Analytic Tableaux Calculi for KLM Logics of Nonmonotonic Reasoning

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18

We present tableau calculi for the logics of nonmonotonic reasoning defined by Kraus, Lehmann and Magidor (KLM). We give a tableau proof procedure for all KLM logics, namely preferential, loop-cumulative, cumulative, and rational logics. Our calculi are obtained by introducing suitable modalities to interpret conditional assertions. We provide a decision procedure for the logics considered and we study their complexity.

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1. INTRODUCTION

In the early 90s, Kraus, Lehmann, and Magidor [Kraus et al. 1990] (from now on KLM) proposed a formalization of nonmonotonic reasoning that was early recognized as a landmark. Their work stemmed from two sources: the theory of nonmonotonic consequence relations initiated by Gabbay [1985] and the preferential semantics proposed by Shoham [1987] as a generalization of circumscription. Their work led to a classification of nonmonotonic consequence relations, determining a hierarchy of stronger and stronger systems. The so-called *KLM properties* have been widely accepted as the "conservative core" of default reasoning. The role of KLM logics is similar to the role of AGM postulates in belief revision [Gardenförs 1988]: They give a set of postulates for default reasoning that any concrete reasoning mechanism should satisfy.

According to the KLM framework, defeasible knowledge is represented by a (finite) set of nonmonotonic conditionals or assertions of the form

$$A \mid\sim B$$

whose reading is *normally (or typically) the A's are B's*. The operator "$\mid\sim$" is nonmonotonic, in the sense that $A \mid\sim B$ does not imply $A \wedge C \mid\sim B$. For instance, a knowledge base K may consistently contain the following set of conditionals:

$$adult \mid\sim worker, adult \mid\sim taxpayer, student \mid\sim adult, student \mid\sim \neg worker,$$
$$student \mid\sim \neg taxpayer, retired \mid\sim adult, retired \mid\sim \neg worker$$

whose meaning is that adults typically work, adults typically pay taxes, students are typically adults, but they typically do not work, nor do they pay taxes, and so on. Observe that if $\mid\sim$ were interpreted as classical (or intuitionistic) implication, we would simply get $student \mid\sim \bot$, $retired \mid\sim \bot$, that is, typically there are no students, nor retired people, thereby obtaining a trivial knowledge base.

It is possible to derive new conditional assertions from the knowledge base by means of a set of inference rules. In KLM framework, the set of adopted inference rules defines some fundamental types of inference systems, namely, from the weakest to the strongest: Cumulative (C), Loop-Cumulative (CL), Preferential (P), and Rational (R) logic. All these systems allow to infer new assertions from a given knowledge base K without incurring the trivializing conclusions of classical logic. In our example, in none of them, we can infer $student \mid\sim worker$ or $retired \mid\sim worker$. In cumulative logics (both C and CL) we can infer $adult \wedge student \mid\sim \neg worker$ (giving preference to more specific information), in Preferential logic P we can also infer that $adult \mid\sim \neg retired$ (i.e., typical adults are not retired). In the rational case R, if we further know that $\neg(adult \mid\sim \neg married)$ (i.e., it is not the case that adults are typically unmarried), we can also infer that $adult \wedge married \mid\sim worker$.

From a semantic point of view, to each logic (C, CL, P, R) there corresponds one kind of model, namely a class of possible-world structures equipped with a preference relation among worlds or states. More precisely, for P we have models with a preference relation (an irreflexive and transitive relation) on worlds. For the stronger R the preference relation is further assumed to be *modular*. For the weaker logic CL, the transitive and irreflexive preference relation is defined on *states*, where a state can be identified, intuitively, with a set of worlds. In the weakest case of C, the preference relation is on states, as for CL, but it is no longer assumed to be transitive. In all cases, the meaning of a conditional assertion $A \mid\sim B$ is that B holds in the *most preferred* worlds/states where A holds.

In KLM framework the operator "$\mid\sim$" is considered as a metalanguage operator, rather than as a connective in the object language. However, it has been readily observed that KLM systems P and R coincide to a large extent with the flat (i.e. unnested) fragments of well-known conditional logics, once we interpret the operator "$\mid\sim$" as a binary connective [Crocco and Lamarre 1992; Boutilier 1994; Katsuno and Sato 1991].

A recent result by Friedman and Halpern [2001] has shown that preferential and rational logic are natural and general systems: Surprisingly enough, the axiom system of preferential (likewise of rational logic) is complete with respect to a wide spectrum of semantics, from ranked models, to parametrized probabilistic structures, ϵ-semantics, and possibilistic structures. The reason is that all these structures are examples of *plausibility structures* and the truth in them is captured by the axioms of preferential (or rational) logic. These results, and their extensions to the first-order setting [Friedman et al. 2000], are the source of a renewed interest in KLM framework. A considerable amount of research in the area has then concentrated in developing concrete mechanisms for plausible reasoning in accordance with KLM systems (P and R mostly). These mechanisms are defined by exploiting a variety of models of reasoning under uncertainty (ranked models, belief functions, possibilistic logic, etc., [Benferhat et al. 2000, 1997; Weydert 2003; Pearl 1990; Makinson 2005, 2003]) that provide, as we observed, alternative semantics to KLM systems. These mechanisms are based on the restriction of the semantics to preferred classes of models of KLM logics; this is also the case of Lehmann's notion of rational closure introduced in Lehmann and Magidor [1992] (not to be confused with the logic R). More recent research has also explored the integration of KLM framework with paraconsistent logics [Arieli and Avron 2000]. Finally, there has been some recent investigation on the relation between KLM systems and decision theory [Dubois et al. 2002, 2003].

Even if KLM was born as an inferential approach to nonmonotonic reasoning, curiously enough, there has not been much investigation on deductive mechanisms for these logics. In short, the state of the art is as follows.[1]

—Lehmann and Magidor [1992] have proved that validity in P is coNP-complete. Their decision procedure for P is more a theoretical tool than a practical

[1] More details about related work will be discussed in the Conclusions, Section 7.

algorithm. They have also provided another procedure for P that exploits its reduction to R. However, the reduction of P to R breaks down if Boolean combinations of conditionals are allowed; indeed, it is exactly when such combinations are allowed that the difference between P and R arises.

—A fairly complicated tableau proof procedure for C has been given in Artosi et al. [2002].

—In Giordano et al. [2005b, 2003] some labeled tableaux calculi have been defined for the conditional logic CE and its main extensions, including CV. The flat fragment (i.e., without nested conditionals) of CE and of CV corresponds respectively to P and to R. These calculi need a rather complex loop-checking mechanism to ensure termination. It is not clear if they match complexity bounds and if they can be adapted in a simple way to CL and to C.

—Finally, decidability of P and R has also been obtained by interpreting them into standard modal logics, as done by Boutilier [1994]. However, his mapping is not very direct and natural, as we discuss next.

—To the best of our knowledge, for CL no decision procedure and complexity bound was known before the present work.

In this work we introduce tableau procedures for all KLM logics. We consider first the preferential logic P. Our approach is based on a novel interpretation of P into modal logics. As a difference with previous approaches (e.g., Crocco and Lamarre [1992] and Boutilier [1994]), that take S4 as the modal counterpart of P, we consider here Gödel-Löb modal logic of provability G (see, for instance, Hughes and Cresswell [1984]).

Our tableau method provides a sort of runtime translation of P into modal logic G. The idea is simply to interpret the preference relation as an accessibility relation: A conditional $A \mid\sim B$ holds in a model if B is true in all minimal A-worlds, where a world w is an A-world if it satisfies A, and it is a minimal A-world if there is no A-world w' preferred to w. The relation with modal logic G is motivated by the fact that we assume, following KLM, the so-called *smoothness condition*, which is related to the well-known *limit assumption*. This condition ensures that minimal A-worlds exist whenever there are A-worlds, by preventing infinitely descending chains of worlds. This condition therefore corresponds to the finite-chain condition on the accessibility relation (as in modal logic G). Therefore, our interpretation of conditionals is different from the one proposed by Boutilier, who rejects the smoothness condition and then gives a less natural (and more complicated) interpretation of P into modal logic S4.

We do not give a formal translation of P into G. Rather, we directly provide a tableau calculus for P. It is possible to notice some similarities between some of the rules for P and some of the rules for G. This is due to the correspondence between the semantics of the two logics. For deductive purposes, we believe that our approach is more direct, intuitive, and efficient than translating P into G and then using a calculus for G.

We are able to extend our approach to the cases of CL and C by using a second modality which takes care of states. Regarding CL, we show that we can map CL-models into P-models with an additional modality. The very fact that it is possible to interpret CL into P by means of an additional modality does not seem to be previously known and might be of independent interest. In both cases, P and CL, we can define a decision procedure and obtain also a complexity bound for these logics, namely that they are both coNP-complete. In case of CL this bound is new, to the best of our knowledge.

We treat C in a similar way: We can establish a mapping between cumulative models and a kind of bimodal models. However, because of the lack of transitivity, the target modal logic is no longer G. The reason is that the *smoothness condition* (for any formula A, if a state satisfies A, then either it is minimal or it admits a smaller minimal state satisfying A) can no longer be identified with the finite-chain condition of G. As a matter of fact, the smoothness condition for C cannot be identified with any property of the accessibility relation, as it involves unavoidably the evaluation of formulas in worlds. We can still derive a tableau calculus based on our semantic mapping. But we pay a price: As a difference with P and CL the calculus for C requires an analytic cut rule to account for the smoothness condition. This calculus gives nonetheless a decision procedure for C.

Finally, we consider the strongest logic R; similarly to the other cases, our calculus is based on an interpretation of R into a strengthening of modal logic G, where the preference relation is assumed modular (previous approaches [Crocco and Lamarre 1992; Boutilier 1994] take S4.3 as the modal counterpart of R). As a difference with the tableau calculi introduced for P, CL, and C, here we develop a *labeled* tableau calculus, which seems to be the most natural approach in order to capture the modularity of the preference relation. The calculus defines a systematic procedure which allows the satisfiability problem for R to be decided in nondeterministic polynomial time, in accordance with the known complexity results for this logic.

From the completeness of our calculi we get for free the finite model property for all the logics considered. With the exception of the calculus for C, in order to ensure termination, our tableau procedures for KLM logics do not need any loop-checking, nor blocking, nor caching machinery. Termination is guaranteed only by adopting a restriction on the order of application of the rules.

All the calculi presented in this article have been implemented in SICStus Prolog. To the best of our knowledge, our theorem prover, called KLMLean, is the first one for KLM logics.[2]

The plan of the article is as follows: In Section 2, we recall KLM logics (from the strongest to the weakest): R, P, CL, and C, and we show how their semantics can be represented by standard Kripke models. In Section 3 we give a terminating tableau calculus for P. We then propose a refinement of the calculus that gives a coNP decision procedure. The latter is based on a tighter semantics of P in terms of *multilinear models*. In Section 4 we propose similar calculi for CL. In Section 5, we give a tableau calculus for C. As mentioned earlier, the calculus requires a form of cut rule. We prove, however, that we can restrict its

[2]The theorem prover KLMLean is not presented here. A description can be found in Olivetti and Pozzato [2005] and in Giordano et al. [2007b]. KLMLean can be downloaded at http://www.di.unito.it/∼pozzato/klmlean2.0.

REF.	$A \mid\!\sim A$
LLE.	If $\vdash_{PC} A \leftrightarrow B$, then $\vdash (A \mid\!\sim C) \rightarrow (B \mid\!\sim C)$
RW.	If $\vdash_{PC} A \rightarrow B$, then $\vdash (C \mid\!\sim A) \rightarrow (C \mid\!\sim B)$
CM.	$((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \wedge B \mid\!\sim C)$
AND.	$((A \mid\!\sim B) \wedge (A \mid\!\sim C)) \rightarrow (A \mid\!\sim B \wedge C)$
OR.	$((A \mid\!\sim C) \wedge (B \mid\!\sim C)) \rightarrow (A \vee B \mid\!\sim C)$
RM.	$((A \mid\!\sim B) \wedge \neg(A \mid\!\sim \neg C)) \rightarrow ((A \wedge C) \mid\!\sim B)$
LOOP.	$(A_0 \mid\!\sim A_1) \wedge (A_1 \mid\!\sim A_2)...(A_{n-1} \mid\!\sim A_n) \wedge (A_n \mid\!\sim A_0) \rightarrow (A_0 \mid\!\sim A_n)$
CUT.	$((A \mid\!\sim B) \wedge (A \wedge B \mid\!\sim C)) \rightarrow (A \mid\!\sim C)$

Fig. 1. Axioms and rules of KLM logics. We use $\vdash_{PC}$ to denote provability in the propositional calculus, whereas $\vdash$ is used to denote provability in a given KLM logic.

application in an analytic way. In Section 6 we describe a labeled, terminating tableau calculus for R, then we refine it in order to describe a coNP decision procedure.

2. KLM LOGICS

We briefly recall the axiomatizations and semantics of the KLM systems. For the sake of exposition, we present the systems in the order from strongest to weakest: R, P, CL, and C. For a complete picture of KLM systems, see Kraus et al. [1990] and Lehmann and Magidor [1992]. The language of KLM logics consists just of conditional assertions $A \mid\!\sim B$. We consider a richer language allowing Boolean combinations of assertions and propositional formulas. Our language $\mathcal{L}$ is defined from a set of propositional variables ATM, the Boolean connectives, and the conditional operator $\mid\!\sim$. We use $A, B, C, \ldots$ to denote propositional formulas (that do not contain conditional formulas), whereas $F, G, \ldots$ are used to denote all formulas (including conditionals); $\Gamma, \Delta, \ldots$ represent sets of formulas, whereas $X, Y, \ldots$ denote sets of sets of formulas. The formulas of $\mathcal{L}$ are defined as follows: if A is a propositional formula, $A \in \mathcal{L}$; if A and B are propositional formulas, $A \mid\!\sim B \in \mathcal{L}$; if F is a Boolean combination of formulas of $\mathcal{L}$, $F \in \mathcal{L}$.

2.1 Rational Logic R

The axiomatization of R consists of all axioms and rules of propositional calculus together with the axioms and rules REF, LLE, RW, CM, AND, OR, and RM, shown in Figure 1. REF (reflexivity) states that A is always a default conclusion of A. LLE (left logical equivalence) states that the syntactic form of the antecedent of a conditional formula is irrelevant. RW (right weakening) describes a similar property of the consequent. This allows to combine default and logical reasoning [Friedman and Halpern 2001]. CM (cautious monotonicity) states that if B and C are two default conclusions of A, then adding one of the two conclusions to A will not cause the retraction of the other conclusion. AND states that it is possible to combine two default conclusions. OR states that it is allowed to reason by cases: If C is the default conclusion of two premises A and B, then it is also the default conclusion of their disjunction. RM is the rule of rational monotonicity, which characterizes the logic R. This rule allows a conditional to be inferred from a set of conditionals in absence of other information. More precisely, "it says that an agent should not have to retract any previous defeasible conclusion when learning about a new fact the negation of which was not previously derivable" [Lehmann and Magidor 1992].

The semantics of R is defined by considering possible world structures with a strict partial order $<$, namely, an irreflexive and transitive relation, that we call *preference relation*. The meaning of $w < w'$ is that w is preferred to w'. The preference relation is also supposed to be *modular*: For all w, w_1 and w_2, if $w_1 < w_2$ then either $w_1 < w$ or $w < w_2$. We have that $A \mid\!\sim B$ holds in a model $\mathcal{M}$ if B holds in all *minimal worlds* (with respect to the relation $<$) where A holds. This definition makes sense provided minimal worlds for A exist, whenever there are A-worlds. This is ensured by the smoothness condition in the next definition.

Definition 2.1 (*Semantics of R*) [*Lehmann and Magidor* 1992*, Definition* 14]. A rational model is a triple

$$\mathcal{M} = \langle \mathcal{W}, <, V \rangle$$

where:

—$\mathcal{W}$ is a nonempty set of items called worlds;

—$<$ is an irreflexive, transitive, and modular relation on $\mathcal{W}$;

—V is a function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the set of atoms holding in that world.

We define the truth conditions for a formula F as follows.

—If F is a Boolean combination of formulas, $\mathcal{M}, w \models F$ is defined as for propositional logic;

—Let A be a propositional formula; we define $Min_<(A) = \{w \in \mathcal{W} \mid \mathcal{M}, w \models A$ and $\forall w', w' < w$ implies $\mathcal{M}, w' \not\models A\}$;

—$\mathcal{M}, w \models A \mid\!\sim B$ if for all w', if $w' \in Min_<(A)$ then $\mathcal{M}, w' \models B$.

(Smoothness Condition). The relation $<$ satisfies the following condition, called *smoothness*: if $\mathcal{M}, w \models A$, then $w \in Min_<(A)$ or $\exists w' \in Min_<(A)$ such that $w' < w$.

We say that a formula F is satisfiable if there exists a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and a world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models F$. A formula F is *valid in a model* $\mathcal{M}$, denoted with $\mathcal{M} \models F$, if $\mathcal{M}, w \models F$ for every $w \in \mathcal{W}$. A formula is *valid* if it is valid in every model $\mathcal{M}$.

Observe that the preceding definition of rational model extends the one given by KLM to Boolean combinations of formulas. Notice also that the truth conditions for conditional formulas are given with respect to single possible worlds for sake of uniformity. Since the truth value of a conditional only depends on global properties of $\mathcal{M}$, we have that: $\mathcal{M}, w \models A \mathrel{|\!\sim} B$ just in case for all worlds w' of the model, it holds that $\mathcal{M}, w' \models A \mathrel{|\!\sim} B$, namely $\mathcal{M} \models A \mathrel{|\!\sim} B$.

It is easy to see that the smoothness condition together with the transitivity of $<$ implies the following *strong smoothness condition*, namely that for all A and w, independently from whether $\mathcal{M}, w \models A$ or not, if there is a world w' preferred to w that satisfies A (i.e., if $\exists w' : w' < w$ and $\mathcal{M}, w' \models A$), then there is also a *minimal* such world (i.e., $\exists w'' : w'' \in Min_<(A)$ and $w'' < w$). This follows immediately: by the smoothness condition, since $\mathcal{M}, w' \models A$, either $w' \in Min_<(A)$ (and the property immediately follows) or $\exists w''$ such that $w'' < w'$ and $w'' \in Min_<(A)$; in turn, by transitivity $w'' < w$, hence the property follows.[3] Observe also that by the modularity of $<$ it follows that possible worlds of $\mathcal{W}$ are *clustered* into equivalence classes, each class consisting of worlds that are incomparable to one another; the classes are totally ordered.[4] In other words, the property of modularity determines a *ranking* of worlds so that the semantics of R can be specified equivalently in terms of *ranked* models [Lehmann and Magidor 1992].

In our tableau calculus for R that we will introduce in Section 6, we need a slightly extended language $\mathcal{L}_R$. $\mathcal{L}_R$ extends $\mathcal{L}$ by formulas of the form $\Box A$, where A is propositional, whose intuitive meaning is that $\Box A$ holds in a world w if A holds in all the worlds preferred to w (i.e., in all w' such that $w' < w$). We extend the notion of rational model to provide an evaluation of boxed formulas as follows.

Definition 2.2 (*Truth Condition of Modality* $\Box$). We define the truth condition of a boxed formula as follows:

$$\mathcal{M}, w \models \Box A \text{ if, for every } w' \in \mathcal{W}, \text{ if } w' < w \text{ then } \mathcal{M}, w' \models A$$

From definition of $Min_<(A)$ in Definition 2.1 before, and Definition 2.2, it follows that for any formula A, $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \land \Box\neg A$.

Notice that by the strong smoothness condition, it holds that if $\mathcal{M}, w \not\models \Box\neg A$, then $\exists w' < w$ such that $\mathcal{M}, w' \models A \land \Box\neg A$. If we regard the relation $<$ as the inverse of the accessibility relation R (thus xRy if $y < x$), it immediately follows that the strong smoothness condition is an instance of the property G restricted to A propositional ($\neg\Box\neg A \rightarrow \Diamond(\Box\neg A \land A)$). Hence it turns out that the modality $\Box$ has the properties of modal system G, in which the accessibility relation is transitive and does not have infinite ascending chains.

[3]It is easy to see that in *frames* $F = \langle \mathcal{W}, < \rangle$ where the $<$ is irreflexive and transitive, the smoothness condition entails that $<$ does not have infinite descending chains and here is a proof: Suppose that F contains an infinite descending chain $\sigma = w_0, w_1, \ldots, w_i \ldots$, with $w_{i+1} < w_i$. Let P be an atom and let V be a propositional evaluation such that $P \in V(w)$ if and only if $w \in \sigma$, then the smoothness condition on P is violated by each world in σ, since for each $w_i \in \sigma$, we have $w_i \models P$ but there is not a *minimal* $w' < w_i$ such that $w' \models P$. Observe that this is a property on frames and not on models.

[4]Notice that the worlds themselves may be incomparable, since the relation $<$ is not assumed to be (weakly) connected.

Since we have introduced boxed formulas for capturing the notion of minimality among worlds, in the rest of the article we will only use this modality in front of negated formulas. Hence, to be precise, the language $\mathcal{L}_R$ of our tableau extends $\mathcal{L}$ with modal formulas of the form $\Box\neg A$.

2.2 Preferential Logic P

The axiomatization of P can be obtained from the axiomatization of R by removing the axiom RM, that is, it consists of all axioms and rules of propositional calculus together with the axioms and rules REF, LLE, RW, CM, AND, and OR (shown in Figure 1). As for R, the semantics of P is defined by considering possible world structures with a preference relation (an irreflexive and transitive relation), which is no longer assumed modular.

Definition 2.3 (*Semantics of P*), [*Kraus et al.* 1990, *Definition* 16]. A preferential model is a triple $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ where $\mathcal{W}$ and V are defined as for rational models in Definition 2.1, and $<$ is an irreflexive and transitive relation on $\mathcal{W}$. We assume that $<$ satisfies the smoothness condition, defined as in Definition 2.1. The truth conditions for a formula F and the notions of satisfiability and validity of a formula are defined as for rational models in Definition 2.1.

As for rational models, we have extended the definition of preferential models given by KLM in order to deal with Boolean combinations of formulas.

We define the satisfiability of conditional formulas with respect to worlds rather than with respect to models for sake of uniformity . As for R, by the transitivity of $<$, the smoothness condition is equivalent to the strong smoothness condition, corresponding to the finite-chain condition for $<$.

Here again, we consider the language $\mathcal{L}_P$ of the calculus introduced in Section 3; $\mathcal{L}_P$ corresponds to the language $\mathcal{L}_R$, that is, it extends $\mathcal{L}$ by boxed formulas of the form $\Box\neg A$. It follows that, even in P, we can prove that for any formula A, $w \in Min_<(A)$ iff $\mathcal{M}, w \models A \wedge \Box\neg A$.

2.2.1 *Multilinear models for* P. In the rest of the article we will need a special kind of preferential model, that we call *multilinear*. As we will see, these models will be useful in order to provide an optimal calculus for P. Indeed, as we will see in Section 3.1, our calculus for P based on multilinear models will allow us to define proof search procedures for testing the satisfiability of a set of formulas in P in nondeterministic polynomial time. This result matches the known complexity results for P, according to which the problem of validity for P is in coNP.

Definition 2.4. A finite preferential model $\mathcal{M} = (\mathcal{W}, <, V)$ is *multilinear* if $\mathcal{W}$ can be partitioned into a set of components $\mathcal{W}_i$ for $i = 1, \ldots, n$, that is $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$, and: (1) the relation $<$ is a total order on each $\mathcal{W}_i$; (2) the elements in two different components $\mathcal{W}_i$ and $\mathcal{W}_j$ are incomparable with respect to $<$.

The following theorem shows that we can restrict our consideration to multilinear models and generalizes Lemma 8 in Lehmann and Magidor [1992].

The proof can be found in the electronic appendix accessible through the ACM Digital Library.

THEOREM 2.5. *Let Γ be a set of formulas, if Γ is satisfiable with respect to the logic* P*, then it has a multilinear model.*

2.3 Loop-Cumulative Logic CL

The next KLM logic we consider is CL, weaker than P. The axiomatization of CL can be obtained from the axiomatization of P by removing the axiom OR and by adding the infinite set of LOOP axioms and the axiom CUT (see Figure 1), namely it consists of all axioms and rules of propositional calculus together with the axioms and rules REF, LLE, RW, CM, AND, LOOP, and CUT. Notice that LOOP and CUT are derivable in P (and therefore in R).

The following definition is essentially the same as Definition 13 in Kraus et al. [1990], but it is extended to Boolean combinations of conditionals.

Definition 2.6 (*Semantics of* CL). A loop-cumulative model is a tuple

$$\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$$

where:

—S is a nonempty set, whose elements are called states;

—$\mathcal{W}$ is a nonempty set, whose elements are called worlds;

—$l : S \mapsto pow(\mathcal{W})$ is a function that labels every state with a nonempty set of worlds;

—$<$ is an irreflexive and transitive relation on S;

—V is a valuation function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the atoms holding in that world.

For $s \in S$ and A propositional, we let $\mathcal{M}, s \models\!\equiv A$ if $\forall w \in l(s)$, $\mathcal{M}, w \models A$, where $\mathcal{M}, w \models A$ is defined as in propositional logic. Let $Min_<(A)$ be the set of minimal states s such that $\mathcal{M}, s \models\!\equiv A$. We define $\mathcal{M}, s \models\!\equiv A \mid\!\sim B$ if $\forall s' \in Min_<(A)$, $\mathcal{M}, s' \models\!\equiv B$. The relation $\models\!\equiv$ can be extended to Boolean combinations of conditionals in the standard way. We assume that $<$ satisfies the smoothness condition (relatively to states and $\models\!\equiv$) .

The previous notion of cumulative model extends the one given by KLM to Boolean combinations of conditionals. A further extension to arbitrary Boolean combinations will be provided by the notion of CL-preferential model next.

Here again, we define satisfiability of conditionals with respect to states rather than with respect to models for uniformity reasons. Indeed, a conditional is satisfied by a state of a model if and only if it is satisfied by all the states of that model, hence by the whole model.

As for P and R, by the transitivity of $<$, the smoothness condition is equivalent to the strong smoothness condition. In turn, this entails that $<$ does not have infinite descending chains.

We show that we can map loop-cumulative models into preferential models extended with an additional accessibility relation R. We call these preferential models *CL-preferential models*. The idea is to represent states as sets of possible worlds related by R in such a way that a formula is satisfied in a state s just in case it is satisfied in all possible worlds w' accessible from a world w corresponding to s. The syntactic counterpart of the extra accessibility relation R is a modality L. Given a loop-cumulative model $\mathcal{M}$ and the corresponding CL-preferential model $\mathcal{M}'$, $\mathcal{M}, s \models A$ iff for a world $w \in \mathcal{M}'$ corresponding to s, we have that $\mathcal{M}', w \models LA$. As we will see, this mapping enables us to use a variant of the tableau calculus for P to deal with system CL. As for P, the tableau calculus for CL will use boxed formulas. In addition, it will also use L-formulas. Thus, the formulas that appear in the tableaux for CL belong to the language $\mathcal{L}_L$ obtained from $\mathcal{L}$ as follows: (i) If A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$; (ii) if A, B are propositional, then $A \mid\sim B \in \mathcal{L}_L$; (iii) if F is a Boolean combination of formulas of $\mathcal{L}_L$, then $F \in \mathcal{L}_L$. Observe that the only allowed combination of $\Box$ and L is in formulas of the form $\Box\neg LA$, where A is propositional.

We can map loop-cumulative models into preferential models with an additional accessibility relation as defined in the following.

Definition 2.7 (*CL-Preferential Models*). A CL-preferential model has the form

$$\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$$

where:

—$\mathcal{W}$ and V are defined as for preferential models in Definition 2.3;

—$<$ is an irreflexive and transitive relation on $\mathcal{W}$;

—R is a serial relation on $\mathcal{W}$.

We add to the truth conditions for preferential models in Definition 2.3 the following clause.

$$\mathcal{M}, w \models LA \text{ if, for all } w', wRw' \text{ implies } \mathcal{M}, w' \models A$$

The relation $<$ satisfies the following *smoothness condition*: If $\mathcal{M}, w \models LA$, then $w \in Min_<(LA)$ or $\exists w' \in Min_<(LA)$ such that $w' < w$.
Moreover, we write $\mathcal{M}, w \models A \mid\sim B$ if for all $w' \in Min_<(LA)$ we have $\mathcal{M}, w' \models LB$.

We can prove that, given a loop-cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$ satisfying a Boolean combination of conditional formulas, it is possible to build a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V' \rangle$ satisfying the same combination of conditionals. We build a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V' \rangle$ as follows:

—$\mathcal{W}' = \{(s, w) : s \in S \text{ and } w \in l(s)\}$;

—$(s, w)R(s, w')$ for all $(s, w), (s, w') \in \mathcal{W}'$;

—$(s, w) <' (s', w')$ if $s < s'$;

—$V'(s, w) = V(w)$.

Vice versa, given a CL-preferential model $\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$ satisfying a Boolean combination of conditional formulas, it is possible to build a loop-cumulative model $\mathcal{M}' = \langle S, \mathcal{W}, l, <', V' \rangle$ satisfying the same combination of conditional formulas. The model $\mathcal{M}'$ is defined as follows (we define $Rw = \{w' \in \mathcal{W} \mid (w, w') \in R\}$):

—$S = \{(w, Rw) \mid w \in \mathcal{W}\}$;
—$l((w, Rw)) = Rw$;
—$(w, Rw) <' (w', Rw')$ if $w < w'$;
—$V'(w) = V(w)$.

Therefore, we obtain the following proposition.

PROPOSITION 2.8 [POZZATO 2007; GIORDANO ET AL. 2007A]. *A Boolean combination of conditional formulas is satisfiable in a loop-cumulative model $\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$ iff it is satisfiable in a CL-preferential model $\mathcal{M}' = \langle \mathcal{W}', R, <', V' \rangle$.*

2.3.1 *Multilinear Models for* CL. Similarly to the case of P, we can define multilinear CL-preferential models as follows.

Definition 2.9. A finite CL-preferential model $\mathcal{M} = (\mathcal{W}, R, <, V)$ is *multilinear* if the set of worlds $\mathcal{W}$ can be partitioned into a set of components $\mathcal{W}_i$ for $i = 1, \ldots, n$ (that is $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$), and in each $\mathcal{W}_i$:

(1) there is a totally ordered chain of worlds $w_1, w_2, \ldots, w_h$ with respect to $<$ (i.e., $w_1 < w_2 < \ldots < w_h$) such that all other worlds $w \in \mathcal{W}_i$ are R-accessible from some w_l in the chain, namely $\forall w \in \mathcal{W}_i$ such that $w \neq w_1, w_2, \ldots, w_h$, there is w_i, $i = 1, 2, \ldots, h$ such that $w_i R w$;
(2) for all w', w'', w''', if $w' < w''$ and $w'' R w'''$, then $w' < w'''$.

Moreover, the elements of different $\mathcal{W}_i$ are incomparable with respect to $<$.

Observe that the worlds in each $\mathcal{W}_i$ are partitioned into two sets of worlds $\mathcal{W}_i^1$ and $\mathcal{W}_i^2$. Worlds in $\mathcal{W}_i^1$ are organized in a totally ordered chain, whereas worlds in $\mathcal{W}_i^2$ are reachable from those in $\mathcal{W}_i^1$ through R. Similarly to the case of P, we can prove the following theorem.

THEOREM 2.10. *Let Γ be any set of formulas, if Γ is satisfiable in a CL-preferential model, then it has a multilinear CL-preferential model.*

2.4 Cumulative Logic C

The weakest logical system considered by KLM [Kraus et al. 1990] is cumulative logic C. System C is weaker than CL considered before, since it does not have the set of (LOOP) axioms. The axiomatization of C consists of all axioms and rules of propositional calculus together with the axioms and rules REF, LLE, RW, CM, AND, and CUT (see Figure 1). At a semantic level, the difference between CL models and C models is that in CL models the relation $<$ is transitive, whereas in C it is not. Thus, cumulative C models are defined as follows.

Definition 2.11 (*Semantics of* C) [*Kraus et al.* 1990, *Definitions* 5, 6, 7]. A cumulative model is a tuple $\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$, where S, $\mathcal{W}$, l, and V are defined as in Definition 2.6, whereas $<$ is an irreflexive relation on S. The truth definitions of formulas are as for loop-cumulative models in Definition 2.6. We assume that $<$ satisfies the smoothness condition.

Since $<$ is not transitive, we can no longer show that the smoothness condition is equivalent to the strong smoothness condition; furthermore, $<$ may have infinite descending chains. As a matter of fact, in C there can be sets of formulas which can be satisfied only by models containing infinite descending chains (or cycles, if the model is finite). As an example, consider the following set of formulas: $\Gamma = \{\neg(C \mid\!\sim B), C \mid\!\sim A, A \mid\!\sim B, B \mid\!\sim C\}$. This is an instance of the negation of (LOOP). Γ can be satisfied by a C model just in case it contains a cycle $w_1, w_2, w_3, w_1, w_2, w_3, \ldots$, such that w_1 is a minimal C-world, w_2 a minimal A-world, w_3 a minimal B-world; in turn w_3 must be preceded again by a minimal C-world as w_1, and so on. In case of a finite C model, Γ can be satisfied only if the model contains cycles made out of the same sequence of worlds. On the contrary, it is easy to see that this model does not satisfy the strong smoothness condition.

Similarly to what we have done for loop-cumulative models, we can establish a correspondence between cumulative models and preferential models augmented with an accessibility relation in which the preference relation $<$ is an irreflexive relation satisfying the smoothness condition. We call these models C-preferential models.

Definition 2.12 (*C-Preferential Models*). A C-preferential model has the form $\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$ where: $\mathcal{W}$ is a nonempty set of items called worlds R is a serial accessibility relation; $<$ is an irreflexive relation on $\mathcal{W}$ satisfying the smoothness condition for L-formulas; V is a function $V : \mathcal{W} \longmapsto pow(ATM)$, which assigns to every world w the atomic formulas holding in that world. The truth conditions for the boolean cases are defined in the obvious way. Truth conditions for modal and conditional formulas are the same as in CL-preferential models in Definition 2.7:

—$M, w \models LA$ if for all w', wRw' implies $\mathcal{M}, w' \models A$;
—$\mathcal{M}, w \models A \mid\!\sim B$ if for all $w' \in Min_<(LA)$, we have $\mathcal{M}, w' \models LB$.

The correspondence between cumulative and preferential models is established by the following proposition. Its proof is the same as the proof of Proposition 2.8 (except for transitivity).

PROPOSITION 2.13 [POZZATO 2007; GIORDANO ET AL. 2007A]. *A Boolean combination of conditional formulas is satisfiable in a cumulative model* $\mathcal{M} = \langle S, \mathcal{W}, l, <, V \rangle$ *iff it is satisfiable in a C-preferential model* $\mathcal{M}' = \langle \mathcal{W}', R, <', V' \rangle$.

In the following sections we present the tableaux calculi for the four KLM logics. For the purpose of exposition, we present first the calculus for P, which is the simplest one. Then we present the calculi for CL, C, and R.

3. THE TABLEAU CALCULUS FOR PREFERENTIAL LOGIC P

In this section we present a terminating tableau calculus for P called $\mathcal{T}\mathrm{P}^{\mathbf{T}}$. We then analyze it in order to give an optimal decision procedure for P, matching the known complexity results for this logic.

$$(\mathbf{AX})\ \Gamma, P, \neg P; \Sigma \quad \text{with } P \in ATM \qquad (\neg)\frac{\Gamma, \neg\neg F; \Sigma}{\Gamma, F; \Sigma} \qquad (\wedge^+)\frac{\Gamma, F \wedge G; \Sigma}{\Gamma, F, G; \Sigma}$$

$$(\wedge^-)\frac{\Gamma, \neg(F \wedge G); \Sigma}{\Gamma, \neg F; \Sigma \qquad \Gamma, \neg G; \Sigma} \qquad (\vee^+)\frac{\Gamma, F \vee G; \Sigma}{\Gamma, F; \Sigma \qquad \Gamma, G; \Sigma} \qquad (\vee^-)\frac{\Gamma, \neg(F \vee G); \Sigma}{\Gamma, \neg F, \neg G; \Sigma}$$

$$(\rightarrow^+)\frac{\Gamma, F \rightarrow G; \Sigma}{\Gamma, \neg F; \Sigma \qquad \Gamma, G; \Sigma} \qquad (\rightarrow^-)\frac{\Gamma, \neg(F \rightarrow G); \Sigma}{\Gamma, F, \neg G; \Sigma} \qquad (\Box^-)\frac{\Gamma, \neg\Box\neg A; \Sigma}{A, \Box\neg A, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\!\sim^{\pm}}, \Sigma; \emptyset}$$

$$(\mid\!\sim^+)\frac{\Gamma, A \mid\!\sim B; \Sigma}{\Gamma, \neg A; \Sigma, A \mid\!\sim B \qquad \Gamma, \neg\Box\neg A; \Sigma, A \mid\!\sim B \qquad \Gamma, B; \Sigma, A \mid\!\sim B} \qquad (\mid\!\sim^-)\frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{A, \Box\neg A, \neg B, \Gamma^{\mid\!\sim^{\pm}}; \emptyset}$$

Fig. 2. Tableau calculus $\mathcal{T}\mathrm{P}^{\mathbf{T}}$.

As already mentioned in Section 2.2, we consider the language $\mathcal{L}_P$, which extends $\mathcal{L}$ by boxed formulas of the form $\Box\neg A$.

Definition 3.1 (*The Calculus* $\mathcal{T}\mathrm{P}^{\mathbf{T}}$). The rules of the calculus manipulate sets of formulas Γ. We write Γ, F as a shorthand for $\Gamma \cup \{F\}$. Moreover, given Γ we define the following sets.

—$\Gamma^{\Box} = \{\Box\neg A \mid \Box\neg A \in \Gamma\}$
—$\Gamma^{\Box^{\downarrow}} = \{\neg A \mid \Box\neg A \in \Gamma\}$
—$\Gamma^{\mid\!\sim^{+}} = \{A \mid\!\sim B \mid A \mid\!\sim B \in \Gamma\}$
—$\Gamma^{\mid\!\sim^{-}} = \{\neg(A \mid\!\sim B) \mid \neg(A \mid\!\sim B) \in \Gamma\}$
—$\Gamma^{\mid\!\sim^{\pm}} = \Gamma^{\mid\!\sim^{+}} \cup \Gamma^{\mid\!\sim^{-}}$

The tableau rules are given in Figure 2. A tableau is a tree whose nodes are pairs $\Gamma; \Sigma$, where Γ is a set of formulas and Σ is a set of conditional formulas $A \mid\!\sim B$. A branch is a sequence of nodes $(\Gamma_1; \Sigma_1), (\Gamma_2; \Sigma_2), \ldots, (\Gamma_n; \Sigma_n), \ldots$ Each node $\Gamma_i; \Sigma_i$ is obtained from its immediate predecessor $\Gamma_{i-1}; \Sigma_{i-1}$ by applying a rule of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$, having $\Gamma_{i-1}; \Sigma_{i-1}$ as the premise and $\Gamma_i; \Sigma_i$ as one of its conclusions. A branch is closed if one of its nodes is an instance of (AX), otherwise it is open. We say that a tableau is closed if all its branches are closed. A node $\Gamma; \Sigma$ is consistent if no tableau for $\Gamma; \Sigma$ is closed. We call *tableau for* $\Gamma; \Sigma$ a tableau having $\Gamma; \Sigma$ as a root.

In order to check whether a set of formulas Γ is unsatisfiable, we verify if there is a closed tableau for $\Gamma; \emptyset$. First of all, we distinguish between static and dynamic rules. The rules $(\mid\!\sim^-)$ and $(\Box^-)$ are called *dynamic*, since their conclusions represent another world with respect to the one represented by the premise; the other rules are called *static*, since the world represented by premise and conclusion(s) is the same. The rules for the Boolean propositions are the usual ones. According to the rule $(\mid\!\sim^+)$, if a positive conditional $A \mid\!\sim B$ holds in a world, then either the world falsifies A or it is not minimal for A (i.e., $\neg\Box\neg A$ holds) or it is a B-world. According to the rule $(\mid\!\sim^-)$, if a negated conditional $\neg(A \mid\!\sim B)$ holds in a world, then there is a minimal A-world (i.e., in which A and $\Box\neg A$ hold) which falsifies B. According to the rule $(\Box^-)$, if a world

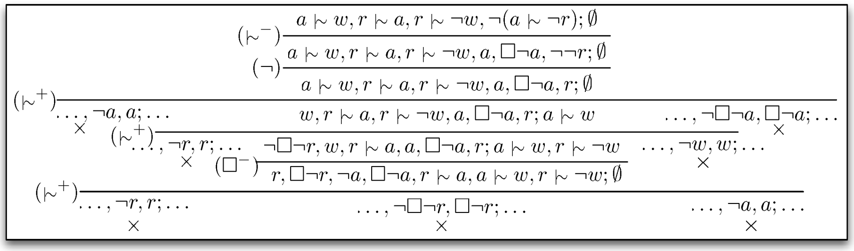

Fig. 3. A derivation of *adult* $\mid\!\sim$ *worker*, *retired* $\mid\!\sim$ *adult*, *retired* $\mid\!\sim$ $\neg$*worker*, $\neg$(*adult* $\mid\!\sim$ $\neg$*retired*). For readability, we use a to denote *adult*, r for *retired*, and w for *worker*.

satisfies $\neg\Box\neg A$, by the strong smoothness condition there must be a preferred minimal A-world, namely a world in which A and $\Box\neg A$ hold.

In order to ensure termination, the set Σ is used to keep track of positive conditionals $A \mid\!\sim B$ to which the rule $(\mid\!\sim^+)$ has already been applied in the current world. The idea is that it is not necessary to apply $(\mid\!\sim^+)$ on the same conditional formula $A \mid\!\sim B$ *more than once in the same world*. When $(\mid\!\sim^+)$ is applied to a formula $A \mid\!\sim B \in \Gamma$, then $A \mid\!\sim B$ is moved from Γ to Σ in the conclusions of the rule, so that it is no longer available for further applications in the current world. The dynamic rules reintroduce formulas from Σ to Γ in order to allow further applications of $(\mid\!\sim^+)$ in new worlds. This is a well-known machinery.

In our calculus $\mathcal{T}\mathrm{P}^{\mathbf{T}}$, axioms are restricted to atomic formulas only. It is easy to extend axioms to a generic formula F, as stated by the following proposition.

Proposition 3.2. *Given a formula F and a set of formulas Γ, then $\Gamma, F, \neg F; \Sigma$ has a closed tableau.*

Proof. By an easy inductive argument on the structure of the formula F. $\Box$

As an example, we show that *adult* $\mid\!\sim$ $\neg$*retired* can be inferred from a knowledge base containing the following assertions: *adult* $\mid\!\sim$ *worker*, *retired* $\mid\!\sim$ *adult*, *retired* $\mid\!\sim$ $\neg$*worker*. Figure 3 shows a derivation for the initial node *adult* $\mid\!\sim$ *worker*, *retired* $\mid\!\sim$ *adult*, *retired* $\mid\!\sim$ $\neg$ *worker*, $\neg$(*adult* $\mid\!\sim$ $\neg$ *retired*); $\emptyset$.

Our tableau calculus $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ is based on a runtime translation of conditional assertions into modal logic G. As we have seen in Section 2.2, this allows a characterization of the minimal worlds satisfying a formula A (i.e., the worlds in $Min_<(A)$) as the worlds w satisfying the formula $A \wedge \Box\neg A$. It is tempting to provide a full translation of the conditionals in the logic G, and then to use the standard tableau calculus for G. To this purpose, we can exploit the transitivity properties of G frames in order to capture the fact that conditionals are global to all worlds by the formula $\Box(A \wedge \Box\neg A \rightarrow B)$. Hence, the overall translation of a conditional formula $A \mid\!\sim B$ could be the following one: $(A \wedge \Box\neg A \rightarrow B) \wedge \Box(A \wedge \Box\neg A \rightarrow B)$. However, there would be significant differences between the calculus resulting from the translation and our calculus.

Using the standard tableau rules for G on the translation, we would get the rule $(\mid\!\sim^+)$ as a derived rule. Instead, the rule for dealing with negated conditionals (which would be translated in G as a disjunction of two formulas, namely $(A \wedge \Box\neg A \wedge \neg B) \vee \Diamond\,(A \wedge \Box\neg A \wedge \neg B)$), would be rather different.

Let us first observe that the rule $(\mid\!\sim^-)$ we have introduced precisely captures the intuition that conditionals are global, hence: (1) All conditionals are kept in the conclusion of the rule; and (2) when moving to a new minimal world, all the boxed formulas (positive and negated) are removed. Conversely, when the tableau rules for G are applied to the translation of the negated conditionals, we get two branches (due to the disjunction). None of the branches can be eliminated. In both branches all the boxed formulas are kept, while negated conditionals are erased. This is quite different from our rule $(\mid\!\sim^-)$, and it is not that obvious that the calculus obtained by the translation of conditionals in G is equivalent to $\mathcal{T}\mathrm{P}^{\mathbf{T}}$. Roughly speaking, point (2) can be explained as follows: When a negated conditional $\neg(A \mid\!\sim B)$ is evaluated in a world w, this corresponds to finding a minimal A-world w' satisfying $\neg B$ (a world satisfying $A, \Box\neg A, \neg B$). w' does not depend on w (since conditionals are global), hence boxed formulas, keeping information about w, can be removed.

Furthermore, a translation to G would not be applicable to the cumulative logic C, as the relation $<$ is not transitive in C. Moreover, the treatment of both the logics C and CL would in any case require the addition to the language of a new modality to deal with states. The advantage of the runtime translation we have adopted is that of providing a uniform approach to deal with the different logics. Moreover, in case of P the target logic G is well established and clear, whereas for the other cases the target logic would be more complicated (as it may combine different modalities).

The system $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ is sound and complete with respect to the semantics.

THEOREM 3.3 (SOUNDNESS OF $\mathcal{T}\mathrm{P}^{\mathbf{T}}$). *The system $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ is sound with respect to preferential models, namely if there is a closed tableau for $\Gamma; \emptyset$, then Γ is unsatisfiable.*

PROOF. As usual, we proceed by induction on the structure of the closed tableau having $\Gamma; \emptyset$ as a root. The base case is when the tableau consists of a single node; in this case, both P and $\neg P$ occur in Γ, therefore Γ is obviously unsatisfiable. For the inductive step, we have to show that, for each rule r, if all the conclusions of r are unsatisfiable, then the premise is unsatisfiable too. We show the contrapositive, that is, we prove that if the premise of r is satisfiable, then at least one of the conclusions is satisfiable. As usual, a node $\Gamma; \Sigma$ is satisfiable if there is a model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \models \Gamma, \Sigma$. Boolean cases are easy and left to the reader. We present the cases for conditional and box rules.

- $(\mid\!\sim^+)$. If $\Gamma, A \mid\!\sim B; \Sigma$ is satisfiable, then there exists a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ with some world $w \in \mathcal{W}$ such that $\mathcal{M}, w \models \Gamma, A \mid\!\sim B, \Sigma$. We have two cases:
  - $\mathcal{M}, w \not\models A$, thus $\mathcal{M}, w \models \neg A$: In this case, the left conclusion of the $(\mid\!\sim^+)$ rule is satisfied;
  - $\mathcal{M}, w \models A$: We consider two subcases:
    - $w \in Min_<(A)$: By definition of $\mathcal{M}, w \models A \mid\!\sim B$, we have that for all $w' \in Min_<(A)$, $\mathcal{M}, w' \models B$. Therefore, we have that $\mathcal{M}, w \models B$ and the right conclusion of $(\mid\!\sim^+)$ is satisfiable;

—$w \not\in Min_<(A)$: By the smoothness condition, there exists a world $w' < w$ such that $w' \in Min_<(A)$; therefore, $\mathcal{M}, w \models \neg\Box\neg A$ by definition of $\Box$. The conclusion in the middle of the $(\mid\!\sim^+)$ rule is then satisfiable.

—$(\mid\!\sim^-)$. If $\Gamma, \neg(A \mid\!\sim B); \Sigma$ is satisfiable, then $\mathcal{M}, w \models \Gamma, \Sigma$ and $(*)\mathcal{M}, w \not\models A \mid\!\sim B$ for some world w. By $(*)$, there is a world w' in the model $\mathcal{M}$ such that $w' \in Min_<(A)$, namely $(1)\mathcal{M}, w' \models A$ and $(2)\mathcal{M}, w' \models \Box\neg A)$ and $(3)\mathcal{M}, w' \not\models B$. By (1), (2) and (3), we have that $\mathcal{M}, w' \models A, \Box\neg A, \neg B$. We conclude $\mathcal{M}, w' \models A, \Box\neg A, \neg B, \Gamma^{\mid\sim\pm}, \Sigma$, since conditionals are global in a model.

—$(\Box^-)$. If $\Gamma, \neg\Box\neg A; \Sigma$ is satisfiable, then there is a model $\mathcal{M}$ and some world w such that $\mathcal{M}, w \models \Gamma, \neg\Box\neg A, \Sigma$, hence $\mathcal{M}, w \not\models \Box\neg A$. By the truth definition of $\Box$, there exists a world w' such that $w' < w$ and $\mathcal{M}, w' \models A$. By the strong smoothness condition, we can assume that w' is a *minimal* A-world. Therefore, $\mathcal{M}, w' \models \Box\neg A$ by the truth definition of $\Box$. We conclude that $\mathcal{M}, w' \models A, \Box\neg A, \Gamma^{\mid\sim\pm}, \Gamma^{\Box\downarrow}, \Gamma^{\Box}, \Sigma$, as follows: (1.) Conditionals are global in a model (hence $\mathcal{M}, w' \models \Gamma^{\mid\sim\pm}, \Sigma$); (2.) Formulas in $\Gamma^{\Box\downarrow}$ are true in w' since $w' < w$; 3. The $<$ relation is transitive, thus boxed formulas holding in w (i.e., $\Gamma^{\Box}$) also hold in w'. □

To prove the completeness of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ we have to show that if Γ is unsatisfiable, then there is a closed tableau starting with $\Gamma; \emptyset$. We prove the contrapositive, that is: If there is no closed tableau for $\Gamma; \emptyset$, then there is a model satisfying Γ. We first introduce the *saturation* of a tableau node $\Gamma; \Sigma$. Intuitively, given a node $\Gamma; \Sigma$, we say that it is saturated if the following condition holds: If the node contains the premise of a static rule, then it also contains one of its conclusions.

Definition 3.4 (*Saturated Node*). A tableau node $\Gamma; \Sigma$ is saturated with respect to the static rules if the following conditions hold.

—If $F \wedge G \in \Gamma$ then $F \in \Gamma$ and $G \in \Gamma$.
—If $\neg(F \wedge G) \in \Gamma$ then $\neg F \in \Gamma$ or $\neg G \in \Gamma$.
—If $\neg\neg F \in \Gamma$ then $F \in \Gamma$.
—If $A \mid\!\sim B \in \Gamma \cup \Sigma$ then $\neg A \in \Gamma$ or $\neg\Box\neg A \in \Gamma$ or $B \in \Gamma$.

The conditions for the other propositional operators are defined similarly.

We can easily prove the following lemma.

LEMMA 3.5. *Given a consistent finite node* $\Gamma; \Sigma$*, there is a consistent, finite, effectively computable, and saturated node* $\Gamma'; \Sigma$ *such that* $\Gamma' \supseteq \Gamma$.

By Lemma 3.5, we can think of having a function which, given a consistent node $\Gamma; \Sigma$, returns one fixed consistent saturated node, denoted by $\mathtt{SAT}(\Gamma; \Sigma)$. Moreover, we denote by $\mathtt{APPLY}((\Gamma; \Sigma), R, F)$ the result of applying the tableau rule (R) to $\Gamma; \Sigma$, with principal formula F. In case (R) has several conclusions (the case of a branching), we suppose that the function APPLY chooses one consistent conclusion in an arbitrary but fixed manner.

THEOREM 3.6 (COMPLETENESS OF $\mathcal{T}\mathrm{P}^{\mathbf{T}}$). $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ *is complete with respect to preferential models, that is, if a set of formulas* Γ *is unsatisfiable, then* $\Gamma; \emptyset$ *has a closed tableau in* $\mathcal{T}\mathrm{P}^{\mathbf{T}}$.

PROOF. We assume that no tableau for $\Gamma;\emptyset$ is closed, then we construct a model for Γ. We build X, the set of worlds of the model, as follows.

---

1. initialize $X = \{\texttt{SAT}(\Gamma;\emptyset)\}$; mark $\texttt{SAT}(\Gamma;\emptyset)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved $\Gamma_i;\Sigma_i$ from X;
  3. **for** each formula $\neg(A \mathrel{|\!\sim} B) \in \Gamma_i$
    3a. let $(\Gamma;\Sigma)_{\neg(A|\sim B)} = \texttt{SAT}(\texttt{APPLY}((\Gamma_i;\Sigma_i), \mathrel{|\!\sim}^-, \neg(A \mathrel{|\!\sim} B)))$;
    3b. **if** $(\Gamma;\Sigma)_{\neg(A|\sim B)} \notin X$ **then** $X = X \cup \{(\Gamma;\Sigma)_{\neg(A|\sim B)}\}$;
  4. **for** each formula $\neg\Box\neg A \in \Gamma_i$, let $(\Gamma;\Sigma)_{\neg\Box\neg A} = \texttt{SAT}(\texttt{APPLY}((\Gamma_i;\Sigma_i), \Box^-, \neg\Box\neg A))$;
    4a. add the relation $(\Gamma;\Sigma)_{\neg\Box\neg A} < (\Gamma_i;\Sigma_i)$;
    4b. **if** $(\Gamma;\Sigma)_{\neg\Box\neg A} \notin X$ **then** $X = X \cup \{(\Gamma;\Sigma)_{\neg\Box\neg A}\}$.
  5. mark $\Gamma_i;\Sigma_i$ as resolved;
endWhile;

This procedure terminates, since the number of possible nodes $\Gamma_i;\Sigma_i$ that can be obtained by applying $\mathcal{TP}^{\mathbf{T}}$'s rules to an initial finite node $\Gamma;\Sigma$ is finite. We construct the model $\mathcal{M} = \langle X, <_X, V\rangle$ for Γ as follows.

—$<_X$ is the transitive closure of the relation $<$;

—for each $\Gamma_i;\Sigma_i \in X$, we have $V(\Gamma_i;\Sigma_i) = \{P \mid P \in \Gamma_i \cap ATM\}$.

In order to show that $\mathcal{M}$ is a preferential model for Γ, we prove the following facts.

FACT 3.7. *The relation $<_X$ is acyclic.*

PROOF OF FACT 3.7. Suppose on the contrary there is a loop $(\Gamma_i;\Sigma_i) < \ldots < (\Gamma_j;\Sigma_j) < (\Gamma_i;\Sigma_i)$. Each $<$ has been introduced by step 4 in the preceding procedure. $\Gamma_j;\Sigma_j$ has been obtained by an application of $(\Box^-)$ to a given $\neg\Box\neg A \in \Gamma_i$. This entails that, for all $(\Gamma_k;\Sigma_k) <_X (\Gamma_i;\Sigma_i)$, we have that $\Box\neg A \in \Gamma_k$, hence $\Box\neg A \in \Gamma_i$, which contradicts the fact that Γ_i is consistent. □ (FACT 3.7)

FACT 3.8. *The relation $<_X$ is irreflexive, transitive, and satisfies the smoothness condition.*

PROOF OF FACT 3.8. Transitivity follows by construction. Irreflexivity follows the acyclicity, since $\Gamma_i;\Sigma_i <_X \Gamma_i;\Sigma_i$ is never added. As there are finitely many worlds, and the relation $<_X$ is acyclic, it follows that there cannot be infinitely descending chains. This fact, together with the transitivity of $<_X$, entails that $<_X$ satisfies the smoothness condition. □ (FACT 3.8)

The only rules introducing a new world in X in the aforesaid procedure are $(\mathrel{|\!\sim}^-)$ and $(\Box^-)$. Since these two rules keep positive conditionals in their conclusions, it follows that any positive conditional $A \mathrel{|\!\sim} B$ belonging to $\texttt{SAT}(\Gamma;\emptyset)$, where Γ is the initial set of formulas, also belongs to each world introduced in X. Furthermore, it can be shown that *only* the conditionals in $\texttt{SAT}(\Gamma;\emptyset)$ belong to possible worlds in X. Indeed, all worlds in X are generated by the application of a dynamic rule, followed by the application of static rules for saturation. It can be shown that this combination of rules does never introduce a new conditional.

FACT 3.9. *Given a world $\Gamma_i;\Sigma_i \in X$ and any positive conditional $A \mathrel{|\!\sim} B$, we have that $A \mathrel{|\!\sim} B \in \Gamma_i \cup \Sigma_i$ iff $A \mathrel{|\!\sim} B \in \texttt{SAT}(\Gamma;\emptyset)$.*

We conclude by proving the following fact.

FACT 3.10. *For all formulas F and for all worlds $\Gamma_i; \Sigma_i \in X$ we have that:* (i) *If $F \in \Gamma_i \cup \Sigma_i$ then $\mathcal{M}, (\Gamma_i; \Sigma_i) \models F$;* (ii) *if $\neg F \in \Gamma_i$ then $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models F$.*

PROOF OF FACT 3.10. By induction on the structure of F. If F is an atom P, then $P \in \Gamma_i$ implies $\mathcal{M}, (\Gamma_i; \Sigma_i) \models P$ by definition of V. Moreover, $\neg P \in \Gamma_i$ implies that $P \notin \Gamma_i$ as $\Gamma_i; \Sigma_i$ is consistent; thus $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models P$ (by definition of V). For the inductive step we only consider the cases of positive and negative box formulas and of positive and negative conditional formulas.

—$\Box\neg A \in \Gamma_i$. Then, for all $\Gamma_j; \Sigma_j <_X \Gamma_i; \Sigma_i$ we have $\neg A \in \Gamma_j$ by definition of $(\Box^-)$, since Γ_j has been generated by a sequence of applications of $(\Box^-)$. By inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \not\models A$ for all $\Gamma_j; \Sigma_j <_X \Gamma_i; \Sigma_i$, whence $\mathcal{M}, (\Gamma_i; \Sigma_i) \models \Box\neg A$.

—$\neg\Box\neg A \in \Gamma_i$. By construction there is a $\Gamma_j; \Sigma_j \in X$ such that $\Gamma_j; \Sigma_j <_X \Gamma_i; \Sigma_i$ and $A \in \Gamma_j$. By inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \models A$. Thus $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models \Box\neg A$.

—$A \mathrel{|\!\sim} B \in \Gamma_i \cup \Sigma_i$. Let $\Gamma_j; \Sigma_j \in Min_{<_X}(A)$; it can be observed that (1)$\neg A \in \Gamma_j$ or (2)$\neg\Box\neg A \in \Gamma_j$ or (3)$B \in \Gamma_j$, since $A \mathrel{|\!\sim} B \in \Gamma_j \cup \Sigma_j$ by Fact 3.9, and since $\Gamma_j; \Sigma_j$ is saturated. (1) cannot be the case, since otherwise by inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \not\models A$, which contradicts the definition of $Min_{<_X}(A)$. If (2), by construction of $\mathcal{M}$ there exists a node $\Gamma_k; \Sigma_k <_X \Gamma_j; \Sigma_j$ such that $A \in \Gamma_k$: By inductive hypothesis $\mathcal{M}, (\Gamma_k; \Sigma_k) \models A$, which contradicts $(\Gamma_j; \Sigma_j) \in Min_{<_X}(A)$. Thus it must be that (3)$B \in \Gamma_j$, and by inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \models B$. Hence we can conclude $\mathcal{M}, (\Gamma_i; \Sigma_i) \models A \mathrel{|\!\sim} B$.

—$\neg(A \mathrel{|\!\sim} B) \in \Gamma_i$. By construction of X, there exists $\Gamma_j; \Sigma_j \in X$ such that $A, \Box\neg A, \neg B \in \Gamma_j$. By inductive hypothesis we have that $\mathcal{M}, (\Gamma_j; \Sigma_j) \models A$ and $\mathcal{M}, (\Gamma_j; \Sigma_j) \models \Box\neg A$. It follows that $(\Gamma_j; \Sigma_j) \in Min_{<_X}(A)$. Furthermore, always by induction, $\mathcal{M}, (\Gamma_j; \Sigma_j) \not\models B$. Hence $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models A \mathrel{|\!\sim} B$. □ (FACT 3.10)

By the previous facts, the proof of the completeness of $\mathcal{T}$P$^{\mathbf{T}}$ is over, since $\mathcal{M}$ is a model for the initial set Γ. □

The construction of the aforementioned model provides a constructive proof of the finite model property of P.

A relevant property of the calculus that will be useful to estimate the complexity of logic P is the so-called *disjunction property* of conditional formulas (the proof can be found in Pozzato [2007] and Giordano et al. [2007a]).

PROPOSITION 3.11 (DISJUNCTION PROPERTY). *If there is a closed tableau for $\Gamma, \neg(A \mathrel{|\!\sim} B), \neg(C \mathrel{|\!\sim} D)$, then there is a closed tableau either for $\Gamma, \neg(A \mathrel{|\!\sim} B)$ or for $\Gamma, \neg(C \mathrel{|\!\sim} D)$.*

The reason why this property holds is that the application of rule $(\mathrel{|\!\sim}^-)$ discards all the other formulas that could have been introduced by its previous applications.

3.1 Decidability and Complexity of P

3.1.1 *Termination of* $\mathcal{T}\mathrm{P}^{\mathbf{T}}$. In general, nontermination in tableau calculi has two different causes: (1.) Some rules copy their principal formula in the conclusion, and can thus be reapplied over the same formula without any control; (2.) dynamic rules may generate infinitely many worlds, creating infinite branches. Concerning the first source of nontermination (point (1)), it cannot occur due to the machinery used to control the application of $(\mid\!\sim^+)$ by means of the additional set Σ. Concerning the second source of nontermination (point (2)), notice that infinitely many worlds cannot be generated on a branch by $(\mid\!\sim^-)$ rule, since this rule can be applied only once to a given negated conditional on a branch. Nonetheless, the interplay between rules $(\mid\!\sim^+)$ and $(\Box^-)$ might lead to generate infinite branches. However, we show that this cannot occur, once we introduce the following restriction on the order of application of the rules.

Definition 3.12 (*Restriction on the Calculus*). Building a tableau for a node $\Gamma; \Sigma$, the application of the $(\Box^-)$ is postponed to the application of the propositional rules and to the verification that Γ is an instance of (AX).

It is easy to observe that, without the preceding restriction, point (2.) could occur; for instance, consider the following trivial example, showing a branch of a tableau starting with $P \mid\!\sim Q$, with $P, Q \in ATM$.

$$
\begin{array}{rcl}
 & P \mid\!\sim Q; \emptyset & \\ \hline
(\mid\!\sim^+) & \neg\Box\neg P; P \mid\!\sim Q & \cdots \\ \hline
(\Box^-) & P, \Box\neg P, P \mid\!\sim Q; \emptyset & \\ \hline
(\mid\!\sim^+) & P, \Box\neg P, \neg\Box\neg P; P \mid\!\sim Q & \cdots \\ \hline
(\Box^-) & (*)\neg P, P, \Box\neg P, P \mid\!\sim Q; \emptyset & \\ \hline
(\mid\!\sim^+) & \neg P, P, \Box\neg P, \neg\Box\neg P; P \mid\!\sim Q & \cdots \\ \hline
(\Box^-) & \neg P, P, \Box\neg P, P \mid\!\sim Q; \emptyset & \\ \hline
(\mid\!\sim^+) & \neg P, P, \Box\neg P, \neg\Box\neg P; P \mid\!\sim Q & \cdots \\
 & \cdots &
\end{array}
$$

In the previous example, the $(\Box^-)$ rule is applied systematically before the other rules, thus generating an infinite branch. However, if the restriction in Definition 3.12 is adopted, as easy to observe, the procedure terminates at the step marked with $(*)$. Indeed, the test that $\neg P, P, \Box\neg P, P \mid\!\sim Q$ is an instance of the axiom (AX) succeeds before applying $(\Box^-)$ again, and the branch is considered closed.

As already mentioned, we show that, given the previous restriction, the calculus $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ always terminates (Theorem 3.18). Notice that this would not work in other systems (for instance, in K4 we need a more sophisticated loop-checking as described in Heuerding et al. [1996]). In more detail, we show (Lemma 3.17 and Theorem 3.18) that the interplay between $(\mid\!\sim^+)$ and $(\Box^-)$ does not generate branches containing infinitely many worlds. Intuitively, the application of $(\Box^-)$ to a formula $\neg\Box\neg A$ (introduced by $(\mid\!\sim^+)$) adds the formula $\Box\neg A$ to the conclusion, so that $(\mid\!\sim^+)$ can no longer consistently introduce $\neg\Box\neg A$. This is due to the properties of $\Box$ in G, and would not hold if $\Box$ had weaker properties (e.g., K4 properties).

Let us introduce a property of the tableau which will be crucial in many of the following proofs. Let us first define the notion of regular node.

Definition 3.13. A node $\Gamma; \Sigma$ is called regular if the following condition holds.

$$\text{If } \neg\Box\neg A \in \Gamma, \text{ then there is } B \text{ such that } A \mid\sim B \in \Gamma \cup \Sigma.$$

It is easy to see that all nodes in a tableau starting from a pair $\Gamma; \emptyset$ are regular, when Γ is a set of formulas of $\mathcal{L}$. This is stated by the following proposition.

PROPOSITION 3.14. *Given a pair $\Gamma; \emptyset$, where Γ is a set of formulas of $\mathcal{L}$, every tableau obtained by applying $\mathcal{TP}^{\mathbf{T}}$'s rules only contains regular nodes.*

From now on, we can assume without loss of generality that only regular nodes occur in a tableau. In order to prove that $\mathcal{TP}^{\mathbf{T}}$ ensures a terminating proof search, we define a complexity measure on a node $\Gamma; \Sigma$, denoted by $m(\Gamma; \Sigma)$, which consists of four measures c_1, c_2, c_3, and c_4 in a lexicographic order. We write $A \mid\sim B \in_+ \Gamma$ (respectively, $A \mid\sim B \in_- \Gamma$) if $A \mid\sim B$ occurs positively (respectively, negatively) in Γ, where positive and negative occurrences are defined in the standard way. We denote by $cp(F)$ the complexity of a formula F, defined as follows.

Definition 3.15 (*Complexity of a Formula*).

—$cp(P) = 1$, where $P \in ATM$;
—$cp(\neg F) = 1 + cp(F)$;
—$cp(F \bigotimes G) = 1 + cp(F) + cp(G)$, where $\bigotimes$ is any binary Boolean operator;
—$cp(\Box\neg A) = 1 + cp(\neg A)$;
—$cp(A \mid\sim B) = 3 + cp(A) + cp(B)$.

Definition 3.16. We define $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$ where:

—$c_1 = \mid \{A \mid\sim B \in_- \Gamma\} \mid$;
—$c_2 = \mid \{A \mid\sim B \in_+ \Gamma \cup \Sigma \mid \Box\neg A \not\in \Gamma\} \mid$;
—$c_3 = \mid \{A \mid\sim B \in_+ \Gamma\} \mid$;
—$c_4 = \sum_{F \in \Gamma} cp(F)$.

We consider the *lexicographic order* given by $m(\Gamma; \Sigma)$, that is to say: Given $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$ and $m(\Gamma'; \Sigma') = \langle c'_1, c'_2, c'_3, c'_4 \rangle$, we say that $m(\Gamma; \Sigma) < m(\Gamma'; \Sigma')$ iff there exists i, $i = 1, 2, 3, 4$, such that the following conditions hold:

—$c_i < c'_i$;
—for all j, $0 < j < i$, we have that $c_j = c'_j$.

Intuitively, c_1 is the number of negated conditionals to which the $(\mid\sim^-)$ rule can still be applied. An application of $(\mid\sim^-)$ reduces c_1. Furhter, c_2 represents the number of positive conditionals *which can still create a new world*. The application of $(\Box^-)$ reduces c_2: Indeed, if $(\mid\sim^+)$ is applied to $A \mid\sim B$, this application introduces a branch containing $\neg\Box\neg A$; when a new world is generated by an application of $(\Box^-)$ on $\neg\Box\neg A$, it contains A and $\Box\neg A$. If $(\mid\sim^+)$ is applied to $A \mid\!\sim B$ once again, then the conclusion where $\neg\Box\neg A$ is introduced leads to a closed branch, by the presence of $\Box\neg A$ in that branch. c_3 is the number of positive conditionals in Γ not yet considered in that branch; an application of $(\mid\!\sim^+)$ reduces c_3. Finally, c_4 is the sum of the complexities of the formulas in Γ; an application of a Boolean rule reduces c_4.

With the restriction on the application of the rules of Definition 3.12 at hand, we prove that the strategy does *not* generate any branch of infinite length. To this purpose we need the following lemma.

LEMMA 3.17. *Let $\Gamma'; \Sigma'$ be obtained by an application of a rule of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ to a premise $\Gamma; \Sigma$. Then, we have that either $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$ or $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ leads to the construction of a closed tableau for $\Gamma'; \Sigma'$.*

PROOF. We consider each rule of the calculus $\mathcal{T}\mathrm{P}^{\mathbf{T}}$.

—$(\mid\!\sim^-)$. We can observe that the conditional formula $\neg(A \mid\!\sim B)$ to which this rule is applied does not belong to the conclusion $\Gamma'; \Sigma'$. Hence c_1 in $m(\Gamma'; \Sigma')$, say c_1', is smaller than c_1 in $m(\Gamma; \Sigma)$, say c_1.

—$(\Box^-)$: No negated conditional is added nor deleted in the conclusions, thus $c_1 = c_1'$. Suppose we are considering an application of $(\Box^-)$ on a formula $\neg\Box\neg A$. We can observe the following facts:

—The formula $\neg\Box\neg A$ has been introduced by an application of $(\mid\!\sim^+)$, since this is the only rule introducing a negated boxed formula in the conclusion; more precisely, it derives from an application of $(\mid\!\sim^+)$ to a conditional formula $A \mid\!\sim B$;

—$A \mid\!\sim B$ belongs to both $\Gamma; \Sigma$ and $\Gamma'; \Sigma'$, since no rule of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ removes positive conditionals (at most, the $(\mid\!\sim^+)$ rule *moves* conditionals from Γ to Σ);

—$A \mid\!\sim B$ does not "contribute" to c_2', since the application of $(\Box^-)$ introduces $\Box\neg A$ in the conclusion Γ' (remember that $c_2' = \mid \{A \mid\!\sim B \in_+ \Gamma' \cup \Sigma' \mid \Box\neg A \not\in \Gamma'\} \mid$).

We distinguish two cases

(1) $\Box\neg A$ does *not* belong to the premise of $(\Box^-)$. In this case, by the aforesaid facts, we can conclude that $c_2' < c_2$, since $\Box\neg A$ belongs only to the conclusion.

(2) $\Box\neg A$ belongs to the premise of $(\Box^-)$. We are considering a derivation in which $\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, \neg A, A, \Box\neg A, \Sigma; \emptyset$ is obtained from $\Gamma, \Box\neg A, \neg\Box\neg A; \Sigma$. In this case, $c_2' = c_2$; however, we can conclude that the tableau built for $\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, \neg A, A, \Box\neg A; \emptyset$ is closed, since:

—A is a propositional formula;

—the restriction in Definition 3.12 leads to a proof in which the propositional rules and (AX) are applied to A and $\neg A$ *before* $(\Box^-)$ is further applied. The resulting tableau is closed.

—$(\mid\!\sim^+)$. We have that $c_1 = c_1'$, since we have the same negated conditionals in the premise as in all the conclusions. The same for c_2, since the formula $A \mid\!\sim B$ to which the rule is applied is also maintained in the conclusions (it moves from unused to already used conditionals). We conclude that $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$, since $c_3' < c_3$. Indeed, the $(\mid\!\sim^+)$ rule moves $A \mid\!\sim B$ from Γ to the set Σ of already considered conditionals, thus $A \mid\!\sim B \in \Gamma$ whereas $A \mid\!\sim B \not\in \Gamma'$.

—*Rules for the Boolean Connectives*. It is easy to observe that c_1, c_2, and c_3 are the same in the premise and in any conclusion, since conditional formulas are side formulas in the application of these rules. We conclude that $m(\Gamma'; \Sigma') < m(\Gamma; \Sigma)$ since $c'_4 < c_4$. Indeed, the complexity of the formula to which the rule is applied is greater than (the sum of) the complexity of its subformula(s) introduced in the conclusion(s). □

Now we have all the elements to prove the following theorem.

THEOREM 3.18 (TERMINATION OF $\mathcal{T}\mathrm{P}^{\mathbf{T}}$). *$\mathcal{T}\mathrm{P}^{\mathbf{T}}$ ensures a terminating proof search.*

PROOF. By Lemma 3.17 we know that in any open branch the value of $m(\Gamma; \Sigma)$ decreases each time a tableau rule is applied. Therefore, a finite number of applications of the rules leads either to build a closed tableau or to nodes $\Gamma; \Sigma$ such that $m(\Gamma; \Sigma)$ is *minimal*. In particular, we observe that when the branch does not close, $m(\Gamma; \Sigma) = \langle 0, 0, 0, c_{4_{min}} \rangle$. In this case, no rule, with the exception of $(\Box^-)$, is applicable to $\Gamma; \Sigma$. Indeed, $(\mid\!\sim^-)$ rule is not applicable, since no negated conditional belongs to Γ (since $c_1 = 0$). If $(\Box^-)$ is applicable, then there is $\neg\Box\neg A \in \Gamma$, to which the rule is applied. However, since $c_2 = 0$, we have also that $\Box\neg A \in \Gamma$. Therefore, the conclusion of an application of $(\Box^-)$ contains both A and $\neg A$ and, by the restriction in Definition 3.12 and since A is propositional, the procedure terminates building a closed tableau. $(\mid\!\sim^+)$ is not applicable, since no positive conditionals belong to Γ (all positive conditionals $A \mid\!\sim B$ have been moved in Σ since $c_3 = 0$). Lastly no rule for the Boolean connectives is applicable, since c_4 assumes its minimal value $c_{4_{min}}$. For a contradiction, suppose one Boolean rule is still applicable: By Lemma 3.17, the sum of the complexity of the formulas in the conclusion(s) decreases, that is, c_4 in the conclusion(s) is smaller than in the premise $\Gamma; \Sigma$, against the minimality of this measure in $\Gamma; \Sigma$. □

3.1.2 *Complexity of* P. We conclude this section with a complexity analysis of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$, in order to define an optimal decision procedure for P based on our tableau calculus. Intuitively, we can obtain a coNP decision procedure by taking advantage of the following facts.

(1) Negated conditionals do not interact with the current world, nor do they interact among themselves (by the disjunction property). Thus they can be handled separately from each other and eliminated always at the first step.
(2) We can replace the $(\Box^-)$, which is responsible for backtracking in the tableau construction, by a stronger rule that does not need backtracking.

Regarding (1), by the disjunction property we can reformulate the $(\mid\!\sim^-)$ rule as follows.

$$\frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{\Sigma, A, \Box\neg A, \neg B, \Gamma^{\mid\sim^+}; \emptyset}\ (\mid\!\sim^-)$$

This rule reduces the length of a branch at the price of making the proof search more nondeterministic.

Regarding (2), we can adopt the following strengthened version of ($\Box^-$). We use $\Gamma^{\Box^-}_{-i}$ to denote $\{\neg\Box\neg A_j \vee A_j \mid \neg\Box\neg A_j \in \Gamma \wedge j \neq i\}$.

$$\frac{\Gamma, \neg\Box\neg A_1, \neg\Box\neg A_2, \ldots, \neg\Box\neg A_n; \Sigma}{\Sigma, \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box\downarrow}, A_1, \Box\neg A_1, \Gamma^{\Box^-}_{-1}; \emptyset \mid \ldots \mid \Sigma, \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box\downarrow}, A_n, \Box\neg A_n, \Gamma^{\Box^-}_{-n}; \emptyset} \, (\Box^-_s)$$

Rule ($\Box^-_s$) contains a branch for each $\neg\Box\neg A_i$ in Γ. On each branch, for a given $\neg\Box\neg A_i$, a new minimal world is created for A_i, where A_i and $\Box\neg A_i$ hold, and for all other $\neg\Box\neg A_j$, either A_j holds in that world or the formula $\neg\Box\neg A_j$ is recorded. The advantage of this rule over the original ($\Box^-$) rule is that no backtracking on the choice of the formula $\neg\Box\neg A_i$ is needed when searching for a closed tableau. The reason is that all alternatives are kept in the conclusion. As we will see next, by using this rule we can provide a tableau construction algorithm with no backtracking.

We call $L\mathcal{T}\text{P}^{\mathbf{T}}$ the calculus obtained by replacing in $\mathcal{T}\text{P}^{\mathbf{T}}$ the initial rules ($\mid\sim^-$) and ($\Box^-$) with the ones reformulated as before. We can prove that $L\mathcal{T}\text{P}^{\mathbf{T}}$ is sound and complete with respect to the preferential models. To prove soundness, we consider the multilinear models introduced in Section 2.2.1.

THEOREM 3.19. *The rule* ($\Box^-_s$) *is sound.*

PROOF. Let $\Gamma = \Gamma', \neg\Box\neg A_1, \neg\Box\neg A_2, \ldots, \neg\Box\neg A_n$. We omit Σ for the sake of readability. We prove that if Γ is satisfiable then also one conclusion of the rule $\Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box\downarrow}, A_i, \Box\neg A_i, \Gamma^{\Box^-}_{-i}$ is satisfiable. By Theorem 2.5, we can assume that Γ is satisfiable in a multilinear model $\mathcal{M} = \langle \mathcal{W}, <, V\rangle$ with $\mathcal{M}, x \models \Gamma$. Then there are $z_1 < x, \ldots, z_n < x$, such that $z_i \in Min_<(A_i)$; thus $\mathcal{M}, z_i \models A_i \wedge \Box\neg A_i$. We easily have also that $\mathcal{M}, z_i \models \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box\downarrow}$. Since $\mathcal{M}$ is a multilinear model, the z_i, $i = 1, 2, \ldots, n$, whenever distinct, are totally ordered: We have that $z_i < x$, so that they must belong to the same component. Let z_k be the maximum of z_i $(1 \leq k \leq n)$, that is, for each z_i, $i = 1, 2, \ldots, n$, we have either: (i) $z_i = z_k$ or (ii) $z_i < z_k$. In case (i), we have that $\mathcal{M}, z_k \models A_i$. In case (ii) we have that $\mathcal{M}, z_k \models \neg\Box\neg A_i$. We have shown that for each $i \neq k$, $\mathcal{M}, z_k \models A_i \vee \neg\Box\neg A_i$. We can conclude that $\mathcal{M}, z_k \models \Gamma^{\Box^-}_{-k}$. Thus $\mathcal{M}, z_k \models \Gamma^{\mid\sim\pm}, \Gamma^{\Box}, \Gamma^{\Box\downarrow}, A_k, \Box\neg A_k, \Gamma^{\Box^-}_{-k}$, which is one of the conclusions of the rule. □

We can prove that the calculus obtained by replacing the ($\Box^-$) rule with its stronger version ($\Box^-_s$) is complete with respect to the semantics.

THEOREM 3.20. *The calculus* $L\mathcal{T}\text{P}^{\mathbf{T}}$ *is complete.*

PROOF. We repeat the same construction as in the proof of Theorem 3.6, in order to build a preferential model, more precisely a multilinear model, of a set of formulas Γ for which there is no closed tableau. As a difference with the construction in Theorem 3.6, we replace point (4.) by the points 4_{strong}, $4a_{strong}$, $4b_{strong}$, and $4c_{strong}$, obtaining the procedure at the top of the next page (X is the set of worlds of the model).

We denote by APPLY$((\Gamma; \Sigma), \Box^-_s)$ the result of applying the rule ($\Box^-_s$) to $\Gamma; \Sigma$; as for the other branching rules, APPLY chooses one consistent conclusion of ($\Box^-_s$) in an arbitrary but fixed manner. Facts 3.7 and 3.8 can be proved as in Theorem 3.6. This holds also for Fact 3.10 with one difference concerning the

1. initialize $X = \{\mathtt{SAT}(\Gamma; \Sigma)\}$; mark $\mathtt{SAT}(\Gamma; \Sigma)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved $\Gamma_i; \Sigma_i$ from X;
  3. **for** each formula $\neg(A \mid\sim B) \in \Gamma_i$
    3a. let $(\Gamma; \Sigma)_{\neg(A \mid\sim B)} = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i; \Sigma_i), \mid\sim^-, \neg(A \mid\sim B)))$;
    3b. **if** $(\Gamma; \Sigma)_{\neg(A \mid\sim B)} \notin X$ **then** $X = X \cup \{(\Gamma; \Sigma)_{\neg(A \mid\sim B)}\}$;
  4_{strong}. **if** there is $\neg\Box\neg A \in \Gamma_i$ **then**
    $4a_{strong}$. let $(\Gamma_j; \Sigma_j) = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i; \Sigma_i), \Box_s^-))$;
    $4b_{strong}$. add the relation $(\Gamma_j; \Sigma_j) < (\Gamma_i; \Sigma_i)$;
    $4c_{strong}$. **if** $(\Gamma_j; \Sigma_j) \notin X$ **then** $X = X \cup \{(\Gamma_j; \Sigma_j)\}$;
  5. mark $\Gamma_i; \Sigma_i$ as resolved;
endWhile;

case in which $\neg\Box\neg A \in \Gamma_i$. In this case, by construction there is a $\Gamma_j; \Sigma_j$ such that $(\Gamma_j; \Sigma_j) <_X (\Gamma_i; \Sigma_i)$. We can prove by induction on the length n of the chain $<_X$ starting from $\Gamma_i; \Sigma_i$ that if $\neg\Box\neg A \in \Gamma_i$, then $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models \Box\neg A$. If $n = 1$, it must be the case that $A \in \Gamma_i$; hence, by inductive hypothesis on the structure of the formula, $\mathcal{M}, (\Gamma_j; \Sigma_j) \models A$, thus $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models \Box\neg A$. If $n > 1$, by the $(\Box_s^-)$ rule, either $A \in \Gamma_j$ and we conclude as in the previous case, or $\neg\Box\neg A \in \Gamma_j$ and the fact holds by inductive hypothesis on the length. From these facts, we can conclude that the model built is preferential and satisfies the initial set Γ, therefore $\mathcal{LT}\mathrm{P}^{\mathbf{T}}$ is complete. □

In the following, we describe a rule application's strategy that allows us to decide the satisfiability of a node $\Gamma; \Sigma$ in P in nondeterministic polynomial time. As a first step, the strategy applies the Boolean rules as long as possible. In case of branching rules, this operation nondeterministically selects (guesses) one of the conclusions of the rules. For each negated conditional, the strategy applies the rule $(\mid\sim^-)$ to it, generating a node $\Gamma'; \Sigma'$ that does not contain any negated conditional.[5] On this node, the strategy applies the static rules as far as possible, then it iterates the following steps until either the current node contains an axiom or it does not contain negated boxed formulas $\neg\Box\neg A$:

(1) apply the $(\Box_s^-)$ rule by guessing a branch;
(2) apply the static rules as far as possible to the obtained node.

If the final node does not contain an axiom, then we conclude that $\Gamma'; \Sigma'$ is satisfiable. If, for each application of $(\mid\sim^-)$, $\Gamma'; \Sigma'$ is satisfiable, then the initial node $\Gamma; \Sigma$ is satisfiable.

The overall complexity of the strategy can be estimated as follows. Consider that $n = \mid \Gamma \cup \Sigma \mid$. The strategy builds a tableau branch for each of the $O(n)$ set of formulas obtained by applying $(\mid\sim^-)$ to each negated conditional (indeed, there are at most $O(n)$ negated conditionals). In order to do this, the strategy alternates applications of $(\Box_s^-)$ and of static rules (to saturate the obtained nodes). In case of branching rules, this saturation nondeterministically selects

[5]By the new version of $(\mid\sim^-)$, we can consider negated conditionals one at a time. Indeed, for $\Gamma, \neg(A \mid\sim B), \neg(C \mid\sim D); \Sigma$ to be satisfiable, it is sufficient that both $\Gamma, \neg(A \mid\sim B); \Sigma$ and $\Gamma, \neg(C \mid\sim D); \Sigma$, separately considered, are satisfiable.

$$(\mid\!\sim^{+})\,\frac{\Gamma, A \mid\!\sim B; \Sigma}{\Gamma, \neg LA; \Sigma, A \mid\!\sim B \qquad \Gamma, \neg\Box\neg LA; \Sigma, A \mid\!\sim B \qquad \Gamma, LB; \Sigma, A \mid\!\sim B} \qquad (\mid\!\sim^{-})\,\frac{\Gamma, \neg(A \mid\!\sim B); \Sigma}{LA, \Box\neg LA, \neg LB, \Gamma^{\mid\sim^{\pm}}, \Sigma; \emptyset}$$

$$(\Box^{-})\,\frac{\Gamma, \neg\Box\neg LA; \Sigma}{LA, \Box\neg LA, \Gamma^{\Box}, \Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim^{\pm}}, \Sigma; \emptyset} \qquad (L^{-})\,\frac{\Gamma, \neg LA; \Sigma}{\Gamma^{L^{\downarrow}}, \neg A; \emptyset}\,;\ \frac{\Gamma; \Sigma}{\Gamma^{L^{\downarrow}}; \emptyset}\ \text{if } \Gamma \text{ does not contain negated } L-\text{formulas}$$

Fig. 4. Tableau calculus $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$. If there are no negated L-formulas $\neg LA$ in the premise of (L^{-}), then the rule allows to step from Γ to $\Gamma^{L^{\downarrow}}$. To save space, the Boolean rules are omitted.

(guesses) one of the conclusions of the rules. The $(\Box_s^{-})$ rule can be applied at most $O(n)$ times, since it can be applied only once for each formula $\neg\Box\neg A$, and the number of different negated box formulas is at most $O(n)$. Moreover, as the number of different subformulas of Γ is at most $O(n)$, in all steps involving the application of static rules, there are at most $O(n)$ applications of these rules. Therefore, the length of the tableau branch built by the strategy is $O(n^2)$. Finally, we can observe that all the nodes of the tableau contain a number of formulas which is polynomial in n, therefore to test that a node contains an axiom has at most complexity polynomial in n.

The aforesaid strategy allows the satisfiability of a set of formulas of logic P to be decided in nondeterministic polynomial time. Given that the problem of deciding validity for preferential logic P is known to be coNP-complete, we can conclude that the previous strategy is optimal for P.

4. THE TABLEAU CALCULUS FOR LOOP-CUMULATIVE LOGIC CL

We develop a tableau calculus $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ for CL, which provides a decision procedure for CL and a coNP-membership upper bound for validity in CL. The calculus $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ can be obtained from the calculus $\mathcal{T}\mathtt{P}^{\mathbf{T}}$ for preferential logics, by adding a suitable rule (L^{-}) for dealing with the modality L introduced in Section 2.3. As already mentioned in Section 2.3, the formulas that appear in the tableaux for CL belong to the language $\mathcal{L}_L$ obtained from $\mathcal{L}$ as follows: (i) If A is propositional, then $A \in \mathcal{L}_L$; $LA \in \mathcal{L}_L$; $\Box\neg LA \in \mathcal{L}_L$; (ii) if A, B are propositional, then $A \mid\!\sim B \in \mathcal{L}_L$; (iii) if F is a Boolean combination of formulas of $\mathcal{L}_L$, then $F \in \mathcal{L}_L$. Observe that the only allowed combination of $\Box$ and L is in formulas of the form $\Box\neg LA$ where A is propositional.

We define

$$\Gamma^{L^{\downarrow}} = \{A \mid LA \in \Gamma\}.$$

Our tableau calculus $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ is shown in Figure 4. Observe that rules $(\mid\!\sim^{+})$ and $(\mid\!\sim^{-})$ have been changed as they introduce the modality L in front of the propositional formulas A and B in their conclusions. This corresponds to the semantics of conditionals in CL-preferential models (see Definition 2.7). The new rule (L^{-}) is a dynamic rule.

As an example, we use $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ in order to show that $A \vee A \mid\!\sim B$ can be inferred from $A \mid\!\sim B \wedge C$. To this aim, a closed tableau for $A \mid\!\sim B \wedge C, \neg(A \vee A \mid\!\sim B); \emptyset$ is presented in Figure 5.

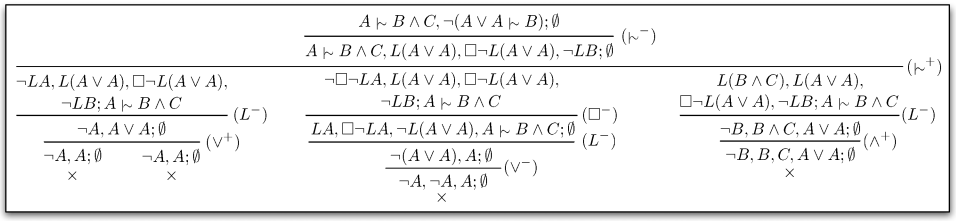

Fig. 5. A derivation in $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$ of $A \mid\sim B \wedge C, \neg(A \vee A \mid\sim B); \emptyset$.

THEOREM 4.1 (SOUNDNESS OF $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$). *The system $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$ is sound with respect to CL-preferential models, namely given a set of formulas Γ, if there is a closed tableau for $\Gamma; \emptyset$, then Γ is unsatisfiable.*

PROOF. We show that for all the rules in $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$, if the premise is satisfiable by a CL-preferential model then also one of the conclusions is. As far as the rules already present in $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ are concerned, the proof is very similar, with the only exception that we have to substitute A in the proof by LA.

Let us now consider the new rule (L^-). Let $\mathcal{M}, w \models \Gamma, \neg LA, \Sigma$ where $\mathcal{M} = \langle \mathcal{W}, R, <, V \rangle$ is a CL-preferential model and $w \in \mathcal{W}$. Then there is $w' \in \mathcal{W}$ such that $(w, w') \in R$ and $\mathcal{M}, w' \models \neg A$. Furthermore, $\mathcal{M}, w' \models \Gamma^{L^{\downarrow}}$. It follows that the conclusion of the rule is satisfiable. If Γ does not contain negated L-formulas, since R is serial, we still have that there exists $w' \in \mathcal{W}$ such that $(w, w') \in R$, and $\mathcal{M}, w' \models \Gamma^{L^{\downarrow}}$. Hence the conclusion is satisfiable. □

Soundness with respect to loop-cumulative models in Definition 2.6 follows from the correspondence established by Proposition 2.8.

The proof of the completeness of the calculus can be done as for the preferential case, provided we suitably modify the procedure for constructing a model for a finite consistent set of formulas Γ of $\mathcal{L}_L$. First of all, we modify the definition of saturated nodes as follows.

—If $A \mid\sim B \in \Gamma \cup \Sigma$ then $\neg LA \in \Gamma$ or $\neg\Box\neg LA \in \Gamma$ or $LB \in \Gamma$.

For this notion of saturated nodes we can still prove Lemma 3.5 for language $\mathcal{L}_L$.

THEOREM 4.2 (COMPLETENESS OF $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$). *$\mathcal{T}\mathrm{CL}^{\mathbf{T}}$ is complete with respect to CL-preferential models, that is, if a set of formulas Γ is unsatisfiable, then $\Gamma; \emptyset$ has a closed tableau in $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$.*

PROOF. We define a procedure for constructing a model satisfying a consistent set of formulas $\Gamma \in \mathcal{L}_L$ by modifying the procedure for the preferential logic P. We add to the procedure used in the proof of Theorem 3.6 the new steps 4' and 4" between step 4 and step 5, obtaining the procedure at the top of the next page.

We denote by APPLY$((\Gamma; \Sigma), L^-)$ the result of applying the rule (L^-) to $\Gamma; \Sigma$, in case Γ does not contain any negated L-formula $\neg LA$. The preceding procedure terminates. The argument is similar to the case of P. In addition, observe that the worlds introduced by an application of (L^-) cannot lead to generate any

1. initialize $X = \{\mathtt{SAT}(\Gamma;\emptyset)\}$; mark $\mathtt{SAT}(\Gamma;\emptyset)$ as unresolved;
while X contains unresolved nodes **do**
  2. choose an unresolved $\Gamma_i;\Sigma_i$ from X;
  3. **for** each formula $\neg(A \mathrel{|\!\sim} B) \in \Gamma_i$
    3a. let $(\Gamma;\Sigma)_{\neg(A|\sim B)} = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i,\Sigma_i), \mathrel{|\!\sim}^-, \neg(A \mathrel{|\!\sim} B)))$;
    3b. **if** $(\Gamma;\Sigma)_{\neg(A|\sim B)} \notin X$ **then** $X = X \cup \{(\Gamma;\Sigma)_{\neg(A|\sim B)}\}$;
  4. **for** each formula $\neg\Box\neg LA \in \Gamma_i$, let $(\Gamma;\Sigma)_{\neg\Box\neg LA} = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i;\Sigma_i), \Box^-, \neg\Box\neg LA))$;
    4a. add the relation $(\Gamma;\Sigma)_{\neg\Box\neg LA} < (\Gamma_i;\Sigma_i)$;
    4b. **if** $(\Gamma;\Sigma)_{\neg\Box\neg LA} \notin X$ **then** $X = X \cup \{(\Gamma;\Sigma)_{\neg\Box\neg LA}\}$.
  4'. **if** $\{\neg LA \mid \neg LA \in \Gamma_i\} \neq \emptyset$ **then**
      for each $\neg LA \in \Gamma_i$, let $(\Gamma;\Sigma_{\neg LA}) = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i;\Sigma_i), L^-, \neg LA))$;
        4' a. add $((\Gamma_i;\Sigma_i), (\Gamma;\Sigma)_{\neg LA})$ to R;
        4' b. **if** $(\Gamma;\Sigma)_{\neg LA} \notin X$ **then** $X = X \cup \{(\Gamma;\Sigma)_{\neg LA}\}$;
  4". **else if** $\Gamma_i^{L^\downarrow} \neq \emptyset$ **then** let $(\Gamma_j;\Sigma_j) = \mathtt{SAT}(\mathtt{APPLY}((\Gamma_i;\Sigma_i), L^-))$;
      4" a. add $((\Gamma_i;\Sigma_i), (\Gamma_j;\Sigma_j))$ to R;
      4" b. **if** $(\Gamma_j;\Sigma_j) \notin X$ **then** $X = X \cup \{(\Gamma_j;\Sigma_j)\}$;
  5. mark $\Gamma_i;\Sigma_i$ as resolved;
endWhile;

further world by means of a dynamic rule, as they do not contain any boxed formula or L-formula.

We construct the model $\mathcal{M} = \langle X, R_X, <_X, V\rangle$ by defining X and V as in the case of P. We then define R_X as the least relation including all pairs in R augmented with the pairs satisfying the following conditions.

(i) If $(\Gamma_i;\Sigma_i) \in X$ and $(\Gamma_i;\Sigma_i)$ has no R-successor, then $((\Gamma_i;\Sigma_i),(\Gamma_i;\Sigma_i)) \in R_X$;
(ii) If $((\Gamma_k;\Sigma_k),(\Gamma_i;\Sigma_i)) \in R_X$ and $((\Gamma_k;\Sigma_k),(\Gamma_j;\Sigma_j)) \in R_X$, then $((\Gamma_i;\Sigma_i),(\Gamma_j;\Sigma_j)) \in R_X$.

Notice that condition (ii) is the Euclidean closure of R. Last, we define $<_X$ as follows:

(iii) If $(\Gamma_j;\Sigma_j) < (\Gamma_i;\Sigma_i)$, then $(\Gamma_j;\Sigma_j) <_X (\Gamma_i;\Sigma_i)$;
(iv) If $(\Gamma_j;\Sigma_j) < (\Gamma_i;\Sigma_i)$, and $(\Gamma_i;\Sigma_i) R_X (\Gamma_k;\Sigma_k)$, then $(\Gamma_j;\Sigma_j) <_X (\Gamma_k;\Sigma_k)$;
(v) If $(\Gamma_j;\Sigma_j) <_X (\Gamma_i;\Sigma_i)$ and $(\Gamma_i;\Sigma_i) <_X (\Gamma_k;\Sigma_k)$, then $(\Gamma_j;\Sigma_j) <_X (\Gamma_k;\Sigma_k)$, namely $<_X$ is transitive.

Notice that the aforementioned conditions on R_X and $<_X$ are needed since the procedure builds two different kinds of worlds:

—*bad* worlds, obtained by an application of (L^-);
—*good* worlds: the other ones.

Bad worlds are those obtained by an application of (L^-). These worlds "forget" the positive conditionals in the initial set of formulas; for instance, if a world $(\neg C, D;\emptyset)$ is a bad world obtained from $(A \mathrel{|\!\sim} B, \neg LC, LD;\Sigma)$, then it is "incomplete" by the absence of $A \mathrel{|\!\sim} B$. The previous supplementary conditions on R_X and $<_X$ are needed to prove that, even in presence of bad worlds, we can build a CL-preferential model satisfying the initial set of formulas, as shown next.

FACT 4.3. *For all* $(\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j) \in X$*, if* $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j)) \in R_X$ *and* $LA \in \Gamma_i$ *then* $A \in \Gamma_j$.

PROOF OF FACT 4.3. Suppose that $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j)) \in R_X$ and $LA \in \Gamma_i$. We distinguish two cases. First, we consider the case in which $(\Gamma_i; \Sigma_i) \neq (\Gamma_j; \Sigma_j)$. In this case, $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j)) \in R$ and it has been added to R by step 4' or step 4" of the previous procedure. Indeed, for all $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j))$ that have been introduced because $((\Gamma_k; \Sigma_k), (\Gamma_i; \Sigma_i)) \in R$ and $((\Gamma_k; \Sigma_k), (\Gamma_j; \Sigma_j)) \in R$, both $(\Gamma_i; \Sigma_i)$ and $(\Gamma_j; \Sigma_j)$ derive from the application of (L^-) to $(\Gamma_k; \Sigma_k)$, hence they only contain propositional formulas and do not contain any LA. Hence, we have two different subcases.

—*The Relation* $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j))$ *has Been Added to* R *by Step 4'*. In this case, we have that $\neg LB \in \Gamma_i$ for some B. We can conclude that $A \in \Gamma_j$ by construction, since for each $LA \in \Gamma_i$ we have that $A \in \Gamma_j$ as a result of the application of $\texttt{SAT(APPLY}(((\Gamma_i; \Sigma_i)), L^-, \neg LB\texttt{))}$;
—*the Relation* $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j))$ *has Been Added to* R *by Step 4"*. Similarly to the previous case, for each $LA \in \Gamma_i$, we have that A is added to Γ_j by construction.

Second, we consider the case in which $(\Gamma_i; \Sigma_i) = (\Gamma_j; \Sigma_j)$. In this case, it must be that $((\Gamma_i; \Sigma_i), (\Gamma_i; \Sigma_i))$ has been added to R_X, as $(\Gamma_i; \Sigma_i)$ has no R-successors. This means that Γ_i does not contain formulas of the form LA or $\neg LA$, otherwise it would have an R-successor. □ (FACT 4.3)

FACT 4.4. *For all formulas* F *and for all worlds* $\Gamma_i; \Sigma_i \in X$ *we have that:* (i) *If* $F \in \Gamma_i \cup \Sigma_i$ *then* $\mathcal{M}, (\Gamma_i; \Sigma_i) \models F$*;* (ii) *if* $\neg F \in \Gamma_i$ *then* $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models F$.

PROOF OF FACT 4.4. The proof is similar to the one for the preferential case. If F is an atom or a Boolean combination of formulas, the proof is easy and left to the reader. We consider the other cases.

—$LA \in \Gamma_i$. We have to show that $\mathcal{M}, (\Gamma_i; \Sigma_i) \models LA$. Let $(\Gamma_j; \Sigma_j)$ such that $((\Gamma_i; \Sigma_i), (\Gamma_j; \Sigma_j)) \in R_X$. By Fact 4.3, we conclude $A \in \Gamma_j$. By inductive hypothesis, then $\mathcal{M}, (\Gamma_j; \Sigma_j) \models A$, and we are done.
—$\neg LA \in \Gamma_i$. We have to show that $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models LA$. By construction (step $4'$ in the procedure) there must be a $\Gamma_j; \Sigma_j \in X$ such that $\neg A \in \Gamma_j$. By inductive hypothesis, $\mathcal{M}, (\Gamma_j; \Sigma_j) \not\models A$, which concludes the proof.
—$\Box\neg LA \in \Gamma_i$. For all $(\Gamma_j; \Sigma_j) <_X (\Gamma_i; \Sigma_i)$ we have $\neg LA \in \Gamma_j$ by the definition of $(\Box^-)$, since $(\Gamma_j; \Sigma_j)$ has been generated by a sequence of applications of $(\Box^-)$ (notice that point (iv) in the previous definition of $<_X$ does not play any role here, since this point only concerns nodes $(\Gamma_i; \Sigma_i)$ that do not contain boxed or negated box formulas). By inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \not\models LA$ for all $(\Gamma_j; \Sigma_j) <_X (\Gamma_i; \Sigma_i)$, whence $\mathcal{M}, (\Gamma_i; \Sigma_i) \models \Box\neg LA$.
—$\neg\Box\neg LA \in \Gamma_i$. By construction there is a $(\Gamma_j; \Sigma_j)$ such that $(\Gamma_j; \Sigma_j) <_X (\Gamma_i; \Sigma_i)$ and $LA \in \Gamma_j$. By inductive hypothesis $\mathcal{M}, (\Gamma_j; \Sigma_j) \models LA$. Thus $\mathcal{M}, (\Gamma_i; \Sigma_i) \not\models \Box\neg LA$.
—$A \mathrel{|\!\sim} B \in \Gamma_i \cup \Sigma_i$. Let $(\Gamma_j; \Sigma_j) \in Min_{<_X}(LA)$. We distinguish two cases:

—$A \mathrel{|\!\sim} B \in \Gamma_j \cup \Sigma_j$, it can be observed that: (1)$\neg LA \in \Gamma_j$ or (2)$\neg\Box\neg LA \in \Gamma_j$ or (3)$LB \in \Gamma_j$, since $(\Gamma_j;\Sigma_j)$ is saturated. (1) cannot be the case, since by inductive hypothesis $\mathcal{M},(\Gamma_j;\Sigma_j) \not\models LA$, which contradicts the definition of $Min_{<_X}(LA)$. If (2), by construction of $\mathcal{M}$ there exists a set $(\Gamma_k;\Sigma_k) <_X (\Gamma_j;\Sigma_j)$ such that $LA \in \Gamma_k$. By inductive hypothesis $\mathcal{M},(\Gamma_k;\Sigma_k) \models LA$, which contradicts $(\Gamma_j;\Sigma_j) \in Min_{<_X}(LA)$. Therefore, it must be that (3)$LB \in \Gamma_j$, and by inductive hypothesis $\mathcal{M},(\Gamma_j;\Sigma_j) \models LB$. We conclude that $\mathcal{M},(\Gamma_i;\Sigma_i) \models A \mathrel{|\!\sim} B$.

—$A \mathrel{|\!\sim} B \not\in \Gamma_j \cup \Sigma_j$. Since all the rules apart from (L^-) preserve the conditionals, $(\Gamma_j;\Sigma_j)$ must have been generated by applying (L^-) to some $(\Gamma_k;\Sigma_k) \in X$, namely, $(\Gamma_j;\Sigma_j)$ is a *bad world*. Hence $((\Gamma_k;\Sigma_k),(\Gamma_j;\Sigma_j)) \in R_X$. In turn, it can be easily shown that $(\Gamma_k;\Sigma_k)$ itself cannot have been generated by (L^-), hence $A \mathrel{|\!\sim} B \in \Gamma_k \cup \Sigma_k$ and, since $(\Gamma_k;\Sigma_k)$ is saturated, either: (1)$\neg LA \in \Gamma_k$ or (2)$\neg\Box\neg LA \in \Gamma_k$ or (3)$LB \in \Gamma_k$. (1) is not possible, since by inductive hypothesis, it would entail that $\mathcal{M},(\Gamma_k;\Sigma_k) \not\models LA$, namely there is $(\Gamma_l;\Sigma_l)$ such that $((\Gamma_k;\Sigma_k),(\Gamma_l;\Sigma_l)) \in R_X$ and $\mathcal{M},(\Gamma_l;\Sigma_l) \not\models A$. By point (ii) in the preceding definition of R_X, also $((\Gamma_j;\Sigma_j),(\Gamma_l;\Sigma_l)) \in R_X$, hence also $\mathcal{M},(\Gamma_j;\Sigma_j) \not\models LA$, which contradicts $(\Gamma_j;\Sigma_j) \in Min_{<_X}(LA)$. If (2), by construction of $\mathcal{M}$ there exists a node $(\Gamma_l;\Sigma_l) <_X (\Gamma_k;\Sigma_k)$ such that $LA \in \Gamma_l$. By point (iv) in the previous definition of $<_X$, we have $(\Gamma_l;\Sigma_l) <_X (\Gamma_j;\Sigma_j)$, which contradicts $(\Gamma_j;\Sigma_j) \in Min_{<_X}(LA)$, since by inductive hypothesis $\mathcal{M},(\Gamma_l;\Sigma_l) \models LA$. It follows that $LB \in \Gamma_k$. By inductive hypothesis $\mathcal{M},(\Gamma_k;\Sigma_k) \models LB$, hence also $\mathcal{M},(\Gamma_j;\Sigma_j) \models LB$. Indeed, since Γ_j does not contain any L-formula, by construction of the model and by point (ii) in the previous definition of R_X, we have $((\Gamma_j;\Sigma_j),(\Gamma_l;\Sigma_l)) \in R_X$ just in case $((\Gamma_k;\Sigma_k),(\Gamma_l;\Sigma_l)) \in R_X$, from which the result follows. Hence we can conclude $\mathcal{M},(\Gamma_i;\Sigma_i) \models A \mathrel{|\!\sim} B$.

—$\neg(A \mathrel{|\!\sim} B) \in \Gamma_i$. By construction of X, there exists $(\Gamma_j;\Sigma_j) \in X$ such that $LA, \Box\neg LA, \neg LB \in \Gamma_j$. By inductive hypothesis we have that $\mathcal{M},(\Gamma_j;\Sigma_j) \models LA$ and $\mathcal{M},(\Gamma_j;\Sigma_j) \models \Box\neg LA$. It follows that $(\Gamma_j;\Sigma_j) \in Min_{<_X}(LA)$. Furthermore, always by induction, $\mathcal{M},(\Gamma_j;\Sigma_j) \not\models LB$. Hence $\mathcal{M},(\Gamma_i;\Sigma_i) \not\models A \mathrel{|\!\sim} B$. □ (FACT 4.4)

Similarly to the case of P, it is easy to prove the following fact.

FACT 4.5. *The relation $<_X$ is irreflexive, transitive, and satisfies the smoothness condition.*

Moreover, we have the next fact.

FACT 4.6. *The relation R_X is serial.*

From the aforesaid facts, we can conclude that $\mathcal{M} = \langle X, R_X, <_X, V\rangle$ is a CL-preferential model satisfying Γ, which concludes the proof of completeness. □

From Theorem 4.2 and Proposition 2.8, it follows that for any Boolean combination of conditionals Γ, if there is no closed tableau for $\Gamma;\emptyset$, then Γ is satisfiable in a loop-cumulative model.

The previous construction allows us to prove the following corollary.

COROLLARY 4.7 (FINITE MODEL PROPERTY). CL *has the finite model property.*

To the best of our knowledge, this property of CL was not previously known.

4.1 Decidability and Complexity of CL

We adopt the same restriction on the order of application of the rules adopted in Definition 3.12. As far as termination is concerned, notice that the rule (L^-) can only be applied a finite number of times. Indeed, if it is applied to a premise $\Gamma, \neg LA; \Sigma$, then the conclusion only contains propositional formulas $\Gamma^{L^\downarrow}, \neg A; \emptyset$, and the rule (L^-) is not further applicable. The same holds in the case (L^-) is applied to a premise $\Gamma; \Sigma$ such that Γ only contains positive L-formulas of the form LA (and does not contain negated L-formulas of the form $\neg LA$). Notice that (L^-) can also be applied to a node *not containing* any formula LA or $\neg LA$: In this case, the conclusion of the rule corresponds to an empty node $\emptyset; \emptyset$.

THEOREM 4.8 (TERMINATION OF $\mathcal{T}$CL$^{\mathbf{T}}$). $\mathcal{T}$CL$^{\mathbf{T}}$ *ensures a terminating proof search.*

PROOF. Exactly as we did for P, we consider a lexicographic order given by $m(\Gamma; \Sigma) = \langle c_1, c_2, c_3, c_4 \rangle$ (Definition 3.16), and easily prove that each application of the rules of $\mathcal{T}$CL$^{\mathbf{T}}$ reduces this measure. Indeed, as far as (L^-) is concerned, then c_1, c_2, and c_3 become 0, since conditional formulas are not kept in the conclusion. If the premise contains only L-formulas, then c_1, c_2, and c_3 are already equal to 0 in both the premise and the conclusion, but c_4 decreases, since (at least) one formula $(\neg)LA$ in the premise is removed, and a formula with a lower complexity ($\neg A$ or A) is introduced in the conclusion. □

Furthermore, the decision algorithm for P described in Section 3.1.2 can be adapted to CL. To this aim, we observe that the disjunction property holds for CL, and this allows us to change the rule for negated conditionals in order to treat them independently as we have done for P. Moreover, we can replace the $(\Box^-)$ rule by a stronger rule that does not require backtracking in the tableau construction. The rule is the following ($\Gamma^{\Box^-}_{-i}$ is used to denote $\{\neg\Box\neg LA_j \vee LA_j \mid \neg\Box\neg LA_j \in \Gamma \wedge j \neq i\}$).

$$\frac{\Gamma, \neg\Box\neg LA_1, \neg\Box\neg LA_2, \ldots, \neg\Box\neg LA_n; \Sigma}{\Sigma, \Gamma^{\mid\!\sim^\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, LA_1, \Box\neg LA_1, \Gamma^{\Box^-}_{-1}; \emptyset \mid \ldots \mid \Sigma, \Gamma^{\mid\!\sim^\pm}, \Gamma^{\Box}, \Gamma^{\Box^\downarrow}, LA_n, \Box\neg LA_n, \Gamma^{\Box^-}_{-n}; \emptyset} \; (\Box^-_s)$$

By reasoning similarly to the case of P, we can show that the calculus in which $(\Box^-)$ is replaced by $(\Box^-_s)$ is sound and complete with respect to multilinear CL-preferential models introduced in Definition 2.9. We get a decision procedure as in the case of P by defining a rule application's strategy that allows us to decide the satisfiability of a set of formulas in CL in nondeterministic polynomial time. As for P, the strategy checks the satisfiability of the nodes $\Gamma; \Sigma$ obtained by the application of the $(\mid\!\sim^-)$ rule to each negated conditional. In each case, we repeatedly apply $(\Box^-_s)$ for each negated boxed formula in Γ. At each step of application of $(\mid\!\sim^-)$ and $(\Box^-_s)$ rules, the obtained node is saturated with respect to the static rules as well as with respect to the (L^-) rule. In case of branching rules, this saturation nondeterministically selects (guesses) one of the conclusions of the rules.

As there is no nesting of box formulas within L modalities, the number of applications of the $(\Box_s^-)$ rule (for each application of $(\mid\!\sim^-)$) is polynomial in n, as in the case of P. As coNP-hardness immediately follows from the fact that CL includes classical logic, we obtain the following result.

THEOREM 4.9 (COMPLEXITY OF CL). *The problem of deciding validity for* CL *is* coNP-*complete.*

To the best of our knowledge, the complexity result for CL is new.

5. THE TABLEAU CALCULUS FOR CUMULATIVE LOGIC C

As for CL, we present a tableau calculus $\mathcal{T}\mathrm{C}^{\mathbf{T}}$ for cumulative logic C based on the mapping between cumulative models and C-preferential models. As mentioned in the Introduction (Section 1), by the lack of transitivity of the preference relation, the smoothness condition can no longer be identified with the finite-chain condition of logic G. The interplay between rules $(\mid\!\sim^+)$ and $(\Box^-)$ may lead to generate infinite branches. Hence, termination cannot be ensured by preventing multiple applications of $(\mid\!\sim^+)$ on the same conditional formula. In this case, we also need a loop-checking mechanism to prevent repeated expansions of the same tableau node. Nodes in $\mathcal{T}\mathrm{C}^{\mathbf{T}}$ are *sets of formulas* Γ, rather than pairs $\Gamma; \Sigma$, since we do not need to keep track of positive conditionals already expanded in a branch.

In order to provide a calculus for C, we have to replace the rule $(\Box^-)$ with the weaker $(\Box^{\mathbf{C}-})$.

$$(\Box^{\mathbf{C}-})\frac{\Gamma, \neg\Box\neg LA}{\Gamma^{\Box^{\downarrow}}, \Gamma^{\mid\sim^{\pm}}, LA}$$

Observe that, if we ignore conditionals, this rule is nothing else than the standard rule of modal logic K. This rule is weaker than the corresponding rule of the two other systems in two respects: (i) Transitivity is not assumed, thus we no longer have $\Gamma^{\Box}$ in the conclusion; (ii) the smoothness condition no longer ensures that if $\neg\Box\neg LA$ is true in one world, then there is a smaller *minimal* world satisfying LA. This only happens if the world itself satisfies LA. Thus $\Box\neg LA$ is dropped from the conclusion as well.

Moreover, we need the following version of the cut rule.

$$(\text{weak-cut})\frac{\Gamma}{\Gamma, \neg LA \qquad \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA \qquad \Gamma, \Box\neg LA}$$

Intuitively, this rule takes care of enforcing the smoothness condition, and it can be applied to all L-formulas.

The (weak-cut) rule is not eliminable, as shown by the following example.[6] Let $\Gamma = \{\neg(A \mid\!\sim C), A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C\}$, which is unsatisfiable in C. Γ has a closed tableau only if we use (weak-cut) in the calculus. Without (weak-cut), the aforementioned set of formulas does not have any closed tableau.

[6]Interestingly enough, this set of formulas corresponds to an instance of the well-known (CSO) axiom of conditional logics $(A \Rightarrow B) \wedge (B \Rightarrow A) \wedge (A \Rightarrow C) \rightarrow (B \Rightarrow C)$.

$$(\mid\!\sim^+)\ \frac{\Gamma, A \mid\!\sim B}{\Gamma, A \mid\!\sim B, \neg LA \qquad \Gamma^{\mid\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, A \mid\!\sim B, LA, \Box\neg LA \qquad \Gamma, A \mid\!\sim B, LA, \Box\neg LA, LB}$$

$$(\mid\!\sim^-)\ \frac{\Gamma, \neg(A \mid\!\sim B)}{LA, \Box\neg LA, \neg LB, \Gamma^{\mid\sim^{\pm}}} \qquad (L^-)\ \frac{\Gamma, \neg LA}{\Gamma^{L^{\downarrow}}, \neg A};\ \frac{\Gamma}{\Gamma^{L^{\downarrow}}}\ \text{if } \Gamma \text{ does not contain negated } L-\text{formulas}$$

Fig. 6. Tableau calculus $\mathcal{T}\mathtt{C}^{\mathbf{T}}$. Boolean rules are omitted.

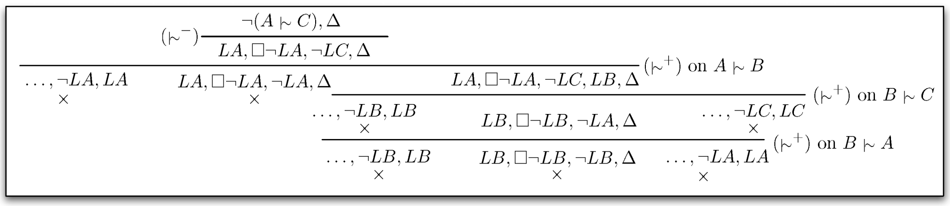

Fig. 7. A derivation in $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ of $\{\neg(A \mid\!\sim C), A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C\}$. To save space, we use Δ to denote the set of positive conditionals $A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C$.

By incorporating the (weak-cut) rule we obtain a nonanalytic calculus. Fortunately, we will show that the rule can be restricted so that it only applies to formulas LA such that A is the antecedent of a positive conditional formula in Γ, thus making the resulting calculus analytic. In order to prove this, we simplify the calculus by incorporating the application of $(\Box^{\mathbf{C}-})$ and the restricted form of (weak-cut) in the $(\mid\!\sim^+)$ rule. The resulting calculus, called $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ and given in Figure 6, is equivalent to the calculus that would be obtained from the calculus $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ by replacing $(\Box^-)$ with $(\Box^{\mathbf{C}-})$ and by introducing (weak-cut) restricted to antecedents of positive conditionals. The advantage of the adopted formulation is not only that it is more compact, but also that it allows a simpler proof of the admissibility of the non-restricted (weak-cut) (see Theorem 5.1 to follow).

Observe that $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ does not contain any rule for negated boxed formulas, as the modified $(\mid\!\sim^+)$ rule no longer introduces formulas of the form $\neg\Box\neg LA$. The resulting language $\mathcal{L}_{L'}$ of the formulas appearing in a tableau for $\mathtt{C}$ extends $\mathcal{L}$ by formulas LA and $\Box\neg LA$ (for A propositional). Notice that negated boxed formulas $\neg\Box\neg LA$ do not appear in $\mathcal{L}_{L'}$, which is therefore a restriction of $\mathcal{L}_L$ used in $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$.

Notice also that, as a difference with $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$, the $(\mid\!\sim^+)$ rule is neither a *static* nor a *dynamic* rule. Indeed, its leftmost and rightmost conclusions represent the same world as the world represented by the premise, whereas the conclusion in the middle represents a world which is different from the one represented by the premise. As an example, a derivation in $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ of the unsatisfiable set of formulas $\{\neg(A \mid\!\sim C), A \mid\!\sim B, B \mid\!\sim A, B \mid\!\sim C\}$ is presented in Figure 7.

We can prove that the (weak-cut) rule is admissible in $\mathcal{T}\mathtt{C}^{\mathbf{T}}$; this is stated by Theorem 5.1 given next. The nontrivial proof of this theorem is moved to the electronic appendix for space reasons.

THEOREM 5.1. *Given a set of formulas Γ and a propositional formula A, if there is a closed tableau for each of the following sets of formulas:*

(1) $\Gamma, \neg LA$
(2) $\Gamma^{\mid\!\sim^{\pm}}, \Gamma^{\Box^{\downarrow}}, LA, \Box\neg LA$
(3) $\Gamma, \Box\neg LA$

then there is also a closed tableau for Γ*, namely the* (weak-cut) *rule is admissible.*

Similarly to the proof of Theorem 4.1 for $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$, we prove that $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ is sound with respect to the semantics. The main difference between $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ and $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ is in rule ($\mid\!\sim^{+}$) that can be easily proven sound and to preserve satisfiability (by the smoothness condition). Therefore, we have the next theorem.

THEOREM 5.2 (SOUNDNESS OF $\mathcal{T}\mathtt{C}^{\mathbf{T}}$). *The system* $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ *is sound with respect to C-preferential models, that is given a set of formulas* Γ*, if there is a closed tableau for* Γ*, then* Γ *is unsatisfiable.*

Soundness with respect to cumulative models follows from the correspondence established by Proposition 2.13.

The completeness of the calculus can be proven by modifying the model construction used to show the completeness of $\mathcal{T}\mathtt{CL}^{\mathbf{T}}$ (Theorem 4.2). The completeness of $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ is a consequence of the admissibility of the (weak-cut) rule. Hence, in the construction of the model, we make use of the (weak-cut) rule. In order to build a *finite* model for an initial set of formulas Γ, we essentially restrict the application of (weak-cut) to formulas built from the finite language of Γ and we control the application of the rule in order to avoid applying this rule to propositionally equivalent formulas.

THEOREM 5.3 (COMPLETENESS OF $\mathcal{T}\mathtt{C}^{\mathbf{T}}$). $\mathcal{T}\mathtt{C}^{\mathbf{T}}$ *is complete with respect to C-preferential models, namely if a set of formulas* Γ *is unsatisfiable, then* Γ *has a closed tableau in* $\mathcal{T}\mathtt{C}^{\mathbf{T}}$.

PROOF. We assume that no tableau for Γ is closed, and we construct a model for Γ. The completeness is a consequence of the admissibility of (weak-cut). Hence, in the construction of the model, we make use of (weak-cut), i.e. we consider $\mathcal{T}\mathtt{C}^{\mathbf{T}}+$ (weak-cut). This is used to ensure the smoothness condition in the resulting model. In order to keep the construction of the model finite, we define a notion of equivalence between formulas with respect to their propositional part (or *p-equivalence*) so to identify those formulas having the same structure and containing equivalent propositional components. Let $\equiv_{PC}$ be logical equivalence in the classical propositional calculus. We define the notion of *p-equivalence* between two formulas F and G (written $F \equiv_p G$) as an equivalence relation satisfying the following conditions.

—If F and G are propositional formulas, then $F \equiv_p G$ iff $F \equiv_{PC} G$.
—$LA \equiv_p LB$ iff $A \equiv_{PC} B$.
—$\neg LA \equiv_p \neg LB$ iff $A \equiv_{PC} B$.
—$\Box\neg LA \equiv_p 2\neg LB$ iff $A \equiv_{PC} B$.

For instance, $LA \equiv_p L(A \wedge A)$, $\Box\neg LA \equiv_p \Box\neg L(A \wedge A)$.

We say that two sets of formulas Γ_i and Γ_j are p-equivalent if for every formula in Γ_i there is a p-equivalent formula in Γ_j, and vice versa.

Observe that this notion of p-equivalence is very weak and, for instance, we do not recognize that the set $\{LA, LB\}$ is equivalent to the set $\{L(A \wedge B)\}$. The notion of p-equivalence has been introduced with the purpose of limiting the application of the rule (weak-cut) so that it will not introduce infinitely many equivalent formulas. Moreover, in the construction of the model, we will not add a set of formulas to the current set of worlds X if X already contains p-equivalent set of formulas.

In our following construction of the model we will identify p-equivalent sets of formulas Γ_i. Before adding a new set of formulas Γ_i to the current set of worlds X, we check that X does not already contain a node Γ_j such that Γ_j is p-equivalent to Γ_i; for short, we will write $\Gamma_i \notin_P X$ in case X does not already contain such a Γ_j.

We define the procedure `SAT'` that extends any Γ_i by applying the transformations described next and, at the same time, produces a set Γ_i^S, initially set to $\emptyset$. Γ_i^S is a set of nonsaturated worlds. These worlds will then be processed at their turn in the model construction. The reason why `SAT'` is different from `SAT` previously defined is that, differently from `SAT`, it has to cope with rules (such as $(\mid\!\sim^+)$ or (weak-cut)) which are neither static nor dynamic. The transformations that follows are performed in sequence.

—Apply to Γ_i the propositional rules, once to each formula, as far as possible. In case of branching, make the choice that leads to an open tableau (this step saturates Γ_i with respect to the static rules).

—For each $A \mid\!\sim B \in \Gamma_i$, apply $(\mid\!\sim^+)$ to it. If the leftmost branch is open, then add $\neg LA$ to Γ_i; otherwise, if the rightmost branch is open, then add $\Box\neg LA, LA, LB$ to Γ_i; if the only open branch is the one in the middle, then add the set $\Delta = \{\Gamma_i^{\mid\sim^\pm}, \Gamma_i^{\Box^\downarrow}, LA, \Box\neg LA\}$ to the support set Γ_i^S associated with Γ_i.

—For all $LA \in \mathcal{L}_\Gamma$, if there is no $A \mid\!\sim B \in \Gamma_i$, and there is no A' propositionally equivalent to A on which (weak-cut) has already been applied (in Γ_i), apply (weak-cut) to it. If the leftmost branch is open, then add $\neg LA$ to Γ_i; otherwise, if the rightmost branch is open, add $\Box\neg LA$ to Γ_i; if the only open branch is the one in the middle, then add $\Delta = \{\Gamma_i^{\mid\sim^\pm}, \Gamma_i^{\Box^\downarrow}, LA, \Box\neg LA\}$ to the support set Γ_i^S.

Observe that `SAT'` terminates, extends Γ_i to a pair of finite sets of formulas, and produces a set Γ_i^S which is a finite set of finite sets of formulas, since: (a) there is only a finite number of conditionals that can lead to create sets of formulas in Γ_i^S; (b) there is only a finite number of formulas which are not *p*-equivalent to each other and that can lead to create a set in Γ_i^S by the third preceding transformation.

We build X, the set of worlds of the model, and $<$, as follows.

1. initialize $X = \{\Gamma\}$; mark Γ as unresolved;
2. **while** X contains unresolved nodes **do**
   3. choose an unresolved Γ_i from X;
   4. **for** each $\Delta \in \Gamma_i^S$, associated with Γ_i,
      let $\Gamma_\Delta =$ `SAT'`(Δ);
      4a. **for** all $\Gamma_j \in X$ s.t. Γ_j is p-equivalent to Γ_Δ
         add the relation $\Gamma_j < \Gamma_i$;
      4b. **if** $\Gamma_\Delta \notin_P X$ **then** let $X = X \cup \{\Gamma_\Delta\}$

5. **for** each formula $\neg LA \in \Gamma_i$, let $\Gamma_{i,\neg LA} =$ SAT'(APPLY($\Gamma_i, L^-, \neg LA$));
 5a. **for** all $\Gamma_j \in X$ s.t. Γ_j is p-equivalent to $\Gamma_{i,\neg LA}$
  add the relation (Γ_j, Γ_i) to R;
 5b. **if** $\Gamma_{i,\neg LA} \notin_P X$ **then** let $X = X \cup \{\Gamma_{i,\neg LA}\}$
6. **for** each formula $\neg(A \mid\!\sim B) \in \Gamma_i$
 6a. let $\Gamma_{i,\neg(A\mid\sim B)} =$SAT'(APPLY($\Gamma_i, \mid\!\sim^-, \neg(A \mid\!\sim B)$));
 6b. **if** $\Gamma_{i,\neg(A\mid\sim B)} \notin_P X$ **then** $X = X \cup \{\Gamma_{i,\neg(A\mid\sim B)}\}$;
7. mark Γ_i as resolved;

endWhile;

If Γ is finite, the procedure terminates. Indeed, it can be seen that SAT' terminates, as there is only a finite number of propositional evaluations. Furthermore, the whole procedure terminates, since the number of possible different sets of formulas that can be added to X starting from a finite set Γ is finite. Indeed, the number of non-p-equivalent sets of formulas that can be introduced in X is finite, as the number of p-equivalent classes is finite.

We construct the model $\mathcal{M} = \langle X, R_X, <_X, V\rangle$ as follows.

— X,V and R_X are defined as in the completeness proof for $\mathcal{T}$CL$^{\mathbf{T}}$ (Theorem 4.2);
— $<_X$ is defined as in the proof of Theorem 4.2, with the exception of condition (v), namely we no longer force $<_X$ to be transitive.

In order to show that $\mathcal{M}$ is a C-preferential model for Γ, we prove the following facts, by reasoning similarly to what was done in the proof of Theorem 4.2.

(1) *The relation $<_X$ is irreflexive.* By the previous procedure, $\Gamma_j <_X \Gamma_i$ only in two cases: (1) the relation has been introduced by step 4a in the procedure. In this case, it can be seen that $\Gamma_i \neq \Gamma_j$. (2) the relation has been introduced when completing $<_X$ starting from $<$, namely $\Gamma_j < \Gamma_k$ and $\Gamma_k R_X \Gamma_i$. Also in this case, it can be seen that $\Gamma_i \neq \Gamma_j$, since Γ_i does not contain L-formulas, whereas Γ_j does.

(2) *For all formulas F and for all $\Gamma_i \in X$ we have that:*
(i) *if $F \in \Gamma_i$ then $\mathcal{M}, \Gamma_i \models F$;* (ii) *if $\neg F \in \Gamma_i$ then $\mathcal{M}, \Gamma_i \not\models F$.* The proof is very similar to the one for CL. Obviously, the case of negated boxed formulas disappears, and the case of positive conditional formulas is slightly different.

(3) *The relation $<_X$ satisfies the smoothness condition on L-formulas.* Let $\mathcal{M}, \Gamma_i \models LA$. Then by Fact 2 as just describe, $\neg LA \notin \Gamma_i$. By definition of SAT' and point 4 in the preceding procedure, either $\Box\neg LA \in \Gamma_i$ or there is $\Gamma_j \in X$, such that $\{LA, \Box\neg LA\} \subseteq \Gamma_j$, and $\Gamma_j <_X \Gamma_i$. In the first case, by preceding Fact 2, $\mathcal{M}, \Gamma_i \models \Box\neg LA$, and it is minimal with respect to the set of LA-worlds. In the second case $\mathcal{M}, \Gamma_j \models \Box\neg LA$, it is minimal with respect to the set of LA-worlds, and $\Gamma_j <_X \Gamma_i$.

From the aforesaid facts, we can conclude that $\mathcal{M}$ is a C-preferential model for Γ, which concludes the completeness proof. □

From this theorem and Proposition 2.13, it follows that for any Boolean combination of conditionals Γ, if Γ does not have any closed tableau, then Γ

is satisfiable in a cumulative model. Furthermore, the following corollary follows.

COROLLARY 5.4 (FINITE MODEL PROPERTY). **C** *has the finite model property.*

As far as we know, this property of C was not previously known.

By Theorem 5.1, only formulas occurring in the initial set Γ can occur on a branch. Hence, the number of possibly different sets of formulas Γ on the branch is finite (and they are exponentially many in the size of Γ). A loop-checking procedure can be used in order to avoid that a given set of formulas is expanded again on a branch, so to ensure the termination of the procedure.

To check wether Γ is satisfiable, we must check that all the tableaux for Γ have an open branch. As there are exponentially many tableaux that have to be taken into consideration, for each one of exponential size with respect to the size of the initial set of formulas, our tableau method provides immediately an hyperexponential procedure to check the satisfiability. In further investigations it might be considered if this bound can be improved. For this, a more accurate analysis of the structure of a derivation (and, in particular, an analysis of permutability of the rules) might be required.

6. THE TABLEAU CALCULUS FOR RATIONAL LOGIC R

In this section we present $\mathcal{T}\mathrm{R}^{\mathbf{T}}$, a tableau calculus for rational logic R. We have already mentioned that, as a difference with the calculi presented for the other weaker logics, the calculus for R is a *labeled* calculus; the use of labels seems the more natural approach. Indeed, in order to capture the modularity condition of the preference relation, intuitively we must keep all worlds generated by $(\Box^-)$ and we need to propagate formulas among them according to all possible modular orderings. In an unlabeled calculus, this might be achieved, for instance, by introducing an ad hoc modal operator (that acts as a marker) or by adding additional structures to tableau nodes similarly to hypersequents calculi (see, e.g., Avron [1996]). However, the resulting calculus would be unavoidably rather cumbersome. In contrast, by using world labels, we can easily keep track of multiple worlds and their relations. This provides a much simpler, intuitive, and natural tableau calculus. On the other hand, even if we use labels, we do not run into problems with complexity and termination, so that we are able to define an optimal decision procedure for R.

The calculus makes use of labels to represent worlds. We consider a language $\mathcal{L}_R$ and a denumerable alphabet of labels $\mathcal{A}$, whose elements are denoted by *x, y, z, ...*. $\mathcal{L}_R$ extends $\mathcal{L}$ by formulas of the form $\Box\neg A$ as for the other logics.

Our tableau calculus includes two kinds of labeled formulas:

—*world formulas* $x : F$, whose meaning is that F holds in the possible world represented by x;

—*relation formulas* $x < y$, where $x, y \in \mathcal{A}$, used to represent the relation $<$.

We denote by $\alpha, \beta \ldots$ a world formula or a relation formula. We define

$$\Gamma^M_{x \rightarrow y} = \{y : \neg A,\ y : \Box\neg A \mid x : \Box\neg A \in \Gamma\}.$$

$$(\mathbf{AX})\ \Gamma, x : P, x : \neg P \quad \text{with } P \in ATM \qquad\qquad (\mathbf{AX})\ \Gamma, x < y, y < x$$

$$(\wedge^+)\frac{\Gamma, x : F \wedge G}{\Gamma, x : F, x : G} \qquad (\wedge^-)\frac{\Gamma, x : \neg(F \wedge G)}{\Gamma, x : \neg F \qquad \Gamma, x : \neg G} \qquad (\neg)\frac{\Gamma, x : \neg\neg F}{\Gamma, x : F}$$

$$(\mid\!\sim^+)\frac{\Gamma, u : A \mid\!\sim B^L}{\Gamma, x : \neg A, u : A \mid\!\sim B^{L,x} \qquad \Gamma, x : \neg\Box\neg A, u : A \mid\!\sim B^{L,x} \qquad \Gamma, x : B, u : A \mid\!\sim B^{L,x}}\ \begin{array}{l}x \text{ occurs in } \Gamma\\ \text{and } x \notin L\end{array}$$

$$(\mid\!\sim^-)\frac{\Gamma, u : \neg(A \mid\!\sim B)}{\Gamma, x : A, x : \Box\neg A, x : \neg B}\ x \text{ new} \qquad (\Box^-)\frac{\Gamma, x : \neg\Box\neg A}{\Gamma, y < x, \Gamma^M_{x \to y}, y : A, y : \Box\neg A}\ y \text{ new}$$

$$(<)\frac{\Gamma, x < y}{\Gamma, x < y, x < z, \Gamma^M_{z \to x} \qquad \Gamma, x < y, z < y, \Gamma^M_{y \to z}}\ \begin{array}{l}z \text{ occurs in } \Gamma \text{ and}\\ \{x < z, z < y\} \cap \Gamma = \emptyset\end{array}$$

Fig. 8. The calculus $\mathcal{T}\mathrm{R}^{\mathbf{T}}$. To save space, rules for $\rightarrow$ and $\vee$ are omitted.

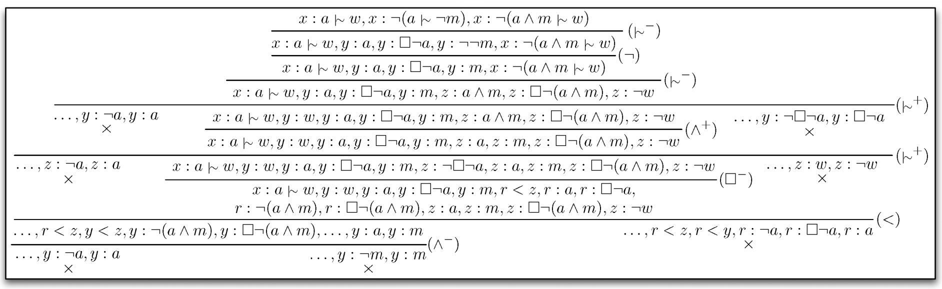

Fig. 9. A derivation in $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ of $\{adult \mid\!\sim worker, \neg(adult \mid\!\sim \neg married), \neg(adult \wedge married \mid\!\sim worker)\}$. To increase readability, we use a for $adult$, m for $married$, and w for $worker$.

The calculus $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is presented in Figure 8. As for P, the rules $(\mid\!\sim^-)$ and $(\Box^-)$ that introduce new labels in their conclusion are called *dynamic rules*; all the other rules are called *static rules*.

Definition 6.1 (*Truth Conditions of Formulas of* $\mathcal{T}\mathrm{R}^{\mathbf{T}}$). Given a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and a label alphabet $\mathcal{A}$, we consider a mapping $I : \mathcal{A} \mapsto \mathcal{W}$. Given a formula α of the calculus $\mathcal{T}\mathrm{R}^{\mathbf{T}}$, we define $\mathcal{M} \models_I \alpha$ as follows:

—$\mathcal{M} \models_I x : F$ iff $\mathcal{M}, I(x) \models F$;
—$\mathcal{M} \models_I x < y$ iff $I(x) < I(y)$.

We say that a set of labeled formulas Γ is satisfiable if, for all formulas $\alpha \in \Gamma$, we have that $\mathcal{M} \models_I \alpha$, for some model $\mathcal{M}$ and some mapping I.

In order to verify that a set of formulas Γ is unsatisfiable, we label all the formulas in Γ with a new label x, and verify that the resulting set of labeled formulas has a closed tableau. For instance, in order to verify that the set $\{adult \mid\!\sim worker, \neg(adult \mid\!\sim \neg married), \neg(adult \wedge married \mid\!\sim worker)\}$ is unsatisfiable (and thus $adult \wedge married \mid\!\sim worker$ is entailed by $\{adult \mid\!\sim worker, \neg(adult \mid\!\sim \neg married)\}$), we can build the closed tableau in Figure 9.

6.1 Soundness, Termination, and Completeness of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$

In this section we prove that the calculus $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is sound and complete with respect to the semantics and guarantees termination.

First of all, we can show that the calculus for R always terminates. As for the calculi for P and CL, every tableau built by the calculus is finite.

Similarly to the other cases, it is easy to observe that it is useless to apply the rule ($\mid\!\sim^+$) on the same conditional formula more than once in the same world, that is, by using the same label x. We prevent redundant applications of ($\mid\!\sim^+$) by keeping track of labels (worlds) in which a conditional $u : A \mid\!\sim B$ has already been applied in the current branch. To this purpose, we add to each positive conditional a list of *used* labels L; we then restrict the application of ($\mid\!\sim^+$) only to labels not occurring in the corresponding list L.

Notice that also the rule ($<$) copies its principal formula $x < y$ in the conclusion; however, this rule will be applied only a finite number of times. This is a consequence of the side condition of the rule application and the fact that the number of labels occurring in a tableau is finite.

It is easy to prove the following structural properties of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$.

LEMMA 6.2. *For any set of formulas Γ and any world formula $x : F$, there is a closed tableau for $\Gamma, x : F, x : \neg F$.*

PROOF. By induction on the complexity of the formula F. □

LEMMA 6.3 (HEIGHT-PRESERVING ADMISSIBILITY OF WEAKENING). *Given any set of formulas Γ and any formula α, if Γ has a closed tableau of height h then Γ, α has a closed tableau whose height is no greater than h.*

PROOF. By induction on the height of the closed tableau for Γ. □

Moreover, it is possible to easily prove that all the rules of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ are height-preserving invertible (the detailed proof can be found in Pozzato [2007], Giordano et al. [2007a]):

THEOREM 6.4 (HEIGHT-PRESERVING INVERTIBILITY OF THE RULES OF $\mathcal{T}\mathrm{R}^{\mathbf{T}}$). *Given any rule of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$, whose premise is Γ and whose conclusions are Γ_i, with $i \leq 3$, if Γ has a closed tableau of height h, then there is a closed tableau, of height no greater than h, for each Γ_i, namely the rules of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ are height-preserving invertible.*

By Theorem 6.4, we have that in $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ the order of application of the rules is not relevant. Hence, no backtracking is required in the tableau construction, and we can assume without loss of generality that a given set of formulas Γ has a unique tableau.

Let us now prove that $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is sound.

THEOREM 6.5 (SOUNDNESS). *$\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is sound with respect to rational models, namely if there is a closed tableau for a set of formulas Γ, then Γ is unsatisfiable.*

PROOF. By induction on the height of the closed tableau for Γ. Let Γ be an axiom: If: (i) $x : P \in \Gamma$ and $x : \neg P \in \Gamma$, then obviously there is no I such that, given a model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$, we have $I(x) \in \mathcal{W}$ and $\mathcal{M}, I(x) \models P$ and $\mathcal{M}, I(x) \not\models P$. If: (ii) $x < y \in \Gamma$ and $y < x \in \Gamma$, suppose on the contrary that Γ be satisfiable. Then there is a rational model $\mathcal{M} = \langle \mathcal{W}, <, V \rangle$ and a mapping I such that $\mathcal{M} \models_I x < y$ and $\mathcal{M} \models_I y < x$. Thus we have $I(x) < I(y)$ and $I(y) < I(x)$ against the fact that $<$ is irreflexive and transitive.

For the inductive step, we prove as usual the contrapositive, that is to say, we prove for each rule that if the premise is satisfiable, so is (at least) one of its conclusions. We only present the case of ($\Box^-$). Since the premise is satisfiable, then there is a model $\mathcal{M}$ and a mapping I such that $\mathcal{M} \models_I \Gamma, x : \neg\Box\neg A$. Let $w \in \mathcal{W}$ such that $I(x) = w$; this means that $\mathcal{M}, w \not\models \Box\neg A$, hence there exists a world $w' < w$ such that $\mathcal{M}, w' \models A$. By the strong smoothness condition, we have that there exists a *minimal* such world, so we can assume that $w' \in Min_<(A)$, thus $\mathcal{M}, w' \models \Box\neg A$. In order to prove that the conclusion of the rule is satisfiable, we construct a mapping I' as follows: Let y be a new label, not occurring in the current branch; we define: (1) $I'(u) = I(u)$ for all $u \neq y$ and (2) $I'(y) = w'$. Since y does not occur in Γ, it follows that $\mathcal{M} \models_{I'} \Gamma$. By Definition 6.1, we have that $\mathcal{M} \models_{I'} y < x$ since $w' < w$. Moreover, since $I'(y) = w'$, we have that $\mathcal{M} \models_{I'} y : A$ and $\mathcal{M} \models_{I'} y : \Box\neg A$. Finally, $\mathcal{M} \models_{I'} \Gamma^M_{x \to y}$ follows from the fact that $I'(y) < I'(x)$ and from the transitivity of $<$. The single conclusion of the rule is then satisfiable in $\mathcal{M}$ via I'. □

In order to prove the completeness of the calculus, similarly to what done for the other logics, we introduce the notion of saturated branch and we show that $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ ensures a terminating proof search. As a consequence, we will observe that the calculus introduces a finite number of labels in a tableau, and this result will be used to prove the completeness of the calculus.

Definition 6.6 (*Saturated Branch*). We say that a branch $\mathtt{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ of a tableau is *saturated* if the following conditions hold:

(1) For the Boolean connectives, the condition of saturation is defined in the usual way. For instance, if $x : A \wedge B \in \Gamma_i$ in $\mathtt{B}$, then there exists Γ_j in $\mathtt{B}$ such that $x : A \in \Gamma_j$ and $x : B \in \Gamma_j$.
(2) If $x : A \mid\!\sim B \in \Gamma_i$, then for any label y in $\mathtt{B}$, there exists Γ_j in $\mathtt{B}$ such that either $y : \neg A \in \Gamma_j$ or $y : \neg\Box\neg A \in \Gamma_j$ or $y : B \in \Gamma_j$.
(3) If $x : \neg(A \mid\!\sim B) \in \Gamma_i$, then there is a Γ_j in $\mathtt{B}$ such that, for some y, $y : A \in \Gamma_j$, $y : \Box\neg A \in \Gamma_j$, and $y : \neg B \in \Gamma_j$.
(4) If $x : \neg\Box\neg A \in \Gamma_i$, then there exists Γ_j in $\mathtt{B}$ such that, for some y, $y < x \in \Gamma_j$, $y : A \in \Gamma_j$ and $y : \Box\neg A \in \Gamma_j$.
(5) If $x < y \in \Gamma_i$, then for all labels z in $\mathtt{B}$, there exists Γ_j in $\mathtt{B}$ such that either $z < y \in \Gamma_j$ or $x < z \in \Gamma_j$.

We say that a set of formulas Γ is *closed* if it is an instance of $(\mathtt{AX})$. A branch $\mathtt{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ is closed if it contains some Γ_i which is closed. A branch is *open* if it is not closed. We can prove the following lemma.

LEMMA 6.7. *Given a tableau starting with $x_0 : F$, for any open, saturated branch $\mathtt{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$, we have that:*

(1) *if* $z < y \in \Gamma_i$ *in* B *and* $y < x \in \Gamma_j$ *in* B*, then there exists* Γ_k *in* B *such that* $z < x \in \Gamma_k$*;*
(2) *if* $x : \Box\neg A \in \Gamma_i$ *in* B *and* $y < x \in \Gamma_j$ *in* B*, then there exists* Γ_k *in* B *such that* $y : \neg A \in \Gamma_k$ *and* $y : \Box\neg A \in \Gamma_k$*;*
(3) *for no* Γ_i *in* B*,* $x < x \in \Gamma_i$.

Proof. Let us consider an open, saturated branch $\mathtt{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$. We consider the three claims of the lemma separately.

(1) We are considering the case when $z < y \in \Gamma_i$ in B and $y < x \in \Gamma_j$ in B. Since B is saturated, as $z < y \in \Gamma_i$, there exists Γ_k in B such that either $z < x \in \Gamma_k$ or $x < y \in \Gamma_k$. If $x < y \in \Gamma_k$, then the branch is closed (Γ_k is an instance of (AX)). Thus, we conclude that $z < x \in \Gamma_k$.
(2) A relation formula $y < x$ can only be introduced by an application of either $(\Box^-)$ or $(<)$; in both cases, $\Gamma^M_{x\to y}$ is added to the current branch of the tableau. Consider any $x : \Box\neg A \in \Gamma_i$; if $j > i$, namely $y < x$ is introduced in the branch *after* $x : \Box\neg A$, then we are done, since $y : \neg A \in \Gamma^M_{x\to y}$ and $y : \Box\neg A \in \Gamma^M_{x\to y}$. Otherwise, if $y < x$ is introduced in the branch *before* $x : \Box\neg A$, then we are considering the case such that $x : \Box\neg A$ is introduced by an application of $(<)$, namely $x : \Box\neg A \in \Gamma^M_{k\to x}$ (by the presence of some $k : \Box\neg A$ in the branch) for some k and $x < k$ is also introduced in the branch. Since the branch is saturated, then either $(*)$ $x < y$ or $(**)$ $y < k$ is introduced in the branch: $(*)$ cannot be, otherwise the branch would be closed. If $(**)$ is introduced after $k : \Box\neg A$, then we are done since $y : \neg A \in \Gamma^M_{k\to y}$ and $y : \Box\neg A \in \Gamma^M_{k\to y}$; otherwise, $(**)$ has also been introduced by an application of $(<)$, and we can repeat the argument. This process must terminate. Indeed, we can observe the following facts: a boxed formula $u : \Box\neg A$ must be initially introduced in a branch by an application of either $(\Box^-)$ or $(\mid\!\sim^-)$, further applications of $(<)$ and $(\Box^-)$ may only "propagate" it to other worlds. In both cases, $(\Box^-)$ and $(\mid\!\sim^-)$, u is a new label not occurring in the branch, so that all formulas $v < u$ will necessarily be introduced when $u : \Box\neg A$ already belongs to the branch. Let therefore u be a label such that $y < u$ is introduced in the branch *after* $u : \Box\neg A$; by saturation we have that $y : \neg A$ and $y : \Box\neg A$ belong to the branch.
(3) A relation $x < x$ cannot be introduced by rule $(\Box^-)$, since this rule establishes a relation between x in B and a label distinct from x. On the other hand, it cannot be introduced by modularity. Indeed, for rule $(<)$ to introduce a relation $x < x$, there must be in B some relation $y < x$ (respectively $x < y$) for some y. But in this case the side condition of the rule would not be fulfilled, and the rule could not be applied. □

Also in $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ we introduce the restriction on the order of application of the rules, as adopted for the other systems (see Definition 3.12), namely, the application of the $(\Box^-)$ rule is postponed to the application of all propositional rules and to the test of whether Γ is an instance of (AX) or not.

Similarly to the case of $\mathcal{T}\mathrm{P}^{\mathbf{T}}$ and $\mathcal{T}\mathrm{CL}^{\mathbf{T}}$, we can show that $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ ensures termination. In particular, the rule $(\mid\!\sim^-)$ can be applied only once for each negated conditional Γ. Furthermore, the generation of infinite branches due to the interplay between rules $(\mid\!\sim^+)$ and $(\Box^-)$ cannot occur. Indeed, each application of $(\Box^-)$ to a formula $x : \neg\Box\neg A$ (introduced by $(\mid\!\sim^+)$) adds the formula $y : \Box\neg A$ to the conclusion, so that $(\mid\!\sim^+)$ can no longer consistently introduce $y : \neg\Box\neg A$ (a detailed proof can be found in Pozzato [2007] and Giordano et al. [2007a]).

THEOREM 6.8 (TERMINATION OF $\mathcal{T}\mathrm{R}^{\mathbf{T}}$). *Let Γ be a finite set of formulas, then any tableau generated by $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is finite.*

As a consequence of Theorem 6.8, a tableau for a given set of formulas Γ only contains a finite number of labels.
Let us now show that $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is complete with respect to the semantics.

THEOREM 6.9 (COMPLETENESS). *$\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is complete with respect to rational models, namely if a set of formulas Γ is unsatisfiable, then it has a closed tableau in $\mathcal{T}\mathrm{R}^{\mathbf{T}}$.*

PROOF. We show the contrapositive, that is, if there is no closed tableau for Γ, then Γ is satisfiable. Consider the tableau starting with the set of formulas $\{x : F$ such that $F \in \Gamma\}$ and any open, saturated branch $\mathtt{B} = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ in it. Starting from $\mathtt{B}$, we build a canonical model $\mathcal{M} = \langle \mathcal{W}_B, <, V\rangle$ satisfying Γ, where:

—$\mathcal{W}_B$ is the set of labels that appear in the branch $\mathtt{B}$;
—for each $x, y \in \mathcal{W}_B$, $x < y$ iff there exists Γ_i in $\mathtt{B}$ such that $x < y \in \Gamma_i$;
—for each $x \in \mathcal{W}_B$, $V(x) = \{P \in ATM \mid$ there is Γ_i in $\mathtt{B}$ such that $x : P \in \Gamma_i\}$.

We can prove that the following holds.

(i) $\mathcal{W}_B$ is finite.
(ii) $<$ is an irreflexive, transitive, and modular relation on $\mathcal{W}_B$ satisfying the smoothness condition. Irreflexivity, transitivity, and modularity are obvious, given Definition 6.6 and Lemma 6.7. Since $<$ is irreflexive and transitive, it can be easily shown that it is also acyclic. This property together with the finiteness of $\mathcal{W}_B$ entails that $<$ cannot have infinite descending chains. In turn this last property together with the transitivity of $<$ entails the smoothness condition.
(iii) We show that, for all formulas F and for all Γ_i in $\mathtt{B}$: (*i*) If $x : F \in \Gamma_i$ then $\mathcal{M}, x \models F$ and (*ii*) if $x : \neg F \in \Gamma_i$ then $\mathcal{M}, x \not\models F$. The proof is by induction on the complexity of the formulas. If $F \in ATM$ this immediately follows from definition of V. For the inductive step, we only present the case of $F = A \mid\!\sim B$. The other cases are similar and then left to the reader. Let $x : A \mid\!\sim B \in \Gamma_i$. By Definition 6.6, we have that, for all y, there is Γ_j in $\mathtt{B}$ such that either $y : \neg A \in \Gamma_j$ or $y : B \in \Gamma_j$ or $y : \neg\Box\neg A \in \Gamma_j$. We show that for all $y \in Min_<(A)$, $\mathcal{M}, y \models B$. Let $y \in Min_<(A)$. This entails that $\mathcal{M}, y \models A$, hence $y : \neg A \notin \Gamma_j$. Similarly, we can show that $y : \neg\Box\neg A \notin \Gamma_j$. It follows that $y : B \in \Gamma_j$, and by inductive hypothesis $\mathcal{M}, y \models B$. (ii) If $x : \neg(A \mid\!\sim B) \in \Gamma_i$, since $\mathtt{B}$ is saturated, there is a label y in some Γ_j such that $y : A \in \Gamma_j$, $y : \Box\neg A \in \Gamma_j$, and $y : \neg B \in \Gamma_j$. By

inductive hypothesis we can easily show that $\mathcal{M}, y \models A$, $\mathcal{M}, y \models \Box\neg A$, hence $y \in Min_<(A)$, and $\mathcal{M}, y \not\models B$, hence $\mathcal{M}, x \not\models A \mathrel{|\!\sim} B$. □

The construction of the model done in the proof of Theorem 6.9 provides a constructive proof of the finite model property of R.

6.2 Complexity of R

In this section we define a systematic procedure which allows the satisfiability problem for R to be decided in nondeterministic polynomial time, in accordance with the known complexity results for this logic.

Let n be the size of the starting set Γ of which we want to verify the satisfiability. The number of applications of the rules is proportional to the number of labels introduced in the tableau. In turn, this is $O(2^n)$ due to the interplay between the rules $(\mathrel{|\!\sim}^+)$ and $(\Box^-)$. Hence, the complexity of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ is exponential in n.

In order to obtain a better complexity bound for validity in R we present the following procedure. Intuitively, we do not apply $(\Box^-)$ to all negated boxed formulas, but only to formulas $y : \neg\Box\neg A$ not already expanded, that is, such that $z : A, z : \Box\neg A$ do not belong to the current branch. As a result, we build a *small* model for the initial set of formulas (see Giordano et al. [2006, Theorem 1]).

Let us define a nondeterministic procedure CHECK(Γ) to decide whether a given set of formulas Γ is satisfiable. We make use of a procedure EXPAND(Γ) that returns one saturated expansion of Γ with respect to all static rules. In case of a branching rule, EXPAND nondeterministically selects (guesses) one conclusion of the rule.

CHECK(Γ)
1. $\Gamma \longleftarrow$ EXPAND(Γ);
2. **if** Γ contains an axiom **then return** unsat;
3. **for each** $x : \neg(A \mathrel{|\!\sim} B) \in \Gamma$ **do** $\Gamma \longleftarrow$APPLY($\Gamma, \mathrel{|\!\sim}^-, x : \neg(A \mathrel{|\!\sim} B)$);
4. $\Gamma \longleftarrow$ EXPAND(Γ);
5. **if** Γ contains an axiom **then return** unsat;

while Γ contains a $y : \neg\Box\neg A$ not marked as CONSIDERED **do**
  6. select $y : \neg\Box\neg A \in \Gamma$ not already marked as CONSIDERED;
    6a. **if** there is z in Γ such that $z : A \in \Gamma$ and $z : \Box\neg A \in \Gamma$
      then 6a'. add $z < y$ and $\Gamma^{M}_{y \to z}$ to Γ;
      else 6a". $\Gamma \longleftarrow$ APPLY($\Gamma, \Box^-, y : \neg\Box\neg A$);
    6b. mark $y : \neg\Box\neg A$ as CONSIDERED;
  7. $\Gamma \longleftarrow$ EXPAND(Γ);
  8. **if** Γ contains an axiom **then return** unsat;

endWhile

9. **return** satisf;

Observe that the addition of the set of formulas $z < y, \Gamma^{M}_{y \to z}$ in step 6a' could be omitted and it has been added mostly to enhance the understanding of the procedure. Indeed, the rule $(<)$, which is applied at each iteration to assure modularity, already takes care of adding such formulas. The procedure CHECK nondeterministically builds an open branch for Γ.

THEOREM 6.10 (SOUNDNESS AND COMPLETENESS OF THE PROCEDURE). *The previous procedure is sound and complete with respect to the semantics.*

Proof. (*Soundness*). We prove that if the initial set of formulas Γ is satisfiable, then the aforesaid procedure returns `satisf`. More precisely, we prove that each step of the procedure preserves the satisfiability of Γ. As far as EXPAND is concerned, notice that it only applies the static rules of $\mathcal{T}\mathrm{R}^{\mathbf{T}}$ and these rules preserve satisfiability (Theorem 6.5). Consider now step 6. Let $y : \neg\Box\neg A$ the formula selected in this step. If $(\Box^-)$ is applied to $y : \neg\Box\neg A$ (step 6a") we are done, since $(\Box^-)$ preserves satisfiability (Theorem 6.5). If Γ already contains $z : A, z : \Box\neg A$, then step 6a' is executed, and the relation $z < y$ is added. In this case we reason as follows. Since Γ is satisfiable, we have that there is a model $\mathcal{M}$ and a mapping I such that (1) $\mathcal{M}, I(y) \models \neg\Box\neg A$ and (2) $\mathcal{M}, I(z) \models A$ and $\mathcal{M}, I(z) \models \Box\neg A$. We can observe that $I(z) < I(y)$ in $\mathcal{M}$. Indeed, by the truth condition of $\neg\Box\neg A$ and by the strong smoothness condition, we have that there exists w such that $w < I(y)$ and $\mathcal{M}, w \models A, \Box\neg A$. By modularity of $<$, either: (1) $w < I(z)$ or (2) $I(z) < I(y)$. (1) is impossible, since otherwise we would have $\mathcal{M}, w \models \neg A$, which contradicts $\mathcal{M}, w \models A$. Hence, (2) holds. Therefore, we can conclude that step 6a' preserves satisfiability.
(*Completeness*). It can be easily shown that in case the given procedure returns `satisf`, then the branch generated by the procedure is saturated (see Definition 6.6). Therefore, we can build a model for the initial Γ as shown in the proof of Theorem 6.9. □

Theorem 6.11 (Complexity of the Check Procedure). *By means of the procedure* CHECK*, the satisfiability of a set of formulas of logic* R *can be decided in nondeterministic polynomial time.*

Proof. Observe that the procedure generates at most *O(n)* labels by applying the rule $(\mid\!\sim^-)$ (step 3) and that the while loop generates at most one new label for each $\neg\Box\neg A$ formula. Indeed, the rule $(\Box^-)$ is applied to a labeled formula $y : \neg\Box\neg A$ to generate a new world only if there is not a label z such that $z : A \in \Gamma$ and $z : \Box\neg A \in \Gamma$ are already on the branch. In essence, the procedure does not add a new minimal A-world on the branch if there is already one. As the number of different $\neg\Box\neg A$ formulas is at most $O(n)$, then the while-loop can add at most $O(n)$ new labels on the branch. Moreover, for each different label x, the expansion step can add at most $O(n)$ formulas $x : \neg\Box\neg A$ on the branch, one for each positive conditional $A \mid\!\sim B$ occurring in the set Γ. We can therefore conclude that the while-loop can be executed at most $O(n^2)$ times.

As the number of generated labels is at most $O(n)$, by the subformula property, the number of labeled formulas on the branch is at most $O(n^2)$. Hence, the execution of step 6a has complexity $O(n^2)$. The execution of the nondeterministic procedure EXPAND has complexity $O(n^2)$, including a guess of size $O(n^2)$, whereas to check if Γ is an axiom is of a polynomial complexity in the size of Γ. We can therefore conclude that the execution of the procedure CHECK has a polynomial complexity. □

Given that the validity problem for R is known to be in coNP, the procedure CHECK is optimal for R.

7. CONCLUSIONS

In this article, we have presented tableau calculi for all KLM logical systems of default reasoning.[7] We have given a tableau calculus for rational logic R, preferential logic P, loop-cumulative logic CL, and cumulative logic C. The calculi presented give a decision procedure for the respective logics. Moreover, for R, P, and CL we have shown that we can obtain coNP decision procedures by refining the rules of the respective calculi. In case of C, we obtain a decision procedure by adding a suitable loop-checking mechanism. Our procedure gives an hyperexponential upper bound. Further investigation is needed to get a more efficient procedure. On the other hand, we are not aware of any tighter complexity bound for this logic. Furthermore, by means of our tableau calculi, we obtain a constructive proof of the finite model property for each logic.

All the calculi presented in this article have been implemented by a theorem prover called KLMLean [Olivetti and Pozzato 2005; Giordano et al. 2007b]. KLMLean (not presented here) is a SICStus Prolog implementation inspired to the “lean” methodology [Beckert and Posegga 1995; Fitting 1998; Beckert and Posegga 1996], whose basic idea is to write short programs that exploit the power of Prolog’s engine as much as possible. To the best of our knowledge, KLMLean is the first theorem prover for KLM logics.

Artosi, Governatori, and Rotolo [Artosi et al. 2002] develop a labeled tableau calculus for C. Their calculus is based on the interpretation of C as a conditional logic with a selection function semantics. As a major difference from our approach, their calculus makes use of labeled formulas, where the labels represent possible worlds or sets of possible worlds. World labels in turn are annotated by formulas to express minimality assumptions, for example, they represent by a label w^A the fact that w is a minimal A-world, or in terms of the selection function, belongs to $f(A, u)$. They use then a sophisticated unification mechanism on the labels to match two annotated worlds, for example, w^A, w^B; observe that by CSO (which is equivalent to CUT+CM), the equivalence of A and B might also be enforced by the conditionals contained in a tableau branch. Even if they do not discuss decidability and complexity issues, their tableau calculus should give a decision procedure for C. Moreover, the authors suggest how to extend the system to loop-cumulative logic CL and discuss some ways to extend it to the other logics.

In Giordano et al. [2003, 2005b] it is defined a labeled tableau calculus for the logic CE and some of its extensions. The flat fragment of CE corresponds to the system P. The similarity between the two calculi lies in the fact that both approaches use a modal interpretation of conditionals. The major difference is that the calculus presented here does not use labels, whereas the one proposed in Giordano et al. [2003] does. A further difference is that in Giordano et al. [2003] the termination is obtained by means of a loop-checking machinery, and it is not clear if it matches complexity bounds and if it can be adapted in a simpler way to CL and to C.

[7] Some preliminary results have been presented in Giordano et al. [2006, 2005a]

Lehmann and Magidor [1992] propose a nondeterministic algorithm that, given a finite set K of conditional assertions $C_i \mid\sim D_i$ and a conditional assertion $A \mid\sim B$, checks if $A \mid\sim B$ is not entailed by K in the logic P. This is an abstract algorithm useful for theoretical analysis, but practically unfeasible, as it requires to guess sets of indexes and propositional evaluations. They conclude that entailment in P is coNP, thus obtaining a complexity result similar to ours. However, it is not easy to compare their algorithm with our calculus, since the two approaches are radically different. As far as the complexity result is concerned, notice that our result is more general than theirs, since our language is richer: We consider Boolean combinations of conditional assertions (and also combinations with propositional formulas), whereas they do not. As remarked by Boutilier [Boutilier 1994], this more general result is not an obvious consequence of the more restricted one. Moreover, we prove the coNP result also for the system CL. At the best of our knowledge, this result was unknown up to now.

We plan to extend our calculi to first-order case. The starting point will be the analysis of first-order preferential and rational logics by Friedman, Halpern, and Koller in Friedman et al. [2000].

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