ANALYTIC TENSOR AND ITS GENERALIZATION

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(Received September 20, 1959)

In our previous papers [7], [8],[1] the notion of almost-analytic vector was introduced in certain almost-Hermitian spaces. In this paper we shall deal with tensors and obtain the notion of Φ-tensors which contains, as special cases, the one of analytic tensors and decomposable tensors.

1. Let us consider an n-dimensional space[2] which admits a tensor field $\varphi_i{}^j$ of type (1, 1). Let $\xi_{(i)}{}^{(j)} \equiv \xi_{i_p \cdots i_1}{}^{j_q \cdots j_1}$ be a tensor of type (q, p). If it commutes with $\varphi_i{}^j$, then we shall say that $\xi_{(i)}{}^{(j)}$ is pure with respect to the corrresponding indices, namely it is pure with respect to i_k and j_h, if

$$\xi_{i_p \cdots r \cdots i_1}{}^{(j)} \varphi_{i_k}{}^r = \xi_{(i)}{}^{j_q \cdots r \cdots j_1} \varphi_r{}^j{}_h \tag{1}$$

and pure with respect to i_k and i_h, if

$$\xi_{i_p \cdots r \cdots i_k \cdots i_1}{}^{(j)} \varphi_{i_k}{}^r = \xi_{i_p \cdots i_k \cdots r \cdots i_1}{}^{(j)} \varphi_{i_h}{}^r.$$

If $\xi_{(i)}{}^{(j)}$ anti-commutes with $\varphi_i{}^j$ then we shall say that it is hybrid with respect to the corresponding indices. Thus if

$$\xi_{i_p \cdots r \cdots i_1}{}^{(j)} \varphi_{i_k}{}^r = -\xi_{(i)}{}^{j_q \cdots r \cdots j_1} \varphi_r{}^{j_k}, \tag{2}$$

for example, holds good, then it is hybrid with respect to i_k and j_h. $\xi_{(i)}{}^{(j)}$ is called pure (resp. hybrid) if it is pure (resp. hybrid) with respect to all its indices.

$\varphi_i{}^j$ itself and $\delta_i{}^j$ are examples of the pure tensor. If $\varphi_i{}^j$ is a regular tensor i.e. $\det(\varphi_i{}^j) \neq 0$, then the tensor whose components are given by the elements of the inverse matrix of $(\varphi_i{}^j)$ is also pure.

LEMMA 1. *If $\xi_{(i)}{}^{(j)}$ is pure (hybrid) with respect to some indices, then so is $\overset{*}{\xi}_{(i)}{}^{(j)} = \xi_{i_p \cdots i_2 r}{}^{(j)} \varphi_{i_1}{}^r$.*

We shall prove only the case when $\xi_{(i)}{}^{(j)}$ is pure with respect to i_1 and i_k $(k \neq 1)$. In fact, we have

$$\overset{*}{\xi}_{i_p \cdots r \cdots i_1}{}^{(j)} \varphi_{i_k}{}^r = \xi_{i_p \cdots r \cdots i_2 t}{}^{(j)} \varphi_{i_k}{}^r \varphi_{i_1}^t = \xi_{p \cdots i_k \cdots i_2 r}{}^{(j)} \varphi_t{}^r \varphi_{i_1}^t$$

1) The number in brackets refers to the bibliography at the end of the paper.

2) We shall mean by a space a differentiable manifold of class C^∞, and denote by x^i its local coordinates. Indices run over 1, 2, $\cdots\cdots n$.

$$= \overset{*}{\xi}_{i_p \cdots i_2 t}{}^{(j)} \varphi_{i_1}{}^{t}. \qquad \text{q. e. d.}$$

LEMMA 2. *If a skew-symmetric tensor* $\xi_{(i)}$ *is pure, then* $\overset{*}{\xi}_{(i)} = \xi_{i_p \cdots i_2 r} \varphi_{i_1}{}^{r}$ *is also a skew-symmetric pure tensor.*

In fact, $\overset{*}{\xi}_{(i)}$ is pure by virtue of Lemma 1. It is evident that it is skew-symmetric with respect to i_k and i_h $(k,h \neq 1)$. For $k \neq 1$, we have

$$\begin{aligned} \overset{*}{\xi}_{i_p \cdots i_k \cdots i_1} &= \xi_{i_p \cdots i_k \cdots i_2 r} \varphi_{i_1}{}^{r} = \xi_{i_p \cdots r \cdots i_1} \varphi_{i_k}{}^{r} \\ &= (-1)^{k-1} \xi_{i_p \cdots \hat{i}_k \cdots i_1 r} \varphi_{i_k}{}^{r} = (-1)^{k-1} \overset{*}{\xi}_{i_p \cdots \hat{i}_k \cdots i_1 i_k} \\ &= - \overset{*}{\xi}_{i_p \cdots i_1 \cdots i_2 i_k}. \qquad \text{q. e. d.} \end{aligned}$$

If $\xi_{(i)}{}^{t(j)} \equiv \xi_{(i)}{}^{t j_q \cdots j_1}$ is a pure tensor of type $(q+1, p)$, then $u_t \xi_{(i)}{}^{t(j)}$ is also pure for a (covariant) vector u_i. Generalizing this fact, we have easily

LEMMA 3. *Let* $\xi_{(i)}{}^{(j)}$ *and* $\eta_{(a)}{}^{t(b)}$ *be pure tensors of type* $(q, p+1)$ *and* $(q'+1, p')$ *respectively. Then* $\xi_{t(i)}{}^{(j)} \eta_{(a)}{}^{t(b)}$ *is also a pure tensor of type* $(q+q', p+p')$, *provided that* $p+p' \neq 0$ *or* $q+q' \neq 0$.

A tensor $\varphi_i{}^j$ is called an almost-product structure, if it satisfies $\varphi_i{}^r \varphi_r{}^j = \delta_i^j$, and is called an almost-complex structure, if it satisfies $\varphi_i{}^r \varphi_r{}^j = -\delta_i^j$, [1], [2], [4], [12].

In these cases, we can verify the following lemmas.

LEMMA 4. *Let* $\varphi_i{}^j$ *be a tensor such that* $\varphi_i{}^r \varphi_r{}^j = \varepsilon \delta_i^j$.[3] *Then we have* $\xi_r{}^r = 0$ *for a hybrid tensor* $\xi_i{}^j$.

LEMMA 5. *Let* $\varphi_i{}^j$ *be a tensor such that* $\varphi_i{}^r \varphi_r{}^j = \varepsilon \delta_i^j$. *If* $\xi_i{}^j$ (ξ^{ij}) *is pure and* $\eta_j{}^t (\eta_{ij})$ *is hybrid, then we have*

$$\xi_i{}^j \eta_j{}^t = 0, \qquad (\xi^{tj} \eta_{ij} = 0).$$

LEMMA 6. *Let* $\varphi_i{}^j$ *be a regular tensor, i.e.* $\operatorname{rank}(\varphi_i{}^j) = n$. *If* ξ_{kji} $(\xi_{kj}{}^t)$ *is hybrid, then it is a zero tensor.*

In fact, we have

$$\xi_{kri} \varphi_j{}^r = - \xi_{rji} \varphi_k{}^r = \xi_{kjr} \varphi_i{}^r = - \xi_{kri} \varphi_j{}^r,$$

from which we find $\xi_{kji} = 0$. q. e. d.

Now consider an almost-complex structure $\varphi_i{}^j$, then if we choose a suitable frame at a point, $\varphi_i{}^j$ has the following components at the point.

$$\varphi_\alpha{}^\beta = i\,\delta_\alpha^\beta, \qquad \varphi_{\bar\alpha}{}^{\bar\beta} = -i\,\delta_{\bar\alpha}^{\bar\beta}, \qquad \varphi_{\bar\alpha}{}^{\beta} = \varphi_\alpha{}^{\bar\beta} = 0.\text{[4]}$$

3) In this paper, by ε we shall always mean ± 1.

4) Indices α, β, run over $1, \ldots, m (= n/2)$ and $\bar\alpha = m + \alpha$.

With respect to this frame, the equation (1) is equivalent to the equations

$$\xi_{i_p\cdots\alpha_k\cdots i_1}{}^{j_q\cdots\bar{\beta}_h\cdots j_1}=0, \qquad \xi_{i_p\cdots\bar{\alpha}_k\cdots i_1}{}^{j_q\cdots\beta_h\cdots j_1}=0,$$

and the equation (2) is equivalent to

$$\xi_{i_p\cdots\alpha_k\cdots i_1}{}^{j_q\cdots\beta_h\cdots j_1}=0, \qquad \xi_{i_p\cdots\bar{\alpha}_k\cdots i_1}{}^{j_q\cdots\bar{\beta}_h\cdots j_1}=0.$$

In this sense we have used the terminologies "pure" and "hybrid" [6], [11].

2. An almost-Hermitian space admits, by definition, a Riemannian metric tensor g_{ji} and an almost-complex structure $\varphi_i{}^j$ such that $g_{rs}\varphi_j{}^r\varphi_i{}^s = g_{ji}$.

A Kählerian space is an almost-Hermitian one such that the equation

$$\nabla_l\varphi_i{}^h = 0 \tag{3}$$

is valid, where ∇_l denotes the operator of the covariant derivative with respect to the Christoffel's symbol $\left\{ {h \atop ji} \right\}$.

We shall devote this section to a Kählerian space.

A pure tensor $\xi_{(i)}{}^{(j)}$ is called analytic [6], if its covariant derivative $\nabla_l\xi_{(i)}{}^{(j)}$ is also pure, i.e. it satisfies

$$\varphi_l{}^r\,\nabla_r\xi_{(i)}{}^{(j)} = \varphi_{i_1}{}^r\nabla_l\xi_{i_p\cdots i_2 r}{}^{(j)}, \tag{4}$$

or

$$\varphi_l{}^r\,\nabla_r\xi_{(i)}{}^{(j)} = \varphi_r{}^{j_1}\nabla_l\xi_{(i)}{}^{j_q\cdots j_2 r}.$$

In fact, the equation (4) is equivalent to the following one with respect to complex coordinates $(z^\alpha, z^{\bar{\alpha}})$:

$$\frac{\partial}{\partial z^{\bar{\lambda}}}\,\xi_{(\alpha)}{}^{(\beta)}=0, \qquad \frac{\partial}{\partial z^{\lambda}}\,\xi_{(\bar{\alpha})}{}^{(\bar{\beta})}=0. \tag{5}$$

The definition (4) of the analytic tensor contains the Kählerian metric in appearence, but (5) is independent to the metric. Hence it is natural to ask if the notion of the analytic tensor is defined in a complex manifold with respect to real coordinates.

In this point of view, we shall attempt to eliminate the Christoffel's symbols in (4) by making use of (3).

If we write down (4) explicitly, we hvae

$$\begin{aligned}
&\varphi_l{}^r\Big[\partial_r\,\xi_{(i)}{}^{(j)} + \sum_{k=1}^{q}\left\{ {j_k \atop rt} \right\}\xi_{(i)}{}^{j_q\cdots t\cdots j_1} - \sum_{k=1}^{p}\left\{ {t \atop ri_k} \right\}\xi_{i_p\cdots t\cdots i_1}{}^{(j)}\Big] \\
&\qquad = \varphi_{i_1}{}^t\Big[\partial_l\,\xi_{i_p\cdots i_2}{}^{(j)} + \sum_{k=1}^{q}\left\{ {j_k \atop lr} \right\}\xi^{i_p}{}_{\cdots i_2 t}{}^{j_q\cdots r\cdots j_1} \\
&\qquad\quad - \sum_{k=2}^{p}\left\{ {r \atop li_k} \right\}\xi_{i_p\cdots r\cdots i_2 t}{}^{(j)} - \left\{ {r \atop lt} \right\}\xi_{i_p\cdots i_2 r}{}^{(j)}\Big],
\end{aligned} \tag{6}$$

where we have put $\partial_r = \partial/\partial x^r$.
On the other hand, on taking account of (3), we have

$$\partial_t \varphi_l^{j_k} - \partial_l \varphi_t^{j_k} = \left\{ {j_k \atop tr} \right\} \varphi_l^r - \left\{ {j_k \atop lr} \right\} \varphi_t^r,$$

$$\partial_{i_1} \varphi_l^t - \partial_l \varphi_{i_1}^t = \left\{ {t \atop lr} \right\} \varphi_{i_1}^r - \left\{ {t \atop i_1 r} \right\} \varphi_l^r,$$

$$\partial_{i_k} \varphi_l^t = \left\{ {r \atop i_k l} \right\} \varphi_r^t - \left\{ {t \atop i_k r} \right\} \varphi_l^r.$$

If we substitute these equations into (6) and take account of the purity of $\xi_{(i)}^{(j)}$, then we find

$$(7) \qquad \varphi_l^r \partial_r \xi_{(i)}^{(j)} - \partial_l \overset{*}{\xi}_{(i)}^{(j)} + \sum_{k=1}^{p} (\partial_{i_k} \varphi_l^r) \xi_{i_p \cdots r \cdots i_1}^{(j)} + \sum_{k=1}^{q} (\partial_l \varphi_r^{j_k} - \partial_r \varphi_l^{j_k}) \xi_{(i)}^{j_q \cdots r \cdots j_1} = 0,$$

where $\overset{*}{\xi}_{(i)}^{(j)}$ is defined by

$$(8) \qquad \overset{*}{\xi}_{(i)}^{(j)} = \varphi_{i_1}^{r} \xi_{i_p \cdots\cdots i_2 r}^{(j)} = \varphi_r^{j_1} \xi_{(i)}^{j_q \cdots\cdots j_2 r}.$$

3. As we have known in the preceding section, the equation (7) which defines the analytic tensor in a Kählerian space does not contain the Kählerian metric. Following to this fact, we shall introduce an operator in a space which admits a tensor field of type (1, 1). The operator will produce from a pure tensor of type (q, p) a new tensor of type $(q, p+1)$.

Let φ_i^j be a tensor of type (1, 1) and $\xi_{(i)}^{(j)}$ a pure tensor of type (q, p). Now we define an operator Φ by

$$(9) \qquad \Phi_l \xi_{(i)}^{(j)} = \varphi_l^r \partial_r \xi_{(i)}^{(j)} - \partial_l \overset{*}{\xi}_{(i)}^{(j)} + \sum_{k=1}^{p} (\partial_{i_k} \varphi_l^r) \xi_{i_p \cdots r \cdots i_1}^{(j)} + \sum_{k=1}^{q} (\partial_l \varphi_r^{j_k} - \partial_r \varphi_l^{j_k}) \xi_{(i)}^{j_q \cdots r \cdots j_1},$$

where $\overset{*}{\xi}_{(i)}^{(j)}$ is given by (8).

In the rest of the present section, we shall show that $\Phi_l \xi_{(i)}^{(j)}$ is a tensor, if $\xi_{(i)}^{(j)}$ is pure.

Let Γ_{ji}^h be an affine connection, $S_{ji}^{\ h}$ its torsion tensor, i.e. $S_{ji}^{\ h} = (1/2)(\Gamma_{ji}^h - \Gamma_{ij}^h)$ and by ∇_k we shall denote the operator of covariant derivative with respect to Γ_{ji}^h. Hence if v^i is a vector field, then its covariant derivative is given by $\nabla_k v^i = \partial_k v^i + \Gamma_{kr}^i v^r$.

If we represent (9) by terms of covariant derivatives, $\Phi_l \xi_{(i)}^{(j)}$ is the sum of the following five terms $a_1, \ldots\ldots, a_5$:

$$a_1 = \varphi_l^{\ t}\, \partial_t\, \xi_{(i)}^{(j)} = \varphi_l^{\ t}\, [\nabla_t\, \xi_{(i)}^{(j)} - \Sigma\, \Gamma_{tr}^{j_k}\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1} + \Sigma\, \Gamma_{ti_k}^{r}\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)}],$$

$$a_2 = -\, \partial_l \overset{*}{\xi}_{(i)}^{(j)} = -\, \nabla_l\, \overset{*}{\xi}_{(i)}^{(j)} + \Sigma\, \Gamma_{lt}^{j_k}\, \overset{*}{\xi}_{(i)}^{\ j_q \cdots t \cdots j_1} - \Sigma\, \Gamma_{li_k}^{t}\, \overset{*}{\xi}_{i_p \cdots t \cdots i_1}^{\ (j)},$$

$$a_3 = \Sigma\, (\partial_{i_k}\, \varphi_l^{\ r})\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)} = \Sigma\, [\nabla_{i_k}\, \varphi_l^{\ r} - \Gamma_{i_k t}^{r}\, \varphi_l^{\ t} + \Gamma_{i_k l}^{t}\, \varphi_t^{\ r}]\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)},$$

$$a_4 = \Sigma\, (\partial_l\, \varphi_r^{\ j_k})\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1} = \Sigma\, [\nabla_l\, \varphi_r^{\ j_k} - \Gamma_{lt}^{j_k}\, \varphi_r^{\ t} + \Gamma_{lr}^{t}\, \varphi_t^{\ j_k}]\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1},$$

$$a_5 = -\, (\partial_r\, \varphi_l^{\ j_k})\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1} = \Sigma\, [-\, \nabla_r\, \varphi_l^{\ j_k} + \Gamma_{rt}^{j_k}\, \varphi_l^{\ t} - \Gamma_{rl}\, \varphi_t^{\ j_k}]\, \xi_{(i_1)}^{\ j_q \cdots r \cdots j_1}.$$

If we denote the λ-th term of a_μ by $a_{\mu\lambda}$, the following relations hold.

$$a_{12} + a_{52} = 2\, \Sigma\, S_{rt}^{\ \ j_k}\, \varphi_l^{\ t}\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1},$$

$$a_{13} + a_{32} = 2\, \Sigma\, S_{t i_k}^{\ \ r}\, \varphi_l^{\ t}\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)},$$

$$a_{43} + a_{53} = 2\, \Sigma\, S_{lr}^{\ \ t}\, \varphi_t^{\ j_k}\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1},$$

$$a_{22} + a_{42} = 0,$$

$$a_{23} + a_{33} = 2\, \Sigma\, S_{i_k l}^{\ \ t}\, \varphi_t^{\ r}\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)}.$$

Thus we find that

$$(10)\quad \begin{aligned} \Phi_l\, \xi_{(i)}^{(j)} = {} & \varphi_l^{\ r}\, \nabla_r\, \xi_{(i)}^{(j)} - \nabla_l\, \overset{*}{\xi}_{(i)}^{(j)} \\ & + \sum_{k=1}^{p} \{\nabla_{i_k}\, \varphi_l^{\ r} + 2\, (S_{i_k l}^{\ \ t}\, \varphi_t^{\ r} - S_{i_k t}^{\ \ r}\, \varphi_l^{\ t})\}\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)} \\ & + \sum_{k=1}^{q} \{\nabla_l\, \varphi_r^{\ j_k} - \nabla_r\, \varphi_l^{\ j_k} + 2\, (S_{lr}^{\ \ t}\, \varphi_t^{\ j_k} - S_{tr}^{\ \ j_k}\, \varphi_l^{\ t})\}\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1}, \end{aligned}$$

which shows that $\Phi_l\, \xi_{(i)}^{(j)}$ is a tensor.

4. In this section, we shall represent (9) in different forms. Using the notation in § 3, we have

$$(11)\qquad a_3 = \Sigma\, (\partial_{i_k}\, \overset{*}{\xi}_{i_p \cdots l \cdots i_1}^{\ (j)} - \varphi_l^{\ r}\, \partial_{i_k}\, \xi_{i_p \cdots r \cdots i_1}^{\ (j)}),$$

$$(12)\qquad a_4 = q\, \partial_l\, \overset{*}{\xi}_{(i)}^{(j)} - \Sigma\, \varphi_r^{\ j_k}\, \partial_l\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1}.$$

Now if we put

$$\partial_{<l}\, \xi_{(i)>}^{(j)} = \partial_l\, \xi_{(i)}^{(j)} - \sum_{k=1}^{p} \partial_{i_k}\, \xi_{i_p \cdots l \cdots i_1}^{\ (j)},$$

then, substituting (11) into (9), we find that

$$(13)\quad \Phi_l\, \xi_{(i)}^{(j)} = \varphi_l^{\ r}\, \partial_{<r}\, \xi_{(i)>}^{(j)} - \partial_{<l}\, \overset{*}{\xi}_{(i)>}^{(j)} + \sum_{k=1}^{q} (\partial_l\, \varphi_r^{\ j_k} - \partial_r\, \varphi_l^{\ j_k})\xi_{(i)}^{\ j_q \cdots r \cdots j_1}.$$

Hence if $\xi_{(i)}$ is a pure tensor of type $(0, p)$, it holds that

$$\Phi_l\, \xi_{(i)} = \varphi_l^{\ r}\, \partial_{<r}\, \xi_{(i)>} - \partial_{<l}\, \overset{*}{\xi}_{(i)>}.$$

In the next place, if we substitute (12) into (13), then we get

$$\Phi_l\, \xi_{(i)}^{\ (J)} = \varphi_l^{\ r}\, \partial_{<r} \xi_{(i)>}^{\ (J)} - \partial_{<l} \overset{*}{\xi}_{(i)>}^{\ (J)} + q\, \partial_l\, \overset{*}{\xi}_{(i)}^{\ (J)}$$
$$- \sum_{k=1}^{q} (\xi_{(i)}^{\ j_q \cdots r \cdots j_1} \partial_r\, \varphi_l^{\ j_k} + \varphi_r^{\ j_k}\, \partial_l\, \xi_{(i)}^{\ j_q \cdots r \cdots j_1}).$$

Hence if $\xi^{(J)}$ is a pure tensor of type $(q,0)$, then we have

$$\Phi_l\, \xi^{(J)} = \varphi_l^{\ r} \partial_r\, \xi^{(J)} + (q-1)\, \partial_l\, \overset{*}{\xi}^{(J)}$$
$$- \sum_{k=1}^{q} (\xi^{j_q \cdots r \cdots j_1} \partial_r\, \varphi_{l j_k} + \varphi_r^{\ j_k}\, \partial_l\, \xi^{j_q \cdots r \cdots j_1}),$$

from which, in the case when $q = 1$, we find

$$\Phi_l\, \xi^j = -(\xi^r\, \partial_r\, \varphi_l^{\ j} - \varphi_l^{\ r}\, \partial_r \xi^j + \varphi_r^{\ j}\, \partial_l\, \xi^r) = - \underset{\xi}{\mathfrak{L}}\, \varphi_l^{\ j},$$

where $\underset{\xi}{\mathfrak{L}}$ denotes the operator of Lie derivative with respect to ξ^j.

If $\xi_i^{\ j}$ is a pure tensor of type (1, 1), then we have from (9)

$$\Phi_l\, \xi_i^{\ j} = \varphi_l^{\ r}\, \partial_r\, \xi_i^{\ j} - \varphi_r^{\ j}\, \partial_l\, \xi_i^{\ r} + \xi_r^{\ j}\, \partial_i\, \varphi_l^{\ r} - \xi_i^{\ r}\, \partial_r\, \varphi_l^{\ j},$$

which is nothing but $\overset{1,2}{\sum}{}_{li}^{\ j}$ in Nijenhuis' paper [3].

In particular, we have $\Phi_l\, \delta_i^{\ j} = 0$.

5. Let $\xi_{t(i)}^{\ (J)} \equiv \xi_{i_p \cdots i_1}^{\ (J)}$ and $\eta_{(a)}^{\ t(b)} \equiv \eta_{(a)}^{\ t b_{q'} \cdots b_1}$ be pure tensors of type $(q, p+1)$ and type $(q'+1, p')$ respectively. Then we shall verify the following formula:

$$\Phi_l\, (\xi_{t(i)}^{\ (J)} \eta_{(a)}^{\ t(b)}) = (\Phi_l\, \xi_{t(i)}^{\ (J)})\, \eta_{(a)}^{\ t(b)} + \xi_{t(i)}^{\ (J)}\, \Phi_l\, \eta_{(a)}^{\ t(b)}, \tag{14}$$

if $p + p' \neq 0$ or $q + q' \neq 0$.

In fact, the left hand side is the sum of the following six terms b_1, ..., b_6.

$$b_1 = \varphi_l^{\ r}\, \partial_r\, (\xi_{t(i)}^{\ (J)}\, \eta_{(a)}^{\ t(b)}),$$
$$b_2 = - \eta_{(a)}^{\ t(b)}\, \partial_l\, \overset{*}{\xi}_{t(i)}^{\ (J)} - \xi_{t(i)}^{\ (J)} \partial_l \overset{*}{\eta}_{(a)}^{\ t(b)} + \xi_{t(i)}^{\ (J)} \eta_{(a)}^{\ r(b)} \partial_l\, \varphi_r^{\ t},$$
$$b_3 = \Sigma\, (\partial_{i_k}\, \varphi_l^{\ r})\, \xi_{t i_p \cdots r \cdots i_1}^{\ (J)}\, \eta_{(a)}^{\ t(b)},$$
$$b_4 = \Sigma\, (\partial_{a_k}\, \varphi_l^{\ r})\, \xi_{t(i)}^{\ (J)}\, \eta_{a_{p'} \cdots r \cdots a_1}^{\ t(b)},$$
$$b_5 = \Sigma\, (\partial_l\, \varphi_r^{\ j_k} - \partial_r\, \varphi_l^{\ j_k})\, \xi_{t(i)}^{\ j_q \cdots r \cdots j_1} \eta_{(a)}^{\ t(b)},$$

$$b_6 = \Sigma (\partial_l \varphi_r{}^{b_k} - \partial_r \varphi_l{}^{b_k}) \xi_{t(i)}{}^{(j)} \eta_{(a)}{}^{tb_q' \cdots r \cdots b_1},$$

from which we can easily obtain (14).

If a pure tensor (or a vecter) ξ satisfies $\Phi_l \xi = 0$, then we shall say that it is a Φ-tensor (or Φ-vector). If the tensor $\varphi_i{}^j$ is a complex structure, then a Φ-tensor is an analytic tensor. If $\varphi_i{}^j$ is a product structure i.e. an almost-product structure such that its Nijenhuis' tensor vanishes, then a Φ-tensor is decomposable.

From (14) we have

THEOREM 1. *If* $\xi_{t(i)}{}^{(j)}$ *and* $\eta_{(a)}{}^{t(b)}$ *are* Φ*-tensors, then so is* $\xi_{t(i)}{}^{(j)} \eta_{(a)}{}^{t(b)}$ *provided that it is not a scalar.*

6. Let us consider two Riemannian metrics g_{ji} and φ_{ji} which are not necessarily positive definite. Putting $\varphi_i{}^j = \varphi_{ir} g^{rj}$ we shall introduce the operator Φ which is associated to $\varphi_i{}^j$.

Since it holds that $g_{ri}\varphi_j{}^r = \varphi_{ji} = \varphi_{ij} = g_{jr}\varphi_i{}^r$, we know that g_{ji} is pure. Taking account of $\overset{*}{g}_{ji} = \varphi_{ji}$, we obtain

$$\Phi_l g_{ji} = \varphi_l{}^r \partial_r g_{ji} - \partial_l \overset{*}{g}_{ji} + (\partial_j \varphi_l{}^r) g_{ri} + (\partial_i \varphi_l{}^r) g_{jr} = - 2\, \varphi_{lt} [\{{}^{\,t}_{ji}\}_g - \{{}^{\,t}_{ji}\}_\varphi],$$

where $\{{}^{\,t}_{ji}\}_g$ and $\{{}^{\,t}_{ji}\}_\varphi$ are the Christoffel's symbols formed by g_{ji} and φ_{ji} respectively. Thus we have

THEOREM 2. *Let* g_{ji} *and* φ_{ji} *be two Riemannian metrics. Then a necessary and sufficient condition in order that the Christoffel's symbols coincide with each other is that* $\Phi_l g_{ji} = 0$, *where* Φ *is the operator associated to* $\varphi_i{}^j = \varphi_{ir} g^{rj}$.

In the rest of the present section, we shall assume that g_{ji} is a Φ-tensor, and denote by ∇_i the operator of the Riemannian covariant derivative with respect to g_{ji}. From Theorem 2, we know that $\Phi_l g_{ji} = 0$ is equivalent to $\nabla_k \varphi_{ji} = 0$.

Let $R_{kji}{}^h$ and $S_{kji}{}^h$ be the Riemannian curvature tensors formed by g_{ji} and φ_{ji} respectively, then we have $R_{kji}{}^h = S_{kji}{}^h$ by means of the assumption.

Applying the Ricci's identity to $\varphi_i{}^h$, we get $R_{kjr}{}^h \varphi_i{}^r = R_{kji}{}^r \varphi_r{}^h$, which shows that $R_{kji}{}^h$ is pure with respect to i and h. Hence $R_{kjih} = R_{kji}{}^r g_{rh}$ is pure with respect to k and j and also pure with respect to i and h.

On the other hand, $S_{kji}{}^h$ being the Riemannian curvature tensor formed by φ_{ji}, if we put $T_{kjih} = S_{kji}{}^r \varphi_{rh}$, then we have

$$T_{kjih} = T_{ihkj}. \tag{15}$$

Since it holds that $T_{kjih} = R_{kjir}\varphi_h{}^r$ and $T_{ihkj} = R_{krih}\, \varphi_j{}^r$, the equation (15) becomes $R_{kjtr}\,\varphi_h^{\ r} = R_{krth}\varphi_j^{\ r}$, which shows that R_{kjih} is pure with respect to j and h. Therefore R_{kjth} is pure.

Since we have

$$\begin{aligned}\Phi_l R_{kjth} &= \varphi_l^{\ r}\,\nabla_r\,R_{kjth} - \nabla_l\,\overset{*}{R}_{kjih}\\ &= -\,\varphi_l^{\ r}\,(\nabla_k\,R_{jrth} + \nabla_j\,R_{rkth}) - \varphi_h^{\ r}\nabla_l\,R_{kjtr}\\ &= -\,\varphi_h^{\ r}\,(\nabla_k\,R_{jltr} + \nabla_j\,R_{lktr} + \nabla_l\,R_{kjtr}) = 0,\end{aligned}$$

$\nabla_l\,R_{kjth}$ is also pure.

LEMMA 7. *Let us assume that* $\Phi_l g_{ji} = 0$. *If a tensor, say* T, *and its covariant derivative are pure, then we have*

$$\nabla_t\Phi_l T = \Phi_l\nabla_t T.$$

PROOF. Let T be a tensor of type (1, 1), for example. Then we have

$\nabla_t\Phi_l T_i^{\ j} - \Phi_l\nabla_t T_i^{\ j} = \varphi_l^{\ r}(\nabla_t\nabla_r - \nabla_r\nabla_t)\,T_i^{\ j} - (\nabla_t\nabla_l - \nabla_l\nabla_t)\,\overset{*}{T}_i^{\ j}.$

On the other hand, it holds that

$$\begin{aligned}(\nabla_t\nabla_l - \nabla_l\nabla_t)\overset{*}{T}_i^{\ j} &= R_{tlr}^{\ \ \ j}\,\overset{*}{T}_i^{\ r} - R_{tli}^{\ \ \ r}\,\overset{*}{T}_r^{\ j}\\ &= R_{tlr}^{\ \ \ j}\,\varphi_s^{\ r}\,T_i^{\ s} - R_{tli}^{\ \ \ r}\,\varphi_r^{\ s}\,T_s^{\ j}\\ &= R_{trs}^{\ \ \ j}\,\varphi_l^{\ r}\,T_i^{\ s} - R_{tri}^{\ \ \ s}\,\varphi_l^{\ r}\,T_s^{\ j}\\ &= \varphi_l^{\ r}(\nabla_t\nabla_r - \nabla_r\nabla_t)T_i^{\ j}.\end{aligned}$$

From these equations, we find that the lemma is true. q. e. d.

If we apply Lemma 7 to our $R_{kji}^{\ \ \ h}$, then we have $\Phi_l\,\nabla_t\,R_{kjth} = 0$, which shows that $\nabla_l\nabla_t\,R_{kjth}$ is pure. Thus we get

THEOREM 3. *Let* g_{ji} *and* φ_{ji} *be two Riemannian metrics and* Φ *be the operator associated to* $\varphi_i^{\ j} = \varphi_{ir}g^{rj}$. *If* $\Phi_l g_{ji} = 0$ *is valid, then* R_{kjth} *and its successive covariant derivatives are pure.*

Let $\varphi_i^{\ j}$ be an almost-product structure. then there exists a Riemannian metric g_{ji} such that $g_{rs}\varphi_j^{\ r}\varphi_i^{\ s} = g_{ji}$. Then we know that the tensor $\varphi_{ji} = \varphi_j^{\ r}$ g_{ri} is also a Riemannian metric. Thus theorems in this section are applicable to this case.

7. In this section we shall assume that $\varphi_i^{\ r}\,\varphi_r^{\ j} = \varepsilon\delta_i^{\ j}$.

If we put $\Phi_l\varphi_i^{\ j} = N_{li}^{\ \ j}$, then it holds that

$$\begin{aligned}N_{li}^{\ \ j} &= \varphi_l^{\ r}\partial_r\varphi_i^{\ j} + (\partial_i\varphi_l^{\ r})\varphi_r^{\ j} + (\partial_l\varphi_r^{\ j} - \partial_r\varphi_l^{\ j})\,\varphi_i^{\ r}\\ &= \varphi_l^{\ r}(\partial_r\varphi_i^{\ j} - \partial_i\varphi_r^{\ j}) - \varphi_i^{\ r}(\partial_r\varphi_l^{\ j} - \partial_l\varphi_r^{\ j}),\end{aligned}$$

which is nothing but the Nijenhuis' tensor [1], [3], [4], [10], [11], [12]. It satisfies the equations

$$N_{li}{}^{j} = -N_{il}{}^{j}, \qquad N_{lr}{}^{j}\varphi_{i}{}^{r} = -N_{li}{}^{r}\varphi_{r}{}^{j}.$$

The last equation shows that $N_{li}{}^{j}$ is hybrid with respect to i and j, hence taking account of the skew-symmetricity of $N_{li}{}^{j}$, it is pure with respect to i and l. Thus we get

$$N_{lr}{}^{j}\varphi_{i}{}^{r} = N_{ri}{}^{j}\varphi_{l}{}^{r}, \quad N_{lr}{}^{r} = 0, \quad N_{lr}{}^{t}\varphi_{t}{}^{r} = 0,$$

by virtue of Lemma 4 and Lemma 5.

Now we introduce an affine connection Γ_{ji}^{h} such that

$$\nabla_{l}\varphi_{i}{}^{j} = 0, \quad S_{ji}{}^{h} = -(\varepsilon/8)N_{ji}{}^{h},$$

where ∇_{l} denotes the operator of the covariant derivative with respect to Γ_{ji}^{h} and $S_{ji}{}^{h}$ its torsion tensor.

It is known that there exists such a connection, which will be called the canonical connection [12].

If we make use of the canonical connection, the equation (10) becomes

$$\begin{aligned}\Phi_{l}\,\xi_{(i)}{}^{(j)} &= \varphi_{l}{}^{r}\,\nabla_{r}\,\xi_{(i)}{}^{(j)} - \nabla_{l}\,\overset{*}{\xi}_{(i)}{}^{(j)} \\ &\quad + (\varepsilon/2)\,\varphi_{l}{}^{r}\left[\sum_{k=1}^{q} N_{rt}{}^{j_k}\xi_{(i)}{}^{j_q\cdots t\cdots j_1} - \sum_{k=1}^{p} N_{ri_k}{}^{t}\xi_{i_p\cdots t\cdots i_1}{}^{(j)}\right].\end{aligned} \tag{16}$$

Making use of the form (16), we shall obtain some formulas on the operator Φ.

The tensor $\varphi_{i}{}^{j}$ being pure, if we substitute it in the place of ξ or η in (14), then we get the following formulas.

$$\Phi_{l}\,\overset{*}{\xi}_{(i)}{}^{b(j)} = \varphi_{r}{}^{b}\,\Phi_{l}\xi_{(i)}{}^{r(j)} + N_{lr}{}^{b}\,\xi_{(i)}{}^{r(j)},$$

$$\Phi_{l}\overset{*}{\xi}_{a(i)}{}^{(j)} = \varphi_{a}{}^{r}\,\Phi_{l}\xi_{r(i)}{}^{(j)} + N_{la}{}^{r}\,\xi_{r(i)}{}^{(j)}.$$

We can see also that

$$\begin{aligned}\Phi_{l}\,\overset{*}{\xi}_{(i)}{}^{(j)} &= \varphi_{r}{}^{j_k}\,\Phi_{l}\xi_{(i)}{}^{j_q\cdots r\cdots j_1} + N_{lr}{}^{j_k}\xi_{(i)}{}^{j_q\cdots r\cdots j_1}, \text{ if } q \geqq 1, \\ &= \varphi_{i_k}{}^{r}\,\Phi_{l}\xi_{i_p\cdots r\cdots i_1}{}^{(j)} + N_{li_k}{}^{r}\,\xi_{i_p\cdots r\cdots i_1}{}^{(j)}, \text{ if } p \geqq 1,\end{aligned}$$

are valid.

In the next place, we shall prove the following formula:

$$\Phi_{l}\,\overset{*}{\xi}_{(i)}{}^{(j)} = -\varphi_{l}{}^{r}\Phi_{r}\,\xi_{(i)}{}^{(j)} + \sum_{k=1}^{q} N_{lr}{}^{j_k}\xi_{(i)}{}^{j_q\cdots r\cdots j_1}. \tag{17}$$

In fact, we have

$$\Phi_l \overset{*}{\xi}_{(i)}{}^{(j)} = \varphi_l{}^r \nabla_r \overset{*}{\xi}_{(i)}{}^{(j)} - \varepsilon \nabla_l \xi_{(i)}{}^{(j)}$$
$$+ (\varepsilon/2) \varphi_l{}^r [\Sigma N_{rt}{}^{j_k} \overset{*}{\xi}_{(i)}{}^{j_q \cdots t \cdots j_1} - \Sigma N_{ri_k}{}^t \overset{*}{\xi}_{i_p \cdots t \cdots i_1}{}^{(j)}]$$
$$= -\varepsilon \nabla_l \xi_{(i)}{}^{(j)} + \varphi_l{}^r \nabla_r \overset{*}{\xi}_{(i)}{}^{(j)}$$
$$- \varepsilon(\varepsilon/2) [\Sigma N_{lr}{}^{j_k} \xi_{(i)}{}^{j_q \cdots r \cdots j_1} - \Sigma N_{li_k}{}^r \xi_{i_p \cdots r \cdots i_1}{}^{(j)}]$$
$$+ \Sigma N_{lr}{}^{j_k} \xi_{(i)}{}^{j_q \cdots r \cdots j_1}$$
$$= -\varphi_l{}^r \Phi_r \xi_{(i)}{}^{(j)} + \Sigma N_{lr}{}^{j_k} \xi_{(i)}{}^{j_q \cdots r \cdots j_1}. \qquad \text{q. e. d.}$$

Especially, for a pure tensor $\xi_{(i)}$ of type $(0, p)$, we have

$$\Phi_l \overset{*}{\xi}_{(i)} = -\varphi_l{}^r \Phi_r \xi_{(i)}, \tag{18}$$

from which it holds, taking account of (14),

$$N_{lt}{}^r \xi_{r(i)} + \varphi_t{}^r \Phi_l \xi_{r(i)} = -\varphi_l{}^r \Phi_r \xi_{t(i)}. \tag{19}$$

From (18) we have

THEOREM 4. *Let $\varphi_i{}^j$ satisfies $\varphi_i{}^r \varphi_r{}^j = \varepsilon\delta_i{}^j$.* If $\xi_{(i)}$ *is a Φ-tensor, then so is $\overset{*}{\xi}_{(i)}$.*

In this case, we know by virtue of (19) that the relation

$$N_{lj}{}^r \xi_{i_p \cdots r \cdots i_1} = 0$$

holds good.

Next we shall generalize the fact that $\Phi_l \varphi_i{}^j = N_{li}{}^j$ is hybrid with respect to l and j.

THEOREM 5. *Let $\varphi_i{}^j$ satisfies $\varphi_i{}^r \varphi_r{}^j = \varepsilon\,\delta_i{}^j$. If $\xi_{(i)}{}^j$ is a pure tensor of type $(1, p)$, then $\Phi_l \xi_{(i)}{}^j$ is hybrid with respect to l and j.*

In fact, we have by virtue of (17)

$$\Phi_l \overset{*}{\xi}_{(i)}{}^j = -\varphi_l{}^r \Phi_r \xi_{(i)}{}^j + N_{lr}{}^j \xi_{(i)}{}^r.$$

On the other hand, taking account of (14), ws find that

$$\Phi_l \overset{*}{\xi}_{(i)}{}^j = \Phi_l (\xi_{(i)}{}^r \varphi_r{}^j) = \varphi_r{}^j \Phi_l \xi_{(i)}{}^r + N_{lr}{}^j \xi_{(i)}{}^r.$$

From these equations we obtain the theorem. q. e. d.

If we define $A_{tkji}{}^h = N_{tk}{}^a N_{ji}{}^b N_{ab}{}^h$, then it is evidently a pure tensor of type (1,4), hence $\Phi_l A_{tkji}{}^h$ is a tensor which is hybrid with respect to t and

h. It depends only on $\varphi_i^{\ j}$ and contains its second derivatives. From Lemma 4, we have $\Phi_r A_{lkji}^{\ \ \ \ r} = 0$, which may be a new identity on $\varphi_i^{\ j}$.

We denote by C the contraction's operator, i. e., if C means the contraction with respect to i_1 and j_1, for example, then $C\xi_{(i)}^{\ (J)} = \xi_{i_p \cdots i_{2r}}^{\ \ \ \ j_q \cdots j_{2r}}$. If $\xi_{(i)}^{\ (J)}$ is a pure tensor of type (q, p), then the tensor $C\xi_{(i)}^{\ (J)}$ is also pure if it is not a scalar. Making use of (16), we can verify the following relation, after some calculations.

$$C\,\Phi_l\,\xi_{(i)}^{\ (J)} = \Phi_l\,C\,\xi_{(i)}^{\ (J)}. \tag{20}$$

In (20), we assumed that $C\xi_{(i)}^{\ (J)}$ is not a scalar and C operates on the same indices of both sides.

From (20) we have

THEOREM 6. *Let $\varphi_i^{\ j}$ satisfies $\varphi_i^{\ r}\varphi_r^{\ j} = \varepsilon\,\delta_i^j$. If $\xi_{(i)}^{\ (J)}$ is a Φ-tensor, then so is $C\xi_{(i)}^{\ (J)}$ provided that it is not a scalar.*

Let $\xi_{(i)}^{\ (J)}$ and $\eta_{(J)}^{\ (i)}$ be pure tensors of type (q, p) and type (p, q) respectively. Then we have

$$\eta_{(J)}^{\ (i)}\,\Phi_l\,\xi_{(i)}^{\ (J)} = \eta_{(J)}^{\ (i)}\,\varphi_l^{\ r}\,\nabla_r\,\xi_{(i)}^{\ (J)} - \eta_{(J)}^{\ (i)}\,\nabla_l\,\overset{*}{\xi}_{(i)}^{\ (J)},$$

because we have from Lemma 4 and the hybridity of $N_{li}^{\ \ j}$,

$$N_{ri_k}^{\ \ \ t}\,\xi_{i_p \cdots t \cdots i_1}^{\ \ \ \ \ (J)}\eta_{(J)}^{\ (i)} = 0.$$

In the same manner, we get

$$\begin{aligned}\xi_{(i)}^{\ (J)}\,\Phi_l\,\eta_{(J)}^{\ (i)} &= \xi_{(i)}^{\ (J)}\,\varphi_l^{\ r}\,\nabla_r\,\eta_{(J)}^{\ (i)} - \xi_{(i)}^{\ (J)}\,\nabla_l\,\overset{*}{\eta}_{(J)}^{\ (i)} \\ &= \xi_{(i)}^{\ (J)}\,\varphi_l^{\ r}\,\nabla_r\,\eta_{(J)}^{\ (i)} - \overset{*}{\xi}_{(i)}^{\ (J)}\,\nabla_l\,\eta_{(J)}^{\ (i)}.\end{aligned}$$

Hence we obtain

$$\xi_{(i)}^{\ (J)}\,\Phi_l\,\eta_{(J)}^{\ (i)} + \eta_{(J)}^{\ (i)}\,\Phi_l\,\xi_{(i)}^{\ (J)} = \varphi_l^{\ r}\partial_r\,(\xi_{(i)}^{\ (J)}\eta_{(J)}^{\ (i)}) - \partial_l(\overset{*}{\xi}_{(i)}^{\ (J)}\,\eta_{(J)}^{\ (i)}). \tag{21}$$

8. Let us consider an almost-Hermitian space M whose positive definite Riemannian metric is g_{ji} and the almost-complex structure is $\varphi_i^{\ j}$. By definition these tensors satisfy $g_{rs}\,\varphi_j^{\ r}\varphi_i^{\ s} = g_{ji}$, from which $\varphi_{ji} = \varphi_j^{\ r}\,g_{ri}$ is skew-symmetric. Now we assume that $\nabla_r\,\varphi_i^{\ r} = 0$, where ∇_r denotes the operator of the Riemannian covariant derivative. The following lemma is known [7], [8].

LEMMA 8. *Let M be a compact almost-Hermitian space satisfying $\nabla_r\,\varphi_i^{\ r} = 0$. If scalar functions f and g satisfy $\partial_i f = \varphi_i^{\ r}\,\partial_r g$, then they are both constant over M.*

From this lemma and (21) we have

THEOREM 7. *Let M be a compact almost-Hermitian space satisfying $\nabla_r \varphi_i^{\ r} = 0$. If $\xi_{(i)}^{(j)}$ and $\eta_{(j)}^{(i)}$ are Φ-tensors of type (q, p) and of type (p, q) respectively, then the inner product $\xi_{(i)}^{(j)} \eta_{(j)}^{(i)}$ is constant.*

COROLLARY. *Let M be a compact almost-Hermitian space satisfying $\nabla_r \varphi_i^{\ r} = 0$. If $\xi_{(i)}^{(j)}$ is a Φ-tensor of type (q, p) and $\underset{\alpha}{v}^i$ $(\alpha = 1, \ldots\ldots, p)$, $\overset{\alpha}{u}_i$ $(\alpha = 1, \ldots\ldots, q)$ are Φ-vectors, then the inner product $\xi_{(i)}^{(j)} \underset{p}{v}^{i_p} \ldots\ldots \underset{1}{v}^{i_1} \overset{q}{u}_{j_q} \ldots\ldots \overset{1}{u}_{j_1}$ is constant.*

9. Let us consider a Kählerian space M with a positive definite metric. We shall make use of the notation in § 2.

An analytic tensor $\xi_{(i)}^{(j)}$ is by definition a pure tensor such that $\nabla_l \xi_{(i)}^{(j)}$ is also pure.

Now we define, for a pure tensor $\xi_{(i)}^{(j)}$,

$$a_{k(i)}^{(j)}(\xi) = \nabla_k \xi_{(i)}^{(j)} + \varphi_k^{\ l} \varphi_{i_1}^{\ r} \nabla_l \xi_{i_p \cdots i_2 r}^{(j)}$$
$$= \nabla_k \xi_{(i)}^{(j)} + \varphi_k^{\ l} \varphi_r^{\ j_1} \nabla_l \xi_{(i)}^{j_q \cdots j_2 r}.$$

$a_{k(i)}^{(j)}(\xi) = 0$ is equivalent to that the pure tensor $\xi_{(i)}^{(j)}$ is analytic.

On taking account of that the Riemannian curvature tensor $R_{kji}^{\ \ \ k}$ and Ricci tensor R_{ji} of a Kählerian space satisfy

$$-(1/2)\varphi^{lt} R_{lti}^{\ \ \ h} = R_i^{\ r}\varphi_r^{\ h} = R_r^{\ h}\varphi_i^{\ r},$$

we can easily obtain

$$\nabla^r a_{r(i)}^{(j)} = \nabla^r \nabla_r \xi_{(i)}^{(j)} + \Sigma R_{ri_k} \xi_{(i)}^{j_q \cdots r \cdots j_1} - \Sigma R_{i_k}^{\ r} \xi_{i_p \cdots r \cdots i_1}^{(j)},$$

where $\nabla^r = g^{rl}\nabla_l$. Hence if $\xi_{(i)}^{(j)}$ is analytic, then it satisfies

$$\nabla^r \nabla_r \xi_{(i)}^{(j)} + \Sigma R_r^{\ j_k} \xi_{(i)}^{j_q \cdots r \cdots j_1} - \Sigma R_{i_k}^{\ r} \xi_{i_p \cdots r \cdots i_1}^{(j)} = 0.$$

Putting $\xi^{(i)}_{\ (j)} = g^{i_p l_p} \cdots\cdots g^{i_1 l_1} g_{j_q h_q} \cdots\cdots g_{j_1 h_1} \xi_{(l)}^{(h)}$ etc., we have, after some calculations,

$$\nabla^r (a_{r(i)}^{(j)} \xi^{(i)}_{\ (j)}) = (\nabla^r a_{r(i)}^{(j)}) \xi^{(i)}_{\ (j)} + (1/2)\, a^2(\xi),$$

where

$$a^2(\xi) = a_{r(i)}^{(j)} a^{r(i)}_{\ \ (j)}. \tag{22}$$

Thus, by Green's theorem, we have

THEOREM 8.[5] *In a compact Kählerian space M, the integral formula*

5) For a skew-symmetric contravariant pure tensor, see [6].

$$\int_M \left[\left(\nabla^r \nabla_r \xi_{(i)}{}^{(j)} + \sum_{k=1}^{q} R_r{}^{j_k} \xi_{(i)}{}^{j_q \cdots r \cdots j_1} - \sum_{k=1}^{p} R_{i_k}{}^r \xi_{i_p \cdots r \cdots i_1}{}^{(j)} \right) \xi^{(i)}{}_{(j)} + (1/2)\, a^2(\xi) \right] d\sigma = 0$$

is valid for a pure tensor $\xi_{(i)}{}^{(j)}$, *where* $d\sigma$ *is the volume element of* M *and* $a^2(\xi)$ *is given by* (22).

THEOREM 9. *In a compact Kählerian space, a necessary and sufficient condition for a pure tensor* $\xi_{(i)}{}^{(j)}$ *to be analytic is that it satisfies*

$$\nabla^r \nabla_r \xi_{(i)}{}^{(j)} + \sum_{k=1}^{q} R_r{}^{j_k} \xi_{(i)}{}^{j_q \cdots r \cdots j_1} - \sum_{k=1}^{p} R_{i_k}{}^r \xi_{i_p \cdots r \cdots i_1}{}^{(j)} = 0.$$

On the other hand, in a compact orientable Riemannian space, a necessary and sufficient condition for a skew-symmetic tensor $\xi_{(i)}$ to be harmonic is that [13]

$$\nabla^r \nabla_r \xi_{(i)} - \sum R_{i_k}{}^r \xi_{i_p \cdots r \cdots i_1} + \sum_{l>k} R_{i_l i_k}{}^{rs} \xi_{i_p \cdots r \cdots s \cdots i_1} = 0.$$

Let $\xi_{(i)}$ be a skew-symmetric pure tensor, then

$$R_{kj}{}^{rs} \xi_{i_p \cdots r \cdots s \cdots i_1} = 0,$$

by vritue of Lemma 5 and the hybridity of $R_{kj}{}^{rs}$ with respect to r and s. Thus we have

COROLLARY [13]. *In a compact Kählerian space, a necessary and sufficient condition for a skew-symmetric pure tensor to be analytic is that it is harmonic.*

If $\xi_{(i)}$ is skew-symmetric pure tensor, then so is $\overset{*}{\xi}_{(i)}$ by virtue of Lemma 2. Hence taking account of Theorem 4, in a compact Kählerian space, if a pure tensor $\xi_{(i)}$ is harmonic, then so is $\overset{*}{\xi}_{(i)}$.

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