TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 177, March 1973

ANALYTIC WAVE FRONT SETS FOR SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF PRINCIPAL TYPE

BY

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ABSTRACT. The propagation of analyticity for solutions u of $P(x, D)u = f$ is studied, in terms of wave front sets, for a large class of differential operators $P = P(x, D)$ of principal type. In view of a theorem by L. Hörmander [9], the results obtained imply rather precise results about the surjectivity of the mapping $P: C^\infty(\Omega) \to C^\infty(\Omega)$.

Introduction. Let $P = P(x, D)$ be a linear differential operator with C^∞-coefficients in an open set $\Omega \subset R^n$. When P is elliptic, then the classical regularity theorem for elliptic equations says that the distribution u is infinitely differentiable whenever Pu is and, if the coefficients of P are analytic, then u is analytic where Pu is analytic. The corresponding question, for more general operators, of describing the set of singularities of u when that of Pu is given has been much studied lately (see [10] and the references there). The introduction of the concept of wave front sets (see [7] and [14]) has added precision to the statements and has also simplified many proofs. For operators with real principal part $P_m(x, D)$, such that the Hamilton field

$$H_{P_m} = \sum_{1\le j\le n} [(\partial P_m(x, \xi)/\partial \xi_j)\partial/\partial x_i \ - \ (\partial P_m(x, \xi)/\partial x_j)\partial/\partial \xi_j]$$

is nondegenerate when $\xi \in \dot{R}^n = R^n \setminus \{0\}$, and for some operators with complex coefficients, very complete results for the C^∞-case are obtained by Duistermaat and Hörmander in [5] (see also [17]). Corresponding results, concerning analyticity, are proved in [1] (operators with constant coefficients), [9] and [11], under the assumption that P_m is real and $d_\xi P_m(x, \xi) = (\partial P_m(x, \xi)/\partial \xi_1, \dots, \partial P_m(x, \xi)/\partial \xi_n) \neq 0$ in $\Omega \times \dot{R}^n$. The purpose of this paper is to extend the results in the analytic case to certain operators with complex coefficients. In doing so, we shall also give new proofs for operators with real principal part. The argument is modeled on the very elegant proof, in the C^∞-case, which Hörmander gave in his Congress lecture [8] for operators with real principal part. The main results

Received by the editors February 1, 1972.
AMS (MOS) subject classifications (1970). Primary 35A05, 35A20.
Key words and phrases. Analytic wave front sets, pseudo-differential operators.

are stated in §1, and in §2 we define certain sequences of pseudo-differential operators used to localize the problem in $T^*(\Omega)$. §3 contains a proof of the non-characteristic regularity theorem and two lemmas about the existence of sequences of pseudo-differential operators suitable for the localization. To obtain the local regularity theorem, we use a variant of a classical inequality of Calderón [4], proved in §4, and in §5 the proofs of the theorems stated in §1 are completed. The results of this paper have been announced in [2].

1. **Statement of the main results.** Let K be a compact subset of the open set $\Omega \subset R^n$. Then it is well known (see e.g. [1] and the references there) that there are functions $\phi_N \in C_0^\infty(\Omega)$ which are equal to 1 on K and, for some constant C, satisfy

$$|D^\alpha \phi_N(x)| \leq C(CN)^{|\alpha|}, \quad \text{when} \quad |\alpha| \leq N. \tag{1.1}$$

It is easy to see (compare [9]) that a distribution u is analytic in a neighborhood of $x_0 \in \Omega$ if and only if there is such a sequence (ϕ_N) with all $\phi_N = 1$ on a neighborhood of x_0 and

$$|\widehat{(\phi_N u)}(\xi)| \leq C^N(1 + |\xi|/N)^{-N}. \tag{1.2}$$

This motivates the following definition of the analytic wave front set, $\mathrm{WF}_a(u)$, of a distribution u.

Definition 1.1. Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \in \Omega \times \dot{R}^n$. We say that $(x_0, \xi_0) \notin \mathrm{WF}_a(u)$ if and only if there is a neighborhood U of x_0 and a sequence $u_N \in \mathcal{E}'(\Omega)$ such that $u_N = u$ in U and

$$|\hat{u}_N(\xi)| \leq C^N(1 + |\xi|/N)^{-N} \tag{1.3}$$

where ξ belongs to some fixed conic neighborhood of ξ_0.

Remark 1.1. Definition 2.1 is readily seen to be equivalent to the definition of $\mathrm{WF}_a(u)$ given in [9]. There is also available the set supp sp u (see [16]) whose definition, for a hyperfunction u, was indicated by M. Sato in [14] (see also [15]). Probably $\mathrm{WF}_a(u) = \text{supp sp } u$, when u is a distribution, but so far this has not been proved.

$\mathrm{WF}_a(u)$ is thus a subset of $\Omega \times \dot{R}^n$ or, if one wants to emphasize the behavior under coordinate transformations, $\dot{T}^*(\Omega)$, where the dot indicates that the zero-section is removed. More precisely, it is proved in [9] that, if $y = \kappa(x)$ is an analytic change of coordinates in Ω, and β denotes the map on $\Omega \times \dot{R}^n$ defined by $\beta(x, \xi) = (\kappa(x), ({}^t\kappa'(x))^{-1}(\xi))$, then $\mathrm{WF}_a(u \circ \kappa^{-1}) = \beta(\mathrm{WF}_a(u))$. This also follows directly from Theorem 2.3 and the alternative characterization of $\mathrm{WF}_a(u)$ which is given in §3. Denote by π the projection $T^*(\Omega) \to \Omega$ and by a.s. u the complement of the largest open subset of Ω on which u is analytic. Then

(1.4) $$\pi(\mathrm{WF}_a(u)) = \text{a.s. } u$$

(for a proof, see [9]). Now let $P = P(x, D)$ be linear differential operator in Ω with analytic coefficients and denote by $Z(P_m)^a$ the set $\{(x, \xi) \in \Omega \times \dot{R}^n;\ P_m(x, \xi) = 0\}$. Then we have the following generalization of the elliptic regularity theorem.

Theorem 1.1 (Sato [14], Hörmander [9]). $\mathrm{WF}_a(u) \subset \mathrm{WF}_a(Pu) \cup Z(P_m)$.

Note that if P is elliptic and Pu analytic, then the right-hand side is empty, so (1.4) implies that u is analytic. In §3 we shall prove a result, Lemma 3.1, which is slightly stronger than Theorem 1.1.

The more precise results about propagation of singularities inside $Z(P_m)$ will depend on the properties of the bicharacteristics of P. If P_m is real and H_{P_m} is nondegenerate, then $Z(P_m)$ is a manifold and H_{P_m} is tangent to $Z(P_m)$. The integral curves in $Z(P_m)$ corresponding to H_{P_m} are called the bicharacteristic strips of P and their projections on Ω are called bicharacteristic curves. If, in addition, $d_\xi P_m \neq 0$ on a bicharacteristic strip, then the corresponding bicharacteristic curve is regular. We denote by $N_1(P_m)$ the set $\{(x, \xi) \in Z(P_m);\ d_\xi P_m(x, \xi) \neq 0\}$.

Theorem 1.2. *If $P = P(x, D)$ has analytic coefficients and real principal part $P_m(x, D)$, then*

$$\mathrm{WF}_a(u) \textit{ is invariant under } H_{P_m} \textit{ in } N_1(P_m)\setminus \mathrm{WF}_a(Pu).$$

When P is principal type, i.e., $N_1(P_m) = Z(P_m)$, and has real principal part, then Theorem 1.1 and Theorem 1.2 together with (1.4) immediately imply the following result.

Suppose that the distribution u is analytic in $\Omega_0 \subset \Omega$. Then u is analytic in a neighborhood of any point x_0 such that, for every bicharacteristic curve l through x_0, the component of $(\Omega \cap l)\setminus(\text{a.s. } Pu)$ which contains x_0 also contains some point in Ω_0.

This result was proved for operators with constant coefficients in [1]. For general operators with analytic coefficients the more precise Theorem 1.2 has been proved by Hörmander [9] and, for hyperfunctions and Sato's wave front sets, by Kawai-Kashiwara (see [11]). We shall give another proof of Theorem 1.2.

When P_m has complex coefficients, then $Z(P_m)$ is a manifold of codimension 2, provided that the vector-fields $H_{\mathrm{Re}\, P_m}$ and $H_{\mathrm{Im}\, P_m}$ are linearly independent. If, in addition, the Poisson bracket $\{\mathrm{Re}\, P_m, \mathrm{Im}\, P_m\} = H_{\mathrm{Re}\, P_m}(\mathrm{Im}\, P_m)$ vanishes in $Z(P_m)$ then the vector fields $H_{\mathrm{Re}\, P_m}$ and $H_{\mathrm{Im}\, P_m}$ are tangent to $Z(P_m)$ and, because of the Frobenius integrability theorem, they define a 2-dimensional foliation of $Z(P_m)$. This foliation is called the bicharacteristic foliation and its leaves are called the bicharacteristic strips of P. If d_ξ Re P_m and d_ξ Im P_m are linearly independent on a bicharacteristic strip, then its projection on Ω is a regular 2-dimensional manifold. We put

$$N_2(P_m) = N_2'(P_m) \cap N_2''(P_m),$$

where

$$N_2'(P_m) = \{(x, \xi) \in Z(P_m);\ d_\xi \operatorname{Re} P_m(x, \xi) \text{ and } d_\xi \operatorname{Im} P_m(x, \xi) \text{ are linearly independent}\}.$$

$$N_2''(P_m) = \{(x_0, \xi_0) \in Z(P_m);\ \{\operatorname{Re} P_m, \operatorname{Im} P_m\}(x, \xi) = 0 \text{ in a neighborhood, in } Z(P_m), \text{ of } (x_0, \xi_0)\}.$$

Theorem 1.3. *If $P = P(x, D)$ has analytic coefficients, then* $\mathrm{WF}_a(u)$ *is invariant under the bicharacteristic foliation in* $N_2(P_m)\backslash \mathrm{WF}_a(Pu)$.

Remark 1.2. T. Kawai has announced (private communication) that, by extending the theory of Fourier integral operators to the analytic category, he and M. Kashiwara have proved Theorem 1.3 for hyperfunctions. The proof we shall give has quite a different character, since only pseudo-differential operators are used.

A third case which we shall consider is when $P = P(D)$ has constant coefficients. Then

$$H_{P_m} = \sum_{1 \le j \le n} (\partial P_m(\xi)/\partial \xi_j)\partial/\partial x_j$$

is a differential operator in Ω with constant coefficients depending on the parameter ξ. If

$$(x_0, \xi_0) \in N_1(P_m) = \Omega \times \{\xi \in \dot{R}^n;\ P_m(\xi) = 0 \text{ and } d_\xi P_m(\xi) \neq 0\}$$

then this operator is nondegenerate and the bicharacteristic strip through (x_0, ξ_0) is defined to be $\{(x_0 + d_\xi (zP_m)(\xi_0), \xi_0);\ z \in \mathbf{C}\}$. These bicharacteristic strips define a foliation, the bicharacteristic foliation of $N_1(P_m)$. Note that the dimension of the leaves may vary between 1 and 2 depending on whether d_ξ Re P_m and d_ξ Im P_m are linearly dependent or not.

Theorem 1.4. *If $P = P(D)$ has constant coefficients, then* $\mathrm{WF}_a(u)$ *is invariant under the bicharacteristic foliation in* $N_1(P_m)\backslash \mathrm{WF}_a(Pu)$.

Theorems 1.1–1.3 supplement the results of Duistermaat-Hörmander [5] for operators with analytic coefficients, provided that the projections of the bicharacteristic strips are regular. Then it follows directly from Theorems 1.1–

1.3 that the conditions imposed on P and Ω in [5] to prove that P maps $\mathfrak{D}'(\Omega)/C^\infty(\Omega)$ onto itself actually give that $PC^\infty(\Omega) = C^\infty(\Omega)$ and thus $P\mathfrak{D}'(\Omega) = \mathfrak{D}'(\Omega)$. However, for operators with analytic coefficients, $PC^\infty(\Omega) = C^\infty(\Omega)$ under much weaker assumptions. In fact, Hörmander [9] observed that results about propagation of analyticity for wave front sets can be used to derive very precise uniqueness theorems. He proved the following general result.

Theorem 1.5 (Hörmander [9]). *Suppose that Ω_0 is an open subset of Ω with C^1 boundary $\partial\Omega_0$. Denote by N_0 the normal of $\partial\Omega_0$ at $x_0 \in \Omega$ and let F be a conic neighborhood of $(x_0, \pm N_0)$. Then there is a neighborhood Ω' of x_0 such that any $u \in \mathfrak{D}'(\Omega)$, with $\mathrm{WF}_a(u) \cap F = \emptyset$, which vanishes in Ω_0 must also vanish in Ω'.*

To be brief, we just state what this theorem together with Theorem 1.4 implies for operators with constant coefficients.

Theorem 1.6. *Let $\Omega \subset R^n$ be an open set with C^1 boundary $\partial\Omega$ such that all points $x_0 \in \partial\Omega$, characteristic with respect to $P(D)$, are simply characteristic, i.e. $d_\xi P_m(N_0) \neq 0$ for the normal N_0 of $\partial\Omega$ at x_0. Denote by B_{x_0} the projection of the bicharacteristic strip through (x_0, N_0) and by $\hat{K}$ the convex hull of a compact set K. If, for any characteristic $x_0 \in \partial\Omega$ and any compact set $K \subset \Omega$, the component of $B_{x_0} \cap \complement K$ containing x_0 also contains some point in $\complement(\bar{\Omega} \cap \hat{K})$ then $P(D)C^\infty(\Omega) = C^\infty(\Omega)$ and, what is equivalent, $P(D)\mathfrak{D}'_F(\Omega) = \mathfrak{D}'_F(\Omega)$.*

This result improves Theorem 1.3.7 of [10].

2. **A space of sequences of pseudo-differential operators.** Functions ϕ_N satisfying (1.1) are suitable for the localization of problems concerning analytic functions in Ω. We shall now define certain sequences of symbols which will fill the same purpose in the cotangent space $T^*(\Omega)$.

Definition 2.1. A sequence $(a_N(x, \xi))$ is said to belong to $\tau^r(\Omega)$ if, for every compact set $K \subset \Omega$, there are positive constants δ and C such that, if $N \geq \delta^{-2}$, $a_N \in C^{[\delta N]}(K \times R^n)$ and

$$\sup_{x \in K} |D_x^\alpha D_\xi^\beta a_N(x, \xi)| \leq C^N N^{|\alpha|}(1 + |\xi|/N)^{r-|\beta|}, \quad \text{when } |\alpha + \beta| \leq \delta N. \tag{2.1}$$

Remark 2.1. The reason for the presence of the constant δ in Definition 2.1 is purely technical. For example, the right-hand side of (2.6) will in general admit fewer derivatives than a_N and b_N. However, by putting $a'_N = a_{[N \cdot \delta^{-2}]+1}$, we can always obtain a sequence for which $\delta = 1$.

Example 2.1. Let $P(x, D)$ be a linear differential operator of order r with analytic coefficients in Ω and put $a_N(x, \xi) = P(x, \xi)$. Then $(a_N) \in \tau^r(\Omega)$.

Example 2.2. Suppose that the functions $\phi_N(x)$ satisfy (1.1) in Ω and that $\psi_N(\xi)$ are homogeneous functions of degree r, satisfying (1.1) when $|\xi| = 1$. If χ_N vanishes when $|\xi| \le 1$, equals 1 when $|\xi| \ge 2$ and satisfies (1.1) then $(\phi_N(x)\psi_N(\xi)\chi_N(\xi/N)) \in \tau^r(\Omega)$. For convenience, we shall introduce a special notation for sequences of this type. Let F and F' be two open cones in $\Omega \times \dot{R}^n$, i.e., open subsets of $\Omega \times \dot{R}^n$ which are invariant under multiplication of the ξ-coordinate with a positive number. Suppose that $\dot{\overline{F}} \subset F'$ and denote by $\tau^0(F, F')$ the set of sequences $(a_N) \in \tau^0(\Omega)$ such that, for some $C > 0$, $a_N(x, \xi)$ vanishes in $(\mathbf{C}F') \cup \{(x, \xi);\ |\xi| \le CN\}$ and $a_N(x, \xi) = 1$ in $F \cap \{(x, \xi);\ |\xi| \ge 2CN\}$.

Example 2.3. Suppose that $F' \subset \Omega \times \dot{R}^n$ is an open cone such that $\pi(F') = \{x;\ (x, \xi) \in F'\} \subset\subset \Omega$, i.e., $\overline{\pi(F')}$ is a compact subset of Ω, and let p be an analytic symbol of order r in F', in the sense of Boutet de Monvel and Krée [3]. This means that p is a formal sum $p = \Sigma_{-\infty < k \le r} p^k$ with the property that for every cone F, with $\dot{\overline{F}} \subset F'$, there is a constant $C > 0$ such that

$$|D_x^\alpha D_\xi^\beta p^k(x, \xi)| \le C^{|\alpha+\beta|+|k|+1}|k|!|\alpha|!|\beta|\,|\xi|^{k-|\beta|}, \quad \text{when } (x, \xi) \in F. \tag{2.2}$$

Moreover, it is assumed that $p^k(x, \xi)$ is homogeneous of degree k with respect to ξ. If $(\mu_N(x, \xi)) \in \tau^0(F, F')$ for some open cone F, with $\dot{\overline{F}} \subset F'$, then $(p_N(x, \xi)) = (\Sigma_{-N \le k \le r}\, p^k(x, \xi)\mu_N(x, \xi)) \in \tau^r(\Omega)$. Note that it follows from (2.2) that $p^k(x, \xi)$ is an analytic function in F'. In order to obtain nontrivial symbols with support in a compactly generated cone, we therefore have to consider the sequences in $\tau^r(\Omega)$.

If $(a_N) \in \tau^r(\Omega)$, $\Omega' \subset\subset \Omega$ and $\delta = \delta(\Omega')$ is small enough, then a_N defines a mapping $C_0^\infty(\Omega') \to C^{[\delta N]}(\Omega')$ by means of the formula

$$a_N(x, D)u(x) = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle} a_N(x, \xi)u(\xi)\,d\xi, \qquad u \in C_0^\infty(\Omega'). \tag{2.3}$$

By duality this mapping extends to a mapping $\mathcal{E}'_{[\delta N]}(\Omega') \to \mathcal{D}'(\Omega')$. We shall now give conditions on the symbol sequence $(a_N(x, \xi))$ which will ensure that the sequence of operators defined by (2.3) is regularizing in the following sense. For any $u \in \mathcal{E}'(\Omega)$ and any $\Omega' \subset\subset \Omega$ there are constants δ and C such that $a_N(x, D)u(x) \in C^{[\delta N]}(\Omega')$, when $N \ge \delta^{-2}$, and

$$\sup_{x \in \Omega'} |D_x^\alpha a_N(x, D)u(x)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N. \tag{2.4}$$

Definition 2.2. We say that the sequence $(a_N(x, \xi))$ is in $\tau_0(\Omega)$ if, for every compact set $K \subset \Omega$, there are positive constants δ and C such that, if $N \ge \delta^{-2}$, $a_N \in C^{[\delta N]}(K \times R^n)$ and

$$\sup_{x \in K} |D_x^\alpha D_\xi^\beta a_N(x, \xi)| \le C^N N^{|\alpha|}(1 + |\xi|/N)^{-k-|\beta|}, \quad \text{when } |\alpha + \beta| + k \le \delta N. \tag{2.5}$$

If $(a_N), (b_N) \in r^r(\Omega)$ and $(a_N - b_N) \in r_0(\Omega)$, we write $(a_N) \sim (b_N)$.

Remark 2.2. Obviously $r_0(\Omega) \subset \bigcap_r r^r(\Omega)$. However, much more information is provided by (2.3). In particular it is clear that $r_0(\Omega)$ is a proper subset of $\bigcap_r r^r(\Omega)$.

Example 2.4. Let $(\chi_N(\xi))$ be a sequence of functions satisfying (1.1) and vanishing when $|\xi| > C > 0$. Then $(\chi_N(\xi/N)) \in r_0(R^n)$.

We shall now give a calculus for the sequences of pseudo-differential operators associated with the spaces $r^r(\Omega)$. The treatment will contain nothing essentially new beyond "classical" expositions of the calculus of pseudo-differential operators. We shall, therefore, be brief at some points and refer to [6], [7], and [13] for the calculus of pseudo-differential operators.

Theorem 2.1. *Let* $(a_N) \in r^r(\Omega)$ and $(b_N) \in r^s(\Omega)$. *Suppose that* $b_N(x, \xi)$ *vanishes when* x *is outside a fixed compact set* $K \subset \Omega$. *Then there is a sequence* $(a_N \circ b_N) \in r^{r+s}(\Omega)$ *such that when* $u \in C_0^\infty(\Omega)$ *and* N *is large, then* $(a_N \circ b_N)(x, D)u = a_N(x, D)(b_N(x, D)u)$. *If* ϵ *is small enough, we have*

$$(2.6) \qquad ((a_N \circ b_N)(x, \xi)) \sim \Big(\sum_{|\alpha| \le \epsilon N} i^{|\alpha|} (D_\xi^\alpha a_N(x, \xi))(D_x^\alpha b_N(x, \xi))/\alpha! \Big).$$

Proof. Put

$$v_N(x) = b_N(x, D)u(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} b_N(x, \xi)\hat{u}(\xi)\, d\xi.$$

If N is large, $a_N(x, D)v_N(x)$ is well defined and we may operate with $a_N(x, D)$ under the sign of integration. Therefore, $(a_N \circ b_N)(x, \xi) = a_N(x, D + \xi) b_N(x, \xi)$ and we only have to prove that

$$(r_N) = \Big(a_N \circ b_N - \sum_{|\alpha| \le \epsilon N} i^{|\alpha|} (D_\xi^\alpha a_N)(D_x^\alpha b_N)/\alpha! \Big)$$

belongs to $r_0(\Omega)$. If $\hat{b}_N$ denotes the Fourier transform of b_N with respect to x, then Taylor's formula gives that

$$r_N(x, \xi) = (2\pi)^{-n} \int r_N(x, \xi, \eta)\, d\eta,$$

where

$$r_N(x, \xi, \eta) = \sum_{|\gamma| = [\epsilon N]+1} i^{|\gamma|} e^{i\langle x, \eta\rangle} (\eta^\gamma \hat{b}_N(\eta, \xi)/\gamma!) \cdot \Big(\int_0^1 D_\xi^\gamma a_N(x, \xi + t\eta)(1 - t)^{|\gamma| - 1}\, dt \Big).$$

When x belongs to a compact set K and $2|\eta| \le |\xi|$, then (2.1) implies

$$(2.7) \qquad |\xi|^k |D_x^{\alpha'} D_\xi^{\beta' + \gamma} a_N(x, \xi + t\eta)| \le C_1^N N^{|\alpha'| + k} (1 + |\xi|/N)^{r + k - |\gamma| - |\beta'|}$$

and

(2.8) $|\eta^{\alpha''+\gamma} D_\xi^{\beta''} \hat{b}_N(\eta, \xi)/\gamma!| \leq C_2^N N^{|\alpha''|+|\gamma|}/(1 + |\xi|/N)^{|\beta''|}(1 + |\eta|)^{n+1}\gamma!.$

Thus if $|\alpha| = |\alpha'| + |\alpha''|$, $|\beta| = |\beta'| + |\beta''|$ and $r + k \leq |\gamma| = [\epsilon N]$, we have

(2.9) $|\xi|^k |D_x^\alpha D_\xi^\beta r_N(x, \xi, \eta)| \leq (C_3^N N^{|\alpha|+k}/(1 + |\xi|/N)^{|\beta|}(1 + |\eta|)^{n+1}(N^{|\gamma|}/\gamma!)$

when $2|\eta| \leq |\xi|$. On the other hand (2.1) implies the following inequalities when $x \in K$ and $|\xi| \leq 2|\eta|$:

(2.10) $|D_x^{\alpha'} D_\xi^{\beta'+\gamma} a_N(x, \xi + t\eta)| \leq C_4^N N^{|\alpha'|}$;

(2.11) $|\xi|^j |\eta^{\alpha''+\gamma} D_\xi^{\beta''} \hat{b}_N(\eta, \xi)| \leq C_5^N N^{|\alpha''|+j+|\gamma|}/(1 + |\xi|/N)^{|\beta''|}(1 + |\eta|)^{n+1}\gamma!.$

By choosing $j = k + |\beta'|$ and $j = k$ we see that (2.9) is valid also when $|\xi| \leq 2|\eta|$, perhaps with another constant. This proves Theorem 2.1, if we also note that $N^{|\gamma|}/\gamma! \leq N^N/N! \leq C^N$.

Suppose now that $(a_N) \in r^r(\Omega)$ and let $a_N(x, D)$ be the mapping defined by (2.3). The distribution kernel of this mapping is defined by

(2.12) $K_N(w) = (2\pi)^{-n} \iint e^{i\langle x, \xi\rangle} a_N(x, \xi)\hat{w}(x, \xi)\, dx\, d\xi, \qquad w \in C_0^\infty(\Omega \times \Omega),$

where $\hat{w}$ denotes Fourier transform with respect to the second variable. For any compact set M in $\Omega \times \Omega$, which does not intersect the diagonal, there are positive constants δ and C such that $K_N \in C^{[\delta N]}(M)$, when $N \geq \delta^{-2}$, and

(2.13) $\sup_{x,y \in M} |D_x^\alpha D_y^\beta K_N(x, y)| \leq C^N N^{|\alpha|+|\beta|}, \quad \text{when } |\alpha| + |\beta| \leq \delta N.$

For sequences $(a_N) \in r_0(\Omega)$ the kernel K_N belongs to $C^{[\delta N]}(M)$ and satisfies (2.13) for any compact set $M \subset \Omega \times \Omega$, if $N \geq \delta^{-2} = \delta(M)^{-2}$. Conversely, suppose that $(K_N(x, y))$ is a sequence such that, for every compact set $M \subset \Omega \times \Omega$, there are positive constants δ and C such that $K_N \in C^{[\delta N]}(M)$ and satisfies (2.13) when $N \geq \delta^{-2}$. If in addition the functions $K_N(x, y)$ vanish when y is outside a fixed compact set, then the sequence $(a_N(x, \xi)) = (e^{-i\langle x, \xi\rangle} \int K_N(x, y) e^{i\langle y, \xi\rangle} dy)$ belongs to $r_0(\Omega)$ and

$$a_N(x, D)u(x) = \int K_N(x, y)u(y)\, dy.$$

However, even if there is no compact set $L \subset \Omega$ such that all $K_N(x, y)$ vanish when $y \notin L$, the mapping $C_0^\infty(\Omega') \to \mathfrak{D}'(\Omega')$ given by

$$u \mapsto \int K_N(\cdot, y)u(y)\, dy, \qquad u \in C_0^\infty(\Omega'),$$

is well defined for any $\Omega' \subset\subset \Omega$ if N is large enough. We denote by $T_0(\Omega)$ the set of such mappings. More generally we make the following definition.

Definition 2.3. Suppose that (A_N) is a sequence such that if $\Omega' \subset\subset \Omega$ then

A_N is a linear mapping from $C_0^\infty(\Omega')$ to $\mathcal{D}'(\Omega')$, for large N. Let (ϕ_N) be a sequence of functions in $C_0^\infty(\Omega)$ with support in a fixed compact set and satisfying (1.1). Assume now that for every such sequence (ϕ_N) there is a sequence $(a_N) \in \tau^r(\Omega)$ such that, if $u \in C_0^\infty(\Omega)$ and N is large, $A_N(\phi_N u)(x) = a_N(x, D)u(x)$. We then say that $(A_N) \in T^r(\Omega)$.

The formula (2.12) extends to define a sequence of kernels $(K_N(x, y))$ for any sequence $(A_N) \in T^r(\Omega)$. We only have to suppose that $\phi_N(y) = 1$ when (x, y) belongs to the support of w. Clearly $T_0(\Omega) \subset T^r(\Omega)$ for every r. We shall now define another subclass of $T^r(\Omega)$.

Definition 2.4. A sequence $(A_N) \in T^r(\Omega)$ is called properly supported if the corresponding kernels $K_N(x, y)$ have support in a set $M \subset \Omega \times \Omega$ such that both projections $M \longrightarrow \Omega$ are proper.

Every sequence $(A_N) \in T^r(\Omega)$ can now be written as a sum $(A_N' + A_N'')$ where (A_N') is properly supported and $(A_N'') \in T_0(\Omega)$. In fact, let M be a neighborhood of the diagonal in Ω such that both projections $M \longrightarrow \Omega$ are proper. Choose functions $\chi_N \in C^\infty(\Omega \times \Omega)$ such that $\chi_N(x, y) = 1$ in a fixed neighborhood of the diagonal, χ_N has support in M and satisfies (1.1) on compact subsets of $\Omega \times \Omega$. Let (K_N) be the sequence of kernels corresponding to (A_N) and denote by (A_N'') the sequence of operators corresponding to $((1 - \chi_N)K_N)$. Then $(A_N'') \in T_0(\Omega)$ and $(A_N') = (A_N - A_N'')$ is properly supported.

If $(B_N) \in T^r(\Omega)$ is properly supported, it is easy to see that there is a sequence $(b_N) \in \tau^r(\Omega)$ such that $(B_N) = (b_N(x, D))$. Therefore, every sequence $(A_N) \in T^r(\Omega)$ has a symbol in the sense of the following definition.

Definition 2.5. The sequence $(a_N) \in \tau^r(\Omega)$ is called a symbol of $(A_N) \in T^r(\Omega)$ if $(A_N - a_N(x, D)) \in T_0(\Omega)$.

The symbol of a sequence $(A_N) \in T^r(\Omega)$ is uniquely determined modulo $\tau_0(\Omega)$. In fact, we have already shown that if the kernels $K_N(x, y)$ corresponding to $(a_N(x, D)) \in T_0(\Omega)$ vanish when y is outside a fixed compact subset of Ω, then $(a_N) \in \tau_0(\Omega)$. In the general case one just has to multiply $K_N(x, y)$ with a sequence $\phi_N(y)$ satisfying (1.1), with support in a fixed compact set, and apply Theorem 2.1.

Theorem 2.1 immediately extends to give the symbol of the composition of two sequences $(A_N) \in T^r(\Omega)$ and $(B_N) \in T^s(\Omega)$, provided that one of them is properly supported. We are now going to study the effect on the symbol, modulo $\tau_0(\Omega)$, of transposing the sequence of operators and of making an analytic change of coordinates in Ω. The space $T_0(\Omega)$ is invariant under these operations, so we may suppose that the sequence of operators (A_N) is properly supported. Given a properly supported sequence $(A_N) \in T^r(\Omega)$ there is a sequence $(a_N) \in \tau^r(\Omega)$ such that, if $u \in C_0^\infty(\Omega)$ and N is large, then

$$(A_N u)(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} a_N(x, \xi)\hat{u}(\xi)\, d\xi$$
$$= (2\pi)^{-n} \int \left(\int e^{i\langle x-y, \xi\rangle} a_N(x, \xi) u(y)\, dy \right) d\xi.$$

Since (A_N) is properly supported, there are functions $\chi_N \in C^\infty(\Omega \times \Omega)$ satisfying (1.1) on compact subsets and having support in a set M, for which both projections $M \longrightarrow \Omega$ are proper, such that if $a_N'(x, y, \xi) = \chi_N(x, y) a_N(x, \xi)$ then

(2.14) $$(A_N u)(x) = (2\pi)^{-n} \int \left(\int e^{i\langle x-y, \xi\rangle} a_N'(x, y, \xi) u(y)\, dy \right) d\xi.$$

This representation of (A_N) is particularly useful for the study of the transposed sequence and the sequence obtained after a change of variables. Suppose more generally that $b_N'(x, y, \xi)$ are functions such that for every compact set $L \subset \Omega \times \Omega$ there are positive constants δ and C such that $b_N' \in C^{[\delta N]}(L \times R^n)$, when $N \geq \delta^{-2}$, and

(2.15) $$\sup_{x, y \in L} |D_x^\alpha D_y^\beta D_\xi^\gamma b_N'(x, y, \xi)| \leq C^N N^{|\alpha| + |\beta|}(1 + |\xi|/N)^{r - |\gamma|},$$
$$\text{when } |\alpha| + |\beta| + |\gamma| \leq \delta N.$$

Then (2.14) defines a sequence of operators (B_N). We shall show that this sequence still belongs to $T^r(\Omega)$. Clearly we may suppose that $b_N'(x, y, \xi)$ vanishes when (x, y) is outside a set $M \subset \Omega \times \Omega$, such that both projections $M \longrightarrow \Omega$ are proper. In fact, every sequence (B_N) may be written as a sum of a sequence of this type and a sequence in $T_0(\Omega)$. Now put $b_N(x, \xi) = e^{-i\langle x, \xi\rangle} B_N(e^{i\langle x, \xi\rangle})$, i.e.

$$b_N(x, \xi) = (2\pi)^{-n} \int \left(\int b_N'(x, x+y, \xi + \eta) e^{-i\langle y, \eta\rangle}\, dy \right) d\eta,$$

and operate with B_N under the sign of integration in $u(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} u(\xi)\, d\xi$. This gives that $(B_N u)(x) = b_N(x, D)u(x)$, so we have to show that $(b_N) \in \tau^r(\Omega)$. Put $d_N(x, y, \xi) = b_N'(x, x+y, \xi)$ and denote by $\hat{d}_N$ the Fourier transform of d_N with respect to y. Then

$$b_N(x, \xi) = (2\pi)^{-n} \int \hat{d}_N(x, \eta, \xi + \eta)\, d\eta.$$

For sufficiently small ϵ, Taylor expansion gives

$$d_N(x, \eta, \xi + \eta) = \sum_{|\alpha| \leq \epsilon N} (iD_\xi)^\alpha d_N(x, \eta, \xi) \eta^\alpha / \alpha! + r_N(x, \eta, \xi),$$

where

$$r_N(x, \eta, \xi) = \sum_{|\gamma| = [\epsilon N] + 1} (i\eta)^\gamma \int_0^1 D_\xi^\gamma \hat{d}_N(x, \eta, \xi + t\eta)(1 - t)^{|\gamma| - 1}\, dt/\gamma!.$$

Because of the Fourier inversion formula we have

$$b_N(x, \xi) = \sum_{|\alpha| \le \epsilon N} (iD_\xi)^\alpha D_x^\alpha b'_N(x, y, \xi)/\alpha!|_{x=y} + r_N(x, \xi).$$

It remains to show that $(r_N(x, \xi)) = ((2\pi)^{-n}\int r_N(x, \eta, \xi)d\eta) \in r_0(\Omega)$. Like in the proof of Theorem 2.1 we consider two cases. When $2|\eta| \le |\xi|$ we note that (2.15) implies that

$$\begin{aligned} &|\eta^\gamma D_x^\alpha D_\xi^{\beta+\gamma}\hat{a}_N(x, \eta, \xi + t\eta)|/\gamma! \\ &\le C_1^N N^{|\alpha|}(1 + |\xi + t\eta|/N)^{r-|\gamma|-|\beta|}(1 + |\eta|)^{-n-1}N^{|\gamma|}/\gamma! \\ &\le C_2^N N^{|\alpha|}(1 + |\xi|/N)^{r-|\gamma|-|\beta|}(1 + |\eta|)^{-n-1}. \end{aligned} \tag{2.16}$$

And when $|\xi| \le 2|\eta|$ we get from (2.15)

$$|\xi|^j|\eta^\gamma D_x^\alpha D_\xi^{\beta+\gamma}\hat{a}_N(x, \eta, \xi + t\eta)|/\gamma! \le C^N N^{|\alpha|+j}(1 + |\eta|)^{-n-1}. \tag{2.17}$$

It follows immediately from (2.16) and (2.17) that $(r_N(x, \xi)) \in r_0(\Omega)$ and we have proved that

$$(b_N(x, \xi)) \sim \left(\sum_{|\alpha| \le \epsilon N} (iD_\xi)^\alpha D_x^\alpha b'_N(x, y, \xi)/\alpha!|_{x=y}\right). \tag{2.18}$$

From this result one proves the following two theorems in exactly the same way as the corresponding results are proved on pp. 105–109 of [7].

Theorem 2.2. *Suppose that $(A_N) \in T^r(\Omega)$. Then the sequence of transposed operators $({}^tA_N)$ also belongs to $T^r(\Omega)$. If (a_N), $({}^ta_N)$ are symbols of (A_N) and $({}^tA_N)$ respectively, we have*

$$({}^ta_N(x, \xi)) \sim \left(\sum_{|\alpha| \le \epsilon N} i^{|\alpha|} D_x^\alpha D_\xi^\alpha a_N(x, -\xi)/\alpha!\right), \tag{2.19}$$

provided that ϵ is small enough.

If $\kappa: \Omega \longrightarrow \Omega$ is an analytic diffeomorphism and $(A_N) \in T^r(\Omega)$ we put

$$(A_N^\kappa u)(x) = A_N(u \circ \kappa)(\kappa^{-1}(x)). \tag{2.20}$$

Theorem 2.3. *Suppose that $(A_N) \in T^r(\Omega)$ and that $\kappa: \Omega \longrightarrow \Omega$ is an analytic diffeomorphism. Then the sequence (A_N^κ), defined by (2.20), also belongs to $T^r(\Omega)$. Denote by $\psi_\alpha(x, \xi)$ the following polynomial, of degree $\le |\alpha|/2$, in ξ:*

$$\psi_\alpha(x, \xi) = D_y^\alpha \exp(i\langle \kappa(y) - \kappa(x) - \kappa'(x)(y - x), \xi\rangle)|_{x=y}. \tag{2.21}$$

If (a_N), (a_N^κ) are symbols of (A_N) and (A_N^κ) respectively, we have

(2.22) $$(a_N^\kappa(\kappa(x), \xi)) \sim \left(\sum_{|\alpha| \le \epsilon N} a_N^{(\alpha)}(x, {}^t\kappa'(x)\xi)\psi_\alpha(x, \xi)/\alpha! \right),$$

provided that ϵ *is small enough. Here* $a_N^{(\alpha)}(x, \eta) = (iD_\eta)^\alpha a_N(x, \eta)$.

3. **Analytic wave front sets and sequences of pseudo-differential operators.** For (C^∞-) wave front sets there is, besides a definition corresponding to Definition 1.1, an alternative description of $\mathrm{WF}(u)$ based on the elliptic regularity theorem for pseudo-differential operators (see [7] and [8]). We shall now give a similar characterization of $\mathrm{WF}_a(u)$ using the sequences of pseudo-differential operators introduced in the previous section.

Definition 3.1. (x_0, ξ_0) is called noncharacteristic with respect to $(a_N) \in \tau^r(\Omega)$ if and only if there is a conic neighborhood F of (x_0, ξ_0) and a decomposition $a_N = a_N^r + a_N^{r-1}$, with $(a_N^{r-1}) \in \tau^{r-1}(\Omega)$, such that

(3.1) $$|a_N^r(x, \xi)| \ge |\xi|^r/C, \quad \text{when } (x, \xi) \in F \text{ and } |\xi| \ge CN,$$

for some constant $C > 0$.

Lemma 3.1. *Suppose that* $u \in \mathcal{E}'(\Omega)$. *Then* $(x_0, \xi_0) \notin \mathrm{WF}_a(u)$ *if and only if there are positive constants* δ *and* C *and a sequence* $(a_N) \in \tau^r(\Omega)$, *for some* r, *such that* (x_0, ξ_0) *is noncharacteristic with respect to* (a_N) *and*

(3.2) $$|D_x^\alpha a_N(x, D)u(x)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } N \ge \delta^{-2}.$$

Proof. We shall construct a "local parametrix" $(e_N(x, D)) \in T^{-r}(\Omega)$, such that $(e_N \circ a_N)(x, \xi) \sim 1$ in a conic neighborhood F_1 of (x_0, ξ_0) with $\bar{F}_1 \subset F$. To do this we put $e_N^{-r} = 1/a_N^r$ in $F \cap \{\xi;\ |\xi| \ge CN\}$. When $k \le \delta' N$, $|\xi| \ge CN$ and $(x, \xi) \in F$, we then recursively define e_N^{-r-k} by

(3.3) $$e_N^{-r-k} = \sum_{|\alpha|+l=k-j>0} i^{|\alpha|} D_\xi^\alpha e_N^{-r-j} \cdot D_x^\alpha a_N^{r-l} \cdot e_N^{-r}.$$

Finally put

$$e_N(x, \xi) = \sum_{k \le \delta' N} e^{-r-k}(x, \xi) \cdot \mu_N(x, \xi),$$

for some sequence $(\mu_N) \in \tau^0(F_1, F)$, which vanishes when $|\xi| \le CN$. To check that $(e_N) \in \tau^{-r}(\Omega)$, we have to prove that

(3.4) $$|D_x^\alpha D_\xi^\beta e_N^{-r-k}(x, \xi)| \le C_1^N (1 + |\xi|/N)^{-r-k-|\alpha+\beta|}, \quad \text{when } |\alpha + \beta| \le \delta'' N.$$

For $k = 0$ we note that $D_x^\alpha D_\xi^\beta (e_N^{-r} \cdot a_N^r) = 0$, if $|\alpha + \beta| \neq 0$. Therefore (3.4) follows from Leibniz' formula and (3.3). Choose now sequences $(\phi_N(x))$, $(\psi_N(\xi)) \in \tau^0(\Omega)$ such that $\phi_N(x) = 1$ in a neighborhood of x_0, $\psi_N(\xi) = 1$ in a conic neighborhood of ξ_0, when $|\xi| \ge CN$, and $\phi_N(x)\psi_N(\xi)$ vanishes outside F_1. Then

$$(\psi_N(D)\phi_N(x)(e_N \circ a_N)(x, D) - \psi_N(D)\phi_N(x)) \in T_0(\Omega).$$

Because of (3.2), this implies that

$$|D_x^\alpha \psi_N(D)\phi_N(x)u(x)| \leq C^N N^{|\alpha|}, \quad \text{when } |\alpha| \leq \delta N.$$

If we put $\phi_N' = \phi_{[N \cdot \delta^{-1}]+1}$, then $u_N(x) = \phi_N'(x)u(x)$ satisfies (1.3).

For a linear differential operator $P(x, D)$ with analytic coefficients, Lemma 3.1 implies Theorem 1.1. We shall now give two lemmas about the existence of suitable cut-off functions in $\tau^0(\Omega)$, which will be used in the proofs of Theorem 1.2 and Theorem 1.3. Suppose that F and F' are two open cones in $\Omega \times \dot{R}^n$ such that $\bar{F} \subset F'$ and let $(\mu_N) \in \tau^0(F, F')$. If $p = \Sigma_{-\infty < k \leq 1} p^k$ is an analytic symbol of order 1 in F', we put

$$p_N(x, \xi) = \sum_{-N \leq k \leq 1} p^k(x, \xi)\mu_N(x, \xi).$$

Lemma 3.2. *Suppose that $p^1(x, \xi)$ is real and that $d_\xi p^1(x, \xi) \neq 0$ in F'. Let $\mathfrak{b}$ be a bicharacteristic for p^1 in F'. If F' is sufficiently small and F'' is an arbitrary conic neighborhood of $\mathfrak{b}$, then there is a sequence $(a_N) \in \tau^0(\Omega)$ such that $a_N(x, \xi)$ vanishes in $F \setminus F''$, the points on $\mathfrak{b} \cap F$ are noncharacteristic with respect to (a_N) and $([p_N(x, D), a_N(x, D)]) \in T_0(\pi(F))$. Here π denotes the projection $T^*(\Omega) \to \Omega$.*

Proof. We shall choose $a_N(x, \xi)$ as a sum $\Sigma_{-N \leq k \leq 0}\, a_N^k(x, \xi)\chi_N(\xi/N)$, where a_N^k is homogeneous of degree k and $\chi_N(\xi)$ are functions which satisfy (1.1), vanish when $|\xi| \leq CN$ and are equal to 1 when $|\xi| \geq 2CN$. We want the symbol sequence of $([p_N(x, D), a_N(x, D)])$ to be in $\tau_0(\pi(F))$. Since $a_N(x, \xi)$ will vanish when $\mu_N(x, \xi) \neq 1$ and since, in view of Example 2.4, it is irrelevant how the symbol looks for $|\xi| \leq CN$, it will, because of Theorem 2.1, be sufficient to find solutions a_N^k to the equations

$$(3.5) \qquad H_{p^1}(a_N^k) = \Phi_k(p^1, \cdots, p^{k+1}, a_N^0, \cdots, a_N^{k+1}), \qquad k \leq \delta' N,$$

where

$$\Phi_k(p^1, \dots, a_N^{k+1}) = \sum_{j-|\alpha|=k-l>0} i^{(|\alpha|-1)}(D_\xi^\alpha p^j D_x^\alpha a_N^l - D_\xi^\alpha a_N^l D_x^\alpha p^j)/\alpha!$$

with the following properties:

(3.6) $a_N^k(x, \xi)$ is homogeneous of degree k and vanishes in $F \setminus F''$;

(3.7) $a_N^0(x, \xi) = 1$ in a fixed conic neighborhood of $\mathfrak{b} \cap F$;

(3.8) $|D_x^\alpha D_\xi^\beta a_N^k(x, \xi)| \leq C^N N^{|\alpha+\beta|+k}$, when $|\alpha + \beta| + k \leq \delta N$, on compact subsets of F'.

Because of the homogeneity of a_N^k, (3.8) shows that $(a_N) \in r^0(\Omega)$. Moreover, since a_N^k satisfies (3.5), we will have $([p_N(x, D), a_N(x, D)]) \in T_0(\pi(F))$. Now $p^1(x, \xi)$ is a first order real symbol of principal type. If F' is small enough, we can therefore find an analytic change of coordinates $(y, \eta) \mapsto (x(y, \eta), \xi(y, \eta))$ such that $x(y, \eta)$ and $\xi(y, \eta)$ are homogeneous of degree 0 and 1 respectively and H_{p^1} is transformed to $\partial/\partial y_n$. Denote $a_N^k(x(y, \eta), \xi(y, \eta))$ by $b_N^k(y, \eta)$. For $k = 0$, (3.5) reduces to $\partial b_N^0/\partial y_n = 0$. If $y' = (y_1, \cdots, y_{n-1})$ we choose $b_N^0 = b_N^0(y', \eta)$ independent of y_n and so that (3.6)–(3.8) are satisfied by a_N^0. For any function $g(y, \eta) = \Sigma_{0 \leq i} c_i(y', \eta) y_n^i$, which is analytic in y_n, we put

$$(Ig)(y, \eta) = \sum_{0 \leq i} c_i(y', \eta) y_n^{i+1}/(i+1). \tag{3.9}$$

Denoting $\Phi_k(p^1, \cdots, a_N^{k+1})(x(y, \eta), \xi(y, \eta))$ by $g_N^k(y, \eta)$, we now solve the equations (3.5) recursively by putting $b_N^k = Ig_N^k$. Clearly the functions a_N^k obtained in this way, satisfy (3.6) and (3.7). It remains to prove the estimates (3.8). If we put $D = (D_{y_1}, \cdots, D_{y_n})$, it follows from (3.5) that b_N^k is a sum of less that C^k terms b_N^k of the form

$$b_N^k = (I \circ (B_{m_1} D^{\alpha_1}) \circ I \circ (B_{m_2} D^{\alpha_2}) \circ \cdots \circ I \circ (B_{m_l} D^{\alpha_l}))(b_N^0), \tag{3.10}$$

where $\Sigma_{1 \leq i \leq l}(m_i + |\alpha_i|) \leq k + l \leq 2k$ and

$$\sup_{(y, \eta) \in M} |D^\beta B_{m_i}(y, \eta)| \leq C^{(m_i + |\beta|)}(m_i)!\beta!. \tag{3.11}$$

Here M is compact and $C = C(M)$ is independent of i and k. In particular each B_{m_i} is an analytic function. To prove (3.8), we have to prove that, on compact sets $|D^\gamma b_N^k(y, \eta)| \leq C^N N^{|\gamma| + k}$, when $|\gamma| + k \leq \delta N$. Now all derivatives except $\partial/\partial y_n$ commute with I, and $\partial/\partial y_n \circ I$ is the identity. If the differentiations are carried out, then $D^\gamma b_N^k$ may be written as a sum of less than C^N terms of the form

$$v_N^k = (I \circ B'_{m'_1} \circ I \circ B'_{m'_2} \circ \cdots \circ I \circ B'_{m'_{l'}})(D^\beta b_N^0), \tag{3.12}$$

where $|\beta| + \Sigma_{1 \leq i \leq l'} m'_i \leq k + l + |\gamma|$ and the functions $B'_{m'_i}(y, \eta)$ satisfy estimates of the type (3.11). To estimate (3.12), we note that

$$|(I \circ f_1 \circ I \circ f_2 \circ \cdots \circ I \circ f_{l-1} \circ I)(f_l)| \leq \prod_{1 \leq i \leq l} (\sup |f_i|) |y_n|^l / l!. \tag{3.13}$$

Therefore

$$|v_N^k| \leq C^N \left(\prod_{1 \leq i \leq l'} (m'_i!) \right) N^{|\beta|} \Big/ l'! \leq C^N N^{k + |\gamma|}.$$

This proves (3.9) and completes the proof of Lemma 3.2.

Lemma 3.3. *Suppose that d_ξ Re $p^1(x, \xi)$ and d_ξ Im $p^1(x, \xi)$ are linearly independent and that the Poisson bracket* {Re p^1, Im p^1}(x, ξ) *vanishes in F'. Let $\mathfrak{L}$ be a bicharacteristic strip for p^1 in F'. If F' is sufficiently small and F'' is an arbitrary conic neighborhood of $\mathfrak{L}$, then there is a sequence $(a_N) \in \tau^0(\Omega)$ such that a_N vanishes in $F\backslash F''$, the points on $\mathfrak{L} \cap F$ are noncharacteristic with respect to (a_N) and $([p_N(x, D), a_N(x, D)]) \in T_0(\pi(F))$.*

The proof of this lemma parallels the proof of the preceding one. Because of the Frobenius integrability theorem, there is an analytic change of coordinates $(y, \eta) \mapsto (x(y, \eta), \xi(y, \eta))$ such that $x(y, \eta)$ and $\xi(y, \eta)$ are homogeneous of degree 0 and 1 respectively and H_{p^1} is transformed to $(\partial/\partial y_n + i\partial/\partial y_{n-1})/2$. Again we denote $a_N^k(x(y, \eta), \xi(y, \eta))$ by $b_N^k(y, \eta)$. b_N^0 is taken to be independent of y_n and y_{n-1} so the right-hand side of (3.5) will in the new coordinates be a convergent power series

$$g(y'', z, \bar{z}, \eta) = \sum_{0 \le i} c_i(y'', z, \eta)\bar{z}^i,$$

where $z = y_n + iy_{n-1}$, $\bar{z} = y_n - iy_{n-1}$ and $y'' = (y_1, \cdots, y_{n-2})$. Finally the operator I is, in this case, defined by

$$(Ig)(y'', z, \bar{z}, \eta) = \sum_{0 \le i} c_i(y'', z, \eta)\bar{z}^{i+1}/(i+1).$$

With these modifications exactly the same proof as for Lemma 3.2 works for Lemma 3.3.

4. **An inequality.** We start with the following simple lemma:

Lemma 4.1. *Suppose that the functions $b_N(x, \xi) \in C^\infty(R^n \times R^n)$ vanish when x is outside a fixed compact set and satisfy*

$$|D_x^\alpha D_\xi^\beta b_N(x, \xi)| \le C_N(1 + |\xi|)^{-|\beta|}, \quad \text{when } |\alpha| \le n + 1 + j, \ |\beta| \le j. \tag{4.1}$$

Put $B_N u(x) = (2\pi)^{-n}\int e^{i\langle x, \xi\rangle} b_N(x, \xi)\hat{u}(\xi)d\xi$. If (4.1) is satisfied for $j = 0$, then B_N is a bounded operator on $L^2(R^n)$ with norm $\le MC_N$, for some constant M independent of N. Moreover, if (4.1) is satisfied with $j = 1$ and $b_N(x, \xi)$ is real, then $B_N \circ \Lambda - \Lambda \circ B_N^*$ is a bounded operator on $L^2(R^n)$ with norm $\le MC_N$. Here $\Lambda = (1 + |D|^2)^{1/2}$.

Proof. Parseval's formula gives

$$\int \overline{v(x)} \cdot B_N u(x)\, dx = (2\pi)^{-2n} \iint \overline{\hat{v}(\eta)}\hat{b}_N(\eta - \xi, \xi)\hat{u}(\xi)\, d\xi\, d\eta.$$

Because of (4.1), we have $|\hat{b}_N(\eta - \xi, \xi)| \le M_1 C_N(1 + |\eta - \xi|)^{-n-1}$. Therefore

$$\left| \int \overline{v(x)} \cdot B_N u(x)\, dx \right| \le M \cdot C_N \|u\| \cdot \|v\|.$$

This proves the first part of the lemma. For the second part we observe that the adjoint B_N^* of B_N is given by

$$\widehat{(B_N^* u)}(\eta) = \int e^{-i\langle x, \eta\rangle} \overline{b}_N(x, \eta) u(x)\, dx.$$

Thus, if $a_N(x, \xi) = b_N(x, \xi)(1 + |\xi|^2)^{1/2}$, we have

$$\widehat{(\Lambda \circ B_N^* u)}(\eta) = (2\pi)^{-n} \int \hat{\overline{a}}_N(\eta - \xi, \eta) \hat{u}(\xi)\, d\xi$$

and

$$\widehat{(B_N \circ \Lambda u)}(\eta) = (2\pi)^{-n} \int \hat{a}_N(\eta - \xi, \xi) \hat{u}(\xi)\, d\xi.$$

Now a_N is real and, because of (4.1),

$$|\hat{a}_N(\eta - \xi, \eta) - \hat{a}_N(\eta - \xi, \xi)| \leq M_2 \cdot C_N (1 + |\xi - \eta|)^{-n-1}.$$

The same argument as for the first part of the lemma now finishes the proof.

Before we state the main result of this section we have to introduce some notation. Let Ω be a neighborhood of the origin in R^n and Γ an open cone in R^{n-1}. Suppose that $S^k(x, \xi')$, $k = 1, 0, -1, \cdots$, are analytic functions in $\Omega \times \Gamma$, homogeneous of degree k, such that, for some constant C_0 independent of k,

$$|D_x^\alpha D_{\xi'}^\beta S^k(x, \xi')| \leq C_0^{|\alpha+\beta| - k + 2} |k|! |\alpha|! |\beta|! |\xi'|^{k - |\beta|}, \quad \text{when } (x, \xi') \in \Omega \times \Gamma. \tag{4.2}$$

Let F be a closed cone with $\dot{F} \subset \Omega \times \Gamma \times R$ and put $L^1(x, \xi) = i\xi_n + S^1(x, \xi')$. We shall assume that L^1 is principally normal in some conic neighborhood F_1 of F, i.e. there exists a function $\lambda(x, \xi)$, analytic in F_1 and homogeneous of degree zero, such that

$$i\{L^1, L^1\}(x, \xi) = \operatorname{Re}(L^1(x, \xi)\lambda(x, \xi)), \quad \text{when } (x, \xi) \in F_1. \tag{4.3}$$

Now choose functions $\phi_N \in C_0^\infty(\Omega)$, $\psi_N \in C^\infty(\dot{R}^{n-1})$ and $\chi_N \in C^\infty(R^{n-1})$ such that ψ_N is homogeneous of degree zero and has support in Γ, $\phi_N(x)\psi_N(\xi') = 1$ on F_1, χ_N vanishes when $|\xi'| \leq 2C_0$ and equals 1 when $|\xi'| \geq 3C_0$. Put $\rho_N(x, \xi') = \phi_N(x)\psi_N(\xi')\chi_N(\xi'/N)$. If ϕ_N, ψ_N, and χ_N are suitably chosen, we will have

$$|D_x^\alpha D_\xi^\beta \rho_N(x, \xi')| \leq C^{|\alpha+\beta|+1} N^{|\alpha|} (1 + |\xi'|/N)^{-|\beta|}, \quad \text{when } |\alpha + \beta| \leq N, \tag{4.4}$$

and

$$|D_x^\alpha D_\xi^\beta \rho_N(x, \xi')| \leq C(1 + |\xi'|)^{-|\beta|}, \quad \text{when } |\alpha| \leq 2n + 5,\ |\beta| \leq 2. \tag{4.5}$$

We define

$$L_N(x,\xi) = i\xi_n + S_N(x,\xi'),\quad \text{where } S_N(x,\xi') = \sum_{-N\le k\le 1} S^k(x,\xi')\rho_N(x,\xi'). \tag{4.6}$$

$L_N = L_N(x, D) = \partial/\partial x_n + S_N(x, D')$ is, strictly speaking, not a pseudo-differential operator in Ω. It is the sum of a differential operator in x_n and a pseudo-differential operator in x' with "coefficients" depending on x_n. Finally we assume that (a_N) is a sequence in $\tau^0(\Omega)$ such that each a_N has support in F. Put $\Lambda_N^s = (2 + |D'|^2/N^2)^{s/2}$, $g_\sigma(x_n) = ((x_n - \sigma)^2/2 - \sigma^2/4)$ and denote by $\|\cdot\|$ the L^2-norm in x'-space.

Theorem 4.1. *Let L_N and a_N be as above. Suppose that the functions $\phi_N \in C_0^\infty(\Omega)$ satisfy (1.1) and put $u_N(x) = \phi_N(x)a_N(x, D)u(x)$. Then there are positive constants C and σ such that, if the functions $\phi_N(x)$ vanish when $|x_n| \ge \sigma$ and if $N \ge C$, then*

$$N\int \|\Lambda_N^{Ng_\sigma(x_n)}u_N\|^2\,dx_n \le C\int \|\Lambda_N^{Ng_\sigma(x_n)}L_N u_N\|^2\,dx_n + C^N\int \|u\|^2\,dx_n, \qquad u \in C_0^\infty. \tag{4.7}$$

The constant σ does only depend on C_0 and F.

Proof. Put $v_N(x) = \Lambda_N^{Ng_\sigma(x_n)}u_N(x)$. Then

$$L_N v_N = \Lambda_N^{Ng_\sigma(x_n)}L_N u_N + (N(x_n - \sigma)\log \Lambda_N)v_N + R_N u_N,\quad \text{where } R_N = [S_N(x, D'), \Lambda_N^{Ng_\sigma(x_n)}]. \tag{4.8}$$

Denote by $R_N'(x, \xi')$ the sum

$$\sum_{1\le|\alpha|\le\sigma N} i^{|\alpha|} D_{\xi'}^\alpha (2 + |\xi'|^2/N^2)^{Ng_\sigma(x_n)/2} D_{x'}^\alpha S_N(x,\xi')/\alpha!$$

and let $\delta_N(x, \xi')$ be a sequence of functions that satisfies (4.4), (4.5) and vanishes when $(x, \xi', \xi_n) \notin F_1$ for all ξ_n. Suppose also that $\delta_N(x, \xi') = 1$ in F, when $|\xi'| \ge 2C_0N$. We may then write

$$R_N = R_N^1 + R_N^2 + R_N^3, \tag{4.9}$$

where

$$R_N^1(x,\xi') = \delta_N(x,\xi')R_N'(x,\xi'), \tag{4.10}$$

$$R_N^2(x,\xi') = (1 - \delta_N(x,\xi'))R_N'(x,\xi'), \tag{4.11}$$

(4.12) $$R_N^3(x, \xi') = \int e^{i\langle x', \eta'\rangle} f_N(\eta', x_n, \xi')\, d\eta'$$

and

$$f_N(\eta', x_n, \xi') = \sum_{|\alpha|=[\sigma N]+1} i^{|\alpha|}(\eta')^\alpha \hat{S}_N(\eta', x_n, \xi') \cdot \int_0^1 D_{\xi'}^\alpha (2 + |\xi' + t\eta'|^2/N^2)^{Ng_\sigma(x_n)/2}(1-t)^{|\alpha|-1}\, dt/\alpha!.$$

Here $\hat{S}_N$ denotes the Fourier transform of S_N with respect to x'. Now

(4.13) $$|D_{\xi'}^\alpha (2 + |\xi'|^2/N^2)^{s/2}| \le (C\sigma)^{|\alpha|}(2 + |\xi'|^2/N^2)^{(s-|\alpha|)/2}, \quad \text{if } |s| \le N\sigma \text{ and } |\alpha| \le 2N\sigma.$$

Therefore, if $\max_{|x_n|\le\sigma} g_\sigma(x_n) = 7\sigma^2/4$ is less than σ, we immediately get from (4.12) and Lemma 4.1

(4.14) $$\int \|R_N^3 u_N\|^2\, dx_n \le C^N \int \|u_N\|^2\, dx_n \le C_1^N \int \|u\|^2\, dx_n.$$

The last inequality follows from Lemma 4.1 applied to $b_N(x, D) = \phi_N(x) a_N(x, D)$. The same estimate holds for the L^2-norm of $R_N^2 u_N$, i.e.,

(4.15) $$\int \|R_N^2 u_N\|^2\, dx_n \le C^N \int \|u\|^2\, dx_n.$$

In fact, since $R_N^2(x, \xi')$ vanishes in the support of $b_N(x, \xi)$, we have $R_N^2(x, D') \circ b_N(x, D)u = R_N^4(x, D)u$, where

(4.16) $$R_N^4(x, \xi) = \sum_{|\gamma|=[\sigma N]+1} i^{|\gamma|} \int e^{i\langle x, \eta\rangle}(\eta^\gamma \hat{b}_N(\eta, \xi)/\gamma!) \cdot \left(\int_0^1 D_{\xi'}^\gamma R_N^2(x, \xi' + t\eta')(1-t)^{|\gamma|-1}\, dt\right) dy.$$

Because $|D_x^\alpha D_{\xi'}^\gamma R_N^2(x, \xi')| \le C^N$ when $|\alpha| \le n+1$ and $|\gamma| = [\sigma N] + 1$, Lemma 4.1 applies to give (4.15). We now put

(4.17) $$R_N^5 = R_N^1 \circ \Lambda_N^{-Ng_\sigma(x_n)}.$$

Since $S_N(x, \xi') = \sum_{-N\le k\le 1} S^k(x, \xi')\chi_N(\xi'/N)$ in the support of $\delta_N(x, \xi')$ and $\chi_N(\xi')$ vanishes when $|\xi'| \le 2C_0$, it follows from (4.2) and (4.13) that

(4.18) $$|D_x^\gamma R_N^5(x, \xi')| \le N \cdot \sum_{1\le|\alpha|\le\sigma N} (C_2\sigma)^{|\alpha|}, \quad \text{when } |\gamma| \le n+1.$$

Here the constant C_2 only depends on C_0 and the constant C in (4.13). If σ is small enough, we get from (4.18) and Lemma 4.1

(4.19) $$\int \|R_N^5 v_N\|^2\, dx_n \le C \cdot \sigma \cdot N \int \|v_N\|^2\, dx_n.$$

In view of (4.8), (4.9), (4.14), (4.15), (4.17), and (4.19) it will be sufficient to prove

$$N\int \|v_N\|^2\,dx_n \le C_3\int \|L_N v_N - (N(x_n-\sigma)\log\Lambda_N)v_N\|^2\,dx_n + C^N\int\|u\|^2\,dx_n, \tag{4.20}$$

for some constant C_3 which stays bounded as σ tends to zero. Put $S_N = A_N + iB_N$, where A_N and B_N are real and denote by (,) the scalar product in $L^2_{x'}$. Then

$$\begin{aligned}\int\|L_N v_N &- (N(x_n-\sigma)\log\Lambda_N)v_N\|^2\,dx_n\\ &= \int\|\partial v_N/\partial x_n + iB_N v_N\|^2\,dx_n\\ &\quad + \int\|A_N v_N - (N(x_n-\sigma)\log\Lambda_N)v_N\|^2\,dx_n + I_N^1 + I_N^2 + I_N^3,\end{aligned}\tag{4.21}$$

where

$$\begin{aligned} I_N^1 &= -\,2\mathrm{Re}\int(\partial v_N/\partial x_n, (N(x_n-\sigma)\log\Lambda_N)v_N)\,dx_n\\ &= N\int(v_N, (\log\Lambda_N)v_N)\,dx_n,\\ I_N^2 &= -\,2\mathrm{Re}\int(iB_N v_N, (N(x_n-\sigma)\log\Lambda_N)v_N)\,dx_n, \quad\text{and}\\ I_N^3 &= 2\mathrm{Re}\int(\partial v_N/\partial x_n + iB_N v_N, A_N v_N)\,dx_n.\end{aligned}\tag{4.22}$$

If we denote $(\log\Lambda_N)^{1/2}v_N$ by w_N, we have

$$2\mathrm{Re}\,(iB_N v_N, (\log\Lambda_N)v_n) = 2\mathrm{Re}\,(i[(\log\Lambda_N)^{1/2}, B_N]v_N, w_N) + (i(B_N - B_N^*)w_N, w_N).$$

Because of (4.2), (4.5), (4.6), and the fact that $\chi_N(\xi')$ vanishes when $|\xi'| \le 2C_0$, it follows from Lemma 4.1 that $\|[(\log v_N)^{1/2}, B_N]v_N\| \le C\|v_N\|$ and $\|(B_N - B_N^*)w_N\| \le C\|w_N\|$. Since $\|v_N\| \le 2\cdot(\log 2)^{-1/2}\cdot\|w_N\|$, we get

$$|I_N^2| \le C_4\cdot\sigma\cdot N\int\|w_N\|^2\,dx_n. \tag{4.23}$$

To estimate I_N^3, we observe that

$$\begin{aligned} I_N^3 &= \int(\partial v_N/\partial x_n + iB_N v_N, (A_N - A_N^*)v_N)\,dx_n\\ &\quad + \int([A_N, \partial/\partial x_n + iB_N]v_N, v_N)\,dx_n.\end{aligned}\tag{4.24}$$

Let now $\nu_N(x,\xi)$ be a sequence of functions which satisfy (4.4) and (4.5), with ξ' replaced by ξ. Suppose also that ν_N has support in F_1 and equals 1 in F, when $|\xi| \ge 2C_0N$. We then write

$$[A_N, \partial/\partial x_n + iB_N] = E_N^1 + E_N^2 + E_N^3, \tag{4.25}$$

where

$$E_N^1(x, \xi) = \nu_N(x, \xi)\{\xi_n + B_N(x, \xi'), A_N(x, \xi')\},$$

$$E_N^2(x, \xi) = (1 - \nu_N(x, \xi))\{\xi_n + B_N(x, \xi'), A_N(x, \xi')\},$$

$$E_N^3(x, \xi) = \sum_{|\gamma|=2} \int e^{i\langle x', \eta'\rangle}(\eta')^\gamma \left(\int_0^1 f_\gamma(x, \eta', \xi', t)(1-t)\,dt\right) d\eta' \Big/ 2, \quad \text{and}$$

$$f_\gamma(x, \eta', \xi', t) = \hat{A}_N(\eta', x_n, \xi') D^\gamma_{\xi'} B_N(x, \xi' + t\eta') - \hat{B}_N(\eta', x_n, \xi') D^\gamma_{\xi'} A_N(x, \xi' + t\eta').$$

Since $\chi_N(\xi')$ vanishes when $|\xi'| \le 2C_0$, (4.2), (4.5), and (4.6) imply that $|D^\alpha_{x'} E_N^3(x, \xi')| \le C$ when $|\alpha| \le n + 1$. Thus Lemma 4.1 gives

$$\|E_N^3 v_N\| \le \|v_N\|. \tag{4.26}$$

Moreover, because of (4.3), there are functions μ_1 and μ_2 homogeneous of degree zero and analytic in a neighborhood of the support of ν_N such that

$$\{\xi_n + B_N^1(x, \xi'), A_N^1(x, \xi')\} = \mu_1(x, \xi)(i\xi_n + iB_N^1(x, \xi')) + \mu_2(x, \xi) A_N^1(x, \xi').$$

Put $\nu_N^1 = \nu_N \cdot \mu_1$ and $\nu_N^2 = \nu_N \cdot \mu_2$. Then

$$E_N^1 = \nu_N^1(x, D) \circ (\partial/\partial x_n + iB_N) + \nu_N^2(x, D) \circ A_N + E_N^4, \tag{4.27}$$

where

$$\int \|E_N^4 v_N\|^2 \, dx_n \le C \int \|v_N\|^2 \, dx_n. \tag{4.28}$$

We now have

$$\begin{aligned} \int ([A_N, \partial/\partial_n + iB_N] v_N, v_N)\, dx_n &= \int (\nu_N^1(x, D) \circ (\partial/\partial x_n + iB_N) v_N, v_N)\, dx_n \\ &+ \int (\nu_N^2(x, D) \circ (A_N - N(x_n - \sigma)\log \Lambda_N) v_N, v_N)\, dx_n \\ &+ N \int ([\nu_N^2(x, D), (\log \Lambda_N)^{1/2}](x_n - \sigma) w_N, v_N)\, dx_n \\ &+ N \int (\nu_N^2(x, D)(x_n - \sigma) w_N, w_N)\, dx_n + \sum_{i=2}^4 (E_N^i v_N, v_N). \end{aligned} \tag{4.29}$$

Exactly as for (4.15) it follows that

$$\int \|E_N^2 v_N\|^2 \, dx_n \le C^N \int \|u\|^2 \, dx_n. \tag{4.30}$$

Since $(A_N - A_N^*)$, $\nu^1(x, D)$, $[\nu^2(x, D), (\log \Lambda_N)^{1/2}]$, E_N^3 and E_N^4 are bounded operators on L_x^2, it follows from (4.24), (4.29), and (4.30) that, for any $\epsilon > 0$,

$$|I_N^3| \le C_5\Big(\epsilon\int\|\partial v_N/\partial x_n + iB_Nv_N\|^2\,dx_n + \epsilon\int\|A_Nv_N - (N(x_n-\sigma)\log\Lambda_N)v_N\|^2\,dx_n \tag{4.31}$$
$$+ \epsilon^{-1}\int\|v_N\|^2\,dx_n + N\cdot\sigma\int\|w_N\|^2\,dx_n\Big)$$
$$+ C^N\int\|u\|^2\,dx_n.$$

Here C_5 only depends on C_0 and F, while C depends on $b_N(x, D) = \phi_N(x)a_N(x, D)$ and thus on σ. To prove (4.20) it now only remains to combine (4.21), (4.22), (4.23), and (4.31). First choose ϵ so that $C_5\cdot\epsilon \le 1$. Then the first two terms in (4.31) are absorbed by the first two terms in (4.21). After that, choose σ so that $2(C_4 + C_5)\cdot\sigma \le 1$. Then, if N is large enough,

$$I_N^1 - C_3\epsilon^{-1}\int\|v_N\|^2\,dx_n - N(C_4+C_5)\sigma\int\|w_N\|^2\,dx_n$$
$$\ge C_6N\int\|w_N\|^2\,dx_n - C^N\int\|u\|^2\,dx_n.$$

Since $\|v_n\| \le 2(\log 2)^{-1/2}\|w_N\|$, this completes the proof.

Remark 4.1. We have actually proved a slightly stronger inequality that (4.7). On the left-hand side $\Lambda_N^{Ng_\sigma(x_n)}$ could be multiplied by $(\log\Lambda_N)^{1/2}$.

Remark 4.2. The proof of Theorem 4.1 is modelled on the proof of Théorème 1 in [12]. Except for technicalities, the main difference is that the norm of R_N^5 has to be estimated as in (4.19).

Remark 4.3. It was assumed in Theorem 4.1 that $u \in C_0^\infty$. However, the proof works without change for any $u \in C_0$ such that $a_N(x, D)u \in C_0^{[\delta N]}(\Omega)$ for some $\delta > \max_{|x_n|\le\sigma} g_\sigma(x_n) = 7\sigma^2/4$.

5. **Completion of the proofs of Theorems 1.2–1.4.** The proofs of the three theorems are similar but, since there are some differences, we give them one by one.

Proof of Theorem 1.2. Clearly it will be sufficient to prove the following "semilocal" statement. Suppose that $(x_0, \xi_0) \in N_1(P_m)\setminus\mathrm{WF}_a(u)$ and denote by ℓ the bicharacteristic strip through (x_0, ξ_0). Then there is a closed conic neighborhood F of (x_0, ξ_0) such that if $\ell\cap F\cap\complement\mathrm{WF}_a(u) \ne \emptyset$ then $\ell\cap F\cap\mathrm{WF}_a(u) = \emptyset$. We shall prove that there is a positive constant γ, independent of $(\bar{x}_0, \bar{\xi}_0) \in F\cap\ell\cap\{(x, \xi);\ |\xi| = 1\}$, such that if $(\bar{x}_0, \bar{\xi}_0) \notin \mathrm{WF}_a(u)$ then $\ell\cap B_\gamma(\bar{x}_0, \bar{\xi}_0)\cap\mathrm{WF}_a(u) = \emptyset$. Here $B_\gamma(\bar{x}_0, \bar{\xi}_0) = \{(\bar{x}, \bar{\xi}); |(\bar{x} - x_0, \xi - \xi_0)| < \gamma\}$. Since $d_\xi P_m(x, \xi) \ne 0$ in $N_1(P_m)$, we may assume that $\partial P_m/\partial\xi_n \ne 0$ at (x_0, ξ_0). According to a factorization lemma of Hörmander [9, Proposition 6.1] there is a closed conic neighborhood F_1 of (x_0, ξ_0) such that

$$P(x, \xi) = Q(x, \xi)\circ(i\xi_n + S(x, \xi')), \tag{5.1}$$

in a conic neighborhood of $\dot{F}_1$. Here $S(x, \xi')$ is an analytic symbol in (x, ξ') and $Q(x, \xi) = \Sigma_{0 \le j \le m-1}\, {}_jq(x, \xi')\xi_n^j$ for some analytic symbols ${}_jq(x, \xi')$ of order $m - 1 - j$. The right-hand side of (5.1) stands for the formal composition of the symbols. Suppose that $\dot{\bar{F}} \subset F_1$ and that $(\bar{x}_0, \bar{\xi}_0) \in \ell \cap F \cap \complement\mathrm{WF}_a(u)$. Then $(U \times \Gamma) \cap \mathrm{WF}_a(u) = \emptyset$, for some conic neighborhood $U \times \Gamma$ of $(\bar{x}_0, \bar{\xi}_0)$. We may assume that U is of the form $U' \times I$, where $I = \{x_n;\ |x_n - \bar{x}_{0,n}| < \epsilon\}$. In view of Lemma 3.2, there is a sequence $(a_N) \in \tau^0(\Omega)$ such that supp $a_N \subset F_1$, (supp $a_N) \cap (R^{n-1} \times I \times \dot{R}^n) \subset U \times \Gamma$, the points on $\ell \cap F$ are noncharacteristic with respect to (a_N) and

$$\text{(5.2)} \qquad ([\Lambda^{1-m} \circ P(x, D), a_N(x, D)]) \in T_0(\pi(F)).$$

As before, π denotes the projection $T^*(\Omega) \to \Omega$ and $\Lambda^k = (1 + |D|^2)^{k/2}$. Choose functions $\rho_N(x, \xi')$ satisfying (4.4) and (4.5) with support in a set where $S(x, \xi')$ and ${}_jq(x, \xi')$ are defined such that $\rho_N(x, \xi') = 1$ in F_1 when $|\xi'| \ge CN$. Put

$${}_jq_N(x, \xi') = \sum_{-N \le k \le m-1-j} {}_jq^k(x, \xi')\rho_N(x, \xi'),$$

$$Q_N(x, \xi) = \sum_{0 \le j \le m-2} {}_jq_N(x, \xi')\xi_n^j,$$

$$S_N(x, \xi') = \sum_{-N \le k \le 1} S^k(x, \xi')\rho_N(x, \xi') \quad \text{and} \quad L_N(x, \xi) = i\xi_n + S_N(x, \xi').$$

Since $\rho_N = 1$ in F_1, when $|\xi| \ge CN$, it follows from (5.1) and (5.2) that

$$\text{(5.3)} \quad (Q_N(x, D) \circ L_N(x, D) \circ a_N(x, D) - \Lambda^{m-1} \circ a_N \circ \Lambda^{1-m} \circ P(x, D)) \in T_0(\pi(F)).$$

We may assume that $F_1 \cap \mathrm{WF}_a(u) = \emptyset$ and that the points in F_1 are noncharacteristic with respect to (Q_N). Then it follows from (5.3) and Lemma 3.1 that

$$\text{(5.4)} \quad |D_x^\alpha L_N(x, D) \circ a_N(x, D)u(x)| \le C^N N^{|\alpha|}, \qquad \text{when } |\alpha| \le \delta N \text{ and } x \in \pi(F).$$

Moreover, since (supp $a_N) \cap (R^{n-1} \times I \times \dot{R}^n) \subset U \times \Gamma$, we have

$$\text{(5.5)} \qquad |D_x^\alpha a_N(x, D)u(x)| \le C^N N^{|\alpha|},$$

$$\text{when } |\alpha| \le \delta N \text{ and } x \notin \pi(\text{supp } a_N) \cap (R^{n-1} \times I).$$

We are now in the position to apply Theorem 4.1. First of all it is clear that we may suppose that $\bar{x}_0 = 0$. Then $I = \{x_n;\ |x_n| < \epsilon\}$. Let σ be the constant in Theorem 4.1 and let the functions $\phi_N(x_n)$ satisfy (1.1), vanish outside $-\epsilon/2 \le x_n \le \sigma$ and be equal to 1 when $0 \le x_n \le \sigma/2$. Put $b_N(x, \xi) = \phi_N(x_n)a_N(x, \xi)$.

Then all assumptions in Theorem 4.1 are fulfilled, except that we do not know that $u \in C_0^\infty(\Omega)$. However, it follows from the results of [5] that $u_N = b_N(x, D)u \in C_0^{[\delta' N]}(\Omega)$. In fact this, and the corresponding results needed for Theorem 1.3 and Theorem 1.4, could be proved directly using the methods of this paper. If $\phi \in C_0^\infty$ and equals 1 in $\pi(F_1)$, then $v = \phi\Lambda^{-2k}\phi u \in C_0$, if k is large enough, and $P \circ \Lambda^{2k} v = f$ in $\pi(F_1)$. Therefore we may assume that $u \in C_0$ and Remark 4.3 applies. If supp a_N is close enough to $\mathfrak{k}$, it follows from (5.5) that (5.4), with a_N replaced by b_N, is valid when $x_n \le \sigma/2$. Since $g_\sigma(x_n) \le -\sigma^2/8$ when $\sigma/2 \le x_n \le \sigma$, it follows that the right-hand side of (4.7) is bounded by C^N. Thus

$$\text{(5.6)} \quad \int_{0 \le x_n \le \sigma/4} \|\Lambda_N^{\lambda \cdot N} u_N\|^2 \, dx_n \le C^N, \quad \text{where } \lambda = \min_{0 \le x_n \le \sigma/4} g_\sigma(x_n) = \sigma^2/32.$$

Because of Euler's identity, $\langle (d_\xi P_m)(x_0, \xi_0), \xi_0 \rangle = m P_m(x_0, \xi_0) = 0$. We may therefore assume that $|\xi_n| \le \epsilon |\xi'|$ in the support of b_N and it follows from (5.6) together with Sobolev's embedding theorem that

$$\text{(5.7)} \quad |D_x^\alpha a_N(x, D)u(x)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \lambda N \text{ and } 0 \le x_n \le \sigma/8.$$

Lemma 3.1 now gives that $\{(x, \xi) \in \mathfrak{k};\ 0 \le x_n \le \sigma/8\} \cap \mathrm{WF}_a(u) = \emptyset$. It follows in the same way that the points on $\mathfrak{k}$, with $-\sigma/8 \le x_n \le 0$, are in the complement of $\mathrm{WF}_a(u)$. Since σ does not depend on $\bar{x}_0$, this completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let $(x_0, \xi_0) \notin N_2(P_m) \setminus \mathrm{WF}_a(u)$. Then there is an analytic function $\lambda(x, \xi)$ homogeneous of degree $m-1$, in ξ, such that

$$\text{(5.8)} \quad \{P_m, \bar{P}_m\}(x, \xi) = \lambda(x, \xi) P_m(x, \xi) - \bar{\lambda}(x, \xi) \bar{P}_m(x, \xi),$$

in a conic neighborhood of (x_0, ξ_0). We shall multiply P_m with a nonvanishing analytic function $g(x, \xi)$ homogeneous of degree $1-m$ in ξ, such that

$$\text{(5.9)} \quad \{gP_m, \overline{gP_m}\}(x, \xi) = 0$$

in a full conic neighborhood of (x_0, ξ_0). Because of (5.8), we have

$$\{gP_m, \overline{gP_m}\} = 2i \operatorname{Im}(\bar{P}_m(g\{P_m, \bar{g}\} - \bar{\lambda} g \bar{g} + P_m\{g, \bar{g}\}/2)).$$

It will therefore be sufficient to solve the following nonlinear equation

$$\text{(5.10)} \quad \{P_m, \bar{g}\} - \overline{\lambda g} + P_m\{g, \bar{g}\}/2g = 0.$$

Since the last term vanishes at (x_0, ξ_0), we can employ the Cauchy-Kovalevsky Theorem and solve (5.10) in a conic neighborhood of (x_0, ξ_0) with data given on any hyperplane with normal N, provided that $\langle (d_\xi P_m)(x_0, \xi_0), N \rangle \ne 0$. Moreover it follows directly from the uniqueness of the solution that, if we give data homogeneous of degree k, then g is homogeneous of degree k. Denote by $\mathfrak{k}$ the

(2-dimensional) bicharacteristic strip through (x_0, ξ_0). Since $(x_0, \xi_0) \in N_2(P_m)$, we may suppose that there is an open conic neighborhood F_1 of (x_0, ξ_0) such that

$$\partial P_m(x, \xi)/\partial \xi_k \neq 0, \quad \text{when } k = n-1, n \text{ and } (x, \xi) \in F_1. \tag{5.11}$$

We shall also assume that (5.9) is satisfied in a conic neighborhood F_1' of $\dot{\overline{F}}_1$. For some $(\mu_N) \in \tau^0(F_1, F_1')$ we put $g_N(x, \xi) = g(x, \xi) \cdot \mu_N(x, \xi)$. If $F_1' \cap \mathrm{WF}(Pu) = \emptyset$, then

$$|D_x^\alpha g_N(x, D) \circ P(x, D)u(x)| \leq C^N N^{|\alpha|}, \quad \text{when } |\alpha| \leq \delta N. \tag{5.12}$$

Let F and F_2 be open conic neighborhoods of (x_0, ξ_0) such that the $\overline{F} \subset F_2 \subset \dot{\overline{F}}_2 \subset F_1$ and denote by F_3 any conic neighborhood of $\mathfrak{k}$. According to Lemma 3.3 there is a sequence $(a_N) \subset \tau^0(\Omega)$ such that $\operatorname{supp} a_N \subset F_1 \cap F_3$, the points on $\mathfrak{k} \cap F_2$ are noncharacteristic with respect to (a_N) and

$$([g_N(x, D) \circ P(x, D), a_N(x, D)]) \in T_0(\pi(F_2)). \tag{5.13}$$

Since we may assume that the points in $\operatorname{supp} a_N$ are noncharacteristic with respect to (g_N), it follows from (5.12) and (5.13) that

$$|D_x^\alpha P(x, D) \circ a_N(x, D)u(x)| \leq C^N N^{|\alpha|}, \quad \text{when } |\alpha| \leq \delta N \text{ and } x \in \pi(F_2). \tag{5.14}$$

To prove Theorem 1.3, we shall prove that there is a constant $\gamma > 0$, independent of $(\overline{x}_0, \overline{\xi}_0) \in F \cap \mathfrak{k} \cap \{(x, \xi); |\xi| = 1\}$ such that if $(\overline{x}_0, \overline{\xi}_0) \notin \mathrm{WF}_a(u)$ then

$$\mathfrak{k} \cap B_\gamma(\overline{x}_0, \overline{\xi}_0) \cap \mathrm{WF}_a(u) = \emptyset. \tag{5.15}$$

In fact we shall prove that (5.15) is true for any γ such that $B_\gamma(\overline{x}_0, \overline{\xi}_0) \subset F_2$. Suppose that $B_\gamma(\overline{x}_0, \overline{\xi}_0)$ has distance $\geq \epsilon > 0$ to $\complement F_2$ and denote by S_γ the boundary of B_γ. We shall then prove that there is a constant $\epsilon' > 0$, independent of γ, such that if $(\overline{x}, \overline{\xi}) \notin \mathfrak{k} \cap S_\gamma(\overline{x}_0, \overline{\xi}_0)$ then $\mathfrak{k} \cap B_{\epsilon'}(\overline{x}, \overline{\xi}) \cap \mathrm{WF}(u) = \emptyset$. Because of (5.11) the normal of $S_\gamma(\overline{x}_0, \overline{\xi}_0)$ at $(\overline{x}, \overline{\xi})$ is not perpendicular to the (x_{n-1}, x_n)-plane. Suppose that it is not perpendicular to the x_n-axis and that $\overline{x} = 0$. Then there are constants ϵ'' and M such that ϵ'' depends on F_3 and tends to zero as the neighborhood F_3 of $\mathfrak{k}$ becomes smaller, M depends just on $B_\gamma(\overline{x}_0, \overline{\xi}_0)$ and

$$\begin{gathered} D_x^\alpha a_N(x, D)u(x)| \leq C^N N^{|\alpha|}, \\ \text{when } |\alpha| \leq \delta N \text{ and } -1/M \leq x_n \leq -\epsilon'' - M|x'|^2. \end{gathered} \tag{5.16}$$

We now make the following change of coordinates:

$$y_n = x_n + 2M|x'|^2, \quad y' = x'. \tag{5.17}$$

(5.16) is then transformed to

$$|D_y^\alpha a_N(y, D)u(y)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } y \in G(\epsilon'', M). \tag{5.18}$$

Here $G(\epsilon'', M) = \{y; -1/M + 2M|y'|^2 \le y_n \le -\epsilon'' + M|y'|^2\}$ and we have kept the notation a_N for the operator in the new coordinates. The rest of the proof now parallels the proof of Theorem 1.2. By means of the factorization lemma of Hörmander, which works also in the complex case, we obtain an operator $L_N(y, D) = \partial/\partial y_n + S_N(y, D')$ such that

$$|D_y^\alpha L_N(y, D) \circ a_N(y, D)u(y)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \tag{5.19}$$

and y belongs to a fixed neighborhood of the origin. The results of [5] give that $a_N(y, D)u \in C_0^{[\delta' N]}$ and since the constant σ in Theorem 4.1 does not depend on $\bar{x}$, it follows from Theorem 4.1 that (5.18) is actually satisfied in a neighborhood of $y = 0$. Observe that, because of (5.8), L_N satisfies the hypothesis in Theorem 4.1. Theorem 1.3 now follows from Lemma 3.1.

Proof of Theorem 1.4. Suppose that $(x_0, \xi_0) \in N_1(P_m)\backslash \mathrm{WF}_a(Pu)$. Because of Theorem 1.3, we just have to consider the case when the bicharacteristic strip $\mathfrak{b}$ through (x_0, ξ_0) is 1-dimensional. Let $U_0 \times \Gamma_0$ be a conic neighborhood of (x_0, ξ_0) such that $Z(P_m) \cap (\overline{U}_0 \times \overline{\Gamma}_0) \subset N_1(P_m)\backslash \mathrm{WF}_a(u)$. Again we shall prove that there is a constant $\gamma > 0$, independent of $(\bar{x}_0, \bar{\xi}_0) \in (U_0 \times \Gamma_0) \cap \mathfrak{b} \cap \{(x, \xi); |\xi| = 1\}$, such that if $(\bar{x}_0, \bar{\xi}_0) \notin \mathrm{WF}_a(u)$ then $\mathfrak{b} \cap B_\gamma(\bar{x}_0, \bar{\xi}_0) \cap \mathrm{WF}_a(u) = \emptyset$. By making a linear change of coordinates and dividing by a complex constant we can always reduce to the situation that $\bar{x}_0 = 0$ and $d_\xi P_m(\bar{\xi}_0) = (0, \cdots, 0, 1)$, i.e., $\mathfrak{b}$ is the x_n-axis. Since $(0, \bar{\xi}_0) \notin \mathrm{WF}_a(u)$, there is a conic neighborhood $U \times \Gamma$ of $(0, \bar{\xi}_0)$ such that $(U \times \Gamma) \cap \mathrm{WF}_a(u) = \emptyset$. We suppose that U is of the form $\{x; |x_n| \le \epsilon$ and $|x'| < \epsilon\}$. Let $\psi_N'(\xi)$ be homogeneous functions of degree 0 with support in Γ and satisfying (1.1) when $|\xi| = 1$. Suppose also that $\psi_N'(\xi) = 1$ in a conic neighborhood of $\bar{\xi}_0$ and put $\psi_N(\xi) = \psi_N'(\xi)\chi(\xi/N)$, where the functions $\chi_N(\xi)$ satisfy (1.1), vanish when $|\xi| \le 1$ and are equal to 1 when $|\xi| \ge 2$. Then

$$|D_x^\alpha \psi_N(D)u(x)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } x \in U. \tag{5.20}$$

We can always assume that $U \times \Gamma \subset U_0 \times \Gamma_0$ so, since $[P(D), \psi_N(D)] = 0$, it follows that

$$|D_x^\alpha P(D)\psi_N(D)u(x)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } x \in U. \tag{5.21}$$

Assume, for simplicity, that $U = \{x; |x_n| < 1$ and $|x'| < 1\}$. Using (5.20) and (5.21), we shall prove that if $0 < a < 1/2$ then (5.20) is valid in a neighborhood of $(0, \cdots, 0, a)$. To do this we employ the change of coordinates (5.17). In the new coordinates $\psi_N(D)$ has a symbol $a_N(y, \eta)$ given by (2.2). From (5.20) it follows that

$$(5.22)\quad |D_y^\alpha a_N(y, D)u(y)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } y \in G_1(\epsilon, M).$$

Here $G_1(\epsilon, M) = \{y;\ |y'| \le \epsilon,\ 2M|y'|^2 - \epsilon < y_n < 2M|y'|^2 + \epsilon\}$. We assume that M is so large that $2a + \epsilon < 2M\epsilon^2$ and put $G_2(\epsilon, M) = \{y;\ 2M|y'|^2 - \epsilon < y_n < 2a\}$. Then it follows from (5.22) that

$$(5.23)\quad |D_y^\alpha P(y, D) \circ a_N(y, D)u(y)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } y \in G_2(\epsilon, M).$$

Since $d_\xi P_m(\xi_0) = (0, \cdots, 0, 1)$ it follows that if the support of $\psi_N(\xi)$ is small enough, depending on M, then

$$(5.24)\quad \partial P_m(y, \eta)/\partial\eta_n \neq 0, \quad \text{when } (y, \eta) \in (\operatorname{supp} a_N) \cap (\overline{G_2(\epsilon, M)} \times R^n).$$

We can now apply the factorization lemma of Hörmander to obtain operators $Q_N(y, D)$ and $L_N(y, D) = \partial/\partial y_n + S_N(y, D')$ such that

$$(5.25)\quad (Q_N(y, D) \circ L_N(y, D) \circ a_N(y, D) - P(y, D) \circ a_N(y, D)) \in T_0(G_2(\epsilon, M)),$$

and the points in supp a_N are noncharacteristic with respect to (Q_N). From (5.23) and (5.25) we get

$$(5.26)\quad |D_y^\alpha L_N(y, D) \circ a_N(y, D)u(y)| \le C^N N^{|\alpha|}, \quad \text{when } |\alpha| \le \delta N \text{ and } y \in G_2(\epsilon, M).$$

It follows from Theorem 1.6.5 of [10] that $a_N(y, D)u \in C_0^{[\delta' N]}$. Moreover, since $P(D)$ has constant coefficients, L_N satisfies the hypothesis of Theorem 4.1. We therefore conclude from Theorem 4.1, in a finite number of steps, that (5.22) is actually satisfied in $G_2(\epsilon, M)$. Finally $\mathfrak{k}$ is invariant under the coordinate transformation so (5.20) is valid in a neighborhood of $(0, \cdots, 0, a)$. In view of Lemma 3.1, this completes the proof of Theorem 1.4.

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