ANALYTIC TORSION, DYNAMICAL ZETA FUNCTIONS, AND THE FRIED CONJECTURE

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ABSTRACT. We prove the equality of the analytic torsion and the value at zero of a Ruelle dynamical zeta function associated with an acyclic unitarily flat vector bundle on a closed locally symmetric reductive manifold. This solves a conjecture of Fried. This article should be read in conjunction with an earlier paper by Moscovici and Stanton.

CONTENTS

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1. Introduction

The purpose of this article is to prove the equality of the analytic torsion and the value at zero of a Ruelle dynamical zeta function associated with an acyclic unitarily flat vector bundle on a closed locally symmetric reductive manifold, which completes a gap in the proof given by Moscovici and Stanton [MoSt91] and solves a conjecture of Fried [F87].

Let Z be a smooth closed manifold. Let F be a complex vector bundle equipped with a flat Hermitian metric g^F. Let $H^{\cdot}(Z, F)$ be the cohomology of sheaf of locally flat sections of F. We assume $H^{\cdot}(Z, F) = 0$.

The Reidemeister torsion has been introduced by Reidemeister [Re35]. It is a positive real number one obtains via the combinatorial complex with values in F associated with a triangulation of Z, which can be shown not to depend on the triangulation.

Let g^{TZ} be a Riemannian metric on TZ. Ray and Singer [RS71] constructed the analytic torsion $T(F)$ as a spectral invariant of the Hodge Laplacian associated with g^{TZ} and g^F. They showed that if Z is an even dimensional oriented manifold, then $T(F) = 1$. Moreover, if $\dim Z$ is odd, then $T(F)$ does not depend on the metric data.

In [RS71], Ray and Singer conjectured an equality between the Reidemeister torsion and the analytic torsion, which was later proved by Cheeger [C79] and Müller [M78]. Using the Witten deformation, Bismut and Zhang [BZ92] gave an extension of the Cheeger-Müller Theorem which is valid for arbitrary flat vector bundles.

From the dynamical side, in [Mi68a, Section 3], Milnor pointed out a remarkable similarity between the Reidemeister torsion and the Weil zeta function. A quantitative description of their relation was formulated by Fried [F86] when Z is a closed oriented hyperbolic manifold. Namely, he showed that the value at zero of the Ruelle dynamical zeta function, constructed using the closed geodesics in Z and the holonomy of F, is equal to $T(F)^2$. In [F87, p. 66, Conjecture], Fried conjectured that a similar result holds true for general closed locally homogeneous manifolds.

In this article, we prove the Fried conjecture for odd dimensional[1] closed locally symmetric reductive manifolds. More precisely, we show that the dynamical zeta function is meromorphic on $\mathbf{C}$, holomorphic at 0, and that its value at 0 is equal to $T(F)^2$.

The proof of the above result by Moscovici-Stanton [MoSt91], based on the Selberg trace formula and harmonic analysis on reductive groups, does not seem to be complete. We tried to give the proper argument to make it correct. Our proof is based on the explicit formula given by Bismut for semisimple orbital integrals [B11, Theorem 6.1.1].

The results contained in this article was announced in [Sh16]. See also Ma's talk [Ma17] at Séminaire Bourbaki for an introduction.

Now, we will describe our results in more details, and explain the techniques used in their proof.

1.1. **The analytic torsion.** Let Z be a smooth closed manifold, and let F be a complex flat vector bundle on Z.

Let g^{TZ} be a Riemannian metric on TZ, and let g^F be a Hermitian metric on F. To g^{TZ} and g^F, we can associate an L^2-metric on $\Omega^\cdot(Z,F)$, the space of differential forms with values in F. Let $\square^Z$ be the Hodge Laplacian acting on $\Omega^\cdot(Z,F)$. By Hodge theory, we have a canonical isomorphism

$$\ker \square^Z \simeq H^\cdot(Z,F). \tag{1.1}$$

Let $\left(\square^Z\right)^{-1}$ be the inverse of $\square^Z$ acting on the orthogonal space to $\ker \square^Z$. Let $N^{\Lambda^\cdot(T^*Z)}$ be the number operator of $\Lambda^\cdot(T^*Z)$, i.e., multiplication by i on $\Omega^i(Z,F)$. Let $\mathrm{Tr_s}$ denote the supertrace. For $s\in\mathbf{C}$, $\mathrm{Re}(s)>\frac{1}{2}\dim Z$, set

$$\theta(s) = -\mathrm{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)}\left(\square^Z\right)^{-s}\right]. \tag{1.2}$$

By [Se67], $\theta(s)$ has a meromorphic extension to $\mathbf{C}$, which is holomorphic at $s=0$. The analytic torsion is a positive real number given by

$$T(F) = \exp(\theta'(0)/2). \tag{1.3}$$

Equivalently, $T(F)$ is given by the following weighted product of the zeta regularized determinants

$$T(F) = \prod_{i=1}^{\dim Z} \det\left(\square^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i/2}. \tag{1.4}$$

[1]The even dimensional case is trivial.

1.2. **The dynamical zeta function.** Let us recall Fried's general definition of the formal dynamical zeta function associated to a geodesic flow [F87, Section 5].

Let (Z, g^{TZ}) be a connected manifold with nonpositive sectional curvature. Let $\Gamma = \pi_1(Z)$ be the fundamental group of Z, and let $[\Gamma]$ be the set of the conjugacy classes of Γ. We identify $[\Gamma]$ with the free homotopy space of Z. For $[\gamma] \in [\Gamma]$, let $B_{[\gamma]}$ be the set of closed geodesics, parametrized by $[0,1]$, in the class $[\gamma]$. The map $x_\cdot \in B_{[\gamma]} \to (x_0, \dot{x}_0/|\dot{x}_0|)$ induces an identification between $\coprod_{[\gamma]\in[\Gamma]-\{1\}} B_{[\gamma]}$ and the fixed points of the geodesic flow at time $t = 1$ acting on the unit tangent bundle SZ. Then, $B_{[\gamma]}$ is equipped with the induced topology, is connected and compact. Moreover, all the elements in $B_{[\gamma]}$ have the same length $l_{[\gamma]}$. Also, the Fuller index $\mathrm{ind}_F(B_{[\gamma]}) \in \mathbf{Q}$ is well defined (c.f. [F87, Section 4]). Given a finite dimensional representation ρ of Γ, for $\sigma \in \mathbf{C}$, the formal dynamical zeta function is then defined by

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)]\mathrm{ind}_F(B_{[\gamma]})e^{-\sigma l_{[\gamma]}}\right). \tag{1.5}$$

Note that our definition is the inverse of the one introduced by Fried [F87, P. 51].

The Fuller index can be made explicit in many case. If $[\gamma] \in [\Gamma] - \{1\}$, the group $\mathbb{S}^1$ acts locally freely on $B_{[\gamma]}$ by rotation. Assume that the $B_{[\gamma]}$ are smooth manifolds. This is the case if (Z, g^{TZ}) has a negative sectional curvature or if Z is locally symmetric. Then $\mathbb{S}^1\backslash B_{[\gamma]}$ is an orbifold. Let $\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) \in \mathbf{Q}$ be the orbifold Euler characteristic [Sa57]. Denote by

$$m_{[\gamma]} = \left|\ker\left(\mathbb{S}^1 \to \mathrm{Diff}(B_{[\gamma]})\right)\right| \in \mathbf{N}^* \tag{1.6}$$

the multiplicity of a generic element in $B_{[\gamma]}$. By [F87, Lemma 5.3], we have

$$\mathrm{ind}_F(B_{[\gamma]}) = \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}. \tag{1.7}$$

By (1.5) and (1.7), the formal dynamical zeta function is then given by

$$R_\rho(\sigma) = \exp\left(\sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}e^{-\sigma l_{[\gamma]}}\right). \tag{1.8}$$

We will say that the formal dynamical zeta function is well defined if $R_\rho(\sigma)$ is holomorphic for $\mathrm{Re}\,(\sigma) \gg 1$ and extends meromorphically to $\sigma \in \mathbf{C}$.

Observe that if (Z, g^{TZ}) is of negative sectional curvature, then $B_{[\gamma]} \simeq \mathbb{S}^1$ and

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = 1. \tag{1.9}$$

In this case, $R_\rho(\sigma)$ was recently shown to be well defined by Giulietti-Liverani-Pollicott [GiLP13] and Dyatlov-Zworski [DyZw16]. Moreover, Dyatlov-Zworski [DyZw17] showed that, if (Z, g^{TZ}) is a negatively curved surface, the order of the zero of $R_\rho(\sigma)$ at $\sigma = 0$ is related to the genus of Z.

1.3. **The Fried conjecture.** Let us briefly recall Fried's results in [F86]. Assume Z is an odd dimensional connected orientable closed hyperbolic manifold. Take $r \in \mathbf{N}$. Let $\rho : \Gamma \to U(r)$ be a unitary representation of the fundamental group Γ. Let F be the unitarily flat vector bundle on Z associated to ρ.

Using the Selberg trace formula, Fried [F86, Theorem 3] showed that there exist explicit constants $C_\rho \in \mathbf{R}^*$ and $r_\rho \in \mathbf{Z}$ such that as $\sigma \to 0$,

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}). \tag{1.10}$$

Moreover, if $H^\cdot(Z, F) = 0$, then

$$C_\rho = 1, \qquad r_\rho = 0, \tag{1.11}$$

so that

$$R_\rho(0) = T(F)^2. \tag{1.12}$$

In [F87, P. 66, Conjecture], Fried conjectured that the same holds true when Z is a general closed locally homogeneous manifold.

1.4. **The V-invariant.** In this and in the following subsections, we give a formal proof of (1.12) using the V-invariant of Bismut-Goette [BG04].

Let S be a closed manifold equipped with an action of a compact Lie group L, with Lie algebra $\mathfrak{l}$. If $a \in \mathfrak{l}$, let a^S be the corresponding vector field on S. Bismut-Goette [BG04] introduced the V-invariant $V_a(S) \in \mathbf{R}$.

Let f be an a^S-invariant Morse-Bott function on S. Let $B_f \subset S$ be the critical submanifold. Since $a^S|_{B_f} \in TB_f$, $V_a(B_f)$ is also well defined. By [BG04, Theorem 4.10], $V_a(S)$ and $V_a(B_f)$ are related by a simple formula.

1.5. **Analytic torsion and the V-invariant.** Let us argue formally. Let LZ be the free loop space of Z equipped with the canonical $\mathbb{S}^1$-action. Write $LZ = \coprod_{[\gamma]\in[\Gamma]} (LZ)_{[\gamma]}$ as a disjoint union of its connected components. Let a be the generator of the Lie algebra of $\mathbb{S}^1$ such that $\exp(a) = 1$. As explained in [B05, Equation (0.3)], if F is a unitarily flat vector bundle on Z such that $H^\cdot(Z, F) = 0$, at least formally, we have

$$\log T(F) = -\sum_{[\gamma]\in[\Gamma]} \mathrm{Tr}[\rho(\gamma)] V_a\big((LZ)_{[\gamma]}\big). \tag{1.13}$$

Suppose that (Z, g^{TZ}) is an odd dimensional connected closed manifold of nonpositive sectional curvature, and suppose that the energy functional

$$E : x_\cdot \in LZ \to \frac{1}{2}\int_0^1 |\dot{x}_s|^2 ds \tag{1.14}$$

on LZ is Morse-Bott. The critical set of E is just $\coprod_{[\gamma]\in[\Gamma]} B_{[\gamma]}$, and all the critical points are local minima. Applying [BG04, Theorem 4.10] to the infinite dimensional manifold $(LZ)_{[\gamma]}$ equipped with the $\mathbb{S}^1$-invariant Morse-Bott functional E, we has the formal identity

$$V_a\big((LZ)_{[\gamma]}\big) = V_a\big(B_{[\gamma]}\big). \tag{1.15}$$

Since $B_{[1]} \simeq Z$ is formed of the trivial closed geodesics, by the definition of the V-invariant,

$$V_a(B_{[1]}) = 0. \tag{1.16}$$

By [BG04, Proposition 4.26], if $[\gamma] \in [\Gamma] - \{1\}$, then

$$V_a(B_{[\gamma]}) = -\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{2m_{[\gamma]}}. \tag{1.17}$$

By (1.13), (1.15)-(1.17), we get a formal identity

$$\log T(F) = \frac{1}{2} \sum_{[\gamma]\in[\Gamma]-\{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}}, \tag{1.18}$$

which is formally equivalent to (1.12).

1.6. **The main result of the article.** Let G be a linear connected real reductive group [K86, p. 3], and let θ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ. Let $\mathfrak{k}$ and $\mathfrak{g}$ be the Lie algebras of K and G, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition. Let B be a nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action of G and θ. Assume that B is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. Set $X = G/K$. Then B induces a Riemannian metric g^{TX} on the tangent bundle $TX = G \times_K \mathfrak{p}$, such that X is of nonpositive sectional curvature.

Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G. Set $Z = \Gamma\backslash X$. Then Z is a closed locally symmetric manifold with $\pi_1(Z) = \Gamma$. Recall that $\rho : \Gamma \to \mathrm{U}(r)$ is a unitary representation of Γ, and that F is the unitarily flat vector bundle on Z associated with ρ. The main result of this article gives the solution of the Fried conjecture for Z. In particular, this conjecture is valid for all the closed locally symmetric space of the noncompact type.

Theorem 1.1. *Assume* $\dim Z$ *is odd. The dynamical zeta function* $R_\rho(\sigma)$ *is holomorphic for* $\mathrm{Re}(\sigma) \gg 1$ *and extends meromorphically to* $\sigma \in \mathbf{C}$*. Moreover, there exist explicit constants* $C_\rho \in \mathbf{R}^*$ *and* $r_\rho \in \mathbf{Z}$ *(c.f. (7.75)) such that, when* $\sigma \to 0$,

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}\left(\sigma^{r_\rho+1}\right). \tag{1.19}$$

If $H^{\cdot}(Z,F) = 0$*, then*

$$C_\rho = 1, \qquad\qquad r_\rho = 0, \tag{1.20}$$

so that

$$R_\rho(0) = T(F)^2. \tag{1.21}$$

Let $\delta(G)$ be the nonnegative integer defined by the difference between the complex ranks of G and K. Since $\dim Z$ is odd, $\delta(G)$ is odd. If $\delta(G) \neq 1$, Theorem 1.1 is originally due to Moscovici-Stanton [MoSt91] and was recovered by Bismut [B11]. Indeed, it was proved in [MoSt91, Corollary 2.2, Remark 3.7] or [B11, Theorem 7.9.3] that $T(F) = 1$ and $\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right) = 0$ for all $[\gamma] \in [\Gamma] - \{1\}$.

Remark that both of the above two proofs use the Selberg trace formula. However, in the evaluation of the geometric side of the Selberg trace formula and of orbital integrals, Moscovici-Stanton relied on Harish-Chandra's Plancherel theory, while Bismut used his explicit formula [B11, Theorem 6.1.1] obtained via the hypoelliptic Laplacian.

Our proof of Theorem 1.1 relies on Bismut's formula.

1.7. **Our results on $R_\rho(\sigma)$.** Assume that $\delta(G) = 1$. To show that $R_\rho(\sigma)$ extends as a meromorphic function on $\mathbf{C}$ when Z is hyperbolic, Fried [F86] showed that $R_\rho(\sigma)$ is an alternating product of certain Selberg zeta functions. Moscovici-Stanton's idea was to introduce the more general Selberg zeta functions and to get a similar formula for $R_\rho(\sigma)$.

Let us recall some facts about reductive group G with $\delta(G) = 1$. In this case, there exists a unique (up to conjugation) standard parabolic subgroup $Q \subset G$ with Langlands decomposition $Q = M_Q A_Q N_Q$ such that $\dim A_Q = 1$. Let $\mathfrak{m}, \mathfrak{b}, \mathfrak{n}$ be the Lie algebras of M_Q, A_Q, N_Q. Let $\alpha \in \mathfrak{b}^*$ be such that for $a \in \mathfrak{b}$, $\mathrm{ad}(a)$ acts on $\mathfrak{n}$ as a scalar $\langle \alpha, a \rangle \in \mathbf{R}$ (c.f. Proposition 6.3). Let M be the connected component of identity of M_Q. Then M is a connected reductive group with maximal compact subgroup $K_M = M \cap K$ and with Cartan decomposition $\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}$. We have an identity of real K_M-representations

$$\mathfrak{p} \simeq \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{b} \oplus \mathfrak{n}. \tag{1.22}$$

An observation due to Moscovici-Stanton is that $\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]}) \neq 0$ only if γ can be conjugated by an element of G into $A_Q K_M$. For $\sigma \in \mathbf{C}$, we define the formal Selberg zeta function by

(1.23)
$$Z_j(\sigma) = \exp\left(- \sum_{[\gamma] \in [\Gamma] - \{1\}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{m_{[\gamma]}} \frac{\mathrm{Tr}^{\Lambda^j(\mathfrak{n}^*)}[\mathrm{Ad}(k^{-1})]}{\left| \det(1 - \mathrm{Ad}(e^a k^{-1}))|_{\mathfrak{n} \oplus \theta \mathfrak{n}} \right|^{1/2}} e^{-\sigma l_{[\gamma]}} \right),$$

where $a \in \mathfrak{b}, k \in K_M$ are such that γ can be conjugated to $e^a k^{-1}$. We remark that $l_{[\gamma]} = |a|$. To show the meromorphicity of $Z_j(\sigma)$, Moscovici-Stanton tried to identify $Z_j(\sigma)$ with the geometric side of the zeta regularized determinant of the resolvent of some elliptic operator acting on some vector bundle on Z. However, the vector bundle used in [MoSt91], whose construction involves the adjoint representation of K_M on $\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \otimes \Lambda^i(\mathfrak{n}^*)$, does not live on Z, but only on $\Gamma \backslash G / K_M$.

We complete this gap by showing that such an objet exists as a virtual vector bundle on Z in the sense of K-theory. More precisely, let $RO(K), RO(K_M)$ be the real representation rings of K and K_M. We can verify that the restriction $RO(K) \to RO(K_M)$ is injective. Note that $\mathfrak{p}_{\mathfrak{m}}, \mathfrak{n} \in RO(K_M)$. In Subsection 6.3, using the classification theory of real simple Lie algebras, we show $\mathfrak{p}_{\mathfrak{m}}, \mathfrak{n}$ are in the image of $RO(K)$. For $0 \leqslant j \leqslant \dim \mathfrak{n}$, let $E_j = E_j^+ - E_j^- \in RO(K)$ such that the following identity in $RO(K_M)$ holds:

$$\left(\sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} (-1)^i \Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \right) \otimes \Lambda^j(\mathfrak{n}^*) = E_j|_{K_M}. \tag{1.24}$$

Let $\mathcal{E}_j = G \times_K E_j$ be a $\mathbf{Z}_2$-graded vector bundle on X. It descends to a $\mathbf{Z}_2$-graded vector bundle $\mathcal{F}_j$ on Z. Let C_j be a Casimir operator of G action on $C^\infty(Z, \mathcal{F}_j \otimes_{\mathbf{R}} F)$. In Theorem 7.6, we show that there are $\sigma_j \in \mathbf{R}$ and an odd polynomial P_j such that if $\mathrm{Re}(\sigma) \gg 1$, $Z_j(\sigma)$ is holomorphic and

$$Z_j(\sigma) = \det\nolimits_{\mathrm{gr}} \left(C_j + \sigma_j + \sigma^2 \right) \exp\left(r \operatorname{vol}(Z) P_j(\sigma) \right), \tag{1.25}$$

where $\det_{\mathrm{gr}}$ is the zeta regularized $\mathbf{Z}_2$-graded determinant. In particular, $Z_j(\sigma)$ extends meromorphically to $\mathbf{C}$.

By a direct calculation of linear algebra, we have

$$R_\rho(\sigma) = \prod_{j=0}^{\dim \mathfrak{n}} Z_j \left(\sigma + (j - \frac{\dim \mathfrak{n}}{2}) |\alpha| \right)^{(-1)^{j-1}}, \tag{1.26}$$

from which we get the meromorphic extension of $R_\rho(\sigma)$. Note that the meromorphic function

$$T(\sigma) = \prod_{i=1}^{\dim Z} \det\left(\sigma + \Box^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i} \tag{1.27}$$

has a Laurent expansion near $\sigma = 0$,

$$T(\sigma) = T(F)^2 \sigma^{\chi'(X,F)} + \mathcal{O}(\sigma^{\chi'(X,F)+1}), \tag{1.28}$$

where $\chi'(X, F)$ is the derived Euler number (c.f. (2.8)). Note also that the Hodge Laplacian $\Box^Z$ coincides with the Casimir operator acting on $\Omega^\cdot(Z, F)$. The Laurent expansion (1.19) can be deduced form (1.25)-(1.28) and the identity in $RO(K)$,

$$\sum_{i=1}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{\dim \mathfrak{n}} (-1)^j E_j. \tag{1.29}$$

1.8. **Proof of Equation (1.20).** To understand how the acyclicity of F is reflected in the function $R_\rho(\sigma)$, we need some deep results of representation theory. Let $\widehat{p} : \Gamma\backslash G \to Z$ be the natural projection. The enveloping algebra of $U(\mathfrak{g})$ acts on $C^\infty(\Gamma\backslash G, \widehat{p}^* F)$. Let $\mathcal{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let $V^\infty \subset C^\infty(\Gamma\backslash G, \widehat{p}^* F)$ be the subspace of $C^\infty(\Gamma\backslash G, \widehat{p}^* F)$ on which the action of $\mathcal{Z}(\mathfrak{g})$ vanishes, and let V be the closure of V^∞ in $L^2(\Gamma\backslash G, \widehat{p}^* F)$. Then V is a unitary representation of G. The compactness of $\Gamma\backslash G$ implies that V is a finite sum of irreducible unitary representations of G. By standard arguments [BoW00, Ch VII, Theorem 3.2, Corollary 3.4], the cohomology $H^\cdot(Z, F)$ is canonically isomorphic to the $(\mathfrak{g}, K)$-cohomology $H^\cdot(\mathfrak{g}, K; V)$ of V.

Vogan-Zuckerman [VZu84] and Vogan [V84] classified all irreducible unitary representations with nonzero $(\mathfrak{g}, K)$-cohomology. On the other hand, Salamanca-Riba [SR99] showed that any irreducible unitary representation with vanishing $\mathcal{Z}(\mathfrak{g})$-action is in the class specified by Vogan and Zuckerman, which means that it possesses nonzero $(\mathfrak{g}, K)$-cohomology.

By the above considerations, the acyclicity of F is equivalent to $V = 0$. This is essentially the algebraic ingredient in the proof of (1.20). Indeed, in Corollary 8.18, we give a formula for the constants C_ρ and r_ρ, obtained by Hecht-Schmid formula [HeSc83] with the help of the $\mathfrak{n}$-homology of V.

1.9. **The organization of the article.** This article is organized as follows. In Sections 2, we recall the definitions of certain characteristic forms and of the analytic torsion.

In Section 3, we introduce the reductive groups and the fundamental rank $\delta(G)$ of G.

In Section 4, we introduce the symmetric space. We recall basic principles for the Selberg trace formula, and we state formulas by Bismut [B11, Theorem 6.1.1] for semisimple orbital integrals. We recall the proof given by Bismut [B11, Theorem 7.9.1] of a vanishing result of the analytic torsion $T(F)$ in the case $\delta(G) \neq 1$, which is originally due to Moscovici-Stanton [MoSt91, Corollary 2.2].

In Section 5, we introduce the dynamical zeta function $R_\rho(\sigma)$, and we state Theorem 1.1 as Theorem 5.5. We prove Theorem 1.1 when $\delta(G) \neq 1$ or when G has noncompact center.

Sections 6-8 are devoted to establish Theorem 1.1 when G has compact center and when $\delta(G) = 1$.

In Section 6, we introduce geometric objects associated with such reductive groups G.

In Section 7, we introduce Selberg zeta functions, and we prove that $R_\rho(\sigma)$ extends meromorphically, and we establish Equation (1.19).

Finally, in Section 8, after recalling some constructions and results of representation theory, we prove that (1.20) holds.

In the whole paper, we use the superconnection formalism of Quillen [Q85] and [BeGeVe04, Section 1.3]. If A is a $\mathbf{Z}_2$-graded algebra, if $a, b \in A$, the supercommutator $[a, b]$ is given by

$$[a,b] = ab - (-1)^{\deg a \deg b} ba. \tag{1.30}$$

If B is another $\mathbf{Z}_2$-graded algebra, we denote by $A\widehat{\otimes}B$ the super tensor product algebra of A and B. If $E = E^+ \oplus E^-$ is a $\mathbf{Z}_2$-graded vector space, the algebra $\mathrm{End}(E)$ is $\mathbf{Z}_2$-graded. If $\tau = \pm 1$ on $E^\pm$, if $a \in \mathrm{End}(E)$, the supertrace $\mathrm{Tr_s}[a]$ is defined by

$$\mathrm{Tr_s}[a] = \mathrm{Tr}[\tau a]. \tag{1.31}$$

We make the convention that $\mathbf{N} = \{0, 1, 2, \cdots\}$ and $\mathbf{N}^* = \{1, 2, \cdots\}$.

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2. Characteristic forms and analytic torsion

The purpose of this section is to recall some basic constructions and properties of characteristic forms and the analytic torsion.

This section is organized as follows. In Subsection 2.1, we recall the construction of the Euler form, the $\widehat{A}$-form and the Chern character form.

In Subsection 2.2, we introduce the regularized determinant.

Finally, in Subsection 2.3, we recall the definition of the analytic torsion of flat vector bundles.

2.1. **Characteristic forms.** If V is a real or complex vector space of dimension n, we denote by V^* the dual space and by $\Lambda^\cdot(V) = \sum_{i=0}^n \Lambda^i(V)$ its exterior algebra. Let Z be a smooth manifold. If V is a vector bundle on Z, we denote by $\Omega^\cdot(Z, V)$ the space of smooth differential forms with values in V. When $V = \mathbf{R}$, we write $\Omega^\cdot(Z)$ instead.

Let E be a real Euclidean vector bundle of rank m with a metric connection ∇^E. Let $R^E = \nabla^{E,2}$ be the curvature of ∇^E. It is a 2-form with values in antisymmetric endomorphisms of E.

If A is an antisymmetric matrix, denote by $\mathrm{Pf}[A]$ the Pfaffian [BZ92, eq. (3.3)] of A. Then $\mathrm{Pf}[A]$ is a polynomial function of A, which is a square root of $\det[A]$. Let $o(E)$ be the orientation line of E. The Euler form $e\left(E,\nabla^E\right)$ of $\left(E,\nabla^E\right)$ is given by

$$e\left(E,\nabla^E\right)=\mathrm{Pf}\left[\frac{R^E}{2\pi}\right]\in\Omega^m\big(Z,o(E)\big). \tag{2.1}$$

If m is odd, then $e(E,\nabla^E)=0$.

For $x\in\mathbf{C}$, set

$$\widehat{A}(x)=\frac{x/2}{\sinh(x/2)}. \tag{2.2}$$

The form $\widehat{A}\left(E,\nabla^E\right)$ of $\left(E,\nabla^E\right)$ is given by

$$\widehat{A}\left(E,\nabla^E\right)=\left[\det\left(\widehat{A}\left(-\frac{R^E}{2i\pi}\right)\right)\right]^{1/2}\in\Omega^{\cdot}(Z). \tag{2.3}$$

If E' is a complex Hermitian vector bundle equipped with a metric connection $\nabla^{E'}$ with curvature $R^{E'}$, the Chern character form $\mathrm{ch}\left(E',\nabla^{E'}\right)$ of $\left(E',\nabla^{E'}\right)$ is given by

$$\mathrm{ch}\left(E',\nabla^{E'}\right)=\mathrm{Tr}\left[\exp\left(-\frac{R^{E'}}{2i\pi}\right)\right]\in\Omega^{\cdot}(Z). \tag{2.4}$$

The differential forms $e\left(E,\nabla^E\right),\widehat{A}\left(E,\nabla^E\right)$ and $\mathrm{ch}\left(E',\nabla^{E'}\right)$ are closed. They are the Chern-Weil representatives of the Euler class of E, the $\widehat{A}$-genus of E and the Chern character of E'.

2.2. **Regularized determinant.** Let (Z,g^{TZ}) be a smooth closed Riemannian manifold of dimension m. Let (E,g^E) be a Hermitian vector bundle on Z. The metrics g^{TZ}, g^E induce an L^2-metric on $C^\infty(Z,E)$.

Let P be a second order elliptic differential operator acting on $C^\infty(Z,E)$. Suppose that P is formally self-adjoint and nonnegative. Let P^{-1} be the inverse of P acting on the orthogonal space to $\ker(P)$. For $\mathrm{Re}\,(s)>m/2$, set

$$\theta_P(s)=-\,\mathrm{Tr}\left[\left(P^{-1}\right)^s\right]. \tag{2.5}$$

By [Se67] or [BeGeVe04, Proposition 9.35], $\theta(s)$ has a meromorphic extension to $s\in\mathbf{C}$ which is holomorphic at $s=0$. The regularized determinant of P is defined as

$$\det(P)=\exp\left(\theta'_P(0)\right). \tag{2.6}$$

Assume now that P is formally self-adjoint and bounded from below. Denote by $\mathrm{Sp}(P)$ the spectrum of P. For $\lambda\in\mathrm{Sp}(P)$, set

$$m_P(\lambda)=\dim_{\mathbf{C}}\ker(P-\lambda) \tag{2.7}$$

its multiplicity. If $\sigma\in\mathbf{R}$ is such that $P+\sigma>0$, then $\det(P+\sigma)$ is defined by (2.6). Voros [Vo87] has shown that the function $\sigma\to\det(P+\sigma)$, defined for $\sigma\gg1$, extends holomorphically to $\mathbf{C}$ with zeros at $\sigma=-\lambda$ of the order $m_P(\lambda)$, where $\lambda\in\mathrm{Sp}(P)$.

2.3. **Analytic torsion.** Let Z be a smooth connected closed manifold of dimension m with fundamental group Γ. Let F be a complex flat vector bundle on Z of rank r. Equivalently, F can be obtained via a complex representation $\rho : \Gamma \to \mathrm{GL}_r(\mathbf{C})$.

Let $H^\cdot(Z,F) = \oplus_{i=0}^m H^i(Z,F)$ be the cohomology of the sheaf of locally flat sections of F. We define the Euler number and the derived Euler number by

$$\chi(Z,F) = \sum_{i=0}^m (-1)^i \dim_{\mathbf{C}} H^i(Z,F), \qquad \chi'(Z,F) = \sum_{i=1}^m (-1)^i i \dim_{\mathbf{C}} H^i(Z,F). \tag{2.8}$$

Let $\left(\Omega^\cdot(Z,F), d^Z\right)$ be the de Rham complex of smooth sections of $\Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F$ on Z. We have a canonical isomorphism of vector spaces

$$H^\cdot\left(\Omega^\cdot(Z,F), d^Z\right) \simeq H^\cdot(Z,F). \tag{2.9}$$

In the sequel, we will also consider the trivial line bundle $\mathbf{R}$. We denote simply by $H^\cdot(Z)$ and $\chi(Z)$ the corresponding objects. Note that, in this case, the complex dimension in (2.8) should be replaced by the real dimension.

Let g^{TZ} be a Riemmanian metric on TZ, and let g^F be a Hermitian metric on F. They induce an L^2-metric $\langle\,,\rangle_{\Omega^\cdot(Z,F)}$ on $\Omega^\cdot(Z,F)$. Let $d^{Z,*}$ be the formal adjoint of d^Z with respect to $\langle\,,\rangle_{\Omega^\cdot(Z,F)}$. Put

$$D^Z = d^Z + d^{Z,*}, \qquad \Box^Z = D^{Z,2} = \left[d^Z, d^{Z,*}\right]. \tag{2.10}$$

Then, $\Box^Z$ is a formally self-adjoint nonnegative second order elliptic operator acting on $\Omega^\cdot(Z,F)$. By Hodge theory, we have a canonical isomorphism of vector spaces

$$\ker \Box^Z \simeq H^\cdot(Z,F). \tag{2.11}$$

Definition 2.1. The analytic torsion of F is a positive real number defined by

$$T\left(F, g^{TZ}, g^F\right) = \prod_{i=1}^m \det\left(\Box^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i/2}. \tag{2.12}$$

Recall that the flat vector bundle F carries a flat metric g^F if and only if the holonomy representation ρ factors through $\mathrm{U}(r)$. In this case, F is said to be unitarily flat. If Z is an even dimensional orientable manifold and if F is unitarily flat with a flat metric g^F, by Poincaré duality, $T(F, g^{TZ}, g^F) = 1$. If $\dim Z$ is odd and if $H^\cdot(Z,F) = 0$, by [BZ92, Theorem 4.7], then $T(F, g^{TZ}, g^F)$ does not depend on g^{TZ} or g^F. In the sequel, we write instead

$$T(F) = T\left(F, g^{TZ}, g^F\right). \tag{2.13}$$

By Subsection 2.2,

$$T(\sigma) = \prod_{i=1}^{\dim Z} \det\left(\sigma + \Box^Z|_{\Omega^i(Z,F)}\right)^{(-1)^i i} \tag{2.14}$$

is meromorphic on $\mathbf{C}$. When $\sigma \to 0$, we have

$$T(\sigma) = T(F)^2 \sigma^{\chi'(Z,F)} + \mathcal{O}(\sigma^{\chi'(Z,F)+1}). \tag{2.15}$$

3. Preliminaries on reductive groups

The purpose of this section is to recall some basic facts about reductive groups.

This section is organized as follows. In Subsection 3.1, we introduce the reductive group G.

In Subsection 3.2, we introduce the semisimple elements of G, and we recall some related constructions.

In Subsection 3.3, we recall some properties of Cartan subgroups of G. We introduce a nonnegative integer $\delta(G)$, which has paramount importance in the whole article. We also recall Weyl's integral formula on reductive groups.

Finally, in Subsection 3.4, we introduce the regular elements of G.

3.1. **The reductive group.** Let G be a linear connected real reductive group [K86, p. 3], that means G is a closed connected group of real matrices that is stable under transpose. Let $\theta \in \mathrm{Aut}(G)$ be the Cartan involution. Let K be the maximal compact subgroup of G of the points of G that are fixed by θ.

Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebra of K. The Cartan involution θ acts naturally as a Lie algebra automorphism of $\mathfrak{g}$. Then $\mathfrak{k}$ is the eigenspace of θ associated with the eigenvalue 1. Let $\mathfrak{p}$ be the eigenspace with the eigenvalue -1, so that

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{3.1}$$

Then we have

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{3.2}$$

Put

$$m = \dim \mathfrak{p}, \qquad n = \dim \mathfrak{k}. \tag{3.3}$$

By [K86, Proposition 1.2], we have the diffeomorphism

$$(Y, k) \in \mathfrak{p} \times K \to e^{Y} k \in G. \tag{3.4}$$

Let B be a real-valued nondegenerate bilinear symmetric form on $\mathfrak{g}$ which is invariant under the adjoint action Ad of G on $\mathfrak{g}$, and also under θ. Then (3.1) is an orthogonal splitting of $\mathfrak{g}$ with respect to B. We assume B to be positive on $\mathfrak{p}$, and negative on $\mathfrak{k}$. The form $\langle \cdot, , \cdot \rangle = -B(\cdot, \theta \cdot)$ defines an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$ such that the splitting (3.1) is still orthogonal. We denote by $|\cdot|$ the corresponding norm.

Let $Z_G \subset G$ be the center of G with Lie algebra $\mathfrak{z}_{\mathfrak{g}} \subset \mathfrak{g}$. Set

$$\mathfrak{z}_{\mathfrak{p}} = \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{p}, \qquad \mathfrak{z}_{\mathfrak{k}} = \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{k}. \tag{3.5}$$

By [K86, Corollary 1.3], Z_G is reductive. As in (3.1) and (3.4), we have the Cartan decomposition

$$\mathfrak{z}_{\mathfrak{g}} = \mathfrak{z}_{\mathfrak{p}} \oplus \mathfrak{z}_{\mathfrak{k}}, \qquad Z_G = \exp(\mathfrak{z}_{\mathfrak{p}})(Z_G \cap K). \tag{3.6}$$

Let $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification of $\mathfrak{g}$ and let $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$ be the compact form of $\mathfrak{g}$. Let $G_{\mathbf{C}}$ and U be the connected group of complex matrices associated with the Lie algebras $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{u}$. By [K86, Propositions 5.3 and 5.6], if G has compact center, i.e., its center Z_G is compact, then $G_{\mathbf{C}}$ is a linear connected complex reductive group with maximal compact subgroup U.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let $\mathcal{Z}(\mathfrak{g}) \subset U(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let $C^{\mathfrak{g}} \in U(\mathfrak{g})$ be the Casimir element. If $e_1, \cdots, e_m$ is an orthonormal basis of $\mathfrak{p}$, and if $e_{m+1}, \cdots, e_{m+n}$ is an orthonormal basis of $\mathfrak{k}$, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m} e_i^2 + \sum_{i=m+1}^{n+m} e_i^2. \tag{3.7}$$

Classically, $C^{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$.

We define $C^{\mathfrak{k}}$ similarly. Let τ be a finite dimensional representation of K on V. We denote by $C^{\mathfrak{k},V}$ or $C^{\mathfrak{k},\tau} \in \mathrm{End}(V)$ the corresponding Casimir operator acting on V, so that

$$C^{\mathfrak{k},V} = C^{\mathfrak{k},\tau} = \sum_{i=m+1}^{m+n} \tau(e_i)^2. \tag{3.8}$$

3.2. **Semisimple elements.** If $\gamma \in G$, we denote by $Z(\gamma) \subset G$ the centralizer of γ in G, and by $\mathfrak{z}(\gamma) \subset \mathfrak{g}$ its Lie algebra. If $a \in \mathfrak{g}$, let $Z(a) \subset G$ be the stabilizer of a in G, and let $\mathfrak{z}(a) \subset \mathfrak{g}$ be its Lie algebra.

An element $\gamma \in G$ is said to be semisimple if γ can be conjugated to $e^a k^{-1}$ such that

$$a \in \mathfrak{p}, \qquad k \in K, \qquad \mathrm{Ad}(k)a = a. \tag{3.9}$$

Let $\gamma = e^a k^{-1}$ such that (3.9) holds. By [B11, eq. (3.3.4), (3.3.6)], we have

$$Z(\gamma) = Z(a) \cap Z(k), \qquad \mathfrak{z}(\gamma) = \mathfrak{z}(a) \cap \mathfrak{z}(k). \tag{3.10}$$

Set

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \qquad \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{3.12}$$

By [K02, Proposition 7.25], $Z(\gamma)$ is a reductive subgroup of G with maximal compact subgroup $K(\gamma) = Z(\gamma) \cap K$, and with Cartan decomposition (3.12). Let $Z^0(\gamma)$ be the connected component of the identity in $Z(\gamma)$. Then $Z^0(\gamma)$ is a reductive subgroup of G, with maximal compact subgroup $Z^0(\gamma) \cap K$. Also, $Z^0(\gamma) \cap K$ coincides with $K^0(\gamma)$, the connected component of the identity in $K(\gamma)$.

An element $\gamma \in G$ is said to be elliptic if γ is conjugated to an element of K. Let $\gamma \in G$ be semisimple and nonelliptic. Up to conjugation, we can assume $\gamma = e^a k^{-1}$ such that (3.9) holds and that $a \neq 0$. By (3.10), $a \in \mathfrak{p}(\gamma)$. Let $\mathfrak{z}^{a,\perp}(\gamma), \mathfrak{p}^{a,\perp}(\gamma)$ be respectively the orthogonal spaces to a in $\mathfrak{z}(\gamma), \mathfrak{p}(\gamma)$, so that

$$\mathfrak{z}^{a,\perp}(\gamma) = \mathfrak{p}^{a,\perp}(\gamma) \oplus \mathfrak{k}(\gamma). \tag{3.13}$$

Moreover, $\mathfrak{z}^{a,\perp}(\gamma)$ is a Lie algebra. Let $Z^{a,\perp,0}(\gamma)$ be the connected subgroup of $Z^0(\gamma)$ that is associated with the Lie algebra $\mathfrak{z}^{a,\perp}(\gamma)$. By [B11, eq. (3.3.11)], $Z^{a,\perp,0}(\gamma)$ is reductive with maximal compact subgroup $K^0(\gamma)$ with Cartan decomposition (3.13), and

$$Z^0(\gamma) = \mathbf{R} \times Z^{a,\perp,0}(\gamma), \tag{3.14}$$

so that e^{ta} maps into $t|a|$.

3.3. **Cartan subgroups.** A Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$. A Cartan subgroup of G is the centralizer of a Cartan subalgebra.

By [K86, Theorem 5.22], there is only a finite number of nonconjugate (via K) θ-stable Cartan subalgebras $\mathfrak{h}_1, \cdots, \mathfrak{h}_{l_0}$. Let $H_1, \cdots, H_{l_0}$ be the corresponding Cartan subgroup. Clearly, the Lie algebra of H_i is $\mathfrak{h}_i$. Set

$$\mathfrak{h}_{i\mathfrak{p}} = \mathfrak{h}_i \cap \mathfrak{p}, \qquad \mathfrak{h}_{i\mathfrak{k}} = \mathfrak{h}_i \cap \mathfrak{k}. \tag{3.15}$$

We call $\dim \mathfrak{h}_{i\mathfrak{p}}$ the noncompact dimension of $\mathfrak{h}_i$. By [K86, Theorem 5.22 (c)] and [K02, Proposition 7.25], H_i is an abelian reductive group with maximal compact subgroup $H_i \cap K$, and with Cartan decomposition

$$\mathfrak{h}_i = \mathfrak{h}_{i\mathfrak{p}} \oplus \mathfrak{h}_{i\mathfrak{k}}, \qquad H_i = \exp(\mathfrak{h}_{i\mathfrak{p}})(H_i \cap K). \tag{3.16}$$

Note that in general, H_i is not necessarily connected.

Let $W(H_i, G)$ be the Weyl group. If $N_K(\mathfrak{h}_i) \subset K$ and $Z_K(\mathfrak{h}_i) \subset K$ are the normalizer and centralizer of $\mathfrak{h}_i$ in K, then

$$W(H_i, G) = N_K(\mathfrak{h}_i)/Z_K(\mathfrak{h}_i). \tag{3.17}$$

In the whole paper, we fix a maximal torus T of K. Let $\mathfrak{t} \subset \mathfrak{k}$ be the Lie algebra of T. Set

$$\mathfrak{b} = \{Y \in \mathfrak{p} : [Y, \mathfrak{t}] = 0\}. \tag{3.18}$$

By (3.5) and (3.18), we have

$$\mathfrak{z}_{\mathfrak{p}} \subset \mathfrak{b}. \tag{3.19}$$

Put

$$\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}. \tag{3.20}$$

By [K86, Theorem 5.22 (b)], $\mathfrak{h}$ is the θ-stable Cartan subalgebra of $\mathfrak{g}$ with minimal noncompact dimension. Also, every θ-stable Cartan subalgebra with minimal noncompact dimension is conjugated to $\mathfrak{h}$ by an element of K. Moreover, the corresponding Cartan subgroup $H \subset G$ of G is connected, so that

$$H = \exp(\mathfrak{b})T. \tag{3.21}$$

We may assume that $\mathfrak{h}_1 = \mathfrak{h}$ and $H_1 = H$.

Note that the complexification $\mathfrak{h}_{i\mathbf{C}} = \mathfrak{h}_i \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. All the $\mathfrak{h}_{i\mathbf{C}}$ are conjugated by inner automorphisms of $\mathfrak{g}_{\mathbf{C}}$. Their common complex dimension $\dim_{\mathbf{C}} \mathfrak{h}_{i\mathbf{C}}$ is called the complex rank $\mathrm{rk}_{\mathbf{C}}(G)$ of G.

Definition 3.1. Put

$$\delta(G) = \mathrm{rk}_{\mathbf{C}}(G) - \mathrm{rk}_{\mathbf{C}}(K) \in \mathbf{N}. \tag{3.22}$$

By (3.18) and (3.22), we have

$$\delta(G) = \dim \mathfrak{b}. \tag{3.23}$$

Note that $m - \delta(G)$ is even. We will see that $\delta(G)$ plays an important role in our article.

Remark 3.2. If $\mathfrak{g}$ is a real reductive Lie algebra, then $\delta(\mathfrak{g}) \in \mathbf{N}$ can be defined in the same way as in (3.23). Since $\mathfrak{g}$ is reductive, by [K02, Corolary 1.56], we have

$$\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}], \tag{3.24}$$

where $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra. By (3.6) and (3.24), we have

$$\delta(\mathfrak{g}) = \dim \mathfrak{z}_{\mathfrak{p}} + \delta([\mathfrak{g}, \mathfrak{g}]). \tag{3.25}$$

Proposition 3.3. *The element $\gamma \in G$ is semisimple if and only if γ can be conjugated into $\cup_{i=1}^{l_0} H_i$. In this case,*

$$\delta(G) \leqslant \delta\big(Z^0(\gamma)\big). \tag{3.26}$$

The two sides of (3.26) *are equal if and only if γ can be conjugated into H.*

Proof. If $\gamma \in H_i$, by the Cartan decomposition (3.16), there exist $a \in \mathfrak{h}_{i\mathfrak{p}}$ and $k \in K \cap H_i$ such that $\gamma = e^a k^{-1}$. Since H_i is the centralizer of $\mathfrak{h}_i$, we have $\mathrm{Ad}(\gamma)a = a$. Therefore, $\mathrm{Ad}(k)a = a$, so that γ is semisimple.

Assume that $\gamma \in G$ is semisimple and such that (3.9) holds. We claim that

$$\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}\left(Z^0(\gamma)\right). \tag{3.27}$$

Indeed, let $\mathfrak{h}' \subset \mathfrak{g}$ be a θ-invariant Cartan subalgebra of $\mathfrak{g}$ containing a. Then, $\mathfrak{h}' \subset \mathfrak{z}(a)$. It implies

$$\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}\left(Z^0(a)\right). \tag{3.28}$$

By choosing a maximal torus T containing k, by (3.20), we have $\mathfrak{h} \subset \mathfrak{z}(k)$. Then

$$\mathrm{rk}_{\mathbf{C}}(G) = \mathrm{rk}_{\mathbf{C}}\left(Z^0(k)\right). \tag{3.29}$$

If we replace G by $Z^0(a)$ in (3.29), by (3.10), we get

$$\mathrm{rk}_{\mathbf{C}}(Z^0(a)) = \mathrm{rk}_{\mathbf{C}}\left(Z^0(\gamma)\right). \tag{3.30}$$

By (3.28) and (3.30), we get (3.27).

Let $\mathfrak{h}(\gamma) \subset \mathfrak{z}(\gamma)$ be the θ-invariant Cartan subalgebra defined as in (3.20) when G is replaced by $Z^0(\gamma)$. By (3.27), $\mathfrak{h}(\gamma)$ is also a Cartan subalgebra of $\mathfrak{g}$. Moreover, γ is an element of the Cartan subgroup of G associated to $\mathfrak{h}(\gamma)$. In particular, γ can be conjugated into some H_i.

By the minimality of noncompact dimension of $\mathfrak{h}$, we have

$$\delta(G) = \dim \mathfrak{h} \cap \mathfrak{p} \leqslant \dim \mathfrak{h}(\gamma) \cap \mathfrak{p} = \delta\big(Z^0(\gamma)\big), \tag{3.31}$$

which completes the proof of (3.26).

It is obvious that if γ can be conjugated into H, the equality in (3.31) holds. If the equality holds in (3.31), by the uniqueness of the Cartan subalgebra with minimal non-compact dimension, there is $k' \in K$ such that

$$\mathrm{Ad}(k')\mathfrak{h}(\gamma) = \mathfrak{h}, \tag{3.32}$$

which implies that $k'\gamma k'^{,-1} \in H$. The proof of our proposition is completed. □

Now we recall the Weyl integral formula on G, which will be used in Section 8. Let dv_{H_i} and $dv_{H_i\backslash G}$ be respectively the Riemannian volumes on H_i and $H_i\backslash G$ induced by $-B(\cdot,\theta\cdot)$. By [K02, Theorem 8.64], for a nonnegative measurable function f on G, we have

$$\int_{g\in G} f(g)dv_G = \sum_{i=1}^{l_0} \frac{1}{|W(H_i,G)|} \int_{\gamma\in H_i} \left(\int_{g\in H_i\backslash G} f(g^{-1}\gamma g)dv_{H_i\backslash G}\right) \left|\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{g}/\mathfrak{h}_i}\right| dv_{H_i}. \tag{3.33}$$

3.4. **Regular elements.** For $0 \leqslant j \leqslant m+n-\mathrm{rk}_{\mathbf{C}}(G)$, let D_j be the analytic function on G such that for $t\in \mathbf{R}$ and $\gamma\in G$, we have

$$\det\big(t+1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{g}} = t^{\mathrm{rk}_{\mathbf{C}}(G)}\left(\sum_{j=0}^{m+n-\mathrm{rk}_{\mathbf{C}}(G)} D_j(\gamma)t^j\right). \tag{3.34}$$

If $\gamma\in H_i$, then

$$D_0(\gamma) = \det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{g}/\mathfrak{h}_i}. \tag{3.35}$$

We call $\gamma\in G$ regular if $D_0(\gamma)\neq 0$. Let $G'\subset G$ be the subset of regular elements of G. Then G' is open in G, such that $G-G'$ has zero measure with respect to the Riemannian volume dv_G on G induced by $-B(\cdot,\theta\cdot)$. For $1\leqslant i\leqslant l_0$, set

$$H_i' = H_i\cap G', \qquad G_i' = \bigcup_{g\in G} g^{-1}H_i'g. \tag{3.36}$$

By [K86, Theorem 5.22 (d)], G_i' is open, and we have the disjoint union

$$G' = \coprod_{1\leqslant i\leqslant l_0} G_i'. \tag{3.37}$$

4. Orbital integrals and Selberg trace formula

The purpose of this section is to recall Bismut's semisimple orbital integral formula [B11, Theorem 6.1.1] and the Selberg trace formula.

This section is organized as follows. In Subsections 4.1, we introduce the Riemannian symmetric space $X=G/K$, and we give a formula for its Euler form.

In Subsection 4.2, we recall the definition of semisimple orbital integrals.

In Subsection 4.3, we recall Bismut's explicit formula for the semisimple orbital integrals associated to the heat operator of the Casimir element.

In Subsection 4.4, we introduce a discrete torsion-free cocompact subgroup Γ of G. We state the Selberg trace formula.

Finally, in Subsection 4.5, we recall Bismut's proof of a vanishing result on the analytic torsion in the case $\delta(G)\neq 1$, which is originally due to Moscovici-Stanton [MoSt91].

4.1. **The symmetric space.** We use the notation of Section 3. Let $\omega^{\mathfrak{g}}$ be the canonical left-invariant 1-form on G with values in $\mathfrak{g}$, and let $\omega^{\mathfrak{p}},\omega^{\mathfrak{k}}$ be its components in $\mathfrak{p},\mathfrak{k}$, so that

$$\omega^{\mathfrak{g}} = \omega^{\mathfrak{p}}+\omega^{\mathfrak{k}}. \tag{4.1}$$

Let $X = G/K$ be the associated symmetric space. Then

$$p : G \to X = G/K \tag{4.2}$$

is a K-principle bundle, equipped with the connection form $\omega^{\mathfrak{k}}$. By (3.2) and (4.1), the curvature of $\omega^{\mathfrak{k}}$ is given by

$$\Omega^{\mathfrak{k}} = -\frac{1}{2}\left[\omega^{\mathfrak{p}}, \omega^{\mathfrak{p}}\right]. \tag{4.3}$$

Let τ be a finite dimensional orthogonal representation of K on the real Euclidean space E_τ. Then $\mathcal{E}_\tau = G \times_K E_\tau$ is a real Euclidean vector bundle on X, which is naturally equipped with a Euclidean connection $\nabla^{\mathcal{E}_\tau}$. The space of smooth sections $C^\infty(X, \mathcal{E}_\tau)$ on X can be identified with the K-invariant subspace $C^\infty(G, E_\tau)^K$ of smooth E_τ-valued functions on G. Let $C^{\mathfrak{g},X,\tau}$ be the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$. Then $C^{\mathfrak{g},X,\tau}$ is a formally self-adjoint second order elliptic differential operator which is bounded from below.

Observe that K acts isometrically on $\mathfrak{p}$. Using the above construction, the tangent bundle $TX = G \times_K \mathfrak{p}$ is equipped with a Euclidean metric g^{TX} and a Euclidean connection ∇^{TX}. Also, ∇^{TX} is the Levi-Civita connection on (TX, g^{TX}) with curvature R^{TX}. Classically, (X, g^{TX}) is a Riemannian manifold of nonpositive sectional curvature. For $x, y \in X$, we denote by $d_X(x, y)$ the Riemannian distance on X.

If $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$, then $C^\infty(X, \mathcal{E}_\tau) = \Omega^\cdot(X)$. In this case, we write $C^{\mathfrak{g},X} = C^{\mathfrak{g},X,\tau}$. By [B11, Proposition 7.8.1], $C^{\mathfrak{g},X}$ coincides with the Hodge Laplacian acting on $\Omega^\cdot(X)$.

Let us state a formula for $e\left(TX, \nabla^{TX}\right)$. Let $o(TX)$ be the orientation line of TX. Let dv_X be the G-invariant Riemnnian volume form on X. If $\alpha \in \Omega^\cdot\left(X, o(TX)\right)$ is of maximal degree and G-invariant, set $[\alpha]^{\max} \in \mathbf{R}$ such that

$$\alpha = [\alpha]^{\max} dv_X. \tag{4.4}$$

Recall that if G has compact center, then U is the compact form of G. If $\delta(G) = 0$, by (3.25), G has compact center. In this case, T are maximal torus of both U and K. Let $W(T, U), W(T, K)$ be respectively the Weyl group. Let $\mathrm{vol}(U/K)$ be the volume of U/K with respecte to the volume form induced by $-B$.

Proposition 4.1. *If $\delta(G) \neq 0$, $\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = 0$. If $\delta(G) = 0$,*

$$\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = (-1)^{\frac{m}{2}} \frac{|W(T,U)|/|W(T,K)|}{\mathrm{vol}(U/K)}. \tag{4.5}$$

Proof. If G has noncompact center (thus $\delta(G) \neq 0$), it is trivial that $\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = 0$. Assume now, G has compact center. By Hirzebruch proportionality (c.f. [Hi66, Theorem 22.3.1] for a proof for Hermitian symmetric spaces, and the proof for general case is identical), we have

$$\left[e\left(TX, \nabla^{TX}\right)\right]^{\max} = (-1)^{\frac{m}{2}} \frac{\chi(U/K)}{\mathrm{vol}(U/K)}. \tag{4.6}$$

Proposition 4.1 is a consequence of (4.6), [Bot65, Theorem II] and Bott's formula [Bot65, p. 175] and of the fact that $\delta(G) = \mathrm{rk}_{\mathbf{C}}(U) - \mathrm{rk}_{\mathbf{C}}(K)$. □

Let $\gamma \in G$ be a semisimple element as in (3.9). Let

$$X(\gamma) = Z(\gamma)/K(\gamma) \tag{4.7}$$

be the associated symmetric space. Clearly,

$$X(\gamma) = Z^0(\gamma)/K^0(\gamma). \tag{4.8}$$

Suppose that γ is nonelliptic. Set

$$X^{a,\perp}(\gamma) = Z^{a,\perp,0}(\gamma)/K^0(\gamma). \tag{4.9}$$

By (3.14), (4.8) and (4.9), we have

$$X(\gamma) = \mathbf{R} \times X^{a,\perp}(\gamma), \tag{4.10}$$

so that the action e^{ta} on $X(\gamma)$ is just the translation by $t|a|$ on $\mathbf{R}$.

4.2. **The semisimple orbital integrals.** Recall that τ is a finite dimensional orthogonal representation of K on the real Euclidean space E_τ, and that $C^{\mathfrak{g},X,\tau}$ is the Casimir element of G acting on $C^\infty(X, \mathcal{E}_\tau)$.

Let $p_t^{X,\tau}(x,x')$ be the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to the Riemannian volume dv_X on X. Classically, for $t > 0$, there exist $c > 0$ and $C > 0$ such that for $x, x' \in X$,

$$\left| p_t^{X,\tau}(x,x') \right| \leqslant C \exp\left(-c\, d_X^2(x,x')\right). \tag{4.11}$$

Set

$$p_t^{X,\tau}(g) = p_t^{X,\tau}(p1, pg). \tag{4.12}$$

For $g \in G$ and $k, k' \in K$, we have

$$p_t^{X,\tau}(kgk') = \tau(k) p_t^{X,\tau}(g) \tau(k'). \tag{4.13}$$

Also, we can recover $p_t^{X,\tau}(x,x')$ by

$$p_t^{X,\tau}(x,x') = p_t^{X,\tau}(g^{-1}g'), \tag{4.14}$$

where $g, g' \in G$ are such that $pg = x, pg' = x'$.

In the sequel, we do not distinguish $p_t^{X,\tau}(x,x')$ and $p_t^{X,\tau}(g)$. We refer to both of them as being the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$.

Let $dv_{K^0(\gamma)\backslash K}$ and $dv_{Z^0(\gamma)\backslash G}$ be the Riemannian volumes on $K^0(\gamma)\backslash K$ and $Z^0(\gamma)\backslash G$ induced by $-B(\cdot, \theta\cdot)$. Let $\mathrm{vol}(K^0(\gamma)\backslash K)$ be the volume of $K^0(\gamma)\backslash K$ with respect to $dv_{K^0(\gamma)\backslash K}$.

Definition 4.2. Let $\gamma \in G$ be semisimple. The orbital integral of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ is defined by

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right] = \frac{1}{\mathrm{vol}(K^0(\gamma)\backslash K)} \int_{g \in Z^0(\gamma)\backslash G} \mathrm{Tr}^{E_\tau}\left[p_t^{X,\tau}(g^{-1}\gamma g)\right] dv_{Z^0(\gamma)\backslash G}. \tag{4.15}$$

Remark 4.3. Definition 4.2 is equivalent to [B11, Definition 4.2.2], where the volume forms are normalized such that $\mathrm{vol}(K^0(\gamma)\backslash K) = 1$.

Remark 4.4. As the notation $\mathrm{Tr}^{[\gamma]}$ indicates, the orbital integral only depends on the conjugacy class of γ in G. However, the notation $[\gamma]$ (c.f. Subsection 4.4) will be used later for the conjugacy class in the discrete group Γ.

Remark 4.5. We will also consider the case where E_τ is a $\mathbf{Z}_2$-graded or virtual representation of K. We will use the notation $\mathrm{Tr_s}^{[\gamma]}[q]$ when the trace on the right-hand side of (4.15) is replaced by the supertrace on E_τ.

4.3. **Bismut's formula for semisimple orbital integrals.** Let us recall the explicit formula for $\mathrm{Tr}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\tau}/2)\right]$, for any semisimple element $\gamma\in G$, obtained by Bismut [B11, Theorem 6.1.1].

Let $\gamma=e^a k^{-1}\in G$ be semisimple as in (3.9). Set

$$\mathfrak{z}_0=\mathfrak{z}(a),\qquad \mathfrak{p}_0=\mathfrak{z}(a)\cap\mathfrak{p},\qquad \mathfrak{k}_0=\mathfrak{z}(a)\cap\mathfrak{k}. \tag{4.16}$$

Then

$$\mathfrak{z}_0=\mathfrak{p}_0\oplus\mathfrak{k}_0. \tag{4.17}$$

By (3.10), (3.11) and (4.16), we have $\mathfrak{p}(\gamma)\subset\mathfrak{p}_0$ and $\mathfrak{k}(\gamma)\subset\mathfrak{k}_0$. Let $\mathfrak{p}_0^\perp(\gamma)$, $\mathfrak{k}_0^\perp(\gamma)$, $\mathfrak{z}_0^\perp(\gamma)$ be the orthogonal spaces of $\mathfrak{p}(\gamma)$, $\mathfrak{k}(\gamma)$, $\mathfrak{z}(\gamma)$ in $\mathfrak{p}_0$, $\mathfrak{k}_0$, $\mathfrak{z}_0$. Let $\mathfrak{p}_0^\perp$, $\mathfrak{k}_0^\perp$, $\mathfrak{z}_0^\perp$ be the orthogonal spaces of $\mathfrak{p}_0$, $\mathfrak{k}_0$, $\mathfrak{z}_0$ in $\mathfrak{p}$, $\mathfrak{k}$, $\mathfrak{z}$. Then we have

$$\mathfrak{p}=\mathfrak{p}(\gamma)\oplus\mathfrak{p}_0^\perp(\gamma)\oplus\mathfrak{p}_0^\perp,\qquad \mathfrak{k}=\mathfrak{k}(\gamma)\oplus\mathfrak{k}_0^\perp(\gamma)\oplus\mathfrak{k}_0^\perp. \tag{4.18}$$

Recall that $\widehat{A}$ is the function defined in (2.2).

Definition 4.6. For $Y\in\mathfrak{k}(\gamma)$, put

$$J_\gamma(Y)=\frac{1}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}}\frac{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{p}(\gamma)}\big)}{\widehat{A}\big(i\,\mathrm{ad}(Y)|_{\mathfrak{k}(\gamma)}\big)}\left[\frac{1}{\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\frac{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp(-i\,\mathrm{ad}(Y))\,\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}. \tag{4.19}$$

As explained in [B11, Section 5.5], there is a natural choice for the square root in (4.19). Moreover, J_γ is an $\mathrm{Ad}\left(K^0(\gamma)\right)$-invariant analytic function on $\mathfrak{k}(\gamma)$, and there exist $c_\gamma>0, C_\gamma>0$, such that for $Y\in\mathfrak{k}(\gamma)$,

$$|J_\gamma(Y)|\leqslant C_\gamma\exp\left(c_\gamma|Y|\right). \tag{4.20}$$

By (4.19), we have

$$J_1(Y)=\frac{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{p}})}{\widehat{A}(i\,\mathrm{ad}(Y)|_{\mathfrak{k}})}. \tag{4.21}$$

For $Y\in\mathfrak{k}(\gamma)$, let dY be the Lebesgue measure on $\mathfrak{k}(\gamma)$ induced by $-B$. Recall that $C^{\mathfrak{k},\mathfrak{p}}$ and $C^{\mathfrak{k},\mathfrak{k}}$ are defined in (3.8). The main result of [B11, Theorem 6.1.1] is the following.

Theorem 4.7. *For $t>0$, we have*

(4.22)
$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]=\frac{1}{(2\pi t)^{\dim\mathfrak{z}(\gamma)/2}}\exp\left(-\frac{|a|^2}{2t}+\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)\int_{Y\in\mathfrak{k}(\gamma)}J_\gamma(Y)\,\mathrm{Tr}^{E_\tau}\left[\tau\left(k^{-1}\right)\exp(-i\tau(Y))\right]\exp\left(-|Y|^2/2t\right)dY.$$

4.4. **A discrete subgroup of G.** Let $\Gamma \subset G$ be a discrete torsion-free cocompact subgroup of G. By [Sel60, Lemma 1], Γ contains the identity element and nonelliptic semisimple elements. Also, Γ acts isometrically on the left on X. This action lifts to all the homogeneous Euclidean vector bundles $\mathcal{E}_\tau$ constructed in Subsection 4.1, and preserves the corresponding connections.

Take $Z = \Gamma\backslash X = \Gamma\backslash G/K$. Then Z is a connected closed orientable Riemannian locally symmetric manifold with nonpositive sectional curvature. Since X is contractible, $\pi_1(Z) = \Gamma$ and X is the universal cover of Z. We denote by $\widehat{p} : \Gamma\backslash G \to Z$ and $\widehat{\pi} : X \to Z$ the natural projections, so that the diagram

$$\begin{array}{ccc} G & \longrightarrow & \Gamma\backslash G \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \widehat{p}} \\ X & \xrightarrow{\widehat{\pi}} & Z \end{array} \tag{4.23}$$

commutes.

The Euclidean vector bundle $\mathcal{E}_\tau$ descends to a Euclidean vector bundle $\mathcal{F}_\tau = \Gamma\backslash\mathcal{E}_\tau$ on Z. Take $r \in \mathbf{N}^*$. Let $\rho : \Gamma \to \mathrm{U}(r)$ be a unitary representation of Γ. Let (F, ∇^F, g^F) be the unitarily flat vector bundle on Z associated to ρ. Let $C^{\mathfrak{g},Z,\tau,\rho}$ be the Casimir element of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{C}} F)$. As in Subsection 4.1, when $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$, we write $C^{\mathfrak{g},Z,\rho} = C^{\mathfrak{g},Z,\tau,\rho}$. Then,

$$\square^Z = C^{\mathfrak{g},Z,\rho}. \tag{4.24}$$

Recall that $p_t^{X,\tau}(x,x')$ is the smooth kernel of $\exp(-tC^{\mathfrak{g},X,\tau}/2)$ with respect to dv_X.

Proposition 4.8. *There exist $c > 0$, $C > 0$ such that for $t > 0$ and $x \in X$, we have*

$$\sum_{\gamma\in\Gamma-\{1\}} \left|p_t^{X,\tau}(x,\gamma x)\right| \leqslant C\exp\left(-\frac{c}{t} + Ct\right). \tag{4.25}$$

Proof. By [Mi68b, Remark p.1, Lemma 2] or [MaMar15, eq. (3.19)], there is $C > 0$ such that for all $r \geqslant 0$, $x \in X$, we have

$$\left|\left\{\gamma \in \Gamma : d_X(x,\gamma x) \leqslant r\right\}\right| \leqslant Ce^{Cr}. \tag{4.26}$$

We claim that there exist $c > 0, C > 0$ and $N \in \mathbf{N}$ such that for $t > 0$ and $x, x' \in X$, we have

$$\left|p_t^{X,\tau}(x,x')\right| \leqslant \frac{C}{t^N}\exp\left(-c\frac{d_X^2(x,x')}{t} + Ct\right). \tag{4.27}$$

Indeed, if $\tau = \mathbf{1}$, $p_t^{X,\mathbf{1}}(x,x')$ is the heat kernel for the Laplace-Beltrami operator. In this case, (4.27) is a consequence of the Li-Yau estimate [LiY86, Corollary 3.1] and of the fact that X is a symmetric space. For general τ, using the Itô formula as in [BZ92, eq. (12.30)]), we can show that there is $C > 0$ such that

$$\left|p_t^{X,\tau}(x,x')\right| \leqslant Ce^{Ct}p_t^{X,\mathbf{1}}(x,x'), \tag{4.28}$$

from which we get (4.27)[2].

[2]See [MaMar15, Theorem 4] for another proof of (4.27) using finite propagation speed of solutions of hyperbolic equations.

Note that there exists $c_0 > 0$ such that for all $\gamma \in \Gamma - \{1\}$ and $x \in X$,

$$d_X(x, \gamma x) \geqslant c_0. \tag{4.29}$$

By (4.27) and (4.29), there exist $c_1 > 0$, $c_2 > 0$ and $C > 0$ such that for $t > 0$, $x \in X$ and $\gamma \in \Gamma - \{1\}$, we have

$$\left| p_t^{X,\tau}(x, \gamma x) \right| \leqslant C \exp\left(-\frac{c_1}{t} - c_2 \frac{d_X^2(x, \gamma x)}{t} + Ct \right). \tag{4.30}$$

By (4.26) and(4.30), for $t > 0$ and $x \in X$, we have

(4.31)

$$\begin{aligned}
\sum_{\gamma \in \Gamma - \{1\}} \left| p_t^{X,\tau}(x, \gamma x) \right| &\leqslant C \sum_{\gamma \in \Gamma} \exp\left(-\frac{c_1}{t} - c_2 \frac{d_X^2(x, \gamma x)}{t} + Ct \right) \\
&= c_2 C \exp\left(-\frac{c_1}{t} + Ct \right) \sum_{\gamma \in \Gamma} \int_{d_X^2(x,\gamma x)/t}^{\infty} \exp(-c_2 r) dr \\
&= c_2 C \exp\left(-\frac{c_1}{t} + Ct \right) \int_0^{\infty} \left| \{ \gamma \in \Gamma : d_X(x, \gamma x) \leqslant \sqrt{rt} \} \right| \exp(-c_2 r) dr \\
&\leqslant C' \exp\left(-\frac{c_1}{t} + Ct \right) \int_0^{\infty} \exp\left(-c_2 r + C\sqrt{rt} \right) dr.
\end{aligned}$$

From (4.31), we get (4.25). The proof of our proposition is completed. □

For $\gamma \in \Gamma$, set

$$\Gamma(\gamma) = Z(\gamma) \cap \Gamma. \tag{4.32}$$

Let $[\gamma]$ be the conjugacy class of γ in Γ. Let $[\Gamma]$ be the set of all the conjugacy classes of Γ.

The following proposition is [Sel60, Lemma 2]. We include a proof for the sake of completeness.

Proposition 4.9. *If $\gamma \in \Gamma$, then $\Gamma(\gamma)$ is cocompact in $Z(\gamma)$.*

Proof. Since Γ is discrete, $[\gamma]$ is closed in G. The inverse image of $[\gamma]$ by the continuous map $g \in G \to g\gamma g^{-1} \in G$ is $\Gamma \cdot Z(\gamma)$. Then $\Gamma \cdot Z(\gamma)$ is closed in G. Since $\Gamma \backslash G$ is compact, the closed subset $\Gamma \backslash \Gamma \cdot Z(\gamma) \subset \Gamma \backslash G$ is then compact.

The group $Z(\gamma)$ acts transitively on the right on $\Gamma \backslash \Gamma \cdot Z(\gamma)$. The stabilizer at $[1] \in \Gamma \backslash \Gamma \cdot Z(\gamma)$ is $\Gamma(\gamma)$. Hence $\Gamma(\gamma) \backslash Z(\gamma) \simeq \Gamma \backslash \Gamma \cdot Z(\gamma)$ is compact. The proof of our proposition is completed. □

Let $\mathrm{vol}\left(\Gamma(\gamma) \backslash X(\gamma) \right)$ be the volume of $\Gamma(\gamma) \backslash X(\gamma)$ with respect to the volume form induced by $dv_{X(\gamma)}$. Clearly, $\mathrm{vol}\left(\Gamma(\gamma) \backslash X(\gamma) \right)$ depends only on the conjugacy class $[\gamma] \in [\Gamma]$.

By the property of heat kernels on compact manifolds, the operator $\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2 \right)$ is trace class. Its trace is given by the Selberg trace formula:

Theorem 4.10. *There exist $c > 0$, $C > 0$ such that for $t > 0$, we have*

$$\sum_{[\gamma] \in [\Gamma] - \{1\}} \mathrm{vol}\left(\Gamma(\gamma) \backslash X(\gamma) \right) \left| \mathrm{Tr}^{[\gamma]} \left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2 \right) \right] \right| \leqslant C \exp\left(-\frac{c}{t} + Ct \right). \tag{4.33}$$

For $t > 0$, the following identity holds:

$$\mathrm{Tr}\left[\exp\left(-tC^{\mathfrak{g},Z,\tau,\rho}/2 \right) \right] = \sum_{[\gamma] \in [\Gamma]} \mathrm{vol}\left(\Gamma(\gamma) \backslash X(\gamma) \right) \mathrm{Tr}[\rho(\gamma)] \, \mathrm{Tr}^{[\gamma]} \left[\exp(-tC^{\mathfrak{g},X,\tau}/2) \right]. \tag{4.34}$$

Proof. Let $F \subset X$ be a fundamental domain of Z in X. By [B11, eq. (4.8.11), (4.8.15)], we have

$$\sum_{\gamma' \in [\gamma]} \int_{x \in F} \mathrm{Tr}^{E_\tau} \left[p_t^{X,\tau}(x, \gamma' x) \right] dx = \mathrm{vol}\left(\Gamma(\gamma) \backslash X(\gamma)\right) \mathrm{Tr}^{[\gamma]} \left[\exp(-tC^{\mathfrak{g},X,\tau}/2) \right]. \tag{4.35}$$

By (4.25) and (4.35), we get (4.33). The proof of (4.34) is well known (c.f. [B11, Section 4.8]). □

4.5. **A formula for** $\mathrm{Tr_s}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-tC^{\mathfrak{g},X}/2\right)\right]$. Let $\gamma = e^a k^{-1} \in G$ be semisimple such that (3.9) holds. Let $\mathfrak{t}(\gamma) \subset \mathfrak{k}(\gamma)$ be a Cartan subalgebra of $\mathfrak{k}(\gamma)$. Set

$$\mathfrak{b}(\gamma) = \{Y \in \mathfrak{p} : \mathrm{Ad}(k)Y = Y, [Y, \mathfrak{t}(\gamma)] = 0\}. \tag{4.36}$$

Then,

$$a \in \mathfrak{b}(\gamma). \tag{4.37}$$

By definition, $\dim \mathfrak{p} - \dim \mathfrak{b}(\gamma)$ is even.

Since k centralizes $\mathfrak{t}(\gamma)$, by [K86, Theorem 4.21], there is $k' \in K$ such that

$$k'\mathfrak{t}(\gamma)k'^{-1} \subset \mathfrak{t}, \qquad k'kk'^{-1} \in T. \tag{4.38}$$

Up to a conjugation on γ, we can assume directly that $\gamma = e^a k^{-1}$ with

$$\mathfrak{t}(\gamma) \subset \mathfrak{t}, \qquad k \in T. \tag{4.39}$$

By (3.18), (4.36), and (4.39), we have

$$\mathfrak{b} \subset \mathfrak{b}(\gamma). \tag{4.40}$$

Proposition 4.11. *A semisimple element $\gamma \in G$ can be conjugated into H if and only if*

$$\dim \mathfrak{b} = \dim \mathfrak{b}(\gamma). \tag{4.41}$$

Proof. If $\gamma \in H$, then $\mathfrak{t}(\gamma) = \mathfrak{t}$. By (4.36), we get $\mathfrak{b} = \mathfrak{b}(\gamma)$, which implies (4.41).

Recall that $\mathfrak{h}(\gamma) \subset \mathfrak{z}(\gamma)$ is defined as in (3.20), when G is replaced by $Z^0(\gamma)$ and $\mathfrak{t}$ is replaced by $\mathfrak{t}(\gamma)$. It is a θ-invariant Cartan subalgebra of both $\mathfrak{g}$ and $\mathfrak{z}(\gamma)$. Let $\mathfrak{h}(\gamma) = \mathfrak{h}(\gamma)_\mathfrak{p} \oplus \mathfrak{h}(\gamma)_\mathfrak{k}$ be the Cartan decomposition. Then,

$$\mathfrak{h}(\gamma)_\mathfrak{p} = \{Y \in \mathfrak{p}(\gamma) : [Y, \mathfrak{t}(\gamma)] = 0\} = \mathfrak{b}(\gamma) \cap \mathfrak{p}(\gamma), \qquad \mathfrak{h}(\gamma)_\mathfrak{k} = \mathfrak{t}(\gamma). \tag{4.42}$$

From (3.26) and (4.42), we get

$$\dim \mathfrak{b} \leqslant \dim \mathfrak{h}(\gamma)_\mathfrak{p} \leqslant \dim \mathfrak{b}(\gamma). \tag{4.43}$$

By (4.43), if $\dim \mathfrak{b} = \dim \mathfrak{b}(\gamma)$, then $\dim \mathfrak{b} = \dim \mathfrak{h}(\gamma)_\mathfrak{p}$. By Proposition 3.3, γ can be conjugated into H. The proof of our proposition is completed. □

The following Proposition extends [B11, Theorem 7.9.1].

Theorem 4.12. *Let $\gamma \in G$ be semisimple such that $\dim \mathfrak{b}(\gamma) \geqslant 2$. For $Y \in \mathfrak{k}(\gamma)$, we have*

$$\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}\left[N^{\Lambda^\cdot(\mathfrak{p}^*)} \mathrm{Ad}(k^{-1}) \exp(-i\,\mathrm{ad}(Y))\right] = 0. \tag{4.44}$$

In particular, for $t > 0$, we have

$$\mathrm{Tr_s}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-tC^{\mathfrak{g},X}/2\right)\right] = 0. \tag{4.45}$$

Proof. Since the left-hand side of (4.44) is $\mathrm{Ad}\left(K^0(\gamma)\right)$-invariant, it is enough to show (4.44) for $Y \in \mathfrak{t}(\gamma)$. If $Y \in \mathfrak{t}(\gamma)$, by [B11, eq. (7.9.1)], we have

(4.46)
$$\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*)}\left[N^{\Lambda^\cdot(\mathfrak{p}^*)}\,\mathrm{Ad}(k^{-1})\exp(-i\,\mathrm{ad}(Y))\right] = \frac{\partial}{\partial b}\Big|_{b=0}\det\left(1-e^b\,\mathrm{Ad}(k)\exp(i\,\mathrm{ad}(Y))\right)|_{\mathfrak{p}}.$$

Since $\dim\mathfrak{b}(\gamma) \geqslant 2$, by (4.46), we get (4.44) for $Y \in \mathfrak{t}(\gamma)$.

By (4.22) and (4.44), we get (4.45). The proof of our theorem is completed. □

In this way, Bismut [B11, Theorem 7.9.3] recover [MoSt91, Corollary 2.2].

Corollary 4.13. *Let F be a unitarily flat vector bundle on Z. Assume that $\dim Z$ is odd and $\delta(G) \neq 1$. Then for any $t>0$, we have*

(4.47)
$$\mathrm{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)}\exp\left(-t\Box^Z/2\right)\right] = 0.$$

In particular,

(4.48)
$$T(F) = 1.$$

Proof. Since $\dim Z$ is odd, $\delta(G)$ is odd. Since $\delta(G) \neq 1$, $\delta(G) \geqslant 3$. By (4.40), $\dim\mathfrak{b}(\gamma) \geqslant \delta(G) \geqslant 3$, so (4.47) is a consequence of (4.24), (4.34) and (4.45). □

Suppose that $\delta(G) = 1$. Up to sign, we fix an element $a_1 \in \mathfrak{b}$ such that $B(a_1,a_1) = 1$. As in Subsection 3.2, set

(4.49)
$$M = Z^{a_1,\perp,0}(e^{a_1}), \qquad K_M = K^0(e^{a_1}),$$

and

(4.50)
$$\mathfrak{m} = \mathfrak{z}^{a_1,\perp}(e^{a_1}), \qquad \mathfrak{p}_{\mathfrak{m}} = \mathfrak{p}^{a_1,\perp}(e^{a_1}), \qquad \mathfrak{k}_{\mathfrak{m}} = \mathfrak{k}(e^{a_1}).$$

As in Subsection 3.2, M is a connected reductive group with Lie algebra $\mathfrak{m}$, with maximal compact subgroup K_M, and with Cartan decomposition $\mathfrak{m} = \mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}$. Let

(4.51)
$$X_M = M/K_M$$

be the corresponding symmetric space. By definition, $T \subset M$ is a compact Cartan subgroup. Therefore $\delta(M) = 0$, and $\dim\mathfrak{p}_{\mathfrak{m}}$ is even.

Assume that $\delta(G) = 1$ and that G has noncompact center, so that $\dim\mathfrak{z}_{\mathfrak{p}} \geqslant 1$. By (3.19), we find that $a_1 \in \mathfrak{z}_{\mathfrak{p}}$, so that $Z^0(a_1) = G$. By (3.14) and (4.10), we have

(4.52)
$$G = \mathbf{R}\times M, \qquad K = K_M, \qquad X = \mathbf{R}\times X_M.$$

Let $\gamma \in G$ be a semisimple element such that $\dim\mathfrak{b}(\gamma) = 1$. By Proposition 4.11, we may assume that $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$ and $k \in T$.

Proposition 4.14. *We have*

(4.53)
$$\mathrm{Tr_s}^{[1]}\left[N^{\Lambda^\cdot(T^*X)}\exp\left(-tC^{\mathfrak{g},X}/2\right)\right] = -\frac{1}{\sqrt{2\pi t}}\left[e\left(TX_M,\nabla^{TX_M}\right)\right]^{\max}.$$

If $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$, $a \neq 0$, and $k \in T$, then

(4.54)
$$\mathrm{Tr}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)}\exp\left(-tC^{\mathfrak{g},X}/2\right)\right] = -\frac{1}{\sqrt{2\pi t}}e^{-|a|^2/2t}\left[e\left(TX^{a,\perp}(\gamma),\nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max}.$$

Proof. By (4.52), for $\gamma = e^a k^{-1}$ with $a \in \mathfrak{b}$ and $k \in T$, we have

$$\operatorname{Tr}_s^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)} \exp(-tC^{\mathfrak{g},X}/2)\right] = -\operatorname{Tr}^{[e^a]}\left[\exp(t\Delta^{\mathbf{R}}/2)\right] \operatorname{Tr}_s^{[k^{-1}]}\left[\exp(-tC^{\mathfrak{m},X_M}/2)\right], \tag{4.55}$$

where $\Delta^{\mathbf{R}}$ is the Laplace-Beltrami operator acting on $C^\infty(\mathbf{R})$.

Clearly,

$$\operatorname{Tr}^{[e^a]}\left[\exp(t\Delta^{\mathbf{R}}/2)\right] = \frac{1}{\sqrt{2\pi t}} e^{-|a|^2/2t}. \tag{4.56}$$

By [B11, Theorem 7.8.13], we have

$$\operatorname{Tr}_s^{[1]}\left[\exp(-tC^{\mathfrak{m},X_M}/2)\right] = \left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max}. \tag{4.57}$$

and

$$\operatorname{Tr}_s^{[k^{-1}]}\left[\exp(-tC^{\mathfrak{m},X_M}/2)\right] = \left[e\left(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max}. \tag{4.58}$$

By (4.55)-(4.58), we get (4.53) and (4.54), which completes the proof of our proposition. □

5. The solution to Fried conjecture

We use the notation in Sections 3 and 4. Also, we assume that $\dim \mathfrak{p}$ is odd. The purpose of this section is to introduce the Ruelle dynamical zeta function on Z and to state our main result, which contains the solution of the Fried conjecture in the case of locally symmetric spaces.

This section is organized as follows. In Subsection 5.1, we describe the closed geodesics on Z.

In Subsection 5.2, we define the dynamical zeta function and state Theorem 5.5, which is the main result of the article.

Finally, in Subsection 5.3, we establish Theorem 5.5 when G has noncompact center and $\delta(G) = 1$.

5.1. The space of closed geodesics.

By [DKoVa79, Proposition 5.15], the set of nontrivial closed geodesics on Z consists of a disjoint union of smooth connected closed submanifolds

$$\coprod_{[\gamma]\in[\Gamma]-[1]} B_{[\gamma]}. \tag{5.1}$$

Moreover, $B_{[\gamma]}$ is diffeomorphic to $\Gamma(\gamma)\backslash X(\gamma)$. All the elements of $B_{[\gamma]}$ have the same length $|a| > 0$, if γ can be conjugated to $e^a k^{-1}$ as in (3.9). Also, the geodesic flow induces a canonical locally free action of $\mathbb{S}^1$ on $B_{[\gamma]}$, so that $\mathbb{S}^1\backslash B_{[\gamma]}$ is a closed orbifold. The $\mathbb{S}^1$-action is not necessarily effective. Let

$$m_{[\gamma]} = \left|\ker\left(\mathbb{S}^1 \to \operatorname{Diff}(B_{[\gamma]})\right)\right| \in \mathbf{N}^* \tag{5.2}$$

be the generic multiplicity.

Following [Sa57], if S is a closed Riemannian orbifold with Levi-Civita connection ∇^{TS}, then $e\big(TS, \nabla^{TS}\big) \in \Omega^{\dim S}\big(S, o(TS)\big)$ is still well define, and the Euler characteristic $\chi_{\mathrm{orb}}(S) \in \mathbf{Q}$ is given by

$$\chi_{\mathrm{orb}}(S) = \int_S e\big(TS, \nabla^{TS}\big). \tag{5.3}$$

Proposition 5.1. *For $\gamma \in \Gamma - \{1\}$, the following identity holds:*

$$\frac{\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} = \frac{\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)}{|a|}\left[e\left(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max}. \tag{5.4}$$

Proof. Take $\gamma \in \Gamma - \{1\}$. We can assume that $\gamma = e^a k^{-1}$ as in (3.9) with $a \neq 0$. By (3.10) and (4.32), for $t \in \mathbf{R}$, e^{ta} commutes with elements of $\Gamma(\gamma)$. Thus, e^{ta} acts on the left on $\Gamma(\gamma)\backslash X(\gamma)$. Since $e^a = \gamma k$, $\gamma \in \Gamma(\gamma)$, $k \in K(\gamma)$ and k commutes with elements of $Z(\gamma)$, we see that e^a acts as identity on $\Gamma(\gamma)\backslash X(\gamma)$. This induces an $\mathbf{R}/\mathbf{Z} \simeq \mathbb{S}^1$ action on $\Gamma(\gamma)\backslash X(\gamma)$ which coincides with the $\mathbb{S}^1$-action on $B_{[\gamma]}$. Therefore,

$$\chi_{\mathrm{orb}}(\mathbb{S}^1\backslash B_{[\gamma]}) = \mathrm{vol}(\mathbb{S}^1\backslash B_{[\gamma]})\left[e\left(TX^{a,\perp}(\gamma), \nabla^{TX^{a,\perp}(\gamma)}\right)\right]^{\max} \tag{5.5}$$

and

$$\frac{\mathrm{vol}(\mathbb{S}^1\backslash B_{[\gamma]})}{m_{[\gamma]}} = \frac{\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)}{|a|}. \tag{5.6}$$

By (5.5) and (5.6), we get (5.4). The proof of our proposition is completed. □

Corollary 5.2. *Let $\gamma \in \Gamma - \{1\}$. If $\dim \mathfrak{b}(\gamma) \geqslant 2$, then*

$$\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right) = 0. \tag{5.7}$$

Proof. By Propositions 4.1 and 5.1, it is enough to show that

$$\delta(Z^{a,\perp,0}(\gamma)) \geqslant 1. \tag{5.8}$$

By (3.14) and (3.26), we have

$$\delta(Z^{a,\perp,0}(\gamma)) = \delta(Z^0(\gamma)) - 1 \geqslant \delta(G) - 1. \tag{5.9}$$

Recall $\dim \mathfrak{p}$ is odd, therefore $\delta(G)$ is odd. If $\delta(G) \geqslant 3$, by (5.9), we get (5.8). If $\delta(G) = 1$, then $\dim \mathfrak{b}(\gamma) \geqslant 2 > \delta(G)$. By Propositions 3.3 and 4.11, the inequality in (5.9) is strict, which implies (5.8). The proof of our corollary is completed. □

Remark 5.3. By Theorem 4.12 and Corollary 5.2, both $\mathrm{Tr_s}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)}\exp\left(-tC^{\mathfrak{g},X}/2\right)\right]$ and $\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)$ vanish when $\dim \mathfrak{b}(\gamma) \geqslant 2$.

5.2. **Statement of the main result.** Recall that $\rho : \Gamma \to \mathrm{U}(r)$ is a unitary representation of Γ and that (F, ∇^F, g^F) is the unitarily flat vector bundle on Z associated with ρ.

Definition 5.4. The Ruelle dynamical zeta function $R_\rho(\sigma)$ is said to be well defined, if the following properties hold:

(1) For $\sigma \in \mathbf{C}, \mathrm{Re}\,(\sigma) \gg 1$, the sum

$$\Xi_\rho(\sigma) = \sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}}\mathrm{Tr}[\rho(\gamma)]e^{-\sigma|a|} \tag{5.10}$$

converges to a holomorphic function.

(2) The function $R_\rho(\sigma) = \exp\left(\Xi_\rho(\sigma)\right)$ has a meromorphic extension to $\sigma \in \mathbf{C}$.

If $\delta(G) \neq 1$, by Corollary 5.2,

$$R_\rho(\sigma) \equiv 1. \tag{5.11}$$

The main result of this article is the solution of the Fried conjecture. We restate Theorem 1.1 as follows.

Theorem 5.5. *The dynamical zeta function $R_\rho(\sigma)$ is well defined. There exist explicit constants $C_\rho \in \mathbf{R}^*$ and $r_\rho \in \mathbf{Z}$ (c.f. (7.75)) such that, when $\sigma \to 0$, we have*

$$R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}\left(\sigma^{r_\rho+1}\right). \tag{5.12}$$

If $H^\cdot(Z, F) = 0$, then

$$C_\rho = 1, \qquad r_\rho = 0, \tag{5.13}$$

so that

$$R_\rho(0) = T(F)^2. \tag{5.14}$$

Proof. When $\delta(G) \neq 1$, Theorem 5.5 is a consequence of (4.48) and (5.11). When $\delta(G) = 1$ and when G has noncompact center, we will show Theorem 5.5 in Subsection 5.3. When $\delta(G) = 1$ and when G has compact center, we will show that $R_\rho(\sigma)$ is well defined such that (5.12) holds in Section 7, and we will show (5.13) in Section 8. □

5.3. **Proof of Theorem 5.5 when G has noncompact center and $\delta(G) = 1$.** We assume that $\delta(G) = 1$ and that G has noncompact center. Let us show the following refined version of Theorem 5.5.

Theorem 5.6. *There is $\sigma_0 > 0$ such that*

$$\sum_{[\gamma] \in [\Gamma] - \{1\}} \frac{\left|\chi_{\mathrm{orb}}\left(\mathbb{S}^1 \backslash B_{[\gamma]}\right)\right|}{m_{[\gamma]}} e^{-\sigma_0 |a|} < \infty. \tag{5.15}$$

The dynamical zeta function $R_\rho(\sigma)$ extends meromorphically to $\sigma \in \mathbf{C}$ such that

$$R_\rho(\sigma) = \exp\left(r \operatorname{vol}(Z) \left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max} \sigma\right) T\left(\sigma^2\right). \tag{5.16}$$

If $\chi'(Z, F) = 0$, then $R_\rho(\sigma)$ is holomorphic at $\sigma = 0$ and

$$R_\rho(0) = T(F)^2. \tag{5.17}$$

Proof. Following (2.5), for $(s, \sigma) \in \mathbf{C} \times \mathbf{R}$ such that $\operatorname{Re}(s) > m/2$ and $\sigma > 0$, put

$$\begin{aligned} \theta_\rho(s, \sigma) &= -\operatorname{Tr}\left[N^{\Lambda^\cdot(T^*Z)} \left(C^{\mathfrak{g}, Z, \rho} + \sigma\right)^{-s}\right] \\ &= -\frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr_s}\left[N^{\Lambda^\cdot(T^*Z)} \exp\left(-t(C^{\mathfrak{g}, Z, \rho} + \sigma)\right)\right] t^{s-1} dt. \end{aligned} \tag{5.18}$$

Let us show that there is $\sigma_0 > 0$ such that (5.15) holds true and that for $\sigma > \sigma_0$, we have

$$\Xi_\rho(\sigma) = \frac{\partial}{\partial s} \theta_\rho(0, \sigma^2) + r \operatorname{vol}(Z) \left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max} \sigma. \tag{5.19}$$

By (4.53), for $(s, \sigma) \in \mathbf{C} \times \mathbf{R}$ such that $\operatorname{Re}(s) > 1/2$ and $\sigma > 0$, the function

$$\theta_{\rho,1}(s, \sigma) = -\frac{r \operatorname{vol}(Z)}{\Gamma(s)} \int_0^\infty \operatorname{Tr_s}^{[1]}\left[N^{\Lambda^\cdot(T^*X)} \exp\left(-t\left(C^{\mathfrak{g}, X} + \sigma\right)\right)\right] t^{s-1} dt \tag{5.20}$$

is well defined so that

$$\theta_{\rho,1}(s, \sigma) = \frac{r \operatorname{vol}(Z)}{2\sqrt{\pi}} \left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sigma^{1/2 - s}. \tag{5.21}$$

Therefore, for $\sigma > 0$ fixed, the function $s \to \theta_{\rho,1}(s, \sigma)$ has a meromorphic extension to $s \in \mathbf{C}$ which is holomorphic at $s = 0$ so that

$$\frac{\partial}{\partial s} \theta_{\rho,1}(0, \sigma) = -r \operatorname{vol}(Z) \left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max} \sigma^{1/2}. \tag{5.22}$$

For $(s,\sigma)\in \mathbf{C}\times\mathbf{R}$ such that $\mathrm{Re}\,(s)>m/2$ and $\sigma>0$, set

(5.23) $$\theta_{\rho,2}(s,\sigma)=\theta_\rho(s,\sigma)-\theta_{\rho,1}(s,\sigma).$$

By (4.45), (4.54), (5.4), and (5.7), for $[\gamma]\in[\Gamma]-\{1\}$, we have

(5.24) $$\mathrm{vol}\left(\Gamma(\gamma)\backslash X(\gamma)\right)\mathrm{Tr}_{\mathrm{s}}^{[\gamma]}\left[N^{\Lambda^\cdot(T^*X)}\exp\left(-tC^{\mathfrak{g},X}\right)\right] = -\frac{1}{2\sqrt{\pi t}}\frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}}|a|\exp\left(-\frac{|a|^2}{4t}\right).$$

By (4.33) and (5.24), there exist $C_1>0$, $C_2>0$, and $C_3>0$ such that for $t>0$, we have

(5.25) $$\sum_{[\gamma]\in[\Gamma]-\{1\}}\frac{\left|\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)\right|}{m_{[\gamma]}}|a|\exp\left(-\frac{|a|^2}{4t}\right)\leqslant C_1\exp\left(-\frac{C_2}{t}+C_3t\right).$$

Take $\sigma_0=\sqrt{2C_3}$. Since ρ is unitary, by (4.34), (5.18), (5.23), and (5.25), for $(s,\sigma)\in \mathbf{C}\times\mathbf{R}$ such that $\mathrm{Re}\,(s)>m/2$ and $\sigma\geqslant\sigma_0$, we have

(5.26)
$$\theta_{\rho,2}\left(s,\sigma^2\right)=\frac{1}{2\sqrt{\pi}\Gamma(s)}\int_0^\infty\sum_{[\gamma]\in[\Gamma]-\{1\}}\mathrm{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}}|a|\exp\left(-\frac{|a|^2}{4t}-\sigma^2t\right)t^{s-3/2}dt.$$

Moreover, for $\sigma\geqslant\sigma_0$ fixed, the function $s\to\theta_{\rho,2}(s,\sigma^2)$ extends holomorphically to $\mathbf{C}$, so that

(5.27)
$$\frac{\partial}{\partial s}\theta_{\rho,2}(0,\sigma^2)=\frac{1}{2\sqrt{\pi}}\int_0^\infty\sum_{[\gamma]\in[\Gamma]-\{1\}}\mathrm{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}}|a|\exp\left(-\frac{|a|^2}{4t}-\sigma^2t\right)\frac{dt}{t^{3/2}}.$$

Using the formula[3] that for $B_1>0, B_2\geqslant 0$,

(5.28) $$\int_0^\infty\exp\left(-\frac{B_1}{t}-B_2t\right)\frac{dt}{t^{3/2}}=\sqrt{\frac{\pi}{B_1}}\exp\left(-2\sqrt{B_1B_2}\right),$$

by (5.25), (5.27), and by Fubini Theorem, we get (5.15). Also, for $\sigma\geqslant\sigma_0$, we have

(5.29) $$\frac{\partial}{\partial s}\theta_{\rho,2}(0,\sigma^2)=\Xi_\rho(\sigma).$$

By (5.22), (5.23), and (5.29), we get (5.19). By taking the exponentials, we get (5.16) for $\sigma\geqslant\sigma_0$. Since the right-hand side of (5.16) is meromorphic on $\sigma\in\mathbf{C}$, then R_ρ has a meromorphic extension to $\mathbf{C}$. By (2.15) and (5.16), we get (5.17). The proof of our theorem is completed. □

[3] We give a proof of (5.28) when $B_1=B_2=1$. Indeed, we have

$$\int_0^\infty\exp\left(-\frac{1}{t}-t\right)\frac{dt}{t^{3/2}}=\frac{1}{2}\int_0^\infty\exp\left(-\frac{1}{t}-t\right)\left(\frac{1}{t^{3/2}}+\frac{1}{t^{1/2}}\right)dt.$$

Using the change of variables $u=t^{1/2}-t^{-1/2}$, we get (5.28).

6. Reductive groups G with compact center and $\delta(G)=1$

In this section, we assume that $\delta(G)=1$ and that G has compact center. The purpose of this section is to introduce some geometric objects associated with G. Their proprieties are proved by algebraic arguments based on the classification of real simple Lie algebras $\mathfrak{g}$ with $\delta(\mathfrak{g})=1$. The results of this section will be used in Section 7, in order to evaluate certain orbital integrals.

This section is organized as follows. In Subsection 6.1, we introduce a splitting $\mathfrak{g}=\mathfrak{b}\oplus\mathfrak{m}\oplus\mathfrak{n}\oplus\overline{\mathfrak{n}}$, associated with the action of $\mathfrak{b}$ on $\mathfrak{g}$.

In Subsection 6.2, we construct a natural compact Hermitian symmetric space $Y_{\mathfrak{b}}$, which will be used in the calculation of orbital integrals in Subection 7.1.

In Subsection 6.3, we state one key result, which says that the action of K_M on $\mathfrak{n}$ lift to K. The purpose of the following subsections is to prove this result.

In Subsection 6.4, we state a classification result of real simple Lie algebras $\mathfrak{g}$ with $\delta(\mathfrak{g})=1$, which asserts that they just contain $\mathfrak{sl}_3(\mathbf{R})$ and $\mathfrak{so}(p,q)$ with $pq>1$ odd. This result has already been used by Moscovici-Stanton [MoSt91].

In Subsections 6.5 and 6.6, we study the Lie groups $\mathrm{SL}_3(\mathbf{R})$ and $\mathrm{SO}^0(p,q)$ with $pq>1$ odd, and the structure of the associated Lie groups M, K_M.

In Subsection 6.7, we study the connected component G_* of the identity of the isometry group of $X=G/K$. We show that G_* has a factor $\mathrm{SL}_3(\mathbf{R})$ or $\mathrm{SO}^0(p,q)$ with $pq>1$ odd.

Finally, in Subsections 6.8-6.12, we show several unproven results stated in Subsections 6.1-6.3. Most of the results are shown case by case for the group $\mathrm{SL}_3(\mathbf{R})$ and $\mathrm{SO}^0(p,q)$ with $pq>1$ odd. We prove the corresponding results for general G using a natural morphism $i_G: G\to G_*$.

6.1. A splitting of $\mathfrak{g}$.

We use the notation in (4.49)-(4.51). Let $Z(\mathfrak{b})\subset G$ be the stabilizer of $\mathfrak{b}$ in G, and let $\mathfrak{z}(\mathfrak{b})\subset\mathfrak{g}$ be its Lie algebra.

We define $\mathfrak{p}(\mathfrak{b})$, $\mathfrak{k}(\mathfrak{b})$, $\mathfrak{p}^\perp(\mathfrak{b})$, $\mathfrak{k}^\perp(\mathfrak{b})$, $\mathfrak{z}^\perp(\mathfrak{b})$ in an obvious way as in Subsection 3.2. By (4.50), we have

$$\mathfrak{p}(\mathfrak{b})=\mathfrak{b}\oplus\mathfrak{p}_{\mathfrak{m}},\qquad \mathfrak{k}(\mathfrak{b})=\mathfrak{k}_{\mathfrak{m}}. \tag{6.1}$$

Also,

$$\mathfrak{p}=\mathfrak{b}\oplus\mathfrak{p}_{\mathfrak{m}}\oplus\mathfrak{p}^\perp(\mathfrak{b}),\qquad \mathfrak{k}=\mathfrak{k}_{\mathfrak{m}}\oplus\mathfrak{k}^\perp(\mathfrak{b}). \tag{6.2}$$

Let $Z^0(\mathfrak{b})$ be the connected component of the identity in $Z(\mathfrak{b})$. By (3.14), we have

$$Z^0(\mathfrak{b})=\mathbf{R}\times M. \tag{6.3}$$

The group K_M acts trivially on $\mathfrak{b}$. It also acts on $\mathfrak{p}_{\mathfrak{m}}$, $\mathfrak{p}^\perp(\mathfrak{b})$, $\mathfrak{k}_{\mathfrak{m}}$ and $\mathfrak{k}^\perp(\mathfrak{b})$, and preserves the splittings (6.2).

Recall that we have fixed $a_1\in\mathfrak{b}$ such that $B(a_1,a_1)=1$. The choice of a_1 fixes an orientation of $\mathfrak{b}$. Let $\mathfrak{n}\subset\mathfrak{z}^\perp(\mathfrak{b})$ be the direct sum of the eigenspaces of $\mathrm{ad}(a_1)$ with the positive eigenvalues. Set $\overline{\mathfrak{n}}=\theta\mathfrak{n}$. Then $\overline{\mathfrak{n}}$ is the direct sum of the eigenspaces with negative eigenvalues, and

$$\mathfrak{z}^\perp(\mathfrak{b})=\mathfrak{n}\oplus\overline{\mathfrak{n}}. \tag{6.4}$$

Clearly, $Z^0(\mathfrak{b})$ acts on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ by adjoint action. Since K_M is fixed by θ, we have isomorphisms of representations of K_M,

$$X \in \mathfrak{n} \to X - \theta X \in \mathfrak{p}^{\perp}(\mathfrak{b}), \qquad X \in \mathfrak{n} \to X + \theta X \in \mathfrak{k}^{\perp}(\mathfrak{b}). \tag{6.5}$$

In the sequel, if $f \in \mathfrak{n}$, we denote $\overline{f} = \theta f \in \overline{\mathfrak{n}}$.

By (6.2) and (6.5), we have $\dim \mathfrak{n} = \dim \mathfrak{p} - \dim \mathfrak{p}_{\mathfrak{m}} - 1$. Since $\dim \mathfrak{p}$ is odd and since $\dim \mathfrak{p}_{\mathfrak{m}}$ is even, $\dim \mathfrak{n}$ is even. Set

$$l = \frac{1}{2} \dim \mathfrak{n}. \tag{6.6}$$

Note that since G has compact center, we have $\mathfrak{b} \not\subset \mathfrak{z}_{\mathfrak{g}}$. Therefore, $\mathfrak{z}^{\perp}(\mathfrak{b}) \neq 0$ and $l > 0$.

Remark 6.1. Let $\mathfrak{q} \subset \mathfrak{g}$ be the direct sum of the eigenspaces of $\mathrm{ad}(a_1)$ with nonnegative eigenvalues. Then $\mathfrak{q}$ is a proper parabolic subalgebra of $\mathfrak{g}$, with Langlands decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{b} \oplus \mathfrak{n}$ [K02, Section VII.7]. Let $Q \subset G$ be the corresponding parabolic subgroup of G, and let $Q = M_Q A_Q N_Q$ be the corresponding Langlands decomposition. Then M is the connected component of the identity in M_Q, and $\mathfrak{b}, \mathfrak{n}$ are the Lie algebras of A_Q and N_Q.

Proposition 6.2. *Any element of $\mathfrak{b}$ acts on $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ as a scalar, i.e., there exists $\alpha \in \mathfrak{b}^*$ such that for $a \in \mathfrak{b}$, $f \in \mathfrak{n}$, we have*

$$[a, f] = \langle \alpha, a \rangle f, \qquad [a, \overline{f}] = -\langle \alpha, a \rangle \overline{f}. \tag{6.7}$$

Proof. The proof of Proposition 6.2, based on the classification theory of real simple Lie algebras, will be given in Subsection 6.8. □

Let $a_0 \in \mathfrak{b}$ be such that

$$\langle \alpha, a_0 \rangle = 1. \tag{6.8}$$

Proposition 6.3. *We have*

$$[\mathfrak{n}, \overline{\mathfrak{n}}] \subset \mathfrak{z}(\mathfrak{b}), \qquad [\mathfrak{n}, \mathfrak{n}] = [\overline{\mathfrak{n}}, \overline{\mathfrak{n}}] = 0. \tag{6.9}$$

Also,

$$B|_{\mathfrak{n} \times \mathfrak{n} = 0}, \qquad B|_{\overline{\mathfrak{n}} \times \overline{\mathfrak{n}}} = 0. \tag{6.10}$$

Proof. By (6.7), $a \in \mathfrak{b}$ acts on $[\mathfrak{n}, \overline{\mathfrak{n}}]$, $[\mathfrak{n}, \mathfrak{n}]$, and $[\overline{\mathfrak{n}}, \overline{\mathfrak{n}}]$ by multiplication by 0, $2\langle \alpha, a \rangle$, and $-2\langle \alpha, a \rangle$. Equation (6.9) follows.

If $f_1, f_2 \in \mathfrak{n}$, by (6.7) and (6.8), we have

$$B(f_1, f_2) = B([a_0, f_1], f_2) = -B(f_1, [a_0, f_2]) = -B(f_1, f_2). \tag{6.11}$$

From (6.11), we get the first equation of (6.10). We obtain the second equation of (6.10) by the same argument. The proof of our proposition is completed. □

Remark 6.4. Clearly, we have

$$[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}(\mathfrak{b})] \subset \mathfrak{z}(\mathfrak{b}). \tag{6.12}$$

Since $\mathfrak{z}(\mathfrak{b})$ preserves B and since $\mathfrak{z}^{\perp}(\mathfrak{b})$ is the orthogonal space to $\mathfrak{z}(\mathfrak{b})$ in $\mathfrak{g}$ with respect to B, we have

$$\left[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}^{\perp}(\mathfrak{b})\right] \subset \mathfrak{z}^{\perp}(\mathfrak{b}) \tag{6.13}$$

By (6.4) and (6.9), we get

$$\left[\mathfrak{z}^{\perp}(\mathfrak{b}), \mathfrak{z}^{\perp}(\mathfrak{b})\right] \subset \mathfrak{z}(\mathfrak{b}). \tag{6.14}$$

We note the similarity between (3.2) and equations (6.12)-(6.14). In the sequel, We call such a pair $(\mathfrak{z}, \mathfrak{z}(\mathfrak{b}))$ a symmetric pair.

For $k \in K_M$, let $M(k) \subset M$ be the centralizer of k in M, and let $\mathfrak{m}(k)$ be its Lie algebra. Let $M^0(k)$ be the connected component of the identity in $M(k)$. Let $\mathfrak{p}_{\mathfrak{m}}(k)$ and $\mathfrak{k}_{\mathfrak{m}}(k)$ be the analogues of $\mathfrak{p}(\gamma)$ and $\mathfrak{k}(\gamma)$ in (3.11), so that

$$\mathfrak{m}(k) = \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}(k). \tag{6.15}$$

Since k is elliptic in M, $M^0(k)$ is reductive with maximal compact subgroup $K^0_M(k) = M^0(k) \cap K$ and with Cartan decomposition (6.15). Let

$$X_M(k) = M^0(k)/K^0_M(k) \tag{6.16}$$

be the corresponding symmetric space. Note that $\delta\big(M^0(k)\big) = 0$ and $\dim X_M(k)$ is even.

Clearly, if $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}$, $a \neq 0$, $k \in T$, then

$$\mathfrak{p}(\gamma) = \mathfrak{p}_{\mathfrak{m}}(k), \qquad \mathfrak{k}(\gamma) = \mathfrak{k}_{\mathfrak{m}}(k), \qquad Z^{a,\perp,0}(\gamma) = M(k), \qquad K^0(\gamma) = K^0_M(k). \tag{6.17}$$

Proposition 6.5. *For $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}, a \neq 0, k \in T$, we have*

$$\begin{aligned} \left|\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{z}_0^{\perp}}\right|^{1/2} &= \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}\left[\mathrm{Ad}\left(k^{-1}\right)\right] e^{(l-j)\langle\alpha, a\rangle} \\ &= \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\mathfrak{n}^*)}\left[\mathrm{Ad}\left(k^{-1}\right)\right] e^{(l-j)|\alpha||a|}. \end{aligned} \tag{6.18}$$

Proof. We claim that

$$\left|\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{z}_0^{\perp}}\right|^{1/2} = e^{l\langle\alpha,a\rangle}\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\overline{\mathfrak{n}}}. \tag{6.19}$$

Indeed, since $\dim \overline{\mathfrak{n}}$ is even, the right-hand side of (6.19) is positive. By (6.4), we have

$$\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{z}_0^{\perp}} = \det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{n}}\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\overline{\mathfrak{n}}}. \tag{6.20}$$

Since $\overline{\mathfrak{n}} = \theta\mathfrak{n}$, we have

$$\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{n}} = \det\big(1-\mathrm{Ad}(\theta\gamma)\big)|_{\overline{\mathfrak{n}}} = \det\big(\mathrm{Ad}(\theta\gamma)\big)|_{\overline{\mathfrak{n}}}\det\big(\mathrm{Ad}(\theta\gamma)^{-1}-1\big)|_{\overline{\mathfrak{n}}}. \tag{6.21}$$

Since $\dim \overline{\mathfrak{n}} = 2l$ is even, and since $(\theta\gamma)^{-1} = e^a k$ and k acts unitarily on $\mathfrak{n}$, by (6.7) and (6.21), we have

$$\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\mathfrak{n}} = e^{2l\langle\alpha,a\rangle}\det\big(1-\mathrm{Ad}(e^a k)\big)|_{\overline{\mathfrak{n}}} = e^{2l\langle\alpha,a\rangle}\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\overline{\mathfrak{n}}}. \tag{6.22}$$

By (6.20) and (6.22), we get (6.19).

Classically,

$$\det\big(1-\mathrm{Ad}(\gamma)\big)|_{\overline{\mathfrak{n}}} = \sum_{j=0}^{2l} (-1)^j \operatorname{Tr}^{\Lambda^j(\overline{\mathfrak{n}})}\left[\mathrm{Ad}\left(k^{-1}\right)\right] e^{-j\langle\alpha,a\rangle}. \tag{6.23}$$

Using the isomorphism of K_M-representations $\mathfrak{n}^* \simeq \overline{\mathfrak{n}}$, by (6.19), (6.23), we get the first equation of (6.18) and the second equation of (6.18) if a is positive in $\mathfrak{b}$. For the case a is negative in $\mathfrak{b}$, it is enough to remark that replacing γ by $\theta\gamma$ does not change the left-hand side of (6.18). The proof of our proposition is completed. □

6.2. **A compact Hermitian symmetric space** $Y_{\mathfrak{b}}$. Let $\mathfrak{u}(\mathfrak{b}) \subset \mathfrak{u}$ and $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{u}$ be the compact form of $\mathfrak{z}(\mathfrak{b})$ and $\mathfrak{m}$. Then,

$$\mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_{\mathfrak{m}}, \qquad \mathfrak{u}_{\mathfrak{m}} = \sqrt{-1}\mathfrak{p}_{\mathfrak{m}} \oplus \mathfrak{k}_{\mathfrak{m}}. \tag{6.24}$$

Since $\delta(M) = 0$, M has compact center. By [K86, Proposition 5.3], let U_M be the compact form of M.

Let $U(\mathfrak{b}) \subset U, A_0 \subset U$ be the connected subgroups of U associated with Lie algebras $\mathfrak{u}(\mathfrak{b}), \sqrt{-1}\mathfrak{b}$. By (6.24), A_0 is in the center of $U(\mathfrak{b})$, and

$$U(\mathfrak{b}) = A_0 U_M. \tag{6.25}$$

By [K02, Corollaire 4.51], the stabilizer of $\mathfrak{b}$ in U is a closed connected subgroup of U, and so it coincides with $U(\mathfrak{b})$.

Proposition 6.6. *The group A_0 is closed in U, and is diffeomorphic to a circle $\mathbb{S}^1$.*

Proof. The proof of Proposition 6.6, based on the classification theory of real simple Lie algebras, will be given in Subsection 6.8. □

Set

$$Y_{\mathfrak{b}} = U/U(\mathfrak{b}). \tag{6.26}$$

We will see that $Y_{\mathfrak{b}}$ is a compact Hermitian symmetric space.

Recall that the bilinear form $-B$ induces an $\mathrm{Ad}(U)$-invariant metric on $\mathfrak{u}$. Let $\mathfrak{u}^{\perp}(\mathfrak{b})$ be the orthogonal space to $\mathfrak{u}(\mathfrak{b})$ in $\mathfrak{u}$, such that

$$\mathfrak{u} = \mathfrak{u}(\mathfrak{b}) \oplus \mathfrak{u}^{\perp}(\mathfrak{b}). \tag{6.27}$$

Also, we have

$$\mathfrak{u}^{\perp}(\mathfrak{b}) = \sqrt{-1}\mathfrak{p}^{\perp}(\mathfrak{b}) \oplus \mathfrak{k}^{\perp}(\mathfrak{b}) \tag{6.28}$$

By (6.12)-(6.14), we have

$$[\mathfrak{u}(\mathfrak{b}), \mathfrak{u}(\mathfrak{b})] \subset \mathfrak{u}(\mathfrak{b}), \qquad [\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^{\perp}(\mathfrak{b})] \subset \mathfrak{u}^{\perp}(\mathfrak{b}), \qquad [\mathfrak{u}^{\perp}(\mathfrak{b}), \mathfrak{u}^{\perp}(\mathfrak{b})] \subset \mathfrak{u}(\mathfrak{b}). \tag{6.29}$$

Thus, $(\mathfrak{u}, \mathfrak{u}(\mathfrak{b}))$ is a symmetric pair.

Set

$$J = \sqrt{-1}\,\mathrm{ad}(a_0)|_{\mathfrak{u}^{\perp}(\mathfrak{b})} \in \mathrm{End}\left(\mathfrak{u}^{\perp}(\mathfrak{b})\right). \tag{6.30}$$

By (6.7)-(6.10), J is a $U(\mathfrak{b})$-invariant complex structure on $\mathfrak{u}^{\perp}(\mathfrak{b})$, which preserves the restriction $B|_{\mathfrak{u}^{\perp}(\mathfrak{b})}$. Moreover, $\mathfrak{n}_{\mathbf{C}} = \mathfrak{n} \otimes_{\mathbf{R}} \mathbf{C}$ and $\overline{\mathfrak{n}}_{\mathbf{C}} = \overline{\mathfrak{n}} \otimes_{\mathbf{R}} \mathbf{C}$ are the eigenspaces of J associated with the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, such that

$$\mathfrak{u}^{\perp}(\mathfrak{b}) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{n}_{\mathbf{C}} \oplus \overline{\mathfrak{n}}_{\mathbf{C}}. \tag{6.31}$$

The bilinear form $-B$ induces a Hermitian metric on $\mathfrak{n}_{\mathbf{C}}$ such that for $f_1, f_2 \in \mathfrak{n}_{\mathbf{C}}$,

$$\langle f_1, f_2 \rangle_{\mathfrak{n}_{\mathbf{C}}} = -B(f_1, \overline{f_2}). \tag{6.32}$$

Since J commutes with the action of $U(\mathfrak{b})$, $U(\mathfrak{b})$ preserves the splitting (6.31). Therefore, $U(\mathfrak{b})$ acts on $\mathfrak{n}_{\mathbf{C}}$ and $\overline{\mathfrak{n}}_{\mathbf{C}}$. In particular, $U(\mathfrak{b})$ acts on $\Lambda^{\cdot}(\overline{\mathfrak{n}}^*_{\mathbf{C}})$. If $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$ is the spinor of $(\mathfrak{u}^{\perp}(\mathfrak{b}), -B)$, by [H74], we have the isomorphism of representations of $U(\mathfrak{b})$,

$$\Lambda^{\cdot}(\overline{\mathfrak{n}}^*_{\mathbf{C}}) \simeq S^{\mathfrak{u}^{\perp}(\mathfrak{b})} \otimes \det(\mathfrak{n}_{\mathbf{C}})^{1/2}. \tag{6.33}$$

Note that M has compact center Z_M. By [K86, Proposition 5.5], M is a product of a connected semisimple Lie group and the connected component of the identity in Z_M. Since both of these two groups act trivially on $\det(\mathfrak{n})$, the same is true for M. Since the action of U_M on $\mathfrak{n}_{\mathbf{C}}$ can be obtained by the restriction of the induced action of $M_{\mathbf{C}}$ on $\mathfrak{n}_{\mathbf{C}}$, U_M acts trivially on $\det(\mathfrak{n}_{\mathbf{C}})$. By (6.33), we have the isomorphism of representations of U_M,

$$\Lambda^{\cdot}(\overline{\mathfrak{n}}_{\mathbf{C}}^{*}) \simeq S^{\mathfrak{u}^{\perp}(\mathfrak{b})}. \tag{6.34}$$

As in Subsection 4.1, let $\omega^{\mathfrak{u}}$ be the canonical left invariant 1-form on U with values in $\mathfrak{u}$, and let $\omega^{\mathfrak{u}(\mathfrak{b})}$ and $\omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}$ be the $\mathfrak{u}(\mathfrak{b})$ and $\mathfrak{u}^{\perp}(\mathfrak{b})$ components of $\omega^{\mathfrak{u}}$, so that

$$\omega^{\mathfrak{u}} = \omega^{\mathfrak{u}(\mathfrak{b})} + \omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}. \tag{6.35}$$

Then, $U \to Y_{\mathfrak{b}}$ is a $U(\mathfrak{b})$-principle bundle, equipped with a connection form $\omega^{\mathfrak{u}(\mathfrak{b})}$. Let $\Omega^{\mathfrak{u}(\mathfrak{b})}$ be the curvature form. As in (4.3), we have

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = -\frac{1}{2}\left[\omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}, \omega^{\mathfrak{u}^{\perp}(\mathfrak{b})}\right]. \tag{6.36}$$

The real tangent bundle

$$TY_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{u}^{\perp}(\mathfrak{b}) \tag{6.37}$$

is equipped with a Euclidean metric and a Euclidean connection $\nabla^{TY_{\mathfrak{b}}}$, which coincides with the Levi-Civita connection. By (6.30), J induces an almost complex structure on $TY_{\mathfrak{b}}$. Let $T^{(1,0)}Y_{\mathfrak{b}}$ and $T^{(0,1)}Y_{\mathfrak{b}}$ be the holomorphic and anti-holomorphic tangent bundles. Then

$$T^{(1,0)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{n}_{\mathbf{C}}, \qquad T^{(0,1)}Y_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \overline{\mathfrak{n}}_{\mathbf{C}}. \tag{6.38}$$

By (6.9) and (6.38), J is integrable.

The form $-B(\cdot, J\cdot)$ induces a Kähler form $\omega^{Y_{\mathfrak{b}}} \in \Omega^2(Y_{\mathfrak{b}})$ on $Y_{\mathfrak{b}}$. Clearly, $\omega^{Y_{\mathfrak{b}}}$ is closed, therefore $(Y_{\mathfrak{b}}, \omega^{Y_{\mathfrak{b}}})$ is a Kähler manifold. Let $f_1, \ldots, f_{2l} \in \mathfrak{n}$ be such that

$$-B(f_i, \overline{f}_j) = \delta_{ij}. \tag{6.39}$$

Then $f_1, \ldots, f_{2l}$ is an orthogonal basis of $\mathfrak{n}_{\mathbf{C}}$ with respect to $\langle\cdot,\cdot\rangle_{\mathfrak{n}_{\mathbf{C}}}$. Let $f^1, \ldots, f^{2l}$ be the dual base of $\mathfrak{n}_{\mathbf{C}}^*$. The Kähler form $\omega^{Y_{\mathfrak{b}}}$ on $Y_{\mathfrak{b}}$ is given by

$$\omega^{Y_{\mathfrak{b}}} = -\sum_{1\leqslant i,j\leqslant 2l} B(f_i, J\overline{f}_j) f^i \overline{f}^j = -\sqrt{-1} \sum_{1\leqslant i\leqslant 2l} f^i \overline{f}^i. \tag{6.40}$$

Let us give a more explicit description of $Y_{\mathfrak{b}}$, although this description will not be needed in the following sections.

Proposition 6.7. *The homogenous space $Y_{\mathfrak{b}}$ is an irreducible compact Hermitian symmetric space of the type AIII or BDI.*

Proof. The proof of Proposition 6.7, based on the classification theory of real simple Lie algebras, will be given in Subsection 6.10. □

Since $U_{\mathfrak{m}}$ acts on $\mathfrak{u}_{\mathfrak{m}}$ and A_0 acts trivially on $\mathfrak{u}_{\mathfrak{m}}$, by (6.25), then $U(\mathfrak{b})$ acts on $\mathfrak{u}_{\mathfrak{m}}$. Put

$$N_{\mathfrak{b}} = U \times_{U(\mathfrak{b})} \mathfrak{u}_{\mathfrak{m}}. \tag{6.41}$$

Then, $N_{\mathfrak{b}}$ is a Euclidean vector bundle on $Y_{\mathfrak{b}}$ equipped with a metric connection $\nabla^{N_{\mathfrak{b}}}$. We equip the trivial connection $\nabla^{\sqrt{-1}\mathfrak{b}}$ on the trivial line bundle $\sqrt{-1}\mathfrak{b}$ on $Y_{\mathfrak{b}}$. Since $U(\mathfrak{b})$ preserves the first splitting in (6.24), we have

$$\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}} = U \times_{U_{\mathfrak{b}}} U(\mathfrak{b}). \tag{6.42}$$

Moreover, the induced connection is given by

$$\nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}} = \nabla^{\sqrt{-1}\mathfrak{b}} \oplus \nabla^{N_{\mathfrak{b}}}. \tag{6.43}$$

By (6.27), (6.37), and (6.42), we have

$$TY_{\mathfrak{b}} \oplus \sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}} = \mathfrak{u}, \tag{6.44}$$

where $\mathfrak{u}$ stands for the corresponding trivial bundle on $Y_{\mathfrak{b}}$.

Proposition 6.8. *The following identity of closed forms holds on $Y_{\mathfrak{b}}$:*

$$\widehat{A}\left(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}}\right)\widehat{A}\left(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}}\right) = 1. \tag{6.45}$$

Proof. Proceeding as in [B11, Proposition 7.1.1], by (6.27), (6.37), and (6.42), we have

$$\widehat{A}\left(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}}\right)\widehat{A}\left(\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}}, \nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}}\right) = 1. \tag{6.46}$$

By (6.43), we have

$$\widehat{A}\left(\sqrt{-1}\mathfrak{b} \oplus N_{\mathfrak{b}}, \nabla^{\sqrt{-1}\mathfrak{b}\oplus N_{\mathfrak{b}}}\right) = \widehat{A}\left(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}}\right). \tag{6.47}$$

By (6.46) and (6.47), we get (6.45). The proof of our proposition is completed. □

Recall that the curvature form $\Omega^{\mathfrak{u}(\mathfrak{b})}$ is a 2-form on $Y_{\mathfrak{b}}$ with values in $U \times_{U(\mathfrak{b})} \mathfrak{u}(\mathfrak{b})$. Recall that $a_0 \in \mathfrak{b}$ is defined in (6.8). Let $\Omega^{\mathfrak{u}_{\mathfrak{m}}}$ be the $\mathfrak{u}_{\mathfrak{m}}$-component of $\Omega^{\mathfrak{u}(\mathfrak{b})}$. By (6.8), (6.36) and (6.40), we have

$$\Omega^{\mathfrak{u}(\mathfrak{b})} = \sqrt{-1}\frac{a_0}{|a_0|^2} \otimes \omega^{Y_{\mathfrak{b}}} + \Omega^{\mathfrak{u}_{\mathfrak{m}}}. \tag{6.48}$$

By (6.48), the curvature of $(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}})$ is given by

$$R^{N_{\mathfrak{b}}} = \mathrm{ad}\left(\Omega^{\mathfrak{u}(\mathfrak{b})}\right)\big|_{\mathfrak{u}_{\mathfrak{m}}} = \mathrm{ad}\left(\Omega^{\mathfrak{u}_{\mathfrak{m}}}\right)\big|_{\mathfrak{u}_{\mathfrak{m}}}. \tag{6.49}$$

Also, $B\left(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}\right)$ and $B\left(\Omega^{\mathfrak{u}_{\mathfrak{m}}}, \Omega^{\mathfrak{u}_{\mathfrak{m}}}\right)$ are well defined 4-forms on $Y_{\mathfrak{b}}$. We have an analogue of [B11, eq. (7.5.19)].

Proposition 6.9. *The following identities hold:*

$$B\left(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}\right) = 0, \qquad B\left(\Omega^{\mathfrak{u}_{\mathfrak{m}}}, \Omega^{\mathfrak{u}_{\mathfrak{m}}}\right) = \frac{\omega^{Y_{\mathfrak{b}},2}}{|a_0|^2}. \tag{6.50}$$

Proof. If $e_1, \cdots, e_{4l}$ is an orthogonal basis of $\mathfrak{u}^{\perp}(\mathfrak{b})$, by (6.36), we have

$$\begin{aligned} B\left(\Omega^{\mathfrak{u}(\mathfrak{b})}, \Omega^{\mathfrak{u}(\mathfrak{b})}\right) &= \frac{1}{4}\sum_{1\leqslant i,j,i',j'\leqslant 4l} B\big([e_i, e_j], [e_{i'}, e_{j'}]\big)e^i e^j e^{i'} e^{j'} \\ &= \frac{1}{4}\sum_{1\leqslant i,j,i',j'\leqslant 4l} B\big([[e_i, e_j], e_{i'}], e_{j'}\big)e^i e^j e^{i'} e^{j'}. \end{aligned} \tag{6.51}$$

Using the Jacobi identity and (6.51), we get the first equation of (6.50).

The second equation of (6.50) is a consequence of (6.48) and the first equation of (6.50). The proof of our proposition is completed. □

6.3. **Auxiliary virtual representations of K.** Let $RO(K_M)$ and $RO(K)$ be the real representation rings of K_M and K. Let $\iota : K_M \to K$ be the injection. We denote by

$$\iota^* : RO(K) \to RO(K_M) \tag{6.52}$$

the induced morphism of rings. Since K_M and K have the same maximal torus T, ι^* is injective.

Proposition 6.10. *The following identity in $RO(K_M)$ holds:*

$$\iota^*\left(\sum_{i=1}^{m}(-1)^{i-1} i\Lambda^i(\mathfrak{p}^*)\right) = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}}\sum_{j=0}^{2l}(-1)^{i+j}\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*)\otimes\Lambda^j(\mathfrak{n}^*). \tag{6.53}$$

Proof. For a representation V of K_M, we use the multiplication notation introduced by Hirzebruch. Put

$$\Lambda_y(V) = \sum_i y^i\Lambda^i(V) \tag{6.54}$$

a polynomial of y with coefficients in $RO(K_M)$. In particular,

$$\Lambda_{-1}(V) = \sum_i(-1)^i\Lambda^i(V), \qquad \Lambda'_{-1}(V) = \sum_i(-1)^{i-1} i\Lambda^i(V). \tag{6.55}$$

Denote by $\mathbf{1}$ the trivial representation. Since $\Lambda_1(\mathbf{1}) = 0$, $\Lambda'_{-1}(\mathbf{1}) = \mathbf{1}$, we get

$$\Lambda'_{-1}(V\oplus\mathbf{1}) = \Lambda_{-1}(V). \tag{6.56}$$

By (6.2), (6.5), and by the fact that K_M acts trivially on $\mathfrak{b}$, we have the isomorphism of K_M-representations

$$\mathfrak{p} \simeq \mathbf{1}\oplus\mathfrak{p}_{\mathfrak{m}}\oplus\mathfrak{n}. \tag{6.57}$$

Taking $V = \mathfrak{p}_{\mathfrak{m}}\oplus\mathfrak{n}$, by (6.56) and (6.57), we get (6.53). The proof of our proposition is completed. □

The following theorem is crucial.

Theorem 6.11. *The adjoint representations of K_M on $\mathfrak{n}$ has a unique lift in $RO(K)$.*

Proof. The injectivity of ι^* implies the uniqueness. The proof of the existence of the lifting of $\mathfrak{n}$, based on the classification theorem of real simple Lie algebras, will be given in Subsection 6.9. □

Corollary 6.12. *For $i, j \in \mathbf{N}$, the adjoint representations of K_M on $\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*)$ and $\Lambda^j(\mathfrak{n}^*)$ have unique lifts in $RO(K)$.*

Proof. As before, it is enough to show the existence of lifts. Since the representation of K_M on $\mathfrak{n}$ lifts to K, the same is true for the $\Lambda^j(\mathfrak{n}^*)$. By (6.57), this extends to the $\Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*)$. The proof of our corollary is completed. □

Denote by η_j the adjoint representation of M on $\Lambda^j(\mathfrak{n}^*)$. Recall that by (6.31), $U(\mathfrak{b})$ acts on $\mathfrak{n}_{\mathbf{C}}$. Recall also that $C^{\mathfrak{u}_{\mathfrak{m}},\eta_j} \in \operatorname{End}\left(\Lambda^j(\mathfrak{n}_{\mathbf{C}}^*)\right)$, $C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})} \in \operatorname{End}(\mathfrak{u}^\perp(\mathfrak{b}))$ are defined in (3.8).

Proposition 6.13. *For $0 \leqslant j \leqslant 2l$, $C^{\mathfrak{u}_{\mathfrak{m}},\eta_j}$ is a scalar so that*

$$C^{\mathfrak{u}_{\mathfrak{m}},\eta_j} = \frac{1}{8}\operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] + (j-l)^2|\alpha|^2. \tag{6.58}$$

Proof. Equation (6.58) was proved in [MoSt91, Lemma 2.5]. We give here a more conceptual proof.

Recall that $(\mathfrak{u}, \mathfrak{u}(\mathfrak{b}))$ is a compact symmetric pair. Let $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ be the $\mathfrak{u}^\perp(\mathfrak{b})$-spinors [B11, Section 7.2]. Let $C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}}$ be the Casimir element of $\mathfrak{u}(\mathfrak{b})$ acting on $S^{\mathfrak{u}^\perp(\mathfrak{b})}$ defined as in (3.8). By [B11, eq. (7.8.6)], $C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}}$ is a scalar such that

$$C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}} = \frac{1}{8}\operatorname{Tr}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right]. \tag{6.59}$$

Let $C^{\mathfrak{u}_\mathfrak{m},\Lambda^\cdot(\overline{\mathfrak{n}}^*_{\mathbf{C}})}$ be the Casimir element of $\mathfrak{u}_\mathfrak{m}$ acting on $\Lambda^\cdot(\overline{\mathfrak{n}}^*_{\mathbf{C}})$. By (3.7), (6.33) and (6.34), we have

$$C^{\mathfrak{u}(\mathfrak{b}),S^{\mathfrak{u}^\perp(\mathfrak{b})}} = C^{\mathfrak{u}_\mathfrak{m},\Lambda^\cdot(\overline{\mathfrak{n}}^*_{\mathbf{C}})} - \left(\operatorname{Ad}(a_1)|_{\Lambda^\cdot(\overline{\mathfrak{n}}^*_{\mathbf{C}})\otimes\det^{-1/2}(\mathfrak{n}_{\mathbf{C}})}\right)^2. \tag{6.60}$$

By (6.7), we have

$$\operatorname{Ad}(a_1)|_{\Lambda^j(\overline{\mathfrak{n}}^*_{\mathbf{C}})\otimes\det^{-1/2}(\mathfrak{n}_{\mathbf{C}})} = (j-l)|\alpha|. \tag{6.61}$$

By (6.59)-(6.61), we get (6.58). The proof of our proposition is completed. □

Let $\gamma = e^a k^{-1} \in G$ be such that (3.9) holds. Since $\Lambda^\cdot(\mathfrak{p}^*_\mathfrak{m}) \in RO(K)$, for $Y \in \mathfrak{k}(\gamma)$, $\operatorname{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*_\mathfrak{m})}\left[k^{-1}\exp(-iY)\right]$ is well defined. We have an analogue of (4.44).

Proposition 6.14. *If* $\dim \mathfrak{b}(\gamma) \geqslant 2$*, then for* $Y \in \mathfrak{k}(\gamma)$*, we have*

$$\operatorname{Tr_s}^{\Lambda^\cdot(\mathfrak{p}^*_\mathfrak{m})}\left[k^{-1}\exp(-iY)\right] = 0. \tag{6.62}$$

Proof. The proof of Proposition 6.14, based on the classification theory of real simple Lie algebras, will be given in Subsection 6.12. □

6.4. A classification of real reductive Lie algebra $\mathfrak{g}$ with $\delta(\mathfrak{g}) = 1$.

Recall that G is a real reductive group with compact center, such that $\delta(G) = 1$.

Theorem 6.15. *We have a decomposition of Lie algebras*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{6.63}$$

where

$$\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbf{R}) \text{ or } \mathfrak{so}(p,q), \tag{6.64}$$

with $pq > 1$ *odd, and* $\mathfrak{g}_2$ *is real reductive with* $\delta(\mathfrak{g}_2) = 0$.

Proof. Since G has compact center, by (3.6), $\mathfrak{z}_\mathfrak{p} = 0$. By (3.25), we have

$$\delta([\mathfrak{g},\mathfrak{g}]) = 1. \tag{6.65}$$

As in [B11, Remark 7.9.2], by the classification theory of real simple Lie algebras, we have

$$[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}_1 \oplus \mathfrak{g}'_2, \tag{6.66}$$

where

$$\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbf{R}) \text{ or } \mathfrak{so}(p,q), \tag{6.67}$$

with $pq > 1$ odd, and where $\mathfrak{g}'_2$ is semisimple with $\delta(\mathfrak{g}'_2) = 0$. Take

$$\mathfrak{g}_2 = \mathfrak{z}_\mathfrak{k} \oplus \mathfrak{g}'_2. \tag{6.68}$$

By (3.24), (6.66)-(6.68), we get (6.63). The proof of our theorem is completed. □

6.5. The group $\mathrm{SL}_3(\mathbf{R})$. In this subsection, we assume that $G = \mathrm{SL}_3(\mathbf{R})$, so that $K = \mathrm{SO}(3)$. We have

$$\begin{aligned}\mathfrak{p} &= \left\{ \begin{pmatrix} x & a_1 & a_2 \\ a_1 & y & a_3 \\ a_2 & a_3 & -x-y \end{pmatrix} : x, y, a_1, a_2, a_3 \in \mathbf{R} \right\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} : a_1, a_2, a_3 \in \mathbf{R} \right\}.\end{aligned} \tag{6.69}$$

Let

$$T = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(2) \right\} \subset K \tag{6.70}$$

be a maximal torus of K.

By (3.18), (6.69) and (6.70), we have

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -2x \end{pmatrix} : x \in \mathbf{R} \right\} \subset \mathfrak{p}. \tag{6.71}$$

By (6.71), we get

(6.72)

$$\mathfrak{p}_{\mathfrak{m}} = \left\{ \begin{pmatrix} x & a_1 & 0 \\ a_1 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} : x, a_1 \in \mathbf{R} \right\}, \quad \mathfrak{p}^{\perp}(\mathfrak{b}) = \left\{ \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ a_2 & a_3 & 0 \end{pmatrix} : a_2, a_3 \in \mathbf{R} \right\}.$$

Also,

$$\mathfrak{k}_{\mathfrak{m}} = \mathfrak{t}, \qquad K_M = T, \qquad M = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SL}_2(\mathbf{R}) \right\}. \tag{6.73}$$

By (6.71), we can orient $\mathfrak{b}$ by $x > 0$. Thus,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} : a_2, a_3 \in \mathbf{R} \right\}. \tag{6.74}$$

By (6.71) and (6.74), since for $x \in \mathbf{R}, a_2 \in \mathbf{R}, a_3 \in \mathbf{R}$,

$$\left[\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & -2x \end{pmatrix}, \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \right] = 3x \begin{pmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6.75}$$

we find that $\mathfrak{b}$ acts on $\mathfrak{n}$ as a scalar.

Denote by $\mathrm{Isom}^0(\mathrm{G/K})$ the connected component of the identity of the isometric group of $X = G/K$. Since G acts isometrically on G/K, we have the morphism of groups

$$i_G : G \to \mathrm{Isom}^0(\mathrm{G/K}). \tag{6.76}$$

Proposition 6.16. *The morphism i_G is an isomrphism, i.e.,*

$$\mathrm{SL}_3(\mathbf{R}) \simeq \mathrm{Isom}^0\big(\mathrm{SL}_3(\mathbf{R})/\,\mathrm{SO}(3)\big). \tag{6.77}$$

Proof. By [He78, Theorem V.4.1], it is enough to show that K acts on $\mathfrak{p}$ effectively. Assume that $k \in K$ acts on $\mathfrak{p}$ as the identity. Thus, k fixes the elements of $\mathfrak{b}$. As in (6.73), there is $A \in \mathrm{GL}_2(\mathbf{R})$ such that

$$k = \begin{pmatrix} A & 0 \\ 0 & \det^{-1}(A) \end{pmatrix}. \tag{6.78}$$

Since k fixes also the elements of $\mathfrak{p}^{\perp}(\mathfrak{b})$, by (6.72) and (6.78), we get $A = 1$. Therefore, $k = 1$. The proof of our proposition is completed. □

6.6. **The group** $G = \mathrm{SO}^0(p,q)$ **with** $pq > 1$ **odd.** In this subsection, we assume that $G = \mathrm{SO}^0(p,q)$, so that $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$, with $pq > 1$ odd.

In the sequel, if $l, l' \in \mathbf{N}^*$, let $\mathrm{Mat}_{l,l'}(\mathbf{R})$ be the space of real matrices of l raws and l' columns. If $L \subset \mathrm{Mat}_{l,l}(\mathbf{R})$ is a matrix group, we denote by σ_l the standard representation of L on $\mathbf{R}^l$. We have

(6.79)
$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix} : B \in \mathrm{Mat}_{p,q}(\mathbf{R}) \right\}, \qquad \mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A \in \mathfrak{so}(p), D \in \mathfrak{so}(q) \right\}.$$

Let

$$T_{p-1} = \left\{ \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_{(p-1)/2} \end{pmatrix} : A_1, \ldots, A_{(p-1)/2} \in \mathrm{SO}(2) \right\} \subset \mathrm{SO}(p-1) \tag{6.80}$$

be a maximal torus of $\mathrm{SO}(p-1)$. Then,

$$T = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & B \end{pmatrix} \in K : A \in T_{p-1}, B \in T_{q-1} \right\} \subset K \tag{6.81}$$

is a maximal torus of K.

By (3.18) and (6.81), we have

$$\begin{aligned}
\mathfrak{b} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p} : x \in \mathbf{R} \right\}, \\
\mathfrak{p}_{\mathfrak{m}} &= \left\{ \begin{pmatrix} 0 & 0 & B \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ B^t & 0 & 0 \end{pmatrix} \in \mathfrak{p} : B \in \mathrm{Mat}_{p-1,q-1}(\mathbf{R}) \right\}, \\
\mathfrak{p}^{\perp}(\mathfrak{b}) &= \left\{ \begin{pmatrix} 0 & \begin{matrix} 0 & v_1 \end{matrix} & 0 \\ \begin{matrix} 0 \\ v_1^t \end{matrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{matrix} v_2^t \\ 0 \end{matrix} \\ 0 & \begin{matrix} v_2 & 0 \end{matrix} & 0 \end{pmatrix} \in \mathfrak{p} : v_1 \in \mathbf{R}^{p-1}, v_2 \in \mathbf{R}^{q-1} \right\},
\end{aligned} \tag{6.82}$$

where v_1, v_2 are considered as column vectors. Also,

$$\mathfrak{k}_{\mathfrak{m}} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in \mathfrak{k} : A \in \mathfrak{so}(p-1), D \in \mathfrak{so}(q-1) \right\}. \tag{6.83}$$

By (6.82) and (6.83), we get

$$\begin{aligned} M &= \left\{ \begin{pmatrix} A & 0 & B \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ C & 0 & D \end{pmatrix} \in G : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}^0(p-1, q-1) \right\}, \\ K_M &= \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in K : A \in \mathrm{SO}(p-1), D \in \mathrm{SO}(q-1) \right\}. \end{aligned} \tag{6.84}$$

By (6.82), we can orient $\mathfrak{b}$ by $x > 0$. Then,

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & -v_1 & v_1 & 0 \\ v_1^t & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & v_2^t \\ v_1^t & & & v_2^t \\ 0 & v_2 & -v_2 & 0 \end{pmatrix} \in \mathfrak{g} : v_1 \in \mathbf{R}^{p-1}, v_2 \in \mathbf{R}^{q-1} \right\}. \tag{6.85}$$

By (6.82) and (6.85), since for $x \in \mathbf{R}$, $v_1 \in \mathbf{R}^{p-1}, v_2 \in \mathbf{R}^{q-1}$,

(6.86)
$$\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -v_1 & v_1 & 0 \\ v_1^t & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & v_2^t \\ v_1^t & & & v_2^t \\ 0 & v_2 & -v_2 & 0 \end{pmatrix} \right] = x \begin{pmatrix} 0 & -v_1 & v_1 & 0 \\ v_1^t & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & v_2^t \\ v_1^t & & & v_2^t \\ 0 & v_2 & -v_2 & 0 \end{pmatrix},$$

we find that $\mathfrak{b}$ acts on $\mathfrak{n}$ as a scalar.

Proposition 6.17. *We have an isomorphism of Lie groups*

$$\mathrm{SO}^0(p, q) \simeq \mathrm{Isom}^0\big(\mathrm{SO}^0(p, q)/\mathrm{SO}(p) \times \mathrm{SO}(q)\big), \tag{6.87}$$

where $pq > 1$ is odd.

Proof. As in the proof of Proposition 6.16, it enough to show that K acts effectively on $\mathfrak{p}$. The representation of $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$ on $\mathfrak{p}$ is equivalent to $\sigma_p \boxtimes \sigma_q$. Assume that $(k_1, k_2) \in \mathrm{SO}(p) \times \mathrm{SO}(q)$ acts on $\mathbf{R}^p \boxtimes \mathbf{R}^q$ as the identity. If λ is any eigenvalue of k_1 and if μ is any eigenvalue of k_2, then

$$\lambda\mu = 1. \tag{6.88}$$

By (6.88), both k_1 and k_2 are scalars. Using the fact that $\det(k_1) = \det(k_2) = 1$ and that p, q are odd, we deduce $k_1 = 1$ and $k_2 = 1$. The proof of our proposition is completed. □

6.7. **The isometry group of X.** We return to the general case, where G is only supposed to be such that $\delta(G)=1$ and have compact center.

Proposition 6.18. *The symmetric space G/K is of the noncompact type.*

Proof. Let Z_G^0 be the connected component of the identity in Z_G, and let $G_{ss} \subset G$ be the connected subgroup of G associated with the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. By [K86, Proposition 5.5], G_{ss} is closed in G, such that

(6.89) $$G = Z_G^0 G_{ss}.$$

Moreover, G_{ss} is semisimple with finite center, with maximal compact subgroup $K_{ss} = G_{ss} \cap K$. Also, the imbedding $G_{ss} \to G$ induces a diffeomorphism

(6.90) $$G_{ss}/K_{ss} \simeq G/K.$$

Therefore, X is a symmetric space of the noncompact type. □

Put

(6.91) $$G_* = \mathrm{Isom}^0(X),$$

and let $K_* \subset G_*$ be the stablizer of $p1 \in X$ fixed. Then G_* is a semisimple Lie group with trivial center, and with maximal compact subgroup K_*. We denote by $\mathfrak{g}_*$ and $\mathfrak{k}_*$ the Lie algebras of G_* and K_*. Let

(6.92) $$\mathfrak{g}_* = \mathfrak{p}_* \oplus \mathfrak{k}_*$$

be the corresponding Cartan decomposition. Clearly,

(6.93) $$G_*/K_* \simeq X.$$

The morphism $i_G : G \to G_*$ defined in (6.76) induces a morphism $i_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}_*$ of Lie algebras. By (3.4) and (6.93), $i_\mathfrak{g}$ induces an isomorphism of vector spaces

(6.94) $$\mathfrak{p} \simeq \mathfrak{p}_*.$$

By the property of $\mathfrak{k}_*$ and by (6.94), we have

(6.95) $$\mathfrak{k}_* = [\mathfrak{p}_*, \mathfrak{p}_*] = i_\mathfrak{g}[\mathfrak{p}, \mathfrak{p}] \subset i_\mathfrak{g}\mathfrak{k}.$$

Thus $i_G, i_\mathfrak{g}$ are surjective.

Proposition 6.19. *We have*

(6.96) $$G_* = G_1 \times G_2$$

where $G_1 = \mathrm{SL}_3(\mathbf{R})$ or $G_1 = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd, and where G_2 is a semisimple Lie group with trivial center with $\delta(G_2) = 0$.

Proof. By [KobN63, Theorem IV.6.2], let $X = \prod_{i=1}^{l_1} X_l$ be the de Rham decomposition of (X, g^{TX}). Then every X_i is an irreducible symmetric space of the noncompact type. By [KobN63, Theorem VI.3.5], we have

(6.97) $$G_* = \prod_{i=1}^{l_1} \mathrm{Isom}^0(X_i),$$

By Theorem 6.15, (6.77), (6.87) and (6.97), Proposition 6.19 follows. □

6.8. **Proof of Proposition 6.2.** By (6.63) and by the definition of $\mathfrak{b}$ and $\mathfrak{n}$, we have

$$\mathfrak{b}, \mathfrak{n} \subset \mathfrak{g}_1. \tag{6.98}$$

Proposition 6.2 follows from (6.75) and (6.86). □

6.9. **Proof of Theorem 6.11.**

The case $G = \mathrm{SL}_3(\mathbf{R})$. By (6.73) and (6.74), the representation of $K_M \simeq \mathrm{SO}(2)$ on $\mathfrak{n}$ is just σ_2. Note that $K = \mathrm{SO}(3)$. We have the identity in $RO(K_M)$:

$$\iota^*\left(\sigma_3 - \mathbf{1}\right) = \sigma_2, \tag{6.99}$$

which says $\mathfrak{n}$ lifts to K.

The case $G = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd. By (6.84) and (6.85), the representation of $K_M \simeq \mathrm{SO}(p-1)\times\mathrm{SO}(q-1)$ on $\mathfrak{n}$ is just $\sigma_{p-1}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\sigma_{q-1}$. Note that $K = \mathrm{SO}(p)\times\mathrm{SO}(q)$. We have the identity in $RO(K_M)$:

$$\iota^*\big((\sigma_p - \mathbf{1})\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes(\sigma_q - \mathbf{1})\big) = \sigma_{p-1}\boxtimes\mathbf{1}\oplus\mathbf{1}\boxtimes\sigma_{q-1}, \tag{6.100}$$

which says $\mathfrak{n}$ lifts to K.

The case for G_*. This is a consequence of Proposition 6.19, (6.98)-(6.100).

The general case. Recall that $i_G : G \to G_*$ is a surjective morphism of Lie groups. Therefore, the restriction $i_K : K \to K_*$ of i_G to K is surjective. By (6.94), we have the identity in $RO(K)$:

$$\mathfrak{p} = i_K^*(\mathfrak{p}_*). \tag{6.101}$$

Set

$$\mathfrak{t}_* = i_{\mathfrak{g}}(\mathfrak{t}) \subset \mathfrak{k}_*. \tag{6.102}$$

Since i_K is surjective, by [BrDi85, Theorem IV.2.9], $\mathfrak{t}_*$ is a Cartan subalgebra of $\mathfrak{k}_*$.

Let $\mathfrak{b}_* \subset \mathfrak{p}_*$ be the analogue of $\mathfrak{b}$ defined by $\mathfrak{t}_*$. Thus,

$$\dim \mathfrak{b}_* = 1, \qquad \mathfrak{b}_* = i_{\mathfrak{g}}(\mathfrak{b}). \tag{6.103}$$

We denote by $K_{*,M}, \mathfrak{p}_*^{\perp}(\mathfrak{b}_*), \mathfrak{n}_*$ the analogues of K_M, $\mathfrak{p}^{\perp}(\mathfrak{b})$, $\mathfrak{n}$. By (6.94), $i_{\mathfrak{g}}$ induces an isomorphism of vector spaces

$$\mathfrak{p}^{\perp}(\mathfrak{b}) \simeq \mathfrak{p}_*^{\perp}(\mathfrak{b}_*). \tag{6.104}$$

Let $i_{K_M} : K_M \to K_{*,M}$ be the restriction of i_G to K_M. We have the identity in $RO(K_M)$:

$$\mathfrak{p}^{\perp}(\mathfrak{b}) = i_{K_M}^*\left(\mathfrak{p}_*^{\perp}(\mathfrak{b}_*)\right). \tag{6.105}$$

Let $\iota' : K_{*,M} \to K_*$ be the embedding. Then the following diagram

$$\begin{array}{ccc} K_M & \xrightarrow{\iota} & K \\ \big\downarrow{\scriptstyle i_{K_M}} & & \big\downarrow{\scriptstyle i_K} \\ K_{*,M} & \xrightarrow{\iota'} & K_* \end{array} \tag{6.106}$$

commutes. It was proved in the previous step that there is $E \in RO(K_*)$ such that the following identity in $RO(K_{*,M})$ holds:

$$\iota'^*(E) = \mathfrak{n}_*. \tag{6.107}$$

By (6.5), (6.105)-(6.107), we have the identity in $RO(K_M)$,

$$\mathfrak{n} = \mathfrak{p}^{\perp}(\mathfrak{b}) = i^*_{K_M}\left(\mathfrak{p}^{\perp}_*(\mathfrak{b}_*)\right) = i^*_{K_M}(\mathfrak{n}_*) = i^*_{K_M}\iota'^*(E) = \iota^* i^*_K(E), \tag{6.108}$$

which completes the proof of our theorem. □

6.10. **Proof of Proposition 6.7.** If $n \in \mathbf{N}$, consider the following closed subgroups:

$$\begin{aligned} &A \in \mathrm{U}(2) \to \begin{pmatrix} A & 0 \\ 0 & \det^{-1}(A) \end{pmatrix} \in \mathrm{SU}(3), \\ &(A,B) \in \mathrm{SO}(n) \times \mathrm{SO}(2) \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathrm{SO}(n+2). \end{aligned} \tag{6.109}$$

We state Proposition 6.7 in a more exact way.

Proposition 6.20. *We have the isomorphism of symmetric spaces*

$$Y_{\mathfrak{b}} \simeq \mathrm{SU}(3)/\mathrm{U}(2) \textit{ or } \mathrm{SO}(p+q)/\,\mathrm{SO}(p+q-2) \times \mathrm{SO}(2), \tag{6.110}$$

with $pq > 1$ odd.

Proof. Let U_* and $U_*(\mathfrak{b}_*)$ be the analogues of U and $U(\mathfrak{b})$ when G and $\mathfrak{b}$ are replaced by G_* and $\mathfrak{b}_*$. It is enough to show that

$$Y_{\mathfrak{b}} \simeq U_*/U_*(\mathfrak{b}_*). \tag{6.111}$$

Indeed, by the explicit constructions given in Subsections 6.5 and 6.6, by Proposition 6.19, and by (6.109), (6.111), we get (6.110).

Let $Z_U \subset U$ be the center of U, and let Z^0_U be the connected component of the identity in Z_U. Let $U_{ss} \subset U$ be the connected subgroup of U associated to the Lie algebra $[\mathfrak{u},\mathfrak{u}] \subset \mathfrak{u}$. By [K86, Proposition 4.32], U_{ss} is compact, and $U = U_{ss}Z^0_U$.

Let $U_{ss}(\mathfrak{b})$ be the analogue of $U(\mathfrak{b})$ when U is replaced by U_{ss}. Then $U(\mathfrak{b}) = U_{ss}(\mathfrak{b})Z^0_U$, and the imbedding $U_{ss} \to U$ induces an isomorphism of homogenous spaces

$$U_{ss}/U_{ss}(\mathfrak{b}) \simeq U/U(\mathfrak{b}). \tag{6.112}$$

Let $\widetilde{U}_{ss}$ be the universal cover of U_{ss}. Since U_{ss} is semisimple, $\widetilde{U}_{ss}$ is compact. We define $\widetilde{U}_{ss}(\mathfrak{b})$ similarly. The canonical projection $\widetilde{U}_{ss} \to U_{ss}$ induces an isomorphism of homogenous spaces

$$\widetilde{U}_{ss}/\widetilde{U}_{ss}(\mathfrak{b}) \simeq U_{ss}/U_{ss}(\mathfrak{b}). \tag{6.113}$$

Similarly, since U_* is semisimple, if $\widetilde{U}_*$ is a universal cover of U_*, and if we define $\widetilde{U}_*(\mathfrak{b})$ in the same way, we have

$$\widetilde{U}_*/\widetilde{U}_*(\mathfrak{b}) \simeq U_*/U_*(\mathfrak{b}). \tag{6.114}$$

The surjective morphism of Lie algebras $i_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}_*$ induces a surjective morphism of the compact forms $i_{\mathfrak{u}} : \mathfrak{u} \to \mathfrak{u}_*$. Since $\mathfrak{u}_*$ is semisimple, the restriction of $i_{\mathfrak{u}}$ to $[\mathfrak{u},\mathfrak{u}]$ is still surjective. It lifts to a surjective morphism of simply connected Lie groups

$$\widetilde{U}_{ss} \to \widetilde{U}_*. \tag{6.115}$$

Since any connected, simply connected, semisimple compact Lie group can be written as a product of connected, simply connected, simple compact Lie groups, we can assume that there is a connected and simply connected semisimple compact Lie group U' such that $\widetilde{U}_{ss} = \widetilde{U}_* \times U'$, and that the morphism (6.115) is the canonical projection. Therefore,

$$\widetilde{U}_{ss}/\widetilde{U}_{ss}(\mathfrak{b}) \simeq \widetilde{U}_*/\widetilde{U}_*(\mathfrak{b}_*). \tag{6.116}$$

From (6.26), (6.112)-(6.114) and (6.116), we get (6.111). The proof of our proposition is completed. □

Remark 6.21. The Hermitian symmetric spaces on the right-hand side of (6.110) are irreducible and respectively of the type AIII and the type BDI in the classification of Cartan [He78, p. 518 Table V].

6.11. **Proof of Proposition 6.6.** We use the notation in Subsection 6.10. By definition, $A_0 \subset U_{ss}$. Let $\widetilde{A}_0 \subset \widetilde{U}_{ss}$ and $A_{*0} \subset U_*$ be the analogues of A_0 when U is replaced by $\widetilde{U}_{ss}$ and U_*. As in the proof of Proposition 6.7, we can show that $\widetilde{A}_0$ is a finite cover of A_0 and A_{*0}.

On the other hand, by the explicit constructions given in Subsections 6.5, 6.6, and by Proposition 6.19, A_{*0} is a circle $\mathbb{S}^1$. Therefore, both $\widetilde{A}_0, A_0$ are circles.

6.12. **Proof of Proposition 6.14.** We use the notation in Subsection 4.5. Let $\gamma \in G$ be such that $\dim \mathfrak{b}(\gamma) \geqslant 2$. As in (4.39), we assume that $\gamma = e^a k^{-1}$ is such that

$$\mathfrak{t}(\gamma) \subset \mathfrak{t}, \qquad k \in T. \tag{6.117}$$

It is enough to show (6.62) for $Y \in \mathfrak{t}(\gamma)$.

For $Y \in \mathfrak{t}(\gamma)$, since $k^{-1}\exp(-iY) \in T$ and $T \subset K^M$, we have

$$\mathrm{Tr_s}^{\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})}\left[k^{-1}\exp(-iY)\right] = \det\big(1 - \mathrm{Ad}(k)\exp(i\,\mathrm{ad}(Y))\big)|_{\mathfrak{p}_\mathfrak{m}}. \tag{6.118}$$

It is enough to show

$$\dim \mathfrak{b}(\gamma) \cap \mathfrak{p}_\mathfrak{m} \geqslant 1. \tag{6.119}$$

Note that $a \neq 0$, otherwise $\dim \mathfrak{b}(\gamma) = 1$. Let

$$a = a^1 + a^2 + a^3 \in \mathfrak{b} \oplus \mathfrak{p}_\mathfrak{m} \oplus \mathfrak{p}^\perp(\mathfrak{b}). \tag{6.120}$$

Since the decomposition $\mathfrak{b}\oplus\mathfrak{p}_\mathfrak{m}\oplus\mathfrak{p}^\perp(\mathfrak{b})$ is preserved by $\mathrm{ad}(\mathfrak{t})$ and $\mathrm{Ad}(T)$, it is also preserved by $\mathrm{ad}(\mathfrak{t}(\gamma))$ and $\mathrm{Ad}(k)$. Since $a \in \mathfrak{b}(\gamma)$, the a_i, $1 \leqslant i \leqslant 3$, all lie in $\mathfrak{b}(\gamma)$. If $a^2 \neq 0$, we get (6.119). If $a^2 = 0$ and $a^3 = 0$, we have $a \in \mathfrak{b}$. Since $a \neq 0$, then $\mathfrak{b}(\gamma) = \mathfrak{b}$, which is impossible since $\dim b(\gamma) \geqslant 2$.

It remains to consider the case

$$a^2 = 0, \qquad a^3 \neq 0. \tag{6.121}$$

We will follow the steps in the proof of Theorem 6.11.

The case $G = \mathrm{SL}_3(\mathbf{R})$. By (6.70) and (6.72), the representation of $T \simeq \mathrm{SO}(2)$ on $\mathfrak{p}^\perp(\mathfrak{b})$ is equivalent to σ_2. A nontrivial element of T never fixes a^3. Therefore,

$$k = 1. \tag{6.122}$$

Since $a \notin \mathfrak{b}$, a does not commute with all the elements of $\mathfrak{t}$. From (6.117), we get

$$\dim \mathfrak{t}(\gamma) < \dim \mathfrak{t} = 1. \tag{6.123}$$

Therefore,

$$\mathfrak{t}(\gamma) = 0. \tag{6.124}$$

By (4.36), (6.122) and (6.124), we see that $\mathfrak{b}(\gamma) = \mathfrak{p}$. Therefore,

$$\dim \mathfrak{b}(\gamma) \cap \mathfrak{p}_{\mathfrak{m}} = \dim \mathfrak{p}_{\mathfrak{m}}. \tag{6.125}$$

By (6.72) and (6.125), we get (6.119).

The case $G = \mathrm{SO}^0(p,q)$ with $pq > 1$ odd. By (6.82) and (6.84), the representations of $K_M \simeq \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ on $\mathfrak{p}_{\mathfrak{m}}$ and $\mathfrak{p}^{\perp}(\mathfrak{b})$ are equivalent to $\sigma_{p-1} \boxtimes \sigma_{q-1}$ and $\sigma_{p-1} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes \sigma_{q-1}$. We identify $a^3 \in \mathfrak{p}^{\perp}(\mathfrak{b})$ with

$$v^1 + v^2 \in \mathbf{R}^{p-1} \oplus \mathbf{R}^{q-1}. \tag{6.126}$$

Then v^1 and v^2 are fixed by $\mathrm{Ad}(k)$ and commute with $\mathfrak{t}(\gamma)$.

If $v^1 \neq 0$ and $v^2 \neq 0$, by (4.36), the nonzero element $v^1 \boxtimes v^2 \in \mathbf{R}^{p-1} \boxtimes \mathbf{R}^{q-1} \simeq \mathfrak{p}_{\mathfrak{m}}$ is in $\mathfrak{b}(\gamma)$. It implies (6.119).

If $v^2 = 0$, we will show that γ can be conjugated into H by an element of K, which implies $\dim \mathfrak{b}(\gamma) = 1$ and contradicts $\dim \mathfrak{b}(\gamma) \geqslant 2$. (The proof for the case $v^1 = 0$ is similar.) Without loss of generality, assume that there exist $s \in \mathbf{N}$ with $1 \leqslant s \leqslant (p-1)/2$ and $\lambda_s, \cdots, \lambda_{(p-1)/2} \in \mathbf{C}$ nonzero complex numbers such that,

$$v^1 = \left(0, \cdots, 0, \lambda_s, \cdots, \lambda_{(p-1)/2}\right) \in \mathbf{C}^{(p-1)/2} \simeq \mathbf{R}^{p-1}. \tag{6.127}$$

Then there exists $x \in \mathbf{R}$ such that

$$a = \begin{pmatrix} 0 & \begin{matrix} 0 & v^1 \end{matrix} & 0 \\ \begin{matrix} 0 \\ v^{1t} \end{matrix} & \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \in \mathfrak{p}. \tag{6.128}$$

By (6.81) and (6.117), there exist $A \in T_{p-1}$ and $D \in T_{q-1}$ such that

$$k = \begin{pmatrix} A & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 0 & D \end{pmatrix} \in T. \tag{6.129}$$

If we identify $T_{p-1} \simeq U(1)^{(p-1)/2}$, there are $\theta_1, \cdots, \theta_{(p-1)/2} \in \mathbf{R}$ such that

$$A = \left(e^{2i\pi\theta_1}, \cdots, e^{2i\pi\theta_{(p-1)/2}}\right). \tag{6.130}$$

Since k fixes a, by (6.127)-(6.130), for $i = s, \cdots, (p-1)/2$, we have

$$e^{2i\pi\theta_i} = 1. \tag{6.131}$$

If $W \in \mathfrak{so}(p - 2s + 2)$, set

$$l(W) = \begin{pmatrix} 0 & \overbrace{0}^{\text{p col.}} & 0 \\ 0 & W & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k}. \tag{6.132}$$

By (6.129)-(6.132), we have

$$kl(W) = l(W)k. \tag{6.133}$$

Put $w = (\lambda_s, \cdots, \lambda_{(p-1)/2}, x) \in \mathbf{C}^{(p-2s+1)/2} \oplus \mathbf{R} \simeq \mathbf{R}^{p-2s+2}$. There exists $W \in \mathfrak{so}(p-2s+2)$ such that

$$\exp(W)w = (0, \cdots, 0, |w|), \tag{6.134}$$

where $|w|$ is the Euclidean norm of w.

Put

$$k' = \exp(l(W)) \in K. \tag{6.135}$$

By (6.82), (6.133) and (6.134), we have

$$\mathrm{Ad}(k')a \in \mathfrak{b}, \qquad k'kk'^{-1} = k. \tag{6.136}$$

Thus, γ is conjugated by k' into H.

The general case. By (6.63), $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbf{R})$ or $\mathfrak{g}_1 = \mathfrak{so}(p,q)$ with $pq > 1$ odd. By (6.98) and (6.121), we have $a \in \mathfrak{g}_1$. The arguments in (6.122)-(6.126) extend directly. We only need to take care of the case $\mathfrak{g}_1 = \mathfrak{so}(p,q)$ and $a^2 = 0, v^1 \neq 0$ and $v^2 = 0$. In this case, the arguments in (6.128)-(6.134) extend to the group of isometries G_*. In particular, there is $W_* \in \mathfrak{k}_*$ such that

$$\mathrm{Ad}\left(\exp(W_*)\right) i_{\mathfrak{g}}(a) \in \mathfrak{b}_*, \qquad \mathrm{Ad}(i_G(k))W_* = W_*. \tag{6.137}$$

By (6.94), $\ker(i_{\mathfrak{g}}) \subset \mathfrak{k}$. Let $\ker(i_{\mathfrak{g}})^\perp$ be the orthogonal space of $\ker(i_{\mathfrak{g}})$ in $\mathfrak{k}$. Then,

$$\mathfrak{k} = \ker(i_{\mathfrak{g}}) \oplus \ker(i_{\mathfrak{g}})^\perp, \qquad \ker(i_{\mathfrak{g}})^\perp \simeq \mathfrak{k}_*. \tag{6.138}$$

Take $W = (0, W_*) \in \mathfrak{k}$. Put

$$k' = \exp(W) \in K. \tag{6.139}$$

By (6.94), (6.137) and (6.139), we get (6.136). Thus, γ is conjugate by k' into H. The proof of (6.62) is completed. □

7. Selberg and Ruelle zeta functions

In this section, we assume that $\delta(G) = 1$ and that G has compact center. The purpose of this section is to establish the first part of our main Theorem 5.5.

In Subsection 7.1, we introduce a class of representations η of M, so that $\eta|_{K_M}$ lifts as an element of $RO(K)$. In particular, η_j is in this class. Take $\widehat{\eta} = \Lambda^\cdot(\mathfrak{p}_{\mathfrak{m}}^*) \otimes \eta \in RO(K)$. Using the explicit formulas for orbital integrals of Theorem 4.7, we give an explicit geometric formula for $\mathrm{Tr_s}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)\right]$, whose proof is given in Subsection 7.2.

In Subsection 7.3, we introduce a Selberg zeta function $Z_{\eta,\rho}$ associated with η and ρ. Using the result in Subsection 7.1, we express $Z_{\eta,\rho}$ in terms of the regularized determinant of the resolvent of $C^{\mathfrak{g},Z,\widehat{\eta},\rho}$, and we prove that $Z_{\eta,\rho}$ is meromorphic and satisfies a functional equation.

Finally, in Subsection 7.4, we show that the dynamical zeta function $R_\rho(\sigma)$ is equal to an alternating product of $Z_{\eta_j,\rho}$, from which we deduce the first part of Theorem 5.5.

7.1. An explicit formula for $\mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right]$.

Now we introduce a class of representations of M.

Assumption 7.1. *Let η be a real finite dimensional representation of M such that*

1. *The restriction $\eta|_{K_M}$ on K_M can be lifted into $RO(K)$;*
2. *The action of the Lie algebra $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{m} \otimes_{\mathbf{R}} \mathbf{C}$ on $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$, induced by complexification, can be lifted to an action of Lie group U_M;*
3. *The Casimir element $C^{\mathfrak{u}_{\mathfrak{m}}}$ of $\mathfrak{u}_{\mathfrak{m}}$ acts on $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$ as the scalar $C^{\mathfrak{u}_{\mathfrak{m}},\eta} \in \mathbf{R}$.*

By Corollary 6.12, let $\widehat{\eta} = \widehat{\eta}^+ - \widehat{\eta}^- \in RO(K)$ be the virtual real finite dimensional representation of K on $E_{\widehat{\eta}} = E^+_{\widehat{\eta}} - E^-_{\widehat{\eta}}$ such that the following identity in $RO(K_M)$ holds:

$$E_{\widehat{\eta}}|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} (-1)^i \Lambda^i(\mathfrak{p}^*_{\mathfrak{m}}) \otimes E_\eta|_{K_M}. \tag{7.1}$$

By Corollary 6.12 and by Proposition 6.13, η_j satisfies Assumption 7.1, so that the following identity in $RO(K)$ holds

$$\sum_{i=1}^{\dim \mathfrak{p}} (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{2l} (-1)^j E_{\widehat{\eta}_j}. \tag{7.2}$$

As in Subsection 4.1, let $\mathcal{E}_{\widehat{\eta}} = G \times_K E_{\widehat{\eta}}$ be the induced virtual vector bundle on X. Let $C^{\mathfrak{g},X,\widehat{\eta}}$ be the corresponding Casimir element of G acting on $C^\infty(X, \mathcal{E}_{\widehat{\eta}})$. We will state an explicit formula for $\mathrm{Tr_s}^{[\gamma]}\left[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)\right]$.

By (6.25), the complex representation of U_M on $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$ extends to a complex representation of $U(\mathfrak{b})$ such that A_0 acts trivially. Set

$$F_{\mathfrak{b},\eta} = U \times_{U(\mathfrak{b})} \left(E_\eta \otimes_{\mathbf{R}} \mathbf{C}\right). \tag{7.3}$$

Then $F_{\mathfrak{b},\eta}$ is a complex vector bundle on $Y_{\mathfrak{b}}$. It is equipped with a connection $\nabla^{F_{\mathfrak{b},\eta}}$, induced by $\omega^{\mathfrak{u}(\mathfrak{b})}$, with curvature $R^{F_{\mathfrak{b},\eta}}$.

Remark 7.2. When $\eta = \eta_j$, the above action of $U(\mathfrak{b})$ on $\Lambda^j(\mathfrak{n}^*_{\mathbf{C}})$ is different from the adjoint action of $U(\mathfrak{b})$ on $\Lambda^j(\mathfrak{n}^*_{\mathbf{C}})$ induced by (6.31).

Recall that T is the maximal torus of both K and U_M. Put

$$c_G = (-1)^{\frac{m-1}{2}} \frac{|W(T, U_M)|}{|W(T, K)|} \frac{\mathrm{vol}(K/K_M)}{\mathrm{vol}(U_M/K_M)}. \tag{7.4}$$

Recall that $X_M = M/K_M$. By Bott's formula [Bot65, p. 175],

$$\chi(K/K_M) = \frac{|W(T,K)|}{|W(T,K_M)|}, \tag{7.5}$$

and by (4.5), (7.4), we have a more geometric expression

$$c_G = (-1)^l \frac{\left[e\left(TX_M, \nabla^{TX_M}\right)\right]^{\max}}{\left[e\left(T(K/K_M), \nabla^{T(K/K_M)}\right)\right]^{\max}}. \tag{7.6}$$

Note that $\dim \mathfrak{u}^\perp(\mathfrak{b}) = 2 \dim \mathfrak{n} = 4l$. If $\beta \in \Lambda^\cdot(\mathfrak{u}^{\perp,*}(\mathfrak{b}))$, let $[\beta]^{\max} \in \mathbf{R}$ be such that

$$\beta - [\beta]^{\max} \frac{\omega^{Y_{\mathfrak{b}},2l}}{(2l)!} \tag{7.7}$$

is of degree smaller than $4l$.

Theorem 7.3. *For $t > 0$, we have*

$$\mathrm{Tr_s}^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{c_G}{\sqrt{2\pi t}} \exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] - \frac{t}{2}C^{\mathfrak{u}_{\mathfrak{m}},\eta}\right) \left[\exp\left(-\frac{\omega^{Y_{\mathfrak{b}},2}}{8\pi^2|a_0|^2 t}\right)\widehat{A}\left(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}}\right)\mathrm{ch}\left(F_{\mathfrak{b},\eta}, \nabla^{F_{\mathfrak{b},\eta}}\right)\right]^{\max}. \tag{7.8}$$

If $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}, a \neq 0, k \in T$, for $t > 0$, we have

$$\operatorname{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{1}{\sqrt{2\pi t}}\left[e\left(TX_M(k), \nabla^{TX_M(k)}\right)\right]^{\max} \exp\left(-\frac{|a|^2}{2t} + \frac{t}{16}\operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] - \frac{t}{2}C^{\mathfrak{u}_\mathfrak{m},\eta}\right)\frac{\operatorname{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}}. \tag{7.9}$$

If $\dim \mathfrak{b}(\gamma) \geqslant 2$, for $t > 0$, we have

$$\operatorname{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = 0. \tag{7.10}$$

Proof. The proof of (7.8) and (7.9) will be given in Subsection 7.2. Equation (7.10) is a consequence of (4.22), (6.62), and (7.1). □

7.2. **The proof of Equations (7.8) and (7.9).** Let us recall some facts on Lie algebra. Let $\Delta(\mathfrak{t},\mathfrak{k}) \subset \mathfrak{t}^*$ be the real root system [BrDi85, Definition V.1.3]. We fix a set of positive roots $\Delta^+(\mathfrak{t},\mathfrak{k}) \subset \Delta(\mathfrak{t},\mathfrak{k})$. Set

$$\rho^\mathfrak{k} = \frac{1}{2}\sum_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})}\alpha. \tag{7.11}$$

By Kostant's strange formula [Ko76] or [B11, Proposition 7.5.1], we have

$$4\pi^2|\rho^\mathfrak{k}|^2 = -\frac{1}{24}\operatorname{Tr}^\mathfrak{k}\left[C^{\mathfrak{k},\mathfrak{k}}\right]. \tag{7.12}$$

Let $\pi_\mathfrak{k} : \mathfrak{t} \to \mathbf{C}$ be the polynomial function such that for $Y \in \mathfrak{t}$,

$$\pi_\mathfrak{k}(Y) = \prod_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})} 2i\pi\langle\alpha, Y\rangle. \tag{7.13}$$

Let $\sigma_\mathfrak{k} : \mathfrak{t} \to \mathbf{C}$ be the denominator in the Weyl character formula. For $Y \in \mathfrak{t}$, we have

$$\sigma_\mathfrak{k}(Y) = \prod_{\alpha\in\Delta^+(\mathfrak{t},\mathfrak{k})}\left(e^{i\pi\langle\alpha,Y\rangle} - e^{-i\pi\langle\alpha,Y\rangle}\right). \tag{7.14}$$

The Weyl group $W(T,K)$ acts isometrically on $\mathfrak{t}$. For $w \in W(T,K)$, set $\epsilon_w = \det(w)|_\mathfrak{t}$. The Weyl denominator formula asserts for $Y \in \mathfrak{t}$, we have

$$\sigma_\mathfrak{k}(Y) = \sum_{w\in W(T,K)}\epsilon_w \exp\left(2i\pi\langle\rho^\mathfrak{k}, wY\rangle\right). \tag{7.15}$$

Let $\widehat{K}$ be the set of equivalence classes of complex irreducible representations of K. There is a bijection between $\widehat{K}$ and the set of dominant and analytic integral elements in $\mathfrak{t}^*$ [BrDi85, Section VI (1.7)]. If $\lambda \in \mathfrak{t}^*$ is dominant and analytic integral, the character χ_λ of the corresponding complex irreducible representation is given by the Weyl character formula: for $Y \in \mathfrak{t}$,

$$\sigma_\mathfrak{k}(Y)\chi_\lambda\left(\exp(Y)\right) = \sum_{w\in W(T,K)}\epsilon_w \exp\left(2i\pi\langle\rho^\mathfrak{k}+\lambda, wY\rangle\right). \tag{7.16}$$

Let us recall the Weyl integral formula for Lie algebras. Let $dv_{K/T}$ be the Riemannian volume on K/T induced by $-B$, and let dY be the Lebesgue measure on $\mathfrak{k}$ or $\mathfrak{t}$ induced by $-B$. By [K86, Lemma 11.4], if $f \in C_c(\mathfrak{k})$, we have

$$\int_{Y\in\mathfrak{k}} f(Y)dY = \frac{1}{|W(T,K)|}\int_{Y\in\mathfrak{t}}|\pi_\mathfrak{k}(Y)|^2\left(\int_{k\in K/T} f\left(\operatorname{Ad}(k)Y\right)dv_{K/T}\right)dY. \tag{7.17}$$

Clearly, the formula (7.17) extends to $L^1(\mathfrak{k})$.

Proof of (7.8). By (3.3), (4.22) and (7.17), we have

$$\operatorname{Tr_s}^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{1}{(2\pi t)^{(m+n)/2}}\exp\left(\frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)$$
$$\frac{\operatorname{vol}(K/T)}{|W(T,K)|}\int_{Y\in\mathfrak{t}}|\pi_{\mathfrak{k}}(Y)|^2 J_1(Y)\operatorname{Tr_s}^{E_{\widehat{\eta}}}\left[\exp(-i\widehat{\eta}(Y))\right]\exp(-|Y|^2/2t)dY. \tag{7.18}$$

As $\delta(M)=0$, $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{u}_{\mathfrak{m}}$. We will use (7.17) again to write the integral on the second line of (7.18) as an integral over $\mathfrak{u}_{\mathfrak{m}}$.

By (6.5), we have the isomorphism of representations of K_M,

$$\mathfrak{p}^{\perp}(\mathfrak{b})\simeq\mathfrak{k}^{\perp}(\mathfrak{b}). \tag{7.19}$$

By (4.21) and (7.19), for $Y\in\mathfrak{t}$, we have

$$J_1(Y)=\frac{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{p}_{\mathfrak{m}}})}{\widehat{A}(i\operatorname{ad}(Y)|_{\mathfrak{k}_{\mathfrak{m}}})}. \tag{7.20}$$

By (7.1), for $Y\in\mathfrak{t}$, we have

$$\operatorname{Tr_s}^{E_{\widehat{\eta}}}\left[\exp(-i\widehat{\eta}(Y))\right]=\det\big(1-\exp(i\operatorname{ad}(Y))\big)|_{\mathfrak{p}_{\mathfrak{m}}}\operatorname{Tr}^{E_{\eta}}\left[\exp(-i\eta(Y))\right]. \tag{7.21}$$

By (7.13), (7.20) and (7.21), for $Y\in\mathfrak{t}$, we have

$$\frac{|\pi_{\mathfrak{k}}(Y)|^2}{|\pi_{\mathfrak{u}_{\mathfrak{m}}}(Y)|^2}J_1(Y)\operatorname{Tr_s}^{E_{\widehat{\eta}}}\left[\exp\left(-i\widehat{\eta}(Y)\right)\right]$$
$$=(-1)^{\frac{\dim\mathfrak{p}_{\mathfrak{m}}}{2}}\det\big(\operatorname{ad}(Y)\big)|_{\mathfrak{k}^{\perp}(\mathfrak{b})}\widehat{A}^{-1}\big(i\operatorname{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}}\big)\operatorname{Tr}^{E_{\eta}}\left[\exp\left(-i\eta(Y)\right)\right]. \tag{7.22}$$

Using (6.5), for $Y\in\mathfrak{t}$, we have

$$\det\big(\operatorname{ad}(Y)\big)|_{\mathfrak{k}^{\perp}(\mathfrak{b})}=\det\big(\operatorname{ad}(Y)\big)|_{\mathfrak{n}_{\mathbf{C}}}. \tag{7.23}$$

By the second condition of Assumption 7.1 and by (7.23), the function on the right-hand side of (7.22) extends naturally to an $\operatorname{Ad}(U_M)$-invariant function defined on $\mathfrak{u}_{\mathfrak{m}}$. By (7.4), (7.17), (7.18), (7.22) and (7.23), we have

$$\operatorname{Tr_s}^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{(-1)^l c_G}{(2\pi t)^{(m+n)/2}}\exp\left(\frac{t}{16}\operatorname{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\operatorname{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)$$
$$\int_{Y\in\mathfrak{u}_{\mathfrak{m}}}\det\big(\operatorname{ad}(Y)\big)|_{\mathfrak{n}_{\mathbf{C}}}\widehat{A}^{-1}\big(i\operatorname{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}}\big)\operatorname{Tr}^{E_{\eta}}\left[\exp(-i\eta(Y))\right]\exp\left(-|Y|^2/2t\right)dY. \tag{7.24}$$

It remains to evaluate the integral on the second line of (7.24). We use the method in [B11, Section 7.5]. For $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$|Y|^2=-B(Y,Y). \tag{7.25}$$

By (6.32), (6.36) and (6.48), for $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$B\left(Y,\Omega^{\mathfrak{u}_{\mathfrak{m}}}\right)=-\sum_{1\leqslant i,j\leqslant 2l}B\left(\operatorname{ad}(Y)f_i,\overline{f}_j\right)f^i\wedge\overline{f}^j=\sum_{1\leqslant i,j\leqslant 2l}\langle\operatorname{ad}(Y)f_i,f_j\rangle_{\mathfrak{n}_{\mathbf{C}}}f^i\wedge\overline{f}^j. \tag{7.26}$$

By (6.40), (7.7) and (7.26), for $Y\in\mathfrak{u}_{\mathfrak{m}}$, we have

$$\frac{\det(\operatorname{ad}(Y))|_{\mathfrak{n}_{\mathbf{C}}}}{(2\pi t)^{2l}}=(-1)^l\left[\exp\left(\frac{1}{t}B\left(Y,\frac{\Omega^{\mathfrak{u}_{\mathfrak{m}}}}{2\pi}\right)\right)\right]^{\max}. \tag{7.27}$$

As $\dim \mathfrak{u}_\mathfrak{m} = \dim \mathfrak{m} = m+n-2l-1$, from (7.24) and (7.27), we get

$$\mathrm{Tr}_s^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{c_G}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right) \exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_\mathfrak{m}}\right)\left\{\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}}\right)\mathrm{Tr}^{E_\eta}\left[\exp(-i\eta(Y))\right]\exp\left(\frac{1}{t}B\left(Y,\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}\right)\right)\right\}^{\max}\Bigg|_{Y=0}. \tag{7.28}$$

Using

$$B\left(Y,\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}\right)+\frac{1}{2}B(Y,Y) = \frac{1}{2}B\left(Y+\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}, Y+\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}\right)-\frac{1}{2}B\left(\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi},\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}\right), \tag{7.29}$$

by (6.50) and (7.28), we have

$$\mathrm{Tr}_s^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{c_G}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right) \left\{\exp\left(-\frac{\omega^{Y_\mathfrak{b},2}}{8\pi^2|a_0|^2t}\right)\exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_\mathfrak{m}}\right)\left(\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}}\right)\mathrm{Tr}^{E_\eta}\left[\exp(-i\eta(Y))\right]\right)\right\}^{\max}\Bigg|_{Y=-\frac{\Omega^{\mathfrak{u}_\mathfrak{m}}}{2\pi}}. \tag{7.30}$$

We claim that the $\mathrm{Ad}\left(U_M\right)$-invariant function

$$Y\in\mathfrak{u}_\mathfrak{m}\to\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}}\right)\mathrm{Tr}^{E_\eta}\left[\exp(-i\eta(Y))\right] \tag{7.31}$$

is an eigenfunction of $\Delta^{\mathfrak{u}_\mathfrak{m}}$ with eigenvalue

$$-C^{\mathfrak{u}_\mathfrak{m},\eta}-\frac{1}{24}\mathrm{Tr}^{\mathfrak{u}_\mathfrak{m}}\left[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}\right]. \tag{7.32}$$

Indeed, if f is an $\mathrm{Ad}(U_M)$-invariant function on $\mathfrak{u}_\mathfrak{m}$, when restricted to $\mathfrak{t}$, it is well known, for example [B11, eq. (7.5.22)], that

$$\Delta^{\mathfrak{u}_\mathfrak{m}}f=\frac{1}{\pi_{\mathfrak{u}_\mathfrak{m}}}\Delta^{\mathfrak{t}}\pi_{\mathfrak{u}_\mathfrak{m}}f. \tag{7.33}$$

Therefore, it is enough to show that the function

$$Y\in\mathfrak{t}\to\pi_{\mathfrak{u}_\mathfrak{m}}(Y)\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)\right)|_{\mathfrak{u}_\mathfrak{m}}\mathrm{Tr}^{E_\eta}\left[\exp(-i\eta(Y))\right] \tag{7.34}$$

is an eigenfunction of $\Delta^{\mathfrak{t}}$ with eigenvalue (7.32). For $Y\in\mathfrak{t}$, we have

$$\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)\right)|_{\mathfrak{u}_\mathfrak{m}}=\frac{\sigma_{\mathfrak{u}_\mathfrak{m}}(iY)}{\pi_{\mathfrak{u}_\mathfrak{m}}(iY)} \tag{7.35}$$

By (7.35), for $Y\in\mathfrak{t}$, we have

$$\pi_{\mathfrak{u}_\mathfrak{m}}(Y)\widehat{A}^{-1}\left(i\,\mathrm{ad}(Y)\right)|_{\mathfrak{u}_\mathfrak{m}}=i^{|\Delta^+(\mathfrak{t},\mathfrak{u}_\mathfrak{m})|}\sigma_{\mathfrak{u}_\mathfrak{m}}(-iY). \tag{7.36}$$

If $E_\eta\otimes_{\mathbf{R}}\mathbf{C}$ is an irreducible representation of U_M with the highest weight $\lambda\in\mathfrak{t}^*$, by the Weyl character formula (7.16), we have

$$\sigma_{\mathfrak{u}_\mathfrak{m}}(-iY)\,\mathrm{Tr_s}^{E_\eta}\left[\exp(-i\eta(Y))\right]=\sum_{w\in W(T,U_M)}\epsilon_w\exp(2\pi\langle\rho^{\mathfrak{u}_\mathfrak{m}}+\lambda,wY\rangle). \tag{7.37}$$

By (7.36) and (7.37), the function (7.34) is an eigenfunction of $\Delta^{\mathfrak{t}}$ with eigenvalue

$$4\pi^2|\rho^{\mathfrak{u}_\mathfrak{m}}+\lambda|^2. \tag{7.38}$$

By Assumption 7.1, the Casimir of $\mathfrak{u}_\mathfrak{m}$ acts as the scalar $C^{\mathfrak{u}_\mathfrak{m},\eta}$. Therefore,

$$-C^{\mathfrak{u}_\mathfrak{m},\eta}=4\pi^2\left(|\rho^{\mathfrak{u}_\mathfrak{m}}+\lambda|^2-|\rho^{\mathfrak{u}_\mathfrak{m}}|^2\right). \tag{7.39}$$

By (7.12) and (7.39), the eigenvalue (7.38) is equal to (7.32). If $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$ is not irreducible, it is enough to decompose $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$ as a sum of irreducible representations of U_M.

Since the function (7.34) and its derivations of any orders satisfy estimations similar to (4.20), by (6.49) and (7.30), we get

$$\begin{aligned}(7.40)\quad &\mathrm{Tr_s}^{[1]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right]\\ &= \frac{c_G}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right] - \frac{t}{48}\mathrm{Tr}^{\mathfrak{u}_\mathfrak{m}}\left[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}\right] - \frac{t}{2}C^{\mathfrak{u}_\mathfrak{m},\eta}\right)\\ &\qquad\left\{\exp\left(-\frac{\omega^{Y_\mathfrak{b},2}}{8\pi^2|a_0|^2t}\right)\widehat{A}^{-1}\left(\frac{R^{N_\mathfrak{b}}}{2i\pi}\right)\mathrm{Tr}^{E_\eta}\left[\exp\left(-\frac{R^{F_\mathfrak{b},\eta}}{2i\pi}\right)\right]\right\}^{\max}.\end{aligned}$$

Since $\widehat{A}$ is an even function, by (2.3), we have

$$(7.41)\qquad \widehat{A}\left(\frac{R^{N_\mathfrak{b}}}{2i\pi}\right) = \widehat{A}\left(N_\mathfrak{b}, \nabla^{N_\mathfrak{b}}\right).$$

We claim that

$$(7.42)\qquad \mathrm{Tr}^{\mathfrak{p}}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \frac{1}{3}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right] - \frac{1}{3}\mathrm{Tr}^{\mathfrak{u}_\mathfrak{m}}\left[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}\right] = \mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right].$$

Indeed, by [B11, Proposition 2.6.1], we have

$$(7.43)\qquad \begin{aligned}\mathrm{Tr}^{\mathfrak{p}}\left[C^{\mathfrak{k},\mathfrak{p}}\right] + \frac{1}{3}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right] &= \frac{1}{3}\mathrm{Tr}^{\mathfrak{u}}\left[C^{\mathfrak{u},\mathfrak{u}}\right],\\ \mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right] + \frac{1}{3}\mathrm{Tr}^{\mathfrak{u}(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})}\right] &= \frac{1}{3}\mathrm{Tr}^{\mathfrak{u}}\left[C^{\mathfrak{u},\mathfrak{u}}\right].\end{aligned}$$

By (6.24), it is trivial that

$$(7.44)\qquad \mathrm{Tr}^{\mathfrak{u}(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}(\mathfrak{b})}\right] = \mathrm{Tr}^{\mathfrak{u}_\mathfrak{m}}\left[C^{\mathfrak{u}_\mathfrak{m},\mathfrak{u}_\mathfrak{m}}\right].$$

From (7.43) and (7.44), we get (7.42)

By (2.4), (6.45), (7.40)-(7.42), we get (7.8). □

Let $U_M(k)$ be the centralizer of k in U_M, and let $\mathfrak{u}_\mathfrak{m}(k)$ be its Lie algebra. Then

$$(7.45)\qquad \mathfrak{u}_\mathfrak{m}(k) = \sqrt{-1}\mathfrak{p}_\mathfrak{m}(k) \oplus \mathfrak{k}_\mathfrak{m}(k).$$

Let $U_M^0(k)$ be the connected component of the identity in $U_M(k)$. Clearly, $U_M^0(k)$ is the compact form of $M^0(k)$.

Proof of (7.9). Since $\gamma \in H$, $\mathfrak{t} \subset \mathfrak{k}(\gamma)$ is a Cartan subalgebra of $\mathfrak{k}(\gamma)$. By (4.22), (6.17) and (7.17), we have

$$\begin{aligned}(7.46)\quad &\mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right] = \frac{1}{(2\pi t)^{\dim\mathfrak{z}(\gamma)/2}}\exp\left(-\frac{|a|^2}{2t} + \frac{t}{16}\mathrm{Tr}^{\mathfrak{p}}[C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48}\mathrm{Tr}^{\mathfrak{k}}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)\\ &\frac{\mathrm{vol}(K_M^0(k)/T)}{|W(T,K_M^0(k))|}\int_{Y\in\mathfrak{t}}|\pi_{\mathfrak{k}_\mathfrak{m}(k)}(Y)|^2 J_\gamma(Y)\,\mathrm{Tr_s}^{E_{\widehat{\eta}}}\left[\widehat{\eta}\left(k^{-1}\right)\exp(-i\widehat{\eta}(Y))\right]\exp\left(-|Y|^2/2t\right)dY.\end{aligned}$$

Since $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{u}_\mathfrak{m}(k)$, as in the proof of (7.8), we will write the integral on the second line of (7.46) as an integral over $\mathfrak{u}_\mathfrak{m}(k)$.

As $k \in T$ and $T \subset K_M$, by (7.1), for $Y \in \mathfrak{t}$, we have

$$\text{(7.47)}\quad \operatorname{Tr_s}^{E_{\widehat{\eta}}}\left[\widehat{\eta}\left(k^{-1}\right)\exp(-i\widehat{\eta}(Y))\right] = \det\left(1-\operatorname{Ad}(k)\exp(i\operatorname{ad}(Y))\right)|_{\mathfrak{p}_\mathfrak{m}} \operatorname{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\exp(-i\eta(Y))\right].$$

By (4.19), (7.13) and (7.47), for $Y \in \mathfrak{t}$, we have

$$\text{(7.48)}\quad \frac{|\pi_{\mathfrak{k}_\mathfrak{m}(k)}(Y)|^2}{|\pi_{\mathfrak{u}_\mathfrak{m}(k)}(Y)|^2} J_\gamma(Y) \operatorname{Tr_s}^{E_{\widehat{\eta}}}\left[\widehat{\eta}\left(k^{-1}\right)\exp(-i\widehat{\eta}(Y))\right] = \frac{(-1)^{\frac{\dim \mathfrak{p}_\mathfrak{m}(k)}{2}}}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}}$$
$$\widehat{A}^{-1}\left(i\operatorname{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}(k)}\right)\left[\frac{\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}}\right]^{1/2} \operatorname{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\exp(-i\eta(Y))\right].$$

Let $\mathfrak{u}_\mathfrak{m}^\perp(k)$ be the orthogonal space to $\mathfrak{u}_\mathfrak{m}(k)$ in $\mathfrak{u}_\mathfrak{m}$. Then

$$\text{(7.49)}\quad \mathfrak{u}_\mathfrak{m}^\perp(k) = \sqrt{-1}\mathfrak{p}_0^\perp(\gamma) \oplus \mathfrak{k}_0^\perp(\gamma).$$

By (7.49), for $Y \in \mathfrak{t}$, we have

$$\text{(7.50)}\quad \frac{\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}}{\det\left(1-\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{z}_0^\perp(\gamma)}} = \frac{\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}{\det\left(1-\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}.$$

By Assumption 7.1 and (7.50), the right-hand side of (7.48) extends naturally to an $\operatorname{Ad}(U_M^0(k))$-invariant function defined on $\mathfrak{u}_\mathfrak{m}(k)$. By (4.5), (7.17), (7.46) and (7.48), we have

$$\text{(7.51)}\quad \operatorname{Tr_s}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\widehat{\eta}}/2\right)\right]$$
$$= \frac{1}{\sqrt{2\pi t}} \frac{\left[e\left(TX_M(k),\nabla^{TX_M(k)}\right)\right]^{\max}}{\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \exp\left(-\frac{|a|^2}{2t}+\frac{t}{16}\operatorname{Tr}^\mathfrak{p}[C^{\mathfrak{k},\mathfrak{p}}]+\frac{t}{48}\operatorname{Tr}^\mathfrak{k}\left[C^{\mathfrak{k},\mathfrak{k}}\right]\right)$$
$$\exp\left(\frac{t}{2}\Delta^{\mathfrak{u}_\mathfrak{m}(k)}\right)\left\{\widehat{A}^{-1}\left(i\operatorname{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}(k)}\right)\left[\frac{\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}{\det\left(1-\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}\right]^{1/2}\right.$$
$$\left.\operatorname{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\exp(-i\eta(Y))\right]\right\}\Big|_{Y=0}.$$

As before, we claim that the function

$$\text{(7.52)}\quad Y \in \mathfrak{u}_\mathfrak{m}(k) \to \widehat{A}^{-1}\left(i\operatorname{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}(k)}\right)\left[\frac{\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}{\det\left(1-\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}}\right]^{1/2}$$
$$\operatorname{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\exp(-i\eta(Y))\right]$$

is an eigenfunction of $\Delta^{\mathfrak{u}_\mathfrak{m}(k)}$ with eigenvalue (7.32). Indeed, it is enough to remark that, as in (7.37), up to a sign, if $k=\exp(\theta_1)$ for some $\theta_1 \in \mathfrak{t}$, we have

$$\text{(7.53)}\quad \pi_{\mathfrak{u}_\mathfrak{m}(k)}(Y)\widehat{A}^{-1}\left(i\operatorname{ad}(Y)|_{\mathfrak{u}_\mathfrak{m}(k)}\right)\left[\det\left(1-\exp(-i\operatorname{ad}(Y))\operatorname{Ad}(k^{-1})\right)|_{\mathfrak{u}_\mathfrak{m}^\perp(k)}\right]^{1/2} = \pm i^{|\Delta^+(\mathfrak{t},\mathfrak{u}_\mathfrak{m})|}\sigma_\mathfrak{u}(-iY-\theta_1).$$

Also, if $E_\eta \otimes_{\mathbf{R}} \mathbf{C}$ is an irreducible representation of U_M with the highest weight $\lambda \in \mathfrak{t}^*$,

(7.54)
$$\sigma_{\mathfrak{u}_\mathfrak{m}}(-iY-\theta_1)\operatorname{Tr}_{\mathrm{s}}^{E_\eta}\left[\eta\left(k^{-1}\right)\exp(-i\eta(Y))\right] = \sum_{w\in W(T,U_M)} \epsilon_w \exp(2\pi\langle \rho_{\mathfrak{u}_\mathfrak{m}}+\lambda, w(Y-i\theta_1)\rangle).$$

Proceeding as in the proof of (7.8), we get (7.9). □

7.3. **Selberg zeta functions.** Recall that $\rho:\Gamma\to \mathrm{U}(r)$ is a unitary representation of Γ and that (F,∇^F,g^F) is the unitarily flat vector bundle on Z associated with ρ.

Definition 7.4. For $\sigma\in\mathbf{C}$, we define a formal sum

$$\Xi_{\eta,\rho}(\sigma) = -\sum_{[\gamma]\in[\Gamma]-\{1\}} \operatorname{Tr}[\rho(\gamma)]\frac{\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)}{m_{[\gamma]}}\frac{\operatorname{Tr}^{E_\eta}\left[\eta(k^{-1})\right]}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}}e^{-\sigma|a|} \tag{7.55}$$

and a formal Selberg zeta function

$$Z_{\eta,\rho}(\sigma) = \exp\left(\Xi_{\eta,\rho}(\sigma)\right). \tag{7.56}$$

The formal Selberg zeta function is said to be well defined if the same conditions as in Definition 5.4 hold.

Remark 7.5. When $G=\mathrm{SO}^0(p,1)$ with $p\geqslant 3$ odd, up to a shift on σ, $Z_{\eta,\rho}$ coincides with Selberg zeta function in [F86, Section 3].

Recall that the Casimir operator $C^{\mathfrak{g},Z,\widehat{\eta},\rho}$ acting on $C^\infty(Z,\mathcal{F}_{\widehat{\eta}}\otimes_{\mathbf{C}} F)$ is a formally self-adjoint second order elliptic operator, which is bounded from below. For $\lambda\in\mathbf{C}$, set

$$m_{\eta,\rho}(\lambda) = \dim_{\mathbf{C}}\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^+,\rho}-\lambda\right) - \dim_{\mathbf{C}}\ker\left(C^{\mathfrak{g},Z,\widehat{\eta}^-,\rho}-\lambda\right). \tag{7.57}$$

Write

$$r_{\eta,\rho} = m_{\eta,\rho}(0). \tag{7.58}$$

As in Subsection 2.2, for $\sigma\in\mathbf{R}$ and $\sigma\gg 1$, set

$$\det\nolimits_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho}+\sigma\right) = \frac{\det(C^{\mathfrak{g},Z,\widehat{\eta}^+,\rho}+\sigma)}{\det(C^{\mathfrak{g},Z,\widehat{\eta}^-,\rho}+\sigma)}. \tag{7.59}$$

Then, $\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho}+\sigma\right)$ extends meromorphically to $\sigma\in\mathbf{C}$. Its zeros and poles belong to the set $\{-\lambda:\lambda\in \mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})\}$. If $\lambda\in\mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})$, the order of the zero at $\sigma=-\lambda$ is $m_{\eta,\rho}(\lambda)$.

Set

$$\sigma_\eta = \frac{1}{8}\operatorname{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right]-C^{\mathfrak{u}_\mathfrak{m},\eta}. \tag{7.60}$$

Set

(7.61)
$$P_\eta(\sigma) = c_G\sum_{j=0}^{l}(-1)^j\frac{\Gamma(-j-\frac{1}{2})}{j!(4\pi)^{2j+\frac{1}{2}}|a_0|^{2j}}\left[\omega^{Y_\mathfrak{b},2j}\widehat{A}\left(TY_\mathfrak{b},\nabla^{TY_\mathfrak{b}}\right)\mathrm{ch}\left(\mathcal{F}_{\mathfrak{b},\eta},\nabla^{\mathcal{F}_{\mathfrak{b},\eta}}\right)\right]^{\max}\sigma^{2j+1}.$$

Then $P_\eta(\sigma)$ is an odd polynomial function of σ. As the notation indicates, σ_η and $P_\eta(\sigma)$ do not depend on Γ or ρ.

Theorem 7.6. *There is $\sigma_0 > 0$ such that*

$$\sum_{[\gamma]\in[\Gamma]-\{1\}} \frac{\left|\chi_{\mathrm{orb}}\left(\mathbb{S}^1\backslash B_{[\gamma]}\right)\right|}{m_{[\gamma]}} \frac{1}{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} e^{-\sigma_0|a|} < \infty. \tag{7.62}$$

The Selberg zeta function $Z_{\eta,\rho}(\sigma)$ has a meromorphic extension to $\sigma \in \mathbf{C}$ such that the following identity of meromorphic functions on $\mathbf{C}$ holds:

$$Z_{\eta,\rho}(\sigma) = \det{}_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho} + \sigma_\eta + \sigma^2\right) \exp\left(r \operatorname{vol}(Z) P_\eta(\sigma)\right). \tag{7.63}$$

The zeros and poles of $Z_{\eta,\rho}(\sigma)$ belong to the set $\{\pm i\sqrt{\lambda+\sigma_\eta} : \lambda \in \mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})\}$. If $\lambda \in \mathrm{Sp}(C^{\mathfrak{g},Z,\widehat{\eta},\rho})$ and $\lambda \neq -\sigma_\eta$, the order of zero at $\sigma = \pm i\sqrt{\lambda+\sigma_\eta}$ is $m_{\eta,\rho}(\lambda)$. The order of zero at $\sigma = 0$ is $2m_{\eta,\rho}(-\sigma_\eta)$. Also,

$$Z_{\eta,\rho}(\sigma) = Z_{\eta,\rho}(-\sigma) \exp\left(2r \operatorname{vol}(Z) P_\eta(\sigma)\right). \tag{7.64}$$

Proof. Proceeding as in the proof of Theorem 5.6, by Proposition 5.1, Corollary 5.2, and Theorem 7.3, we get the first two statements of our theorem. By (7.63), the zeros and poles of $Z_{\eta,\rho}(\sigma)$ coincide with that of $\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta},\rho} + \sigma_\eta + \sigma^2\right)$, from which we deduce the third statement of our theorem. Equation (7.64) is a consequence of (7.63) and of the fact that $P_\eta(\sigma)$ is an odd polynomial. The proof of our theorem is completed. □

7.4. **The Ruelle dynamical zeta function.** We turn our attention to the Ruelle dynamical zeta function $R_\rho(\sigma)$.

Theorem 7.7. *The dynamical zeta function $R_\rho(\sigma)$ is holomorphic for $\mathrm{Re}(\sigma) \gg 1$, and extends meromorphically to $\sigma \in \mathbf{C}$ such that*

$$R_\rho(\sigma) = \prod_{j=0}^{2l} Z_{\eta_j,\rho}\left(\sigma + (j-l)|\alpha|\right)^{(-1)^{j-1}}. \tag{7.65}$$

Proof. Clearly, there is $C > 0$ such that for all $\gamma \in \Gamma$,

$$\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2} \leqslant C\exp(C|a|). \tag{7.66}$$

By (7.62) and (7.66), for $\sigma \in \mathbf{C}$ and $\mathrm{Re}(\sigma) > \sigma_0 + C$, the sum in (5.10) converges absolutely to a holomorphic function. By (5.4), (5.7), (5.10), (6.18) and (7.55), for $\sigma \in \mathbf{C}$ and $\mathrm{Re}(\sigma) > \sigma_0 + C$, we have

$$\Xi_\rho(\sigma) = \sum_{j=0}^{2l} (-1)^{j-1} \Xi_{\eta_j,\rho}\left(\sigma + (j-l)|\alpha|\right). \tag{7.67}$$

By taking exponentials, we get (7.65) for $\mathrm{Re}(\sigma) > \sigma_0 + C$. Since the right-hand side of (7.65) is meromorphic, $R_\rho(\sigma)$ has a meromorphic extension to $\mathbf{C}$, such that (7.65) holds. The proof of our theorem is completed. □

Remark that for $0 \leqslant j \leqslant 2l$, we have the isomorphism of K_M-representations of $\eta_j \simeq \eta_{2l-j}$. By (7.1), we have the isomorphism of K-representations,

$$\widehat{\eta}_j \simeq \widehat{\eta}_{2l-j} \tag{7.68}$$

Note that by (6.58) and (7.60), we have

$$\sigma_{\eta_j} = -(j-l)^2|\alpha|^2. \tag{7.69}$$

By (7.63), (7.68) and (7.69), we have

$$Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) Z_{\eta_{2l-j},\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) = Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) Z_{\eta_j,\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) = \det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_j,\rho}+\sigma^2\right)^2 = \det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_j,\rho}+\sigma^2\right)\det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_{2l-j},\rho}+\sigma^2\right). \tag{7.70}$$

Recall that $T(\sigma)$ is defined in (2.14).

Theorem 7.8. *The following identity of meromorphic functions on* $\mathbf{C}$ *holds:*

$$R_\rho(\sigma) = T(\sigma^2)\exp\left((-1)^{l-1} r \operatorname{vol}(Z) P_{\eta_l}(\sigma)\right) \prod_{j=0}^{l-1}\left(\frac{Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|)Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)}{Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) Z_{\eta_{2l-j},\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right)}\right)^{(-1)^{j-1}}. \tag{7.71}$$

Proof. By (2.14), (4.24), and (7.59), we have the identity of meromorphic functions,

$$T(\sigma) = \prod_{j=0}^{2l} \det_{\mathrm{gr}}\left(C^{\mathfrak{g},Z,\widehat{\eta}_j,\rho}+\sigma\right)^{(-1)^{j-1}}. \tag{7.72}$$

By (7.63), (7.70), and (7.72), we have

$$T(\sigma^2) = Z_{\eta_l,\rho}(\sigma)^{(-1)^{l-1}} \exp\left((-1)^{l} r \operatorname{vol}(Z) P_{\eta_l}(\sigma)\right) \prod_{j=0}^{l-1}\left(Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) Z_{\eta_{2l-j},\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right)\right)^{(-1)^{j-1}}. \tag{7.73}$$

By (7.65) and (7.73), we get (7.71). The proof of our theorem is completed. □

For $0 \leqslant j \leqslant 2l$, as in (7.58), we write $r_j = r_{\eta_j,\rho}$. By (7.68) and (7.72), we have

$$\chi'(Z,F) = 2\sum_{j=0}^{l-1}(-1)^{j-1} r_j + (-1)^{l-1} r_l. \tag{7.74}$$

Set

$$C_\rho = \prod_{j=0}^{l-1}\left(-4(l-j)^2|\alpha|^2\right)^{(-1)^{j-1} r_j}, \qquad r_\rho = 2\sum_{j=0}^{l}(-1)^{j-1} r_j. \tag{7.75}$$

Proof of (5.12). By Proposition 6.13 and Theorem 7.6, for $0 \leqslant j \leqslant l-1$, the orders of the zero at $\sigma = 0$ of the functions $Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|)$ and $Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)$ are equal to r_j. Therefore, for $0 \leqslant j \leqslant l-1$, there are $A_j \neq 0$, $B_j \neq 0$ such that as $\sigma \to 0$,

$$\begin{aligned} Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|) &= A_j \sigma^{r_j} + \mathcal{O}\left(\sigma^{r_j+1}\right), \\ Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|) &= B_j \sigma^{r_j} + \mathcal{O}\left(\sigma^{r_j+1}\right), \end{aligned} \tag{7.76}$$

and

$$\begin{aligned} Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) &= A_j\left(\frac{-\sigma^2}{2(l-j)|\alpha|}\right)^{r_j} + \mathcal{O}\left(\sigma^{2r_j+2}\right) \\ Z_{\eta_{2l-j},\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right) &= B_j\left(\frac{\sigma^2}{2(l-j)|\alpha|}\right)^{r_j} + \mathcal{O}\left(\sigma^{2r_j+2}\right). \end{aligned} \tag{7.77}$$

By (7.76) and (7.77), as $\sigma \to 0$,

$$\frac{Z_{\eta_j,\rho}(\sigma+(j-l)|\alpha|)Z_{\eta_{2l-j},\rho}(\sigma+(l-j)|\alpha|)}{Z_{\eta_j,\rho}\left(-\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right)Z_{\eta_{2l-j},\rho}\left(\sqrt{\sigma^2+(l-j)^2|\alpha|^2}\right)} \to \left(-4(l-j)^2|\alpha|^2\right)^{r_j}\sigma^{-2r_j}+\mathcal{O}(\sigma^{-2r_j+1}). \tag{7.78}$$

By (7.61), (7.71), (7.74), (7.75), and (7.78), we get (5.12). □

Remark 7.9. When $G = \mathrm{SO}^0(p,1)$ with $p \geqslant 3$ odd, we recover [F86, Theorem 3].

Remark 7.10. If we scale the form B with the factor $a > 0$, $R_\rho(\sigma)$ is replaced by $R_\rho(\sqrt{a}\sigma)$. By (5.12), as $\sigma \to 0$,

$$R_\rho(\sqrt{a}\sigma) = a^{r_\rho/2}C_\rho T(F)^2\sigma^{r_\rho}+\mathcal{O}(\sigma^{r_\rho+1}). \tag{7.79}$$

On the other hand, C_ρ should become $a^{\sum_{j=0}^{l-1}(-1)^j r_j}C_\rho$, and $T(F)$ should scale by $a^{\chi'(Z,F)/2}$. This is only possible if

$$r_\rho = 2\sum_{j=0}^{l-1}(-1)^j r_j + 2\chi'(Z,F), \tag{7.80}$$

which just (7.74).

8. A cohomological formula for r_j

The purpose of this section is to establish (5.13) when G is such that $\delta(G) = 1$ and has compact center. We rely on some deep results from the representation theory of reductive Lie groups.

This section is organized as follows. In Subsection 8.1, we recall the constructions of the infinitesimal and global characters of Harish-Chandra modules. We also recall some properties of $(\mathfrak{g},K)$-cohomology and $\mathfrak{n}$-homology of Harish-Chandra modules.

In Subsection 8.2, we give a formula relating r_j with an alternating sum of the dimensions of Lie algebra cohomologies of certain Harish-Chandra modules, and we establish Equation (5.13).

8.1. **Some results from representation theory.**

In this Subsection, we do not assume that $\delta(G) = 1$. We use the notation in Section 3 and the convention of real root systems introduced in Subsection 7.2.

8.1.1. *Infinitesimal characters.* Let $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ be the center of the enveloping algebra $U(\mathfrak{g}_{\mathbf{C}})$ of the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$. A morphism of algebras $\chi : \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}) \to \mathbf{C}$ will be called a character of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$.

Recall that $\mathfrak{h}_1, \cdots, \mathfrak{h}_{l_0}$ form all the nonconjugated θ-stable Cartan subalgebras of $\mathfrak{g}$. Let $\mathfrak{h}_{i\mathbf{C}} = \mathfrak{h}_i \otimes_{\mathbf{R}} \mathbf{C}$ and $\mathfrak{h}_{i\mathbf{R}} = \sqrt{-1}\mathfrak{h}_{i\mathfrak{p}} \oplus \mathfrak{h}_{i\mathfrak{k}}$ are the complexfication and real form of $\mathfrak{h}_i$. For $\alpha \in \mathfrak{h}_{i\mathbf{R}}^*$, we extend α to a $\mathbf{C}$-linear form on $\mathfrak{h}_{i\mathbf{C}}$ by $\mathbf{C}$-linearity. In this way, we identify $\mathfrak{h}_{i\mathbf{R}}^*$ to a subset of $\mathfrak{h}_{i\mathbf{C}}^*$.

For $1 \leqslant i \leqslant l_0$, let $S(\mathfrak{h}_{i\mathbf{C}})$ be the symmetric algebra of $\mathfrak{h}_{i\mathbf{C}}$. The algebraic Weyl group $W(\mathfrak{h}_{i\mathbf{R}},\mathfrak{u})$ acts isometrically on $\mathfrak{h}_{i\mathbf{R}}$. By $\mathbf{C}$-linearity, $W(\mathfrak{h}_{i\mathbf{R}},\mathfrak{u})$ acts on $\mathfrak{h}_{i\mathbf{C}}$. Therefore,

$W(\mathfrak{h}_{i\mathbf{R}}, \mathfrak{u})$ acts on $S(\mathfrak{h}_{i\mathbf{C}})$. Let $S(\mathfrak{h}_{i\mathbf{C}})^{W(\mathfrak{h}_{i\mathbf{R}},\mathfrak{u})} \subset S(\mathfrak{h}_{i\mathbf{C}})$ be the $W(\mathfrak{h}_{i\mathbf{R}}, \mathfrak{u})$-invariant subalgebra of $S(\mathfrak{h}_{i\mathbf{C}})$. Let

$$\gamma_i : \mathcal{Z}(\mathfrak{g}_{\mathbf{C}}) \simeq S(\mathfrak{h}_{i\mathbf{C}})^{W(\mathfrak{h}_{i\mathbf{R}},\mathfrak{u})} \tag{8.1}$$

be the Harish-Chandra isomorphism [K02, Section V.5]. For $\Lambda \in \mathfrak{h}_{i\mathbf{C}}^*$, we can associate to it a character χ_Λ of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ as follows: for $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$,

$$\chi_\Lambda(z) = \langle \gamma_i(z), 2\sqrt{-1}\pi\Lambda \rangle. \tag{8.2}$$

By [K02, Theorem 5.62], every character of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ is of the form χ_Λ, for some $\Lambda \in \mathfrak{h}_{i\mathbf{C}}^*$. Also, Λ is uniquely determined up to an action of $W(\mathfrak{h}_{i\mathbf{R}}, \mathfrak{u})$. Such an element $\Lambda \in \mathfrak{h}_{i\mathbf{C}}^*$ is called the Harish-Chandra parameter of the character. In particular, $\chi_\Lambda = 0$ if and only if there is $w \in W(\mathfrak{h}_{i\mathbf{R}}, \mathfrak{u})$ such that

$$w\Lambda = \rho_i^{\mathfrak{u}}, \tag{8.3}$$

where $\rho_i^{\mathfrak{u}}$ is defined as in (7.11) with respect to $(\mathfrak{h}_{i\mathbf{R}}, \mathfrak{u})$.

Definition 8.1. A complex representation of $\mathfrak{g}_{\mathbf{C}}$ is said to have infinitesimal character χ, if $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$ acts as a scalar $\chi(z) \in \mathbf{C}$.

A complex representation of $\mathfrak{g}_{\mathbf{C}}$ is said to have generalized infinitesimal character χ, if $z - \chi(z)$ acts nilpotently for all $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$, i.e., $(z - \chi(z))^i$ acts like 0 for $i \gg 1$.

If $\lambda \in \mathfrak{h}_{i\mathbf{R}}^*$ is algebraically integral and dominant, let V_λ be the complex finite dimensional irreducible representation of the Lie algebra $\mathfrak{g}_{\mathbf{C}}$ with the highest weight λ. Then V_λ possesses an infinitesimal character with Harish-Chandra parameter $\lambda + \rho_i^{\mathfrak{u}} \in \mathfrak{h}_{i\mathbf{R}}^*$.

8.1.2. *Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-modules and admissible representations of G.* We follow [HeSc83, p. 54-55] and [K86, p. 207].

Definition 8.2. We will say that a complex $U(\mathfrak{g}_{\mathbf{C}})$-module V, equipped with an action of K, is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, if the following conditions hold:

(1) The space V is finitely generated as a $U(\mathfrak{g}_{\mathbf{C}})$-module;
(2) Every $v \in V$ lies in a finite dimensional, $\mathfrak{k}_{\mathbf{C}}$-invariant subspace;
(3) The action of $\mathfrak{g}_{\mathbf{C}}$ and K are compatible;
(4) Each irreducible K-module occurs only finitely many times in V.

Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module. For a character χ of $\mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$, let $V_\chi \subset V$ be the largest submodule of V on which $z - \chi(z)$ acts nilpotently for all $z \in \mathcal{Z}(\mathfrak{g}_{\mathbf{C}})$. Then V_χ is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-submodule of V with generalized infinitesimal character χ. By [HeSc83, eq. (2.4)], we can decompose V as a finite sum of Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-submodules

$$V = \bigoplus_\chi V_\chi. \tag{8.4}$$

Any Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module V has a finite composition series in the following sense: there exist finitely many Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-submodules

$$V = V_{n_1} \supset V_{n_1-1} \supset \cdots \supset V_0 \supset V_{-1} = 0 \tag{8.5}$$

such that each quotient V_i/V_{i-1}, for $0 \leqslant i \leqslant n_1$, is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module. Moreover, the set of all irreducible quotients and their multiplicities are the same for all the composition series.

Definition 8.3. We say that a representation π of G on a Hilbert space is admissible if the followings hold:

(1) when restricted to K, $\pi|_K$ is unitary;
(2) each $\tau \in \widehat{K}$ occurs with only finite multiplicity in $\pi|_K$.

Let π be a finitely generated admissible representation of G on the Hilbert space V_π. If $\tau \in \widehat{K}$, let $V_\pi(\tau) \subset V_\pi$ be the τ-isotopic subspace of V_π. Then $V_\pi(\tau)$ is the image of the evaluation map

$$(f, v) \in \mathrm{Hom}_K(V_\tau, V_\pi) \otimes V_\tau \to f(v) \in V_\pi. \tag{8.6}$$

Let

$$V_{\pi,K} = \bigoplus_{\tau\in\widehat{K}} V_\pi(\tau) \subset V_\pi \tag{8.7}$$

be the algebraic sum of representations of K. By [K86, Proposition 8.5], $V_{\pi,K}$ is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module. It is explained in [V08, Section4] that, by results of Casselman, Harish-Chandra, Lepowsky and Wallach, any Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module V can be constructed in this way and the corresponding V_π is called a Hilbert globalization of V. Moreover, V is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module if and only if V_π is an irreducible admissible representation of G. In this case, V or V_π has an infinitesimal character.

We note that a Hilbert globalization of a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module is not unique.

8.1.3. *Global characters.* We recall the definition of the space of rapidly decreasing functions $\mathcal{S}(G)$ on G [W88, Section 7.1.2].

For $z \in U(\mathfrak{g})$, we denote by z_L and z_R respectively the corresponding left and right invariant differential operators on G. For $r \geqslant 0$, $z_1 \in U(\mathfrak{g})$, $z_2 \in U(\mathfrak{g})$, and $f \in C^\infty(G)$, put

$$\|f\|_{r,z_1,z_2} = \sup_{g\in G} e^{r d_X(p1,pg)} |z_{1L} z_{2R} f(g)|. \tag{8.8}$$

Let $\mathcal{S}(G)$ be the space of all $f \in C^\infty(G)$ such that, for all $r \geqslant 0$, $z_1 \in U(\mathfrak{g})$, $z_2 \in U(\mathfrak{g})$, $\|f\|_{r,z_1,z_2} < \infty$. We endow $\mathcal{S}(G)$ with the topology given by the above semi-norms. By [W88, Theorem 7.1.1], $\mathcal{S}(G)$ is a Fréchet space which contains $C_c^\infty(G)$ as a dense subspace.

Let π be a finitely generated admissible representation of G on the Hilbert space V_π. By [W88, Lemma 2.A.2.2], there exists $C > 0$ such that for $g \in G$, we have

$$\|\pi(g)\| \leqslant C e^{C d_X(p1,pg)}, \tag{8.9}$$

where $\|\cdot\|$ is the operator norm. By (8.9), if $f \in \mathcal{S}(G)$,

$$\pi(f) = \int_G f(g)\pi(g)dg \tag{8.10}$$

is a bounded operator on V_π. By [W88, Lemma 8.1.1], $\pi(f)$ is trace class. The global character Θ_π^G of π is a continuous linear functional on $\mathcal{S}(G)$ such that for $f \in \mathcal{S}(G)$,

$$\mathrm{Tr}[\pi(f)] = \langle \Theta_\pi^G, f\rangle. \tag{8.11}$$

If V is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, we can define the global character Θ^G_V of V by the global character of its Hilbert globalization. We note that the global character does not depend on the choice of Hilbert globalization [HeSc83, p.56].

By Harish-Chandra's regularity theorem [K86, Theorems 10.25], there is an L^1_{loc} and $\mathrm{Ad}(G)$-invariant function $\Theta^G_\pi(g)$ on G, whose restriction to the regular set G' is analytic, such that for $f \in C^\infty_c(G)$, we have

$$\left\langle \Theta^G_\pi, f\right\rangle = \int_{g\in G} \Theta^G_\pi(g) f(g) dv_G. \tag{8.12}$$

Proposition 8.4. *If $f \in \mathcal{S}(G)$, then $\Theta^G_\pi(g)f(g) \in L^1(G)$ such that*

$$\left\langle \Theta^G_\pi, f\right\rangle = \int_{g\in G} \Theta^G_\pi(g) f(g) dv_G. \tag{8.13}$$

Proof. It is enough to show that there exist $C > 0$ and a seminorm $\|\cdot\|$ on $\mathcal{S}(G)$ such that

$$\int_G |\Theta^G_\pi(g)f(g)|dg \leqslant C\|f\|. \tag{8.14}$$

Recall that H' is defined in (3.36). By (3.33), we need to show that there exist $C > 0$ and a semi-norm $\|\cdot\|$ on $\mathcal{S}(G)$ such that for $1 \leqslant i \leqslant l_0$, we have

$$\int_{\gamma\in H'_i} \left|\Theta^G_\pi(\gamma)\right| \left(\int_{g\in H_i\backslash G} |f(g^{-1}\gamma g)| dv_{H_i\backslash G}\right) \left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right| dv_{H_i} \leqslant C\|f\|. \tag{8.15}$$

By [K86, Theorem 10.35], there exist $C > 0$ and $r_0 > 0$ such that, for $\gamma = e^a k^{-1} \in H'_i$ with $a \in \mathfrak{h}_{i\mathfrak{p}}$, $k \in H_i \cap K$, we have

$$\left|\Theta^G_\pi(\gamma)\right| \left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right|^{1/2} \leqslant Ce^{r_0|a|}. \tag{8.16}$$

We claim that there exist $r_1 > 0$ and $C > 0$, such that for $\gamma \in H'_i$, we have

$$\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right|^{1/2} \int_{g\in H_i\backslash G} \exp\left(-r_1 d_X(p1, g^{-1}\gamma g\cdot p1)\right) dv_{H_i\backslash G} \leqslant C. \tag{8.17}$$

Indeed, let $\Xi(g)$ be the Harish-Chandra's Ξ-function [Va77, Section II.8.5]. By [Va77, Section II.12.2, Corollary 5], there exist $r_2 > 0$ and $C > 0$, such that for $\gamma \in H'_i$, we have

$$\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right|^{1/2} \int_{g\in H_i\backslash G} \Xi(g^{-1}\gamma g)\left(1+d_X(p1, g^{-1}\gamma g\cdot p1)\right)^{-r_2} dv_{H_i\backslash G} \leqslant C. \tag{8.18}$$

By [K86, Proposition 7.15 (c)] and by (8.18), we get (8.17).

By [B11, (3.1.10)], for $g \in G$ and $\gamma = e^a k^{-1} \in H_i$, we have

$$d_X\left(p1, g^{-1}\gamma g\cdot p1\right) \geqslant |a|. \tag{8.19}$$

Take $r = 2r_0 + r_1$, $z_1 = z_2 = 1 \in U(\mathfrak{g})$. Since $f \in \mathcal{S}(G)$, by (8.8) and (8.19), for $\gamma = e^a k^{-1} \in H'_i$, we have

$$\begin{aligned}|f(g^{-1}\gamma g)| &\leqslant \|f\|_{r,z_1,z_2} \exp\left(-r d_X(p1, g^{-1}\gamma g\cdot p1)\right)\\ &\leqslant \|f\|_{r,z_1,z_2} \exp(-2r_0|a|) \exp\left(-r_1 d_X(p1, g^{-1}\gamma g\cdot p1)\right).\end{aligned} \tag{8.20}$$

By (8.16), (8.17), and (8.20), for $\gamma \in H'_i$, we have

(8.21)

$$\left|\Theta^G_\pi(\gamma)\right| \left(\int_{g\in H_i\backslash G} |f(g^{-1}\gamma g)| dv_{H_i\backslash G}\right) \left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right| \leqslant C\|f\|_{r,z_1,z_2} \exp(-r_0|a|).$$

By (8.21), we get (8.15). The proof of our proposition is completed. □

Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, and let τ be a real finite dimensional orthogonal representation of K on the real Euclidean space E_τ. Then the invariant subspace $(V \otimes_{\mathbf{R}} E_\tau)^K \subset V \otimes_{\mathbf{R}} E_\tau$ has a finite dimension. We will describe an integral formula for $\dim_{\mathbf{C}}(V \otimes_{\mathbf{R}} E_\tau)^K$, which extends [BaMo83, Corollary 2.2].

Recall that $p_t^{X,\tau}(g)$ is the smooth integral kernel of $\exp\left(-tC^{X,\tau}/2\right)$. By the estimation on the heat kernel or by [BaMo83, Proposition 2.4], $p_t^{X,\tau}(g) \in \mathcal{S}(G) \otimes \mathrm{End}(E_\tau)$. Recall that dv_G is the Riemannian volume on G induced by $-B(\cdot, \theta\cdot)$.

Proposition 8.5. *Let $f \in C^\infty(G, E_\tau)^K$. Assume that there exist $C > 0$ and $r > 0$ such that*

$$|f(g)| \leqslant C \exp\left(r d_X(p1, pg)\right). \tag{8.22}$$

The integral

$$\int_{g \in G} p_t^{X,\tau}(g) f(g) dv_G \in E_\tau \tag{8.23}$$

is well defined such that

$$\begin{aligned} \frac{\partial}{\partial t} \int_{g \in G} p_t^{X,\tau}(g) f(g) dv_G &= -\frac{1}{2} \int_{g \in G} C^{\mathfrak{g}} p_t^{X,\tau}(g) f(g) dv_G, \\ \frac{1}{\mathrm{vol}(K)} \lim_{t \to 0} \int_{g \in G} p_t^{X,\tau}(g) f(g) dv_G &= f(1). \end{aligned} \tag{8.24}$$

Proof. By (8.22), by the property of $\mathcal{S}(G)$ and by $\frac{\partial}{\partial t} p_t^{X,\tau}(g) = -\frac{1}{2} C^{\mathfrak{g}} p_t^{X,\tau}(g)$, the left-hand side of (8.23) and the right-hand side of the first equation of (8.24) are well defined such that the first equation of (8.24) holds true.

It remains to show the second equation of (8.24). Let $\phi_1 \in C_c^\infty(G)^K$ such that $0 \leqslant \phi_1(g) \leqslant 1$ and that

$$\phi_1(g) = \begin{cases} 1, & d_X(p1, pg) \leqslant 1, \\ 0, & d_X(p1, pg) \geqslant 2. \end{cases} \tag{8.25}$$

Set $\phi_2 = 1 - \phi_1$.

Since $\phi_1 f$ has compact support, it descends to an L^2-section on X with values in $G \times_K E_\tau$. We have

$$\frac{1}{\mathrm{vol}(K)} \lim_{t \to 0} \int_{g \in G} p_t^{X,\tau}(g) \phi_1(g) f(g) dv_G = f(1). \tag{8.26}$$

By (4.27), there exist $c > 0$ and $C > 0$ such that for $g \in G$ with $d_X(p1, pg) \geqslant 1$ and for $t \in (0, 1]$, we have

$$\left| p_t^{X,\tau}(g) \right| \leqslant C \exp\left(-c \frac{d_X^2(p1, pg)}{t}\right) \leqslant C e^{-c/2t} \exp\left(-c \frac{d_X^2(p1, pg)}{2t}\right). \tag{8.27}$$

By (8.22) and (8.27), there exist $c > 0$ and $C > 0$ such that for $t \in (0, 1]$, we have

$$\int_{g \in G} \left| p_t^{X,\tau}(g) \phi_2(g) f(g) dv_G \right| \leqslant C e^{-c/2t}. \tag{8.28}$$

By (8.26) and (8.28), we get the second equation of (8.24). The proof of our proposition is completed. □

Proposition 8.6. *Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module with generalized infinitesimal character χ. For $t > 0$, we have*

$$\dim_{\mathbf{C}} \left(V \otimes_{\mathbf{R}} E_\tau\right)^K = \mathrm{vol}(K)^{-1} e^{t\chi(C^{\mathfrak{g}})/2} \int_{g\in G} \Theta_V^G(g) \operatorname{Tr}\left[p_t^{X,\tau}(g)\right] dv_G. \tag{8.29}$$

Proof. Let V_π be a Hilbert globalization of V. Then,

$$\left(V \otimes_{\mathbf{R}} E_\tau\right)^K = \left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K. \tag{8.30}$$

As in (8.10), set

$$\pi\left(p_t^{X,\tau}\right) = \frac{1}{\mathrm{vol}(K)} \int_{g\in G} \pi(g) \otimes_{\mathbf{R}} p_t^{X,\tau}(g) dv_G. \tag{8.31}$$

Then, $\pi\left(p_t^{X,\tau}\right)$ is a bounded operator acting on $V_\pi \otimes_{\mathbf{R}} E_\tau$.

We follow [BaMo83, p. 160-161]. Let $\left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^{K,\perp}$ be the orthogonal space to $\left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K$ in $V_\pi \otimes_{\mathbf{R}} E_\tau$, such that

$$V_\pi \otimes_{\mathbf{R}} E_\tau = \left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K \oplus \left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^{K,\perp}. \tag{8.32}$$

Let $Q_{\pi,\tau}$ be the orthogonal projection from $V_\pi \otimes_{\mathbf{R}} E_\tau$ to $\left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K$. Then,

$$Q_{\pi,\tau} = \frac{1}{\mathrm{vol}(K)} \int_{k\in K} \pi \otimes \tau(k) dv_K. \tag{8.33}$$

By (4.13), (8.31) and (8.33), we get

$$Q_{\pi,\tau} \pi\left(p_t^{X,\tau}\right) Q_{\pi,\tau} = \pi\left(p_t^{X,\tau}\right). \tag{8.34}$$

In particular, $\pi(p_t^{X,\tau})$ is of finite rank.

Take $u \in \left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K$ and $v \in V_\pi$. Define $\langle u, v\rangle \in E_\tau$ be such that for any $w \in E_\tau$,

$$\big\langle \langle u, v\rangle, w \big\rangle = \langle u, v \otimes_{\mathbf{R}} w\rangle. \tag{8.35}$$

By (8.9), the function $g \in G \to \langle \pi(g) \otimes_{\mathbf{R}} \mathrm{id} \cdot u, v\rangle \in E_\tau$ is of class $C^\infty(G, E_\tau)^K$ such that (8.22) holds. By (8.31), we have

$$\left\langle \pi\left(p_t^{X,\tau}\right) u, v\right\rangle = \frac{1}{\mathrm{vol}(K)} \int_{g\in G} p_t^{X,\tau}(g) \langle \pi(g) \otimes_{\mathbf{R}} \mathrm{id} \cdot u, v\rangle dv_G. \tag{8.36}$$

By Proposition 8.5 and (8.36), we have

$$\frac{\partial}{\partial t}\left\langle \pi\left(p_t^{X,\tau}\right) u, v\right\rangle = -\frac{1}{2}\left\langle \pi\left(C^{\mathfrak{g}}\right) \pi\left(p_t^{X,\tau}\right) u, v\right\rangle, \quad \lim_{t\to 0}\left\langle \pi\left(p_t^{X,\tau}\right) u, v\right\rangle = \langle u, v\rangle \tag{8.37}$$

Since $C^{\mathfrak{g}} \in \mathcal{Z}(\mathfrak{g})$ and since $\pi(C^{\mathfrak{g}})$ preserves the splitting (8.32), by (8.34) and (8.37), under the splitting (8.32), we have

$$\pi\left(p_t^{X,\tau}\right) = \begin{pmatrix} e^{-t\pi(C^{\mathfrak{g}})/2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{8.38}$$

Since V has a generalized infinitesimal character χ, by (8.38), we have

$$\operatorname{Tr}\left[\pi(p_t^{X,\tau})\right] = e^{-t\chi(C^{\mathfrak{g}})/2} \dim_{\mathbf{C}} \left(V_\pi \otimes_{\mathbf{R}} E_\tau\right)^K. \tag{8.39}$$

Let $(\xi_i)_{i=1}^{\infty}$ and $(\eta_j)_{j=1}^{\dim E_\tau}$ be orthogonal basis of V_π and E_τ. Then

$$\begin{aligned}\operatorname{Tr}\left[\pi\left(p_t^{X,\tau}\right)\right] &= \frac{1}{\operatorname{vol}(K)}\sum_{i=1}^{\infty}\sum_{j=1}^{\dim E_\tau}\int_{g\in G}\langle p_t^{X,\tau}(g)\eta_j,\eta_j\rangle\langle\pi(g)\xi_i,\xi_i\rangle dv_G\\ &= \frac{1}{\operatorname{vol}(K)}\sum_{i=1}^{\infty}\int_{g\in G}\operatorname{Tr}\left[p_t^{X,\tau}(g)\right]\langle\pi(g)\xi_i,\xi_i\rangle dv_G.\end{aligned} \tag{8.40}$$

Since $\operatorname{Tr}\left[p_t^{X,\tau}(g)\right] \in \mathcal{S}(G)$, by (8.13) and (8.40), we have

$$\operatorname{Tr}\left[\pi\left(p_t^{X,\tau}\right)\right] = \frac{1}{\operatorname{vol}(K)}\int_{g\in G}\operatorname{Tr}\left[p_t^{X,\tau}(g)\right]\Theta_\pi^G(g)dv_G. \tag{8.41}$$

From (8.30), (8.39) and (8.41), we get (8.29). The proof of our proposition is completed. □

Proposition 8.7. *For $1 \leqslant i \leqslant l_0$, the function*

$$\gamma \in H_i' \to \operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]\Theta_\pi^G(\gamma)\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right| \tag{8.42}$$

is almost everywhere well defined and integrable on H_i', so that

$$\begin{aligned}\int_{g\in G}\operatorname{Tr}\left[p_t^{X,\tau}(g)\right]\Theta_\pi^G(g)dv_G &= \sum_{i=1}^{l_0}\frac{\operatorname{vol}(K\cap H_i\backslash K)}{|W(H_i,G)|}\\ &\int_{\gamma\in H_i'}\operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]\Theta_\pi^G(g)\left|\det\left(1-\operatorname{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}_i}\right|dv_{H_i}.\end{aligned} \tag{8.43}$$

Proof. Since $\operatorname{Tr}\left[p_t^{X,\tau}(g)\right]\Theta_\pi^G(g) \in L^1(G)$, by (3.33) and by Fubini Theorem, the function

$$\gamma \in H_i \to \left(\int_{g\in H_i\backslash G}\operatorname{Tr}^{E_\tau}\left[p_t^{X,\tau}\left(g^{-1}\gamma g\right)\right]dv_{H_i\backslash G}\right)\Theta_\pi^G(\gamma)\left|\det(1-\operatorname{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}_i}\right| \tag{8.44}$$

is almost everywhere well defined and integrable on H_i.

Take $\gamma \in H_i'$. Since H_i is abelian, we have

$$Z^0(\gamma) = H_i^0 \subset H_i \subset Z(\gamma). \tag{8.45}$$

We have a finite covering space $H_i^0\backslash G \to H_i\backslash G$. Note that

$$\left[H_i : H_i^0\right] = \left[K\cap H_i : K\cap H_i^0\right]. \tag{8.46}$$

By (4.15), (8.45) and (8.46), if $\gamma \in H_i'$, we have

$$\begin{aligned}\int_{H_i\backslash G}\operatorname{Tr}^{E_\tau}\left[p_t^{X,\tau}(g^{-1}\gamma g)\right]dv_{H_i\backslash G} &= \frac{\operatorname{vol}(K^0(\gamma)\backslash K)}{[H_i:H_i^0]}\operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right]\\ &= \operatorname{vol}(K\cap H_i\backslash K)\operatorname{Tr}^{[\gamma]}\left[\exp\left(-tC^{\mathfrak{g},X,\tau}/2\right)\right].\end{aligned} \tag{8.47}$$

Since $H_i - H_i'$ has zero measure, and by (8.44) and (8.47), the function (8.42) defines an L^1-function on H_i'. By (3.33) and (8.47), we get (8.43). The proof of our proposition is completed. □

8.1.4. *The* $(\mathfrak{g}, K)$*-cohomology.* If V is a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module, let $H^{\cdot}(\mathfrak{g}, K; V)$ be the $(\mathfrak{g}, K)$-cohomology of V [BoW00, Section I.1.2]. The following two theorems are the essential algebraic ingredients in our proof of (5.13).

Theorem 8.8. *Let V be a Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module with generalized infinitesimal character χ. Let W be a finite dimensional $\mathfrak{g}_{\mathbf{C}}$-module with infinitesimal character. Let χ^{W^*} be the infinitesimal character of W^*. If $\chi \neq \chi^{W^*}$, then*

$$H^{\cdot}(\mathfrak{g}, K; V \otimes W) = 0. \tag{8.48}$$

Proof. If χ is the infinitesimal character of V, then (8.48) is a consequence of [BoW00, Theorem I.5.3 (ii)].

In general, let

$$V = V_{n_1} \supset V_{n_1 - 1} \supset \cdots \supset V_0 \supset V_{-1} = 0 \tag{8.49}$$

be the composition series of V. Then for $0 \leqslant i \leqslant n_1$, V_i/V_{i-1} is an irreducible Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-module with infinitesimal character χ. Therefore, for all $0 \leqslant i \leqslant n_1$, we have

$$H^{\cdot}(\mathfrak{g}, K; (V_i/V_{i-1}) \otimes W) = 0. \tag{8.50}$$

We will show by induction that, for all $0 \leqslant i \leqslant n_1$,

$$H^{\cdot}(\mathfrak{g}, K; V_i \otimes W) = 0. \tag{8.51}$$

By (8.50), Equation (8.51) holds for $i = 0$. Assume that (8.51) holds for some i with $0 \leqslant i \leqslant n_1$. Using the short exact sequence of Harish-Chandra $(\mathfrak{g}_{\mathbf{C}}, K)$-modules

$$0 \to V_i \to V_{i+1} \to V_{i+1}/V_i \to 0, \tag{8.52}$$

we get the long exact sequence of cohomologies

$$\cdots \to H^j(\mathfrak{g}, K; V_i \otimes W) \to H^j(\mathfrak{g}, K; V_{i+1} \otimes W) \to H^j(\mathfrak{g}, K; (V_{i+1}/V_i) \otimes W) \to \cdots . \tag{8.53}$$

By (8.50), (8.53) and by the induction hypotheses, Equation (8.51) holds for $i+1$, which completes the proof of (8.51). The proof of our theorem is completed. □

We denote by $\widehat{G}_u$ the unitary dual of G, that is the set of equivalence classes of complex irreducible unitary representations π of G on Hilbert spaces V_π. If $(\pi, V_\pi) \in \widehat{G}_u$, by [K86, Theorem 8.1], π is irreducible admissible. Let χ_π be the corresponding infinitesimal character.

Theorem 8.9. *If $(\pi, V_\pi) \in \widehat{G}_u$, then*

$$\chi_\pi \neq 0 \iff H^{\cdot}(\mathfrak{g}, K; V_{\pi,K}) = 0. \tag{8.54}$$

Proof. The direction $\Longrightarrow$ of (8.54) is (8.48). The direction $\Longleftarrow$ of (8.54) is a consequence of Vogan-Zuckerman [VZu84], Vogan [V84] and Salamanca-Riba [SR99]. Indeed, the irreducible unitary representations with nonvanishing $(\mathfrak{g}, K)$-cohomology are classified and constructed in [VZu84, V84]. By [SR99], the irreducible unitary representations with vanishing infinitesimal character is in the class specified by Vogan and Zuckerman, which implies that their $(\mathfrak{g}, K)$-cohomology are nonvanishing. □

Remark 8.10. The condition that π is unitary is crucial in the (8.54). See [W88, Section 9.8.3] for a counterexample.

8.1.5. *The Hecht-Schmid character formula.* Let us recall the main result of [HeSc83]. Let $Q \subset G$ be a standard parabolic subgroup of G with Lie algebra $\mathfrak{q} \subset \mathfrak{g}$. Let

$$Q = M_Q A_Q N_Q, \qquad \mathfrak{q} = \mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q} \oplus \mathfrak{n}_\mathfrak{q} \tag{8.55}$$

be the corresponding Langlands decompositions [K86, Section V.5].

Put $\Delta^+(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q})$ to be the set of all linear forms $\alpha \in \mathfrak{a}_\mathfrak{q}^*$ such that there exists a nonzero element $Y \in \mathfrak{n}_\mathfrak{q}$ such that for all $a \in \mathfrak{a}_\mathfrak{q}$,

$$\mathrm{ad}(a)Y = \langle \alpha, a\rangle Y. \tag{8.56}$$

Set

$$\mathfrak{a}_\mathfrak{q}^- = \{a \in \mathfrak{a}_\mathfrak{q} : \langle \alpha, a\rangle < 0, \text{ for all } \alpha \in \Delta^+(\mathfrak{a}, \mathfrak{n})\}. \tag{8.57}$$

Put $(M_Q A_Q)^-$ to be the interior in $M_Q A_Q$ of the set

$$\{g \in M_Q A_Q : \det\big(1 - \mathrm{Ad}(ge^a)\big)|_{\mathfrak{n}_\mathfrak{q}} \geqslant 0 \text{ for all } a \in \mathfrak{a}_\mathfrak{q}^-\}. \tag{8.58}$$

If V is a Harish-Chandra $(\mathfrak{g}_\mathbf{C}, K)$-module, let $H_\cdot(\mathfrak{n}_\mathfrak{q}, V)$ be the $\mathfrak{n}_\mathfrak{q}$-homology of V. By [HeSc83, Proposition 2.24], $H_\cdot(\mathfrak{n}_\mathfrak{q}, V)$ is a Harish-Chandra $(\mathfrak{m}_{\mathfrak{q}\mathbf{C}} \oplus \mathfrak{a}_{\mathfrak{q}\mathbf{C}}, K \cap M_Q)$-module. We denote by $\Theta^{M_Q A_Q}_{H_\cdot(\mathfrak{n}_\mathfrak{q}, V)}$ the corresponding global character. Also, $M_Q A_Q$ acts on $\mathfrak{n}_\mathfrak{q}$. We denote by $\Theta^{M_Q A_Q}_{\Lambda^\cdot(\mathfrak{n}_\mathfrak{q})}$ the character of $\Lambda^\cdot(\mathfrak{n}_\mathfrak{q})$. By [HeSc83, Theorem 3.6], the following identity of analytic functions on $(M_Q A_Q)^- \cap G'$ holds:

$$\Theta^G_V|_{(M_Q A_Q)^- \cap G'} = \frac{\sum_{i=0}^{\dim \mathfrak{n}_\mathfrak{q}} (-1)^i \Theta^{M_Q A_Q}_{H_i(\mathfrak{n}_\mathfrak{q}, V)}}{\sum_{i=0}^{\dim \mathfrak{n}_\mathfrak{q}} (-1)^i \Theta^{M_Q A_Q}_{\Lambda^i(\mathfrak{n}_\mathfrak{q})}}\Bigg|_{(M_Q A_Q)^- \cap G'}. \tag{8.59}$$

Take a θ-stable Cartan subalgebra $\mathfrak{h}^{\mathfrak{m}_\mathfrak{q}}$ of $\mathfrak{m}_\mathfrak{q}$. Set $\mathfrak{h}_\mathfrak{q} = \mathfrak{h}^{\mathfrak{m}_\mathfrak{q}} \oplus \mathfrak{a}_\mathfrak{q}$. Then $\mathfrak{h}_\mathfrak{q}$ is a θ-stable Cartan subalgebra of both $\mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q}$ and $\mathfrak{g}$. Put $\mathfrak{u}_\mathfrak{q}$ to be the compact form of $\mathfrak{m}_\mathfrak{q} \oplus \mathfrak{a}_\mathfrak{q}$. Then $\mathfrak{h}_{\mathfrak{q}\mathbf{R}}$, the real form of $\mathfrak{h}_\mathfrak{q}$, is a Cartan subalgebra of both $\mathfrak{u}_\mathfrak{q}$ and $\mathfrak{u}$. The real root system of $\Delta(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u}_\mathfrak{q})$ is a subset of $\Delta(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u})$ consisting of the elements whose restriction to $\mathfrak{a}_\mathfrak{q}$ vanish. The set of positive real roots $\Delta^+(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u}) \subset \Delta(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u})$ determines a set of positive real roots $\Delta^+(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u}_\mathfrak{q}) \subset \Delta(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u}_\mathfrak{q})$. Let $\rho_\mathfrak{q}^\mathfrak{u}$ and $\rho_\mathfrak{q}^{\mathfrak{u}_\mathfrak{q}}$ be the corresponding half sums of positive real roots.

If V possesses an infinitesimal character with Harich-Chandra parameter $\Lambda \in \mathfrak{h}_{\mathfrak{q}\mathbf{C}}^*$, by [HeSc83, Corollary 3.32], $H_\cdot(\mathfrak{n}_\mathfrak{q}, V)$ can be decomposed in the sense of (8.4), where the generalized infinitesimal characters are given by

$$\chi_{w\Lambda + \rho_\mathfrak{q}^\mathfrak{u} - \rho_\mathfrak{q}^{\mathfrak{u}_\mathfrak{q}}}, \tag{8.60}$$

for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u})$.

Also, $H_\cdot(\mathfrak{n}, V)$ is a Harish-Chandra $(\mathfrak{m}_{\mathfrak{q}\mathbf{C}}, K \cap M_Q)$-module. For $\nu \in \mathfrak{a}_{\mathfrak{q}\mathbf{C}}^*$, let $H_\cdot(\mathfrak{n}, V)_{[\nu]}$ be the largest submodule of $H_\cdot(\mathfrak{n}, V)$ on which $z - \langle 2\sqrt{-1}\pi\nu, z\rangle$ acts nilpotently for all $z \in \mathfrak{a}_{\mathfrak{q}\mathbf{C}}$. Then,

$$H_\cdot(\mathfrak{n}, V) = \bigoplus_\nu H_\cdot(\mathfrak{n}, V)_{[\nu]}, \tag{8.61}$$

where $\nu = (w\Lambda + \rho_\mathfrak{q}^\mathfrak{u} - \rho_\mathfrak{q}^{\mathfrak{u}_\mathfrak{q}})|_{\mathfrak{a}_{\mathfrak{q}\mathbf{C}}}$, for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u})$. Let $\Theta^{M_Q}_{H_\cdot(\mathfrak{n}, V)}$ and $\Theta^{M_Q}_{H_\cdot(\mathfrak{n}, V)_{[\nu]}}$ be the corresponding global characters. We have the identities of L^1_{loc}-functions: for $m \in M_Q$

and $a \in \mathfrak{a}_{\mathfrak{q}}$,

$$\Theta^{M_Q A_Q}_{H_\cdot(\mathfrak{n},V)}(me^a) = \sum_{\nu} e^{2\sqrt{-1}\pi\langle \nu, a\rangle} \Theta^{M_Q}_{H_\cdot(\mathfrak{n},V)_{[\nu]}}(m), \quad \Theta^{M_Q}_{H_\cdot(\mathfrak{n},V)}(m) = \sum_{\nu} \Theta^{M_Q}_{H_\cdot(\mathfrak{n},V)_{[\nu]}}(m), \tag{8.62}$$

where $\nu = (w\Lambda + \rho^{\mathfrak{u}}_{\mathfrak{q}} - \rho^{\mathfrak{u}_{\mathfrak{q}}}_{\mathfrak{q}})|_{\mathfrak{a}_{\mathfrak{q}\mathbf{C}}}$, for some $w \in W(\mathfrak{h}_{\mathfrak{q}\mathbf{R}}, \mathfrak{u})$.

Consider now G is such that $\delta(G) = 1$ and has compact center. Use the notation in Subsection 6.1. Take $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{b} \oplus \mathfrak{n}$, and let $Q = M_Q A_Q N_Q \subset G$ be the corresponding parabolic subgroup. Then M is the connected component of the identity in M_Q. Since $K \cap M_Q$ has a finite number of connected components, $H_\cdot(\mathfrak{n}, V)$ is still a Harish-Chandra $(\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}}, K_M)$-module. Also, it is a Harish-Chandra $(\mathfrak{m}_{\mathbf{C}}, K_M)$-module. Let $\Theta^{MA_Q}_{H_\cdot(\mathfrak{n},V)}$ and $\Theta^{M}_{H_\cdot(\mathfrak{n},V)}$ be the respective global characters.

Recall that $H = \exp(\mathfrak{b})T \subset MA_Q$ is the Cartan subgroup of MA_Q.

Proposition 8.11. *We have*

$$\bigcup_{g \in MA_Q} gH'g^{-1} \subset (M_Q A_Q)^- \cap G'. \tag{8.63}$$

Proof. Put $L' = \bigcup_{g \in MA_Q} gH'g^{-1} \subset MA_Q \cap G'$. Then L' is an open subset of MA_Q. It is enough to show that L' is a subset of (8.58).

By (6.19) and (6.22), for $\gamma = e^a k^{-1} \in H$ with $a \in \mathfrak{b}$ and $k \in T$, we have $\det\big(1 - \mathrm{Ad}(\gamma)\big)|_{\mathfrak{n}} \geqslant 0$. Therefore, L' is a subset of (8.58). The proof of our proposition is completed. ☐

8.2. **Formulas for $r_{\eta,\rho}$ and r_j.** Recall that $\widehat{p} : \Gamma\backslash G \to Z$ is the natural projection. The group G acts unitarily on the right on $L^2(\Gamma\backslash G, \widehat{p}^* F)$. By [GeGrPS69, p.23, Theorem], we can decompose $L^2(\Gamma\backslash G, \widehat{p}^* F)$ into a direct sum of unitary representations of G,

$$L^2\left(\Gamma\backslash G, \widehat{p}^* F\right) = \bigoplus_{\pi \in \widehat{G}_u}^{\mathrm{Hil}} n_\rho(\pi) V_\pi, \tag{8.64}$$

with $n_\rho(\pi) < \infty$.

Recall that τ is a real finite dimensional orthogonal representation of K on the real Euclidean space E_τ, and that $C^{\mathfrak{g},Z,\tau,\rho}$ is the Casimir element of G acting on $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbf{C}} F)$. By (8.64), we have

$$\ker C^{\mathfrak{g},Z,\tau,\rho} = \bigoplus_{\pi \in \widehat{G}_u, \chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\big(V_{\pi,K} \otimes_{\mathbf{R}} E_\tau\big)^K. \tag{8.65}$$

By the properties of elliptic operators, the sum on right-hand side of (8.65) is finite.

We will give two applications of (8.65). In our first application, we take $E_\tau = \Lambda^\cdot(\mathfrak{p}^*)$.

Proposition 8.12. *We have*

$$H^\cdot(Z, F) = \bigoplus_{\pi \in \widehat{G}_u, \chi_\pi = 0} n_\rho(\pi) H^\cdot(\mathfrak{g}, K; V_{\pi,K}). \tag{8.66}$$

If $H^\cdot(Z, F) = 0$, then for any $\pi \in \widehat{G}_u$ such that $\chi_\pi = 0$, we have

$$n_\rho(\pi) = 0. \tag{8.67}$$

Proof. By Hodge theory, and by (4.24), (8.65), we have

$$H^{\cdot}(Z,F) = \bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)\big(V_{\pi,K}\otimes_{\mathbf{R}}\Lambda^{\cdot}(\mathfrak{p}^*)\big)^K. \tag{8.68}$$

By Hodge theory for Lie algebras [BoW00, Proposition II.3.1], if $\chi_\pi(C^{\mathfrak{g}})=0$, we have

$$\big(V_{\pi,K}\otimes_{\mathbf{R}}\Lambda^{\cdot}(\mathfrak{p}^*)\big)^K = H^{\cdot}(\mathfrak{g},K;V_{\pi,K}). \tag{8.69}$$

From (8.68) and (8.69), we get

$$H^{\cdot}(Z,F) = \bigoplus_{\pi\in\widehat{G}_u,\chi_\pi(C^{\mathfrak{g}})=0} n_\rho(\pi)H^{\cdot}(\mathfrak{g},K;V_{\pi,K}). \tag{8.70}$$

By (8.48) and (8.70), we get (8.66).

By Theorem 8.9, and by (8.66), we get (8.67). The proof of our proposition is completed. □

Remark 8.13. Equation (8.66) is [BoW00, Proposition VII.3.2]. When ρ is a trivial representation, (8.70) is originally due to Matsushima [Mat67].

In the rest of this section, G is supposed to be $\delta(G)=1$ and has compact center. Recall that η is a real finite dimensional representation of M satisfying Assumption 7.1, and that $\widehat{\eta}$ is defined in (7.1). In our second application of (8.65), we take $\tau=\widehat{\eta}$.

Theorem 8.14. *If* $(\pi,V_\pi)\in\widehat{G}_u$, *then*

$$\begin{aligned}\dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^+\big)^K &- \dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^-\big)^K \\ &= \frac{1}{\chi(K/K_M)}\sum_{i=0}^{\dim\mathfrak{p}_{\mathfrak{m}}}\sum_{j=0}^{2l}(-1)^{i+j}\dim_{\mathbf{C}}H^i\big(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes_{\mathbf{R}}E_\eta\big).\end{aligned} \tag{8.71}$$

Proof. Let $\Lambda(\pi)\in\mathfrak{h}^*_{\mathbf{C}}$ be the Harish-Chandra parameter of the infinitesimal character of π. By (8.29), for $t>0$, we have

$$\begin{aligned}\dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^+\big)^K &- \dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^-\big)^K \\ &= \mathrm{vol}(K)^{-1}e^{t\chi_\pi(C^{\mathfrak{g}})/2}\int_{g\in G}\Theta^G_\pi(g)\,\mathrm{Tr_s}\Big[p_t^{X,\widehat{\eta}}(g)\Big]\,dv_G.\end{aligned} \tag{8.72}$$

By (7.10), by Proposition 8.7 and by $H\cap K=T$, we have

$$\begin{aligned}\int_G\Theta^G_\pi(g)\,\mathrm{Tr_s}\Big[p_t^{X,\widehat{\eta}}(g)\Big]\,dv_G = \frac{\mathrm{vol}(T\backslash K)}{|W(H,G)|} \\ \int_{\gamma\in H'}\Theta^G_\pi(\gamma)\,\mathrm{Tr_s}^{[\gamma]}\big[\exp(-tC^{\mathfrak{g},X,\widehat{\eta}}/2)\big]\,\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}}\big|\,dv_H.\end{aligned} \tag{8.73}$$

Since $\gamma=e^ak^{-1}\in H'$ implies $T=K_M(k)=M^0(k)$, by (7.9), (8.72) and (8.73), we have

$$\begin{aligned}\dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^+\big)^K - \dim_{\mathbf{C}}\big(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^-\big)^K = \frac{1}{|W(H,G)|\,\mathrm{vol}(T)} \\ \frac{1}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\Big[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\Big]-\frac{t}{2}C^{\mathfrak{u}_{\mathfrak{m}},\eta}+\frac{t}{2}\chi_\pi(C^{\mathfrak{g}})\right) \\ \int_{\gamma=e^ak^{-1}\in H'}\Theta^G_\pi(\gamma)\exp\big(-|a|^2/2t\big)\,\mathrm{Tr}^{E_\eta}\big[\eta\left(k^{-1}\right)\big]\,\frac{\big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{g}/\mathfrak{h}}\big|}{\Big|\det(1-\mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp}\Big|^{1/2}}dv_H.\end{aligned} \tag{8.74}$$

By (6.19), for $\gamma = e^a k^{-1} \in H'$, we have

$$\frac{\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{g}/\mathfrak{h}}\right|}{\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{n}}\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} = e^{-l\langle\alpha,a\rangle}\left|\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{m}/\mathfrak{t}}\right|. \tag{8.75}$$

By (8.59), (8.63), (8.74), and (8.75), we have

$$\begin{aligned}\dim_{\mathbf{C}}\left(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^+\right)^K - \dim_{\mathbf{C}}\left(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^-\right)^K = \frac{1}{|W(H,G)|\,\mathrm{vol}(T)}\\ \frac{1}{\sqrt{2\pi t}}\exp\left(\frac{t}{16}\mathrm{Tr}^{\mathfrak{u}^\perp(\mathfrak{b})}\left[C^{\mathfrak{u}(\mathfrak{b}),\mathfrak{u}^\perp(\mathfrak{b})}\right]-\frac{t}{2}C^{\mathfrak{u}_\mathfrak{m},\eta}+\frac{t}{2}\chi_\pi(C^{\mathfrak{g}})\right)\\ \sum_{j=0}^{2l}(-1)^j\int_{\gamma=e^ak^{-1}\in H'}\Theta^{MA_Q}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\exp\left(-|a|^2/2t-l\langle\alpha,a\rangle\right)\\ \mathrm{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\right]\left|\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{m}/\mathfrak{t}}\right|dv_H.\end{aligned} \tag{8.76}$$

By (8.16), there exist $C>0$ and $c>0$ such that for $\gamma = e^a k^{-1}\in H'$, we have

$$\left|\Theta^{MA_Q}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\right|\left|\det\left(1-\mathrm{Ad}(k^{-1})\right)|_{\mathfrak{m}/\mathfrak{t}}\right|^{1/2}\leqslant Ce^{c|a|}. \tag{8.77}$$

By (8.62), (8.76), (8.77), and by letting $t\to 0$, we get

$$\begin{aligned}\dim_{\mathbf{C}}\left(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^+\right)^K - \dim_{\mathbf{C}}\left(V_{\pi,K}\otimes_{\mathbf{R}}\widehat{\eta}^-\right)^K = \frac{1}{|W(H,G)|\,\mathrm{vol}(T)}\\ \sum_{j=0}^{2l}(-1)^j\int_{\gamma\in T'}\Theta^{M}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\,\mathrm{Tr}^{E_\eta}[\eta(\gamma)]\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{m}/\mathfrak{t}}\right|dv_T,\end{aligned} \tag{8.78}$$

where T' is the set of the regular elements of M in T.

We claim that, for $0\leqslant j\leqslant 2l$, we have

$$\begin{aligned}\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}(-1)^i\dim_{\mathbf{C}}\left(H_j(\mathfrak{n},V_{\pi,K})\otimes_{\mathbf{R}}\Lambda^i(\mathfrak{p}_\mathfrak{m}^*)\otimes_{\mathbf{R}}E_\eta\right)^{K_M}\\ =\frac{1}{|W(T,M)|\,\mathrm{vol}(T)}\int_{\gamma\in T'}\Theta^{M}_{H_j(\mathfrak{n},V_{\pi,K})}(\gamma)\,\mathrm{Tr}^{E_\eta}\left[\eta\left(\gamma\right)\right]\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{m}/\mathfrak{t}}\right|dv_T.\end{aligned} \tag{8.79}$$

Indeed, consider $H_j(\mathfrak{n},V_{\pi,K})$ as a Harish-Chandra $(\mathfrak{m}_{\mathbf{C}},K_M)$-module. We can decompose $H_j(\mathfrak{n},V_{\pi,K})$ in the sense of (8.4), where the generalized infinitesimal characters are given by

$$\chi_{(w\Lambda(\pi)+\rho^{\mathfrak{u}}-\rho^{\mathfrak{u}(\mathfrak{b})})|_{\mathfrak{t}_{\mathbf{C}}}}, \tag{8.80}$$

for some $w\in W(\mathfrak{h}_{\mathbf{R}},\mathfrak{u})$. Therefore, it is enough to show (8.79) when $H_j(\mathfrak{n},V_{\pi,K})$ is replaced by any Harish-Chandra $(\mathfrak{m}_{\mathbf{C}},K_M)$-module with generalized infinitesimal character $\chi_{(w\Lambda(\pi)+\rho^{\mathfrak{u}}-\rho^{\mathfrak{u}(\mathfrak{b})})|_{\mathfrak{t}_{\mathbf{C}}}}$. Let (π^M,V_{π^M}) be a Hilbert globalization of such a Harish-Chandra $(\mathfrak{m}_{\mathbf{C}},K_M)$-module. As before, let $C^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}$ be the Casimir element of M acting on $C^\infty(M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta)^{K_M}$, and let $p_t^{X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}(g)$ be the smooth integral kernel of the heat operator $\exp(-tC^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}/2)$. Remark that by [BMaZ17, Proposition 8.4], $C^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}-C^{\mathfrak{m},E_\eta}$ is the Hodge Laplacian on X_M acting on the differential forms with values in the homogenous flat vector bundle $M\times_{K_M}E_\eta$. Proceeding as in [B11, Theorem 7.8.2], if $\gamma\in M$ is semisimple and nonelliptic, we have

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t(C^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}-C^{\mathfrak{m},E_\eta})/2\right)\right]=0. \tag{8.81}$$

Also, if $\gamma=k^{-1}\in K_M$, then

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(-t(C^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}-C^{\mathfrak{m},E_\eta})/2\right)\right]=\mathrm{Tr}^{E_\eta}\left[\eta\left(k^{-1}\right)\right]e\left(X_M(k),\nabla^{TX_M(k)}\right). \tag{8.82}$$

Using (8.81), proceeding as in (8.72) and (8.73), we have

$$\begin{aligned}\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}&(-1)^i\dim_\mathbf{C}\left(V_{\pi^M}\otimes_\mathbf{R}\Lambda^i(\mathfrak{p}^*_\mathfrak{m})\otimes_\mathbf{R}E_\eta\right)^{K_M}\\&=\mathrm{vol}(K_M)^{-1}\exp\left(t\chi_{\pi^M}(C^\mathfrak{m})/2\right)\int_{g\in M}\Theta^M_{\pi^M}(g)\,\mathrm{Tr_s}\left[p_t^{X_M,\Lambda^\cdot(\mathfrak{p}^*_\mathfrak{m})\otimes E_\eta}(g)\right]dv_M\\&=\frac{\exp\left(t\chi_{\pi^M}(C^\mathfrak{m})/2\right)}{|W(T,M)|\,\mathrm{vol}(T)}\int_{\gamma\in T'}\Theta^M_{\pi^M}(\gamma)\,\mathrm{Tr_s}^{[\gamma]}\left[\exp(-tC^{\mathfrak{m},X_M,\Lambda^\cdot(\mathfrak{p}_\mathfrak{m})\otimes E_\eta}/2)\right]\\&\qquad\qquad\left|\det\left(1-\mathrm{Ad}(\gamma)\right)|_{\mathfrak{m}/\mathfrak{t}}\right|dv_T.\end{aligned} \tag{8.83}$$

By (8.82), (8.83), proceeding as in (8.74), and letting $t\to0$, we get the desired equality (8.79).

The Euler formula asserts

$$\begin{aligned}\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}(-1)^i\dim_\mathbf{C}&\left(H_j(\mathfrak{n},V_{\pi,K})\otimes_\mathbf{R}\Lambda^i(\mathfrak{p}^*_\mathfrak{m})\otimes_\mathbf{R}E_\eta\right)^{K_M}\\&=\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}(-1)^i\dim_\mathbf{C}H^i\left(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes_\mathbf{R}E_\eta\right).\end{aligned} \tag{8.84}$$

By (3.17), we have

$$W(H,G)=W(T,K),\qquad\qquad W(T,M)=W(T,K_M). \tag{8.85}$$

By (7.5), (8.78), (8.79), (8.84)-(8.85), we get (8.71). The proof of our theorem is completed. □

Corollary 8.15. *The following identity holds:*

$$r_{\eta,\rho}=\frac{1}{\chi(K/K_M)}\sum_{\pi\in\widehat{G}_u,\chi_\pi(C^\mathfrak{g})=0}n_\rho(\pi)\sum_{i=0}^{\dim\mathfrak{p}_\mathfrak{m}}\sum_{j=0}^{2l}(-1)^{i+j}\dim_\mathbf{C}H^i\left(\mathfrak{m},K_M;H_j(\mathfrak{n},V_{\pi,K})\otimes_\mathbf{R}E_\eta\right). \tag{8.86}$$

Proof. This is a consequence of (7.58), (8.65), and (8.71). □

Remark 8.16. When $G=\mathrm{SO}^0(p,1)$ with $p\geqslant3$ odd, the formula (8.86) is compatible with [J01, Theorem 3.11].

We will apply (8.86) to η_j. The following proposition allows us to reduce the first sum in (8.86) to the one over $\pi\in\widehat{G}_u$ with $\chi_\pi=0$.

Proposition 8.17. *Let $(\pi, V_\pi) \in \widehat{G}_u$. Assume $\chi_\pi(C^{\mathfrak{g}}) = 0$ and*

$$H^{\cdot}\left(\mathfrak{m}, K_M; H_{\cdot}(\mathfrak{n}, V_\pi) \otimes_{\mathbf{R}} \Lambda^j(\mathfrak{n}^*)\right) \neq 0. \tag{8.87}$$

Then the infinitesimal character χ_π vanishes.

Proof. Recall that $\Lambda(\pi) \in \mathfrak{h}_{\mathbf{C}}^*$ is a Harish-Chandra parameter of π. We need to show that there is $w \in W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$ such that

$$w\Lambda(\pi) = \rho^{\mathfrak{u}}. \tag{8.88}$$

Let B^* be the bilinear form on $\mathfrak{g}^*$ induced by B. It extends to $\mathfrak{g}_{\mathbf{C}}^*$ and $\mathfrak{u}^*$ in an obvious way. Since $\chi_\pi(C^{\mathfrak{g},\pi}) = 0$, we have

$$B^*(\Lambda(\pi), \Lambda(\pi)) = B^*(\rho^{\mathfrak{u}}, \rho^{\mathfrak{u}}). \tag{8.89}$$

We identify $\mathfrak{h}_{\mathbf{R}}^* = \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^*$. By definition,

$$\rho^{\mathfrak{u}} = \left(\frac{l\alpha}{2\sqrt{-1}\pi}, \rho^{\mathfrak{u}_{\mathfrak{m}}}\right) \in \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^* \quad \text{and} \quad \rho^{\mathfrak{u}(\mathfrak{b})} = (0, \rho^{\mathfrak{u}_{\mathfrak{m}}}) \in \sqrt{-1}\mathfrak{b}^* \oplus \mathfrak{t}^*. \tag{8.90}$$

By (8.48), (8.80) and (8.87), there exist $w \in W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$, $w' \in W(\mathfrak{t}, \mathfrak{u}_{\mathfrak{m}}) \subset W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$ and the highest real weight $\mu_j \in \mathfrak{t}^*$ of an irreducible subrepresentation of $\mathfrak{m}_{\mathbf{C}}$ on $\Lambda^j(\mathfrak{n}_{\mathbf{C}}) \simeq \Lambda^j(\overline{\mathfrak{n}}_{\mathbf{C}}^*)$ such that

$$w\Lambda(\pi)|_{\mathfrak{t}_{\mathbf{C}}} = w'(\mu_j + \rho^{\mathfrak{u}_{\mathfrak{m}}}). \tag{8.91}$$

By (6.58), (8.89) and (8.91), there exists $w'' \in W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$ such that

$$w''\Lambda(\pi) = \left(\pm\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j + \rho^{\mathfrak{u}_{\mathfrak{m}}}\right) = \left(\pm\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) + \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8.92}$$

In particular, $w''\Lambda(\pi) \in \mathfrak{h}_{\mathbf{R}}^*$.

Clearly, $\left((j-l)\alpha/2\sqrt{-1}\pi, \mu_j\right) \in \mathfrak{h}_{\mathbf{R}}^*$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}}$ on $\Lambda^j(\overline{\mathfrak{n}}_{\mathbf{C}}^*) \otimes_{\mathbf{C}} (\det(\mathfrak{n}_{\mathbf{C}}))^{-1/2}$. By (6.33), $\left((j-l)\alpha/2\sqrt{-1}\pi, \mu_j\right) \in \mathfrak{h}_{\mathbf{R}}^*$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}}$ on $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$. By [BoW00, Lemma II.6.9], there exists $w_1 \in W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$ such that

$$\left(\frac{(j-l)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) = w_1\rho^{\mathfrak{u}} - \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8.93}$$

Similarly, $\left((l-j)\alpha/2\sqrt{-1}\pi, \mu_j\right) \in \mathfrak{h}_{\mathbf{R}}^*$ is the highest real weight of an irreducible subrepresentation of $\mathfrak{m}_{\mathbf{C}} \oplus \mathfrak{b}_{\mathbf{C}}$ on both $\Lambda^{2l-j}(\overline{\mathfrak{n}}_{\mathbf{C}}^*) \otimes_{\mathbf{C}} (\det(\mathfrak{n}_{\mathbf{C}}))^{-1/2}$ and $S^{\mathfrak{u}^{\perp}(\mathfrak{b})}$. Therefore, there exists $w_2 \in W(\mathfrak{h}_{\mathbf{R}}, \mathfrak{u})$ such that

$$\left(\frac{(l-j)\alpha}{2\sqrt{-1}\pi}, \mu_j\right) = w_2\rho^{\mathfrak{u}} - \rho^{\mathfrak{u}(\mathfrak{b})}. \tag{8.94}$$

By (8.92)-(8.94), we get (8.88). The proof of our proposition is completed. □

Corollary 8.18. *For $0 \leqslant j \leqslant 2l$, we have*

(8.95)

$$r_j = \frac{1}{\chi(K/K_M)} \sum_{\pi \in \widehat{G}_u, \chi_\pi = 0} n_\rho(\pi) \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} \sum_{k=0}^{2l} (-1)^{i+k} \dim_{\mathbf{C}} H^i\left(\mathfrak{m}, K_M; H_k(\mathfrak{n}, V_{\pi,K}) \otimes_{\mathbf{R}} \Lambda^j(\mathfrak{n}^*)\right).$$

If $H^{\cdot}(Z, F) = 0$, then for all $0 \leqslant j \leqslant 2l$,

$$r_j = 0. \tag{8.96}$$

Proof. This is a consequence of Proposition 8.12, Corollary 8.15 and Proposition 8.17. □

Remark 8.19. By (7.75) and (8.96), we get (5.13) when G has compact center and $\delta(G) = 1$.

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