consisting of the integral of φ_m^1 round the loop (c) in Fig. 4. It is hoped that such counterterms may be preferred because of the analytic way they have been obtained.

The complex part of the counterterm comes from the integral of φ_m^1 from $-\infty$ to 0. The integral from 1 to ∞ may be mapped back onto the interval 0 to 1, by sending w to w^{-1}, and this then gives a special case of the counterterms in Refs. 3),4). The complex nature of the counterterms is directly related to the Pomeranchukon-like singularity being a cut rather than a pole. This cut seems to violate unitarity. It seems likely that if the theory can be modified to make this cut a pole these counterterms will become real. Such a modification has recently been considered by Lovelace [14].

The prescription of using an alternative contour is not very surprising when it is noticed that

$$B(\alpha,\beta) = \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \qquad \alpha>0,\ \beta>0. \tag{5.1}$$

but

$$B(\alpha,\beta) = \int_0^1 dx\,\{x^{\alpha-1}(1-x)^{\beta-1} - x^{\alpha-1}\} - \int_1^\infty x^{\alpha-1}dx \qquad -1<\alpha<0,\ \beta>0. \tag{5.2}$$

The discussion here has been solely concerned with single loops. For an N loop amplitude there are many new kinds of divergence [15),16)] but their structure is such that they could be removed by similar regularization procedures.

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APPENDIX

TWISTING OPERATORS FOR OFF-MASS SHELL AMPLITUDES

The reversal symmetry of Eq. (2.1) implies that the integrand of (2.3) be unchanged if x_i is replaced by $\bar{x}_i$ and p_i is replaced by $\bar{p}_i = p_{N+2-i}$, where $x_1 x_2 \ldots x_i = 1 - \bar{x}_1 \bar{x}_2 \ldots \bar{x}_{N-i}$ throughout. It then follows in the usual way that [9]

$$d\chi(x)\,\Omega\, V(p_1)\, x_1^{L_0}\, V(p_2)\, x_2^{L_0} \ldots x_{N-1}^{L_0}\, V(p_N)|0\rangle$$

$$= \; d\chi(\bar{x})\, V(\bar{p}_1)\, \bar{x}_1^{L_0}\, V(\bar{p}_2)\, \bar{x}_2^{L_0} \ldots \bar{x}_{N-1}^{L_0}\, V(\bar{p}_N)|0\rangle \tag{A.1}$$

where we used $d\chi(x)$ to indicate the measure in (2.3). From this it is clear that Ω will not do as a twisting operator, because, in a larger chain of operators, $d\chi(x)$ will be multiplied by $(1-x_1 z)^{\alpha_1}(1-z)^{-\alpha_1}$, say, and $d\chi(\bar{x})$ by $(1-\bar{x}_1 z)^{\alpha_{N+1}}(1-z)^{-\alpha_{N+1}}$; the trouble is the measure does not factorize in the usual way off-mass shell. Now, consider $\Theta(z) = \Omega(1-z)^W$ [11),12)]. Using (2.6) we find

$$d\chi(x)(1-x_1 z)^{\alpha_1}(1-z)^{-\alpha_1+W}\, V(p_1)\, x_1^{L_0}\, V(p_2)\, x_2^{L_0} \ldots x_{N-1}^{L_0}\, V(p_N)|0\rangle$$

$$= \; d\chi(y)\,(1-[1-y_1 y_2 \ldots y_{r-1}]z)^{\alpha_{N+1}}(1-z)^{-\alpha_{N+1}}\, V(p_1)\, y_1^{L_0}\, V(p_2)\, y_2^{L_0} \ldots y_{N-1}^{L_0}\, V(p_N)|0\rangle \tag{A.2}$$

with the usual relation between the x's and y's

$$y_i = \frac{x_i(1 - x_{i-1} \ldots x_1 z)}{1 - x_i x_{i-1} \ldots x_1 z} \tag{A.3}$$

Defining $\bar{y}_i$ in a similar way to $\bar{x}_i$ we combine (A.1) and (A.2) to obtain

$$d\chi(x)\,(1-x_1 z)^{\alpha_1}(1-z)^{-\alpha_1}\,\theta(z)\,V(p_1)\,x_1^{L_0}\,V(p_2)\,x_2^{L_0}\ldots x_{N-1}^{L_0}\,V(p_N)|0\rangle$$

$$= \quad d\chi(\bar{y})\,(1-\bar{y}_1 z)^{\alpha_{N+1}}(1-z)^{-\alpha_{N+1}}\,V(\bar{p}_1)\,\bar{y}_1^{L_0}\,V(\bar{p}_2)\,\bar{y}_2^{L_0}\ldots\bar{y}_{N-1}^{L_0}\,V(\bar{p}_N)|0\rangle \tag{A.4}$$

which is the correct relation for the twisting operator to satisfy. Further since the relation between the x's and $\bar{y}$'s is the appropriate change of Chan variables, Θ will have all the usual operator duality properties [12].

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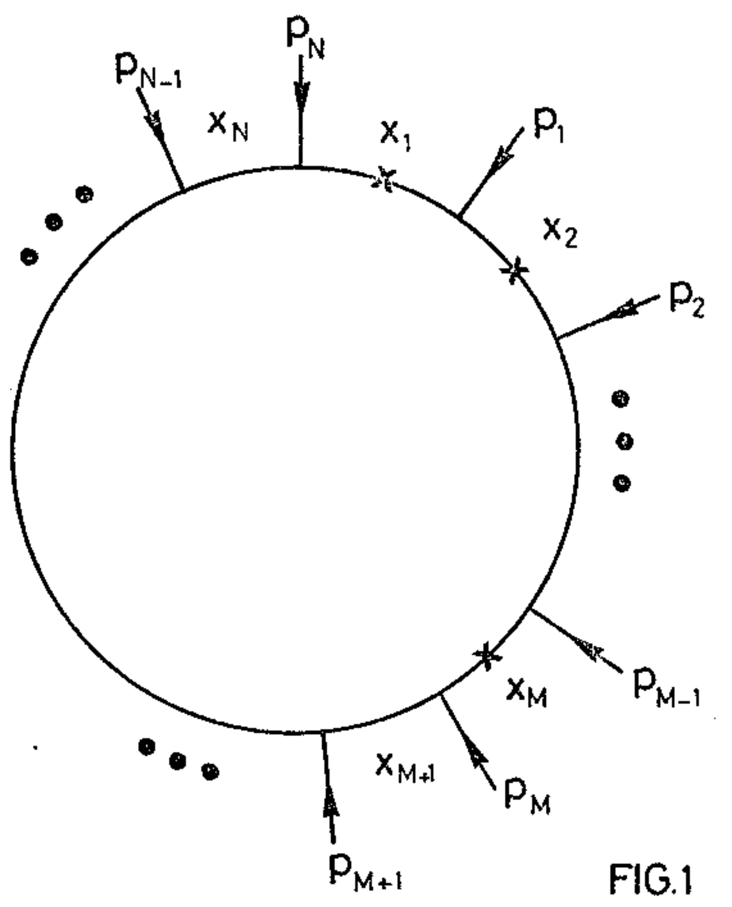
p_N
p_{N-1}
x_N
x_1
p_1
x_2
p_2
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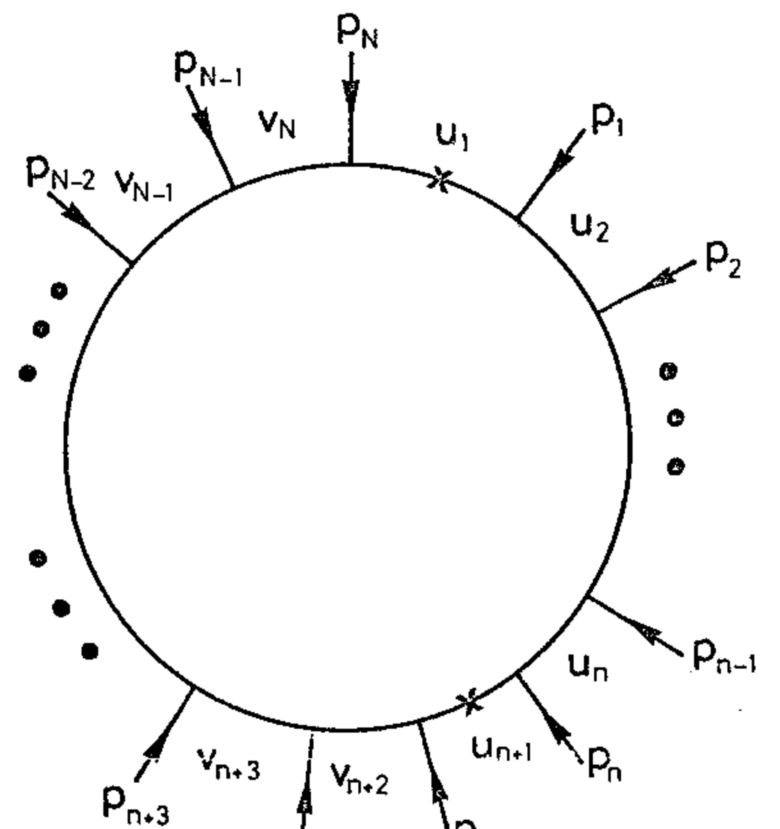
p_M
p_{M+1}

FIG.1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p_{N-1}
v_N
u_1
p_1
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v_{N-1}
u_2
p_2
p_{n-1}
u_n
u_{n+1}
p_n
v_{n+3}
v_{n+2}
p_{n+3}
p_{n+2}
p_{n+1}

FIG. 2

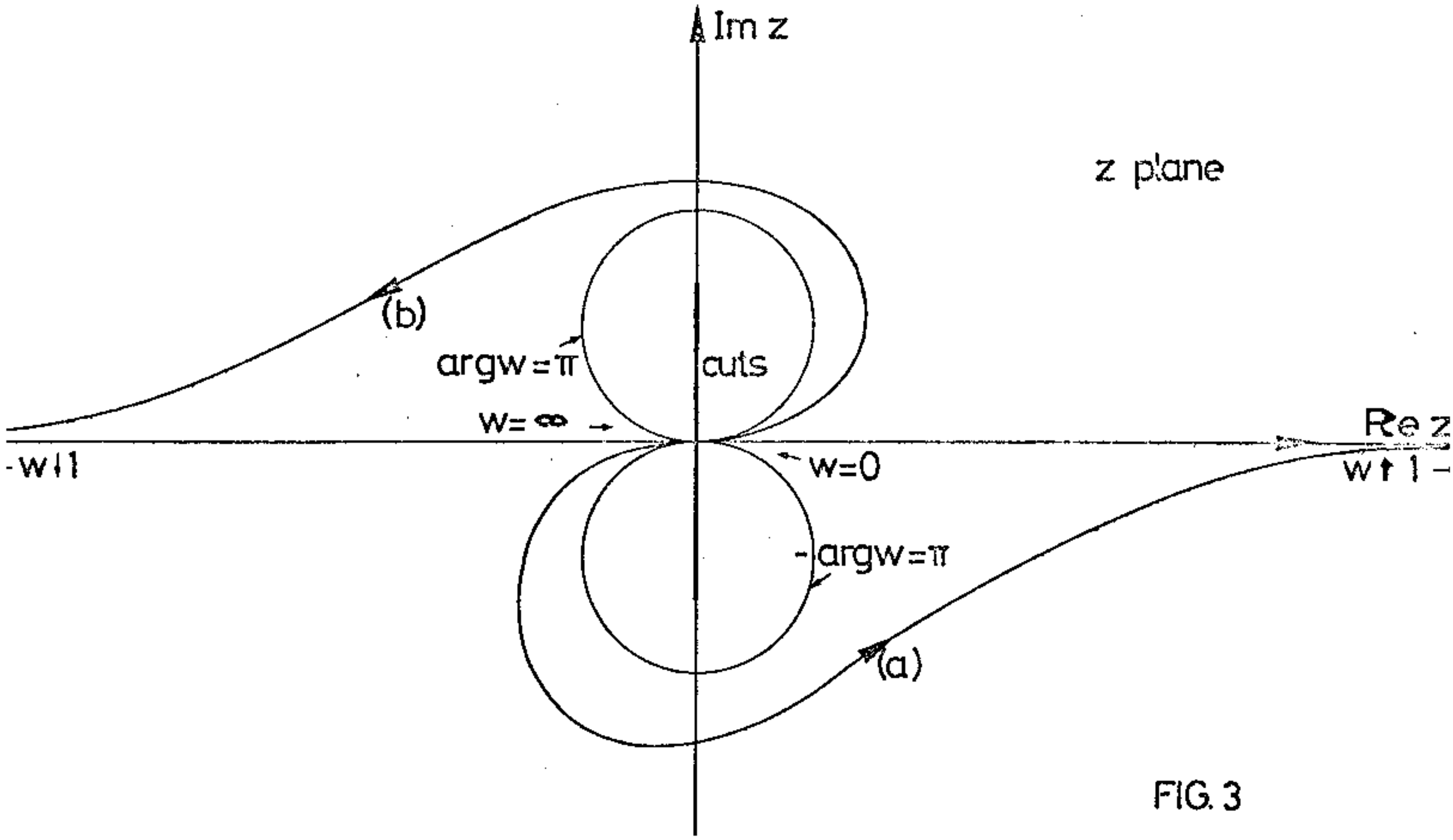

FIG. 3

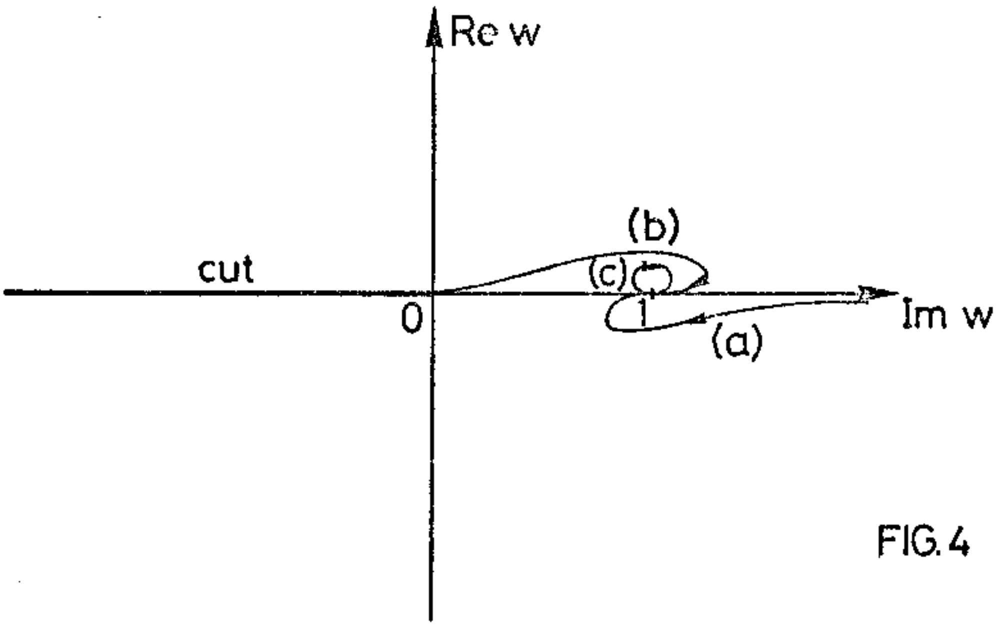

FIG. 4