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Analytical expressions for the envelope correlation of certain narrow-band stimuli

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Analytical solutions are presented for the correlation between the envelopes of certain narrow-band stimuli which are typically used in binaural research. These stimuli are either the sum of an in-phase masker and an out-of-phase signal, or they are two partially correlated noises. Because an envelope has a nonzero mean, the solutions differ depending on whether the envelope correlation is expressed as the normalized cross correlation or as the normalized cross covariance (Pearson product-moment correlation). The envelope correlation depends on the statistics of the masker and the signal whereas the waveform correlation depends on neither. This influence only disappears for the normalized envelope *cross correlation* provided that the correlation is close to one. In this case, the normalized envelope cross correlation is equal to the square root of the waveform correlation. The results for two partially correlated noise bands are also of relevance for experiments dealing with monaural envelope discrimination and comodulation masking release. © *1995 Acoustical Society of America.*

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INTRODUCTION

The interaural cross correlation of a stimulus presented to the two ears of a listener forms an important cue in binaural listening. For example, the masked threshold in conditions leading to a binaural masking level difference can be predicted in many cases from the subject's sensitivity to changes in the interaural cross correlation (Durlach *et al.*, 1986; Bernstein and Trahiotis, 1992). Therefore most modern models of binaural interaction incorporate a central stage in which the interaural cross-correlation functions are calculated within each critical band (e.g., Colburn, 1977; Lindemann, 1986; Raatgever and Bilsen, 1986).

The interaural correlation is usually calculated from the temporal waveforms. In the typical condition of an in-phase masker combined with an out-of-phase signal $(M_0 S_\pi)$,[1] this leads to a very simple relation between the interaural waveform correlation and the relative intensity of masker and signal (cf. Durlach *et al.*, 1986). This correlation measure is, however, of little relevance for experiments that use high-frequency stimuli. Various binaural experiments have revealed that, at high frequencies, the ear is insensitive to interaural differences in the fine structure of the stimuli, but that it can detect interaural differences in the envelopes (cf. Henning, 1974; McFadden and Pasanen, 1978). Furthermore, at frequencies above about 1.5 kHz, the phase locking in the responses of the inner hair cells decreases and, consequently, the coding of the stimulus fine structure is lost (e.g., Palmer and Russell, 1986). Thus, at high frequencies, only the *envelope* of the acoustic stimuli is coded in the neural activity. As a consequence, the envelope correlation is a more interesting measure than the waveform correlation for binaural experiments performed at high frequencies.

The envelope correlation is also of interest for experiments on *monaural* envelope correlation perception (e.g., Richards, 1987). In these experiments, the (monaural) stimulus consists of two narrow-band noises at different center frequencies. Subjects are asked to discriminate between stimuli with different degrees of envelope correlation. For these stimuli, the waveform correlation between the two noise bands is always zero, independent of the correlation between their envelopes.

Despite the relevance of the envelope correlation for binaural and monaural experiments, no analytical expressions have been derived so far (to the best of our knowledge) that relate waveform and envelope correlation to one another. There are, however, some observations available that are derived from numerical simulations. Bernstein (1991) reported an empirical relation between the normalized cross covariance (sometimes referred to as the Pearson product-moment correlation) of the envelopes, r_E, and the waveform correlation, r_W, for an $M_0 S_\pi$ condition and also for conditions where narrow bands of noise are mixed according to the method specified by Licklider and Dzendolet (1948) and Jeffress and Robinson (1962). Provided that r_W was close to one, he found that r_E was approximately equal to r_W^2.

A qualitatively similar result has been published earlier by Richards (1987). Using the cross-covariance measure for the envelope she noted that "waveform correlations are larger than correlations based on envelopes alone (except at $r = 0$ and $r = 1$, where whole waveform and envelope correlations are the same)" (Richards, 1987, pp. 1628–1629).[2]

A different result, again based on numerical simulations, was found by Kohlrausch *et al.* (1995). They derived the normalized *cross correlation* of the envelopes for $M_0 S_\pi$ for a wide range of signal-to-masker ratios (SMR). In this simulation, the envelope correlation was approximately equal to the square root of the waveform correlation. Thus, in numerical simulations, the cross covariance and the cross correlation of the envelopes differ significantly. The different outcome of these numerical simulations was one of the motivations for the investigation presented in this paper.

Analytical expressions are derived that relate the SMR of an $M_0 S_\pi$ configuration to the envelope correlation for a

specific class of stimuli. It is shown how these expressions can be used for any set of partially correlated Gaussian noises as well. The specific class consists of pairs of band-limited waveforms where one waveform is the sum and the other is the difference of a masker $M(t)$ and a signal $S(t)$. The phase and amplitude of the masker and the signal are assumed to be stationary and pointwise independent for each specific instant of time. Furthermore, the differences between the phases of $M(t)$ and $S(t)$ are assumed to be uniformly distributed. The masker and signal can both be noiselike, but the derivation is also applicable to a situation where one or both are sinusoids with a random starting phase. Furthermore, it is shown how the derivation can be applied to multiplied noise which is generated by multiplying a low-pass noise by a sinusoidal carrier. If the masker (or the signal) is such a multiplied noise and the signal (or the masker) is a sinusoid with the same frequency as the sinusoidal carrier and differs only in a constant phase offset, i.e., if it is spectrally centered in the multiplied noise, the differences between the phases of $M(t)$ and $S(t)$ are not uniformly distributed.

All derivations are presented for the normalized cross covariance and the normalized cross correlation. Because this paper concentrates on the mathematical derivations, we do not go into the interesting discussion of which of the two envelope correlation measures is more appropriate for psychoacoustics.

The following section (I) gives some general definitions and assumptions and some assumptions about the masker and signal used. At the end of that section, the well-known relation between the SMR and the waveform correlation (cf. Durlach *et al.*, 1986) is formulated in the notation used throughout this study. In Secs. II and III, analytical expressions are derived for the cross correlation and the cross covariance, respectively, for the condition M_0S_π. Besides the exact solutions, which are quite complex and not suitable for fast calculations, approximations are given for the case that the SMR is small. Section IV shows how the previous results can be applied to the general case of two partially correlated Gaussian noises and Sec. V derives solutions for multiplied noise. The final section (VI) gives an overview of the results and presents some implications and questions for future research.

I. DEFINITIONS AND ASSUMPTIONS

In the psychoacoustical literature, we find that two definitions of correlation are used for the correlation between two random processes x and y (e.g., Lindemann, 1986; Richards, 1987):

(1) the normalized cross correlation (in short, *cross correlation*), which is defined as

$$\rho_{xy} = \frac{\langle xy \rangle}{\sqrt{\langle x^2 \rangle \langle y^2 \rangle}};$$
(1)

(2) the normalized cross covariance (in short, *cross covariance*), which is defined as

$$r_{xy} = \frac{\langle (x - \langle x \rangle)(y - \langle y \rangle) \rangle}{\sqrt{\langle (x - \langle x \rangle)^2 \rangle \langle (y - \langle y \rangle)^2 \rangle}}.$$
(2)

In these equations, $\langle\ \rangle$ is used to denote an expected value. It is obvious that both types of correlation are identical when x and y have zero mean.

For psychoacoustical experiments, the following estimators are commonly used to analyze specific waveform samples:

$$\varrho_{xy} = \frac{(1/T)\int_0^T x(t)y(t)dt}{\sqrt{(1/T)^2 \int_0^T x^2(t)dt \int_0^T y^2(t)dt}}$$
(3)

for the cross correlation, where T is the length of the waveform sample, and

$$\chi_{xy} = \frac{(1/T)\int_0^T (x(t) - \bar{x})(y(t) - \bar{y})dt}{\sqrt{(1/T)^2 \int_0^T (x(t) - \bar{x})^2\ dt \int_0^T (y(t) - \bar{y})^2\ dt}}$$
(4)

for the cross covariance, where $\bar{x}$ and $\bar{y}$ denote the time averages of x and y, respectively.

If x and y result from strongly ergodic processes, the definitions in Eqs. (1) and (2) coincide with the definitions in Eqs. (3) and (4) provided that $T \to \infty$ (Bendat and Piersol, 1971). In real experiments, time intervals are always finite and therefore the estimators are of limited accuracy.

In the following, the maskers and signals are specified to which the derivations in the following sections are applicable. A nonzero masker $M(t)$ and a nonzero signal $S(t)$ are defined. One or both may be a specific realization of a narrow-band random process. The masker and signal are used in the derivations of the pure waveform and the envelope correlation and can be written in terms of their respective Hilbert envelopes $m(t)$ and $s(t)$:

$$M(t) = m(t)\cos(\omega t + \phi_m(t)),$$
(5)

$$S(t) = s(t)\cos(\omega t + \phi_s(t)),$$
(6)

where $\phi_m(t)$, $\phi_s(t)$, $m(t)$, and $s(t)$ are random functions that vary slowly with respect to $\cos \omega t$ (Davenport and Root, 1958).

The amplitude functions $m(t)$ and $s(t)$ and the phase functions $\phi_m(t)$ and $\phi_s(t)$ are assumed to be pointwise independent, i.e., their joint probability factors for each t:

$$P_{ms\phi_m\phi_s}(m(t), s(t), \phi_m(t), \phi_s(t))$$
$$= P_m(m(t))P_s(s(t))P_{\phi_m}(\phi_m(t))P_{\phi_s}(\phi_s(t)).$$
(7)

The difference of the two phase functions, $\phi(t)$, is assumed to be uniformly distributed. Furthermore, the processes generating $M(t)$ and $S(t)$ are assumed to be stationary. These assumptions imply that $\langle M \rangle = \langle S \rangle = 0$.

Using the assumption that the amplitude and phase functions are pointwise independent, we can find the following relations:

$$\langle M^2 \rangle = \tfrac{1}{2}\langle m^2 \rangle,$$
(8)

$$\langle S^2 \rangle = \tfrac{1}{2}\langle s^2 \rangle.$$
(9)

For the masker $M(t)$ and the signal $S(t)$, the SMR is defined as

$$\text{SMR} = \frac{\langle S^2 \rangle}{\langle M^2 \rangle} = \frac{\langle s^2 \rangle}{\langle m^2 \rangle}. \tag{10}$$

Two new waveforms are defined using the masker and signal:

$$L(t) = M(t) + S(t), \tag{11}$$

$$R(t) = M(t) - S(t). \tag{12}$$

$L(t)$ and $R(t)$ denote waveforms that could be presented dichotically in a binaural experiment.

Because the means of $M(t)$ and $S(t)$ are zero, the means of $L(t)$ and $R(t)$ are also equal to zero. This implies that the two definitions of correlation [Eqs. (1) and (2)] give the same value if applied to $L(t)$ and $R(t)$.

In order to derive an expression for the waveform correlation, we will use the definition of the normalized cross correlation in Eq. (1). Using the fact that M and S are uncorrelated and stationary, we get the waveform correlation

$$
r_{LR} = \rho_{LR} = \frac{\langle (M+S)(M-S) \rangle}{\sqrt{\langle (M+S)^2 \rangle \langle (M-S)^2 \rangle}}
$$
$$
= \frac{\langle M^2 \rangle - \langle S^2 \rangle}{\langle M^2 \rangle + \langle S^2 \rangle} = \frac{\langle m^2 \rangle - \langle s^2 \rangle}{\langle m^2 \rangle + \langle s^2 \rangle}, \tag{13}
$$

as was also derived by others, e.g., Durlach *et al.* (1986). The relative power of the masker, M, and the signal, S, is the only determining factor for the correlation; other statistical properties of M and S have no effect.

The Hilbert envelopes[3] of L and R can be derived using Eqs. (11) and (12):

$$E_L = |m(t) e^{i\phi_m(t)} + s(t) e^{i\phi_s(t)}|, \tag{14}$$

$$E_R = |m(t) e^{i\phi_m(t)} - s(t) e^{i\phi_s(t)}|. \tag{15}$$

The frequency ω is not present in the expressions for the envelope. Because all further calculations of the envelope correlation are based on these two expressions, the waveforms L and R may also be located at different places in the frequency domain as long as the functions $m(t)$, $s(t)$, $\phi_m(t)$, and $\phi_s(t)$ remain the same. Therefore the derivations of the envelope correlations are also applicable to stimuli used in comodulation masking release experiments.

II. NORMALIZED CROSS CORRELATION OF THE ENVELOPES

In this section, the normalized cross correlation of the envelopes of two waveforms L and R as defined in the previous section is calculated. Because the definition of the normalized cross correlation [Eq. (1)] is the ratio of two quantities, the derivation splits into two parts. First, the quantity $\langle E_L E_R \rangle$ has to be derived [Eqs. (16)–(24)] and, second, the quantity $\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle}$ [Eq. (25)]. These two steps lead to the exact solution for the envelope cross correlation [Eq. (26)]. In the second part of this section the exact solution of Eq. (26) is approximated by a much simpler expression for the specific case of low SMRs [Eqs. (27)–(30)]. The resulting approximation has a simple relation with the waveform correlation [Eqs. (31) and (32)].

The envelopes of the waveforms L and R are given by Eqs. (14) and (15):

$$
E_L = |m e^{i\phi_m} + s e^{i\phi_s}|
$$
$$
= \sqrt{(m \cos \phi_m + s \cos \phi_s)^2 + (m \sin \phi_m + s \sin \phi_s)^2} \tag{16}
$$

and

$$
E_R = |m e^{i\phi_m} - s e^{i\phi_s}|
$$
$$
= \sqrt{(m \cos \phi_m - s \cos \phi_s)^2 + (m \sin \phi_m - s \sin \phi_s)^2}. \tag{17}
$$

Using the fact that $\cos^2 \alpha + \sin^2 \alpha = 1$ and that $\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\beta - \alpha)$ we can write an expression for the product of the envelopes:

$$
\langle E_L E_R \rangle = \left\langle (m^2 + s^2) \sqrt{1 - \frac{4m^2 s^2}{(m^2 + s^2)^2} \cos^2 \phi} \right\rangle, \tag{18}
$$

where

$$\phi = \phi_s - \phi_m. \tag{19}$$

Using the Taylor series

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} c_n x^n, \tag{20}$$

with

$$c_n = \frac{1}{n!} \prod_{k=1}^{n} \left(\frac{3}{2} - k \right), \tag{21}$$

we obtain the expression

$$
\langle E_L E_R \rangle = \left\langle m^2 + s^2 + \sum_{n=1}^{\infty} c_n (-4)^n \right.
$$
$$
\left. \times \frac{m^{2n} s^{2n}}{(m^2 + s^2)^{2n-1}} \cos^{2n} \phi \right\rangle. \tag{22}
$$

Using the facts that the amplitude and phase functions are pointwise independent, and that ϕ is uniformly distributed, and that for positive integers n

$$
\int_0^{2\pi} \cos^{2n} \theta \, d\theta = \frac{(2n-1)!!}{(2n)!!} 2\pi, \tag{23}
$$

where $n!! = n(n-2)(n-4)\cdots 2$ for n even and $n!! = n(n-2)(n-4)\cdots 1$ for n odd, we get

$$
\langle E_L E_R \rangle = \langle m^2 \rangle + \langle s^2 \rangle + \sum_{n=1}^{\infty} c_n (-4)^n \frac{(2n-1)!!}{(2n)!!}
$$
$$
\times \left\langle \frac{m^{2n} s^{2n}}{(m^2 + s^2)^{2n-1}} \right\rangle. \tag{24}
$$

The denominator for the normalized cross correlation of the envelopes, from Eq. (1), reduces rather straightforwardly to

$$
\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle} = \sqrt{(\langle m^2 \rangle + \langle s^2 \rangle)^2 - (2\langle m \rangle \langle s \rangle \langle \cos \phi \rangle)^2}
$$
$$
= \langle m^2 \rangle + \langle s^2 \rangle, \tag{25}
$$

FIG. 1. The normalized cross correlation as a function of the SMR in the $M_0 S_\pi$ condition with a Gaussian noise masker. The solid line shows the waveform cross correlation; the envelope cross correlation is shown by the dotted line for a sinusoidal signal and by the dashed line for a Gaussian noise signal. Note that from Eq. (13) it follows that the *waveform* cross correlation is only dependent on the SMR and not on the other statistical properties of M and S.

because ϕ is uniform. Now we can write the cross correlation of the envelope as the ratio of Eqs. (24) and (25):

$$\rho_{E_L E_R} = \frac{\langle E_L E_R \rangle}{\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle}} = 1 + \frac{1}{\langle m^2 \rangle + \langle s^2 \rangle}$$

$$\times \sum_{n=1}^{\infty} c_n (-4)^n \frac{(2n-1)!!}{(2n)!!}$$

$$\times \left\langle \frac{m^{2n} s^{2n}}{(m^2 + s^2)^{2n-1}} \right\rangle. \qquad (26)$$

The last factor in Eq. (26) indicates that the cross correlation of the envelopes depends on the statistics of the masker and the signal. In the calculations for Figs. 1–4, it has been assumed that the noises have Gaussian waveform statistics.

In Fig. 1, the cross correlation ρ is plotted for several $M_0 S_\pi$ conditions as a function of the SMR. In all three conditions the masker is a Gaussian noise. The three curves show the waveform correlation (solid line), the envelope cross correlation for a sinusoidal signal (dotted line), and the envelope correlation for a Gaussian signal (dashed line). This latter condition is an example of the more general case of two Gaussian noises with a specified waveform correlation which is discussed in more detail in Sec. IV. The curves are obtained by calculating Eqs. (13) and (26). The expected value in the series of Eq. (26) is evaluated numerically, using the probability distribution functions of $m(t)$ and $s(t)$.
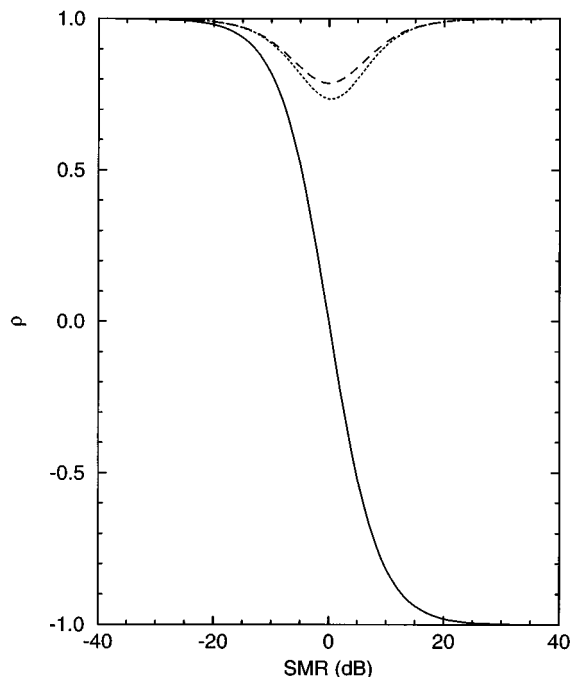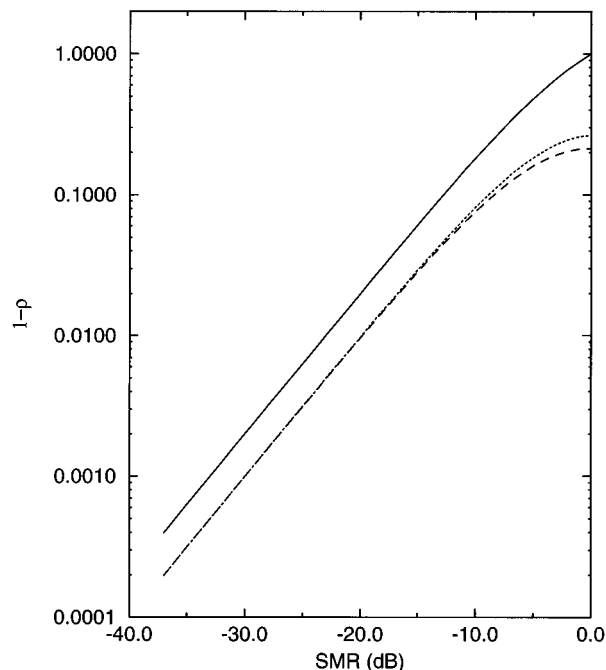

FIG. 2. Change in the normalized cross correlation as a function of the SMR in the $M_0 S_\pi$ condition with a Gaussian noise masker. The solid line shows the waveform cross correlation; the envelope cross correlation is shown by the dotted line for a sinusoidal signal and by the dashed line for a Gaussian noise signal.

We can see that the envelope cross correlation of a Gaussian signal (dashed line) differs from the envelope cross correlation with a sinusoidal signal (dotted line) for SMRs around 0 dB. This illustrates the dependence of the envelope cross correlation on the statistics of the signal $S(t)$. Furthermore, the envelope cross correlation with a sinusoidal signal shows a slightly asymmetrical curve around SMR=0 dB. For a Gaussian signal the (dashed) curve is symmetrical around 0 dB SMR. Because signal and masker are Gaussian noises, changing the sign of the SMR is equivalent to transforming the $M_0 S_\pi$ condition into an $M_\pi S_0$ condition. Since the envelopes in both conditions are identical, the envelope cross correlation must also be identical.

The waveform correlation (solid line) varies from 1 for very small SMRs where the homophasic masker dominates, to -1 for very large SMRs where the antiphasic signal dominates. For the envelope cross correlation we do not observe negative values. This is due to the fact that the envelopes are non-negative functions. The cross-correlation value of nonnegative functions is necessarily non-negative [cf. Eq. (1)].

In binaural experiments, signal thresholds typically correspond to SMRs in the range -10 to -30 dB. In addition, the relevant quantity in detecting the signal in the condition $M_0 S_\pi$ is the change $\Delta\rho = 1 - \rho$ in the cross correlation. This quantity is plotted in Fig. 2 as a function of the SMR for the same conditions as in Fig. 1. Generally, the change in the cross correlation becomes smaller with decreasing SMR. The curves approximate straight lines for small SMRs, a range most relevant for binaural detection experiments. This suggests that a simple approximation must be possible.

Indeed, when $\langle s^2 \rangle \ll \langle m^2 \rangle$, Eq. (26) can be approximated

by a much simpler equation, because the probability of finding $s^2 \ll m^2$ at a specific instant of time is close to unity. In this case, the following approximation can be made at most instants of time:

$$\frac{m^{2n}s^{2n}}{(m^2+s^2)^{2n-1}} \approx \frac{s^{2n}}{m^{2n-2}} \approx \frac{s^{2n}}{m^{2n-2}+s^{2n-2}}, \tag{27}$$

for $n>1$ (for $n=1$, the fraction is approximately equal to s^2). For the rare cases that $s^2 \gg m^2$ or $s^2 \approx m^2$, the approximation of the fraction in Eq. (27) is not valid. This is, however, of little influence on the expected value of Eq. (27), because the fraction has only a limited magnitude under these circumstances and contributes little to the expected value. Therefore the expected value can be written as

$$\left\langle \frac{m^{2n}s^{2n}}{(m^2+s^2)^{2n-1}} \right\rangle \approx \left\langle \frac{s^{2n}}{m^{2n-2}+s^{2n-2}} \right\rangle, \tag{28}$$

for $n>1$ (for $n=1$, the expected value is approximately equal to $\langle s^2 \rangle$). By substituting Eq. (28) into Eq. (26) we can write

$$\rho_{E_L E_R} \approx 1 - \frac{1}{\langle m^2 \rangle + \langle s^2 \rangle} \left(\langle s^2 \rangle + \frac{3}{4} \left\langle \frac{s^4}{m^2+s^2} \right\rangle + \cdots \right). \tag{29}$$

Of the terms within the parenthesis in Eq. (29), only the first is needed for a good approximation. All further terms have successive expected values which are negligible with respect to the preceding values. Using Eqs. (8) and (9), the envelope correlation as derived in Eq. (26) reduces to

$$\rho_{E_L E_R} \approx 1 - \frac{\langle S^2 \rangle}{\langle M^2 \rangle + \langle S^2 \rangle} = \frac{\langle M^2 \rangle}{\langle M^2 \rangle + \langle S^2 \rangle}$$

$$\approx \frac{\langle M^2 \rangle - \frac{1}{2} \langle S^2 \rangle}{\langle M^2 \rangle + \frac{1}{2} \langle S^2 \rangle}. \tag{30}$$

In contrast to the exact formula, the envelope cross correlation is in this approximation only dependent on the SMR and not on the statistics of S and M.

Because $\langle M^2 \rangle \gg \langle S^2 \rangle$, we know that $\rho_{E_L E_R}$ in Eq. (30) is nearly one, differing only by $\Delta \rho_{E_L E_R} \approx \langle S^2 \rangle / \langle M^2 \rangle$. A similar statement can be made for the waveform correlation ρ_{LR} in Eq. (13), for which we find $\Delta \rho_{LR} \approx 2 \langle S^2 \rangle / \langle M^2 \rangle$. This leads to the following relation:

$$2 \Delta \rho_{E_L E_R} \approx \Delta_{\rho_{LR}}. \tag{31}$$

A different way to express this relation is

$$\sqrt{\rho_{LR}} = (1 - \Delta \rho_{LR})^{1/2} \approx 1 - \tfrac{1}{2} \Delta \rho_{LR}$$

$$\approx 1 - \Delta \rho_{E_L E_R} = \rho_{E_L E_R}. \tag{32}$$

Equation (32) indicates that at low SMRs, the envelope cross correlation is approximately equal to the square root of the waveform correlation.

In Table I the accuracy of the approximation in Eq. (30) is shown for the two cases of a sinusoidal and a Gaussian noise signal. For an SMR of -8 dB or less the approximation of the envelope cross correlation differs 5% or less from the exact value. Thresholds obtained in a binaural $M_0 S_\pi$

TABLE I. Maximum SMRs for several error margins in the approximation of the normalized cross correlation of the envelope for the $M_0 S_\pi$ condition with a noise masker. The second column applies to the case that the signal is a sinusoid and the third column applies to the case that the signal is a noise. The noises are assumed to have Gaussian waveform statistics.

	SMR for $M_0 S_\pi$	
Error in $\rho_{E_L E_R}$	S sinusoid	S noise
$< 1\%$	$\leqslant -11$ dB	$\leqslant -12$ dB
$< 5\%$	$\leqslant -8$ dB	$\leqslant -8$ dB
$<10\%$	$\leqslant -6$ dB	$\leqslant -6$ dB

condition with a sinusoidal signal and a narrow-band masker, both at 4 kHz, typically correspond to an SMR of -12 dB or less (cf. Zurek and Durlach, 1987). In this case, the approximation in Eq. (30) is accurate to within 1%. When we compare the result of Eq. (30) with the waveform correlation [Eq. (13)], we can conclude that in order to achieve the same amount of change in interaural correlation for the waveform as for the envelope, the signal level in the latter condition has to be 3 dB higher than in the former. This agrees with the results of numerical simulations published by Kohlrausch et al. (1995).

III. NORMALIZED CROSS COVARIANCE OF THE ENVELOPES

In this section, the cross covariance $r_{E_L E_R}$ of the envelopes of the two waveforms L and R as defined in Sec. I is calculated using Eq. (2). First it will be shown that the only unknown quantity is $\langle E_L \rangle = \langle E_R \rangle$. This quantity is then calculated in Eqs. (33)–(36). In the second part of this section, approximations are derived for low values of the SMR [Eq. (37)–(40)]. The last part of this section shows how the envelope cross covariance is related to the waveform correlation for low SMRs [Eqs. (41)–(44)]. For the case of a masker with a flat temporal envelope (e.g., a sinusoid), the approximation needs more elaborate derivations. These are given in Appendix A and are discussed in Eqs. (45)–(48).

It is easy to see that $\langle E_L \rangle = \langle E_R \rangle$, $\langle E_L^2 \rangle = \langle E_R^2 \rangle$, and $\langle (E_L - \langle E_L \rangle)^2 \rangle = \langle (E_R - \langle E_R \rangle)^2 \rangle$. With this knowledge we can write Eq. (2) slightly differently:

$$r_{E_L E_R} = \frac{\langle E_L E_R \rangle - \langle E_L \rangle^2}{\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle} - \langle E_L \rangle^2}. \tag{33}$$

In order to evaluate the cross covariance, we need to know $\langle E_L E_R \rangle$, $\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle}$, and $\langle E_L \rangle$. The first two terms have been derived in the previous section [cf. Eqs. (24) and (25)]. The average value, $\langle E_L \rangle$, can be expressed using Eq. (16), which is the general expression for the envelope:

$$\langle E_L \rangle = \left\langle \sqrt{m^2+s^2} \sqrt{1 + \frac{2ms}{m^2+s^2} \cos \phi} \right\rangle. \tag{34}$$

Using the Taylor series of Eq. (20) we can expand the second square root:

$$\langle E_L \rangle = \left\langle \sqrt{m^2+s^2} + \sum_{n=1}^{\infty} c_n 2^n \frac{m^n s^n}{(m^2+s^2)^{n-1/2}} \cos^n \phi \right\rangle. \tag{35}$$

FIG. 3. The normalized cross covariance as a function of the SMR in the $M_0 S_\pi$ condition with a Gaussian noise masker. The solid line shows the waveform cross covariance; the envelope cross covariance is shown by the dotted line for a sinusoidal signal and by the dashed line for a Gaussian noise signal.
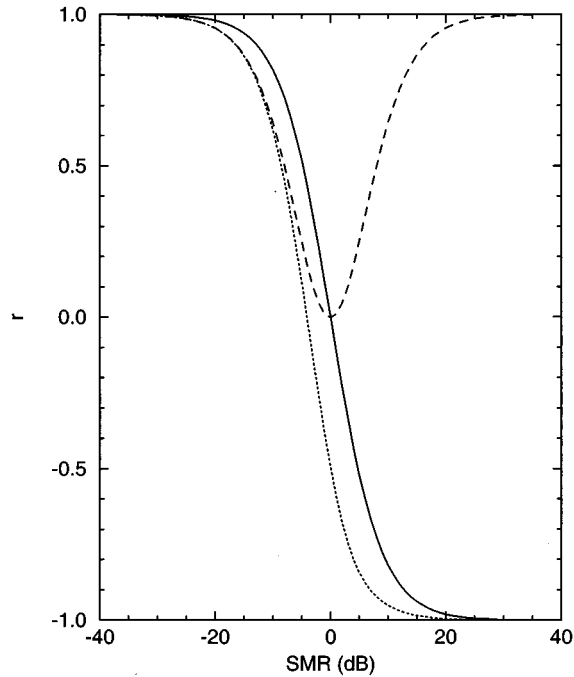

FIG. 4. Change in the normalized cross covariance as a function of the SMR in the $M_0 S_\pi$ condition with a Gaussian noise masker. The solid line shows the waveform cross covariance; the envelope cross covariance is shown by the dotted line for a sinusoidal signal and by the dashed line for a Gaussian noise signal.

Because the amplitude and phase functions are pointwise independent, we can write

$$\langle E_L \rangle = \langle \sqrt{m^2 + s^2} \rangle$$
$$+ \sum_{n=1}^{\infty} c_{2n} 4^n \left\langle \frac{m^{2n} s^{2n}}{(m^2+s^2)^{2n-1/2}} \right\rangle \frac{(2n-1)!!}{2n!!}.$$

(36)

Here, the odd terms of the series in Eq. (35) are omitted since they are equal to zero. With this expression we know all terms needed for the cross covariance as defined in Eq. (33).

In Fig. 3, the cross covariance is shown for the same conditions as in Fig. 1 as a function of the SMR. In contrast to the observation for the cross correlation, the cross covariance of the envelope for $M_0 S_\pi$ with a sinusoidal signal and a noise masker (dotted line) is negative for positive SMRs. This can be understood if we consider that the sign of the cross covariance is determined by the similarity of the envelope *variations* in both ears. When a noise dominates the stimulus (dotted line at small SMRs and dashed line at small negative and at high positive SMRs), the envelope variations in *both* ears will be large and similar to the noise envelope which yields a cross covariance close to 1. When a sinusoid dominates the stimulus (dotted line at high SMRs), the variations will be much smaller, but tend to be out of phase between the two ears. This results in a negative cross covariance.

In Fig. 4, the quantity $\Delta r = 1 - r$ is plotted as a function of the SMR for the same conditions. We can see that the curves for the envelope cross covariance are generally higher
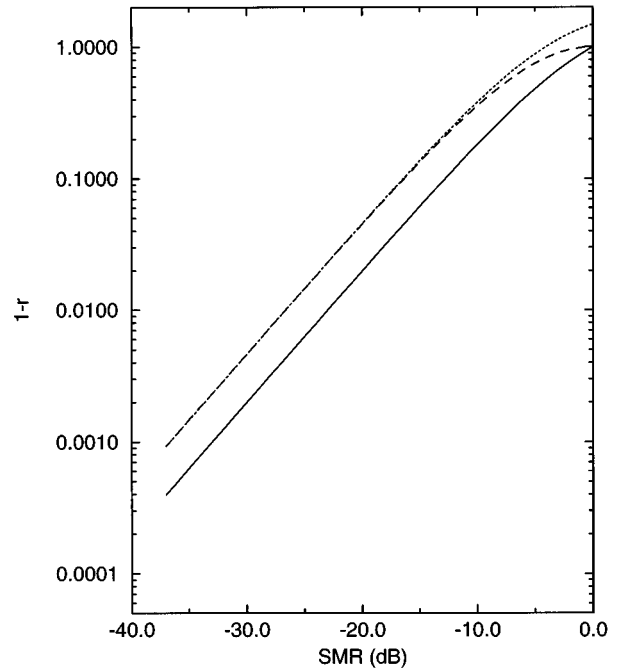
than the curves for the waveform correlation. This is the opposite result from what was found for the envelope cross correlation (cf. Fig. 2). Again the curves approximate straight lines for small SMRs. This suggests that a simple approximation must be possible just as was derived previously for the cross correlation of the envelope.

We first derive an approximation for the case that the masker envelope is not constant, which is equivalent to $\langle m^2 \rangle - \langle m \rangle^2 \neq 0$. When $\langle m^2 \rangle - \langle m \rangle^2 \gg \langle s^2 \rangle$, the first term of the series in Eq. (36) gives a good approximation. The same line of reasoning can be followed here as was used for the cross correlation [cf. Eq. (26)]. We use the fact that

$$\langle \sqrt{m^2 + s^2} \rangle = \langle m \rangle + \epsilon_1,$$

(37)

with $\epsilon_1 \ll \langle m \rangle$ and ϵ_1 of the order $\langle s^2/(m+s) \rangle$. Furthermore, we denote the series in Eq. (36) as ϵ_2, and we know that ϵ_2 is of the order $\langle s^2/(m+s) \rangle$ and thus $\epsilon_2 \ll \langle m \rangle$. This results in the following expression for the mean of the envelope:

$$\langle E_L \rangle = \langle m \rangle + \epsilon_1 + \epsilon_2.$$

(38)

Evaluating Eq. (24), taking into account the first term of the series, yields

$$\langle E_L E_R \rangle = \langle m^2 \rangle + \epsilon_3,$$

(39)

with $\epsilon_3 \ll \langle s^2 \rangle \ll \langle m^2 \rangle - \langle m \rangle^2$. Since, in addition, the terms ϵ_1 and ϵ_2 are both of the order $\langle s^2/(m+s) \rangle$, the cross covariance of the envelopes can be written as

	SMR for $M_0 S_\pi$	
Error in $r_{E_L E_R}$	S sinusoid	S noise
$< 1\%$	≤ -15 dB	≤ -14 dB
$< 5\%$	≤ -12 dB	≤ -11 dB
$< 10\%$	≤ -11 dB	≤ -9 dB

$$r_{E_L E_R} = \frac{\langle E_L E_R \rangle - \langle E_L \rangle^2}{\sqrt{\langle E_L^2 \rangle \langle E_R^2 \rangle} - \langle E_L \rangle^2}$$

$$= \frac{\langle m^2 \rangle + \epsilon_3 - (\langle m \rangle + \epsilon_1 + \epsilon_2)^2}{\langle m^2 \rangle + \langle s^2 \rangle - (\langle m \rangle + \epsilon_1 + \epsilon_2)^2}$$

$$\approx \frac{\langle m^2 \rangle - \langle m \rangle^2}{\langle m^2 \rangle - \langle m \rangle^2 + \langle s^2 \rangle}. \tag{40}$$

In Table II the accuracy of the approximation in Eq. (40) is shown for the two cases of a sinusoidal signal and a Gaussian noise signal.

Because $\langle m^2 \rangle - \langle m \rangle^2 \gg \langle s^2 \rangle$, we find that $r_{E_L E_R}$ is close to one, differing only by

$$\Delta r_{E_L E_R} \approx \frac{\langle s^2 \rangle}{\langle m^2 \rangle - \langle m \rangle^2}. \tag{41}$$

For the waveform covariance we can show that $\Delta r_{LR} \approx 2 \langle s^2 \rangle / \langle m^2 \rangle$. Therefore we find

$$\gamma \Delta r_{LR} \approx \Delta r_{E_L E_R}, \tag{42}$$

with

$$\gamma = \frac{\langle m^2 \rangle}{2 \langle m^2 \rangle - 2 \langle m \rangle^2}. \tag{43}$$

Equation (42) can be reformulated using the fact that $\Delta r_{LR} \ll 1$:

$$r_{LR}^\gamma = (1 - \Delta r_{LR})^\gamma \approx 1 - \gamma \Delta r_{LR} \approx 1 - \Delta r_{E_L E_R} = r_{E_L E_R}. \tag{44}$$

In these approximations the value of the cross covariance depends on the statistics of the masker M, which is not the case for the approximation for the cross correlation of the envelope [cf. Eq. (30)]. When we consider a Gaussian noise as masker M, with $\langle m^2 \rangle = 1$ we know that $\langle m \rangle^2 = \pi/4$. This gives a value for the exponent $\gamma \approx 2.3$, a value that was found by Bernstein (1991) in numerical simulations.

When the masker M has a flat envelope, the approximation of Eq. (40) does not hold. In this case, $\langle m^2 \rangle = \langle m \rangle^2 = m^2$. In Appendix A this specific case is evaluated by also taking into account the second term of the series in Eqs. (24) and (36). This leads to a different expression:

$$\gamma \Delta r_{LR} \approx 1 + r_{E_L E_R}, \tag{45}$$

with

$$\gamma = \frac{1}{2} \frac{(1/2)\langle s^2 \rangle - (7/32 m^2)\langle s^4 \rangle + (1/16 m^2)\langle s^2 \rangle^2}{(1/2)\langle s^2 \rangle - (1/32 m^2)\langle s^4 \rangle - (1/16 m^2)\langle s^2 \rangle^2}. \tag{46}$$

When we take the example of M being a sinusoidal masker and S being a Gaussian noise we find (see Appendix A)

$$\tfrac{1}{4} \Delta r_{LR} \approx 1 + r_{E_L E_R}. \tag{47}$$

If both masker and signal are sinusoids with different frequencies we find (see Appendix A)

$$\tfrac{1}{16} \Delta r_{LR} \approx 1 + r_{E_L E_R}. \tag{48}$$

IV. ENVELOPE CORRELATION BETWEEN GAUSSIAN NOISES

If the masker M and the signal S are Gaussian noises, the two waveforms L and R are Gaussian noises, too, with a waveform correlation between 1 and -1 (cf. Figs. 1 and 3). Another way of creating partially correlated noises uses three independent source waveforms ("three generator case," Licklider and Dzendolet, 1948; Jeffress and Robinson, 1962). A derivation of the general relation between envelope and waveform correlation for Gaussian noises follows.

If we have two Gaussian noises L and R with $\langle L^2 \rangle = \langle R^2 \rangle$, we can rewrite L and R as the sum and the difference of two independent noises M' and S':

$$L = \tfrac{1}{2}(L+R) + \tfrac{1}{2}(L-R) = M' + S', \tag{49}$$

$$R = \tfrac{1}{2}(L+R) - \tfrac{1}{2}(L-R) = M' - S'. \tag{50}$$

Indeed, it is not difficult to see that $r_{M' S'} = 0$. Because the new waveforms M' and S' are Gaussian distributed and uncorrelated, we conclude that we have independent waveforms and we can use these in the same way as the waveforms M and S as defined in Eqs. (5) and (6). The relative intensity of S' and M' (SMR'), which is not known beforehand, can be calculated from their waveform correlation with the help of Eq. (13). This leads to

$$\frac{1 - \rho_{LR}}{1 + \rho_{LR}} = \text{SMR}' = \frac{\langle (S')^2 \rangle}{\langle (M')^2 \rangle} = \frac{\langle (s')^2 \rangle}{\langle (m')^2 \rangle}. \tag{51}$$

Now the general case of two partially correlated Gaussian noises can be translated to an $M_0 S_\pi$ condition with a Gaussian signal and masker. The problem revolves about obtaining the probability distributions of the functions m' and s'. Once these are known it is possible to apply all equations that were derived for the cross correlation and the cross covariance of the envelope. Because the waveforms of S' and M' are Gaussian distributed, their envelope functions s' and m' have a Rayleigh distribution. Without loss of generality we can set $\langle (m')^2 \rangle = 1$. This specifies the exact distribution of m' and also the distribution of s', because the value of $\langle (s')^2 \rangle / \langle (m')^2 \rangle$ is given by Eq. (51). Knowing these distributions, the envelope cross correlation and the envelope cross covariance can be calculated [cf. Eqs. (26), (33), and (36)].
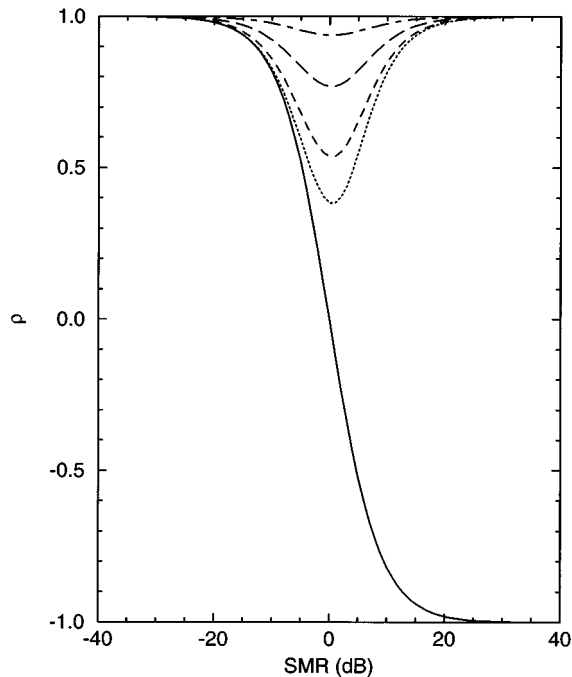
FIG. 5. The normalized cross correlation as a function of the SMR for the waveform for $M_0 S_\pi$ (solid line), the envelope for multiplied noise for $M_0 S_\pi$ with M a multiplied noise (low-pass Gaussian noise multiplied by a sinusoid) and S a sinusoid, with $\vartheta=0$ (dotted line), $\vartheta=\frac{1}{8}\pi$ (dashed line), $\vartheta=\frac{1}{4}\pi$ (long-dashed line), and $\vartheta=\frac{3}{8}\pi$ (dashed–dotted line).
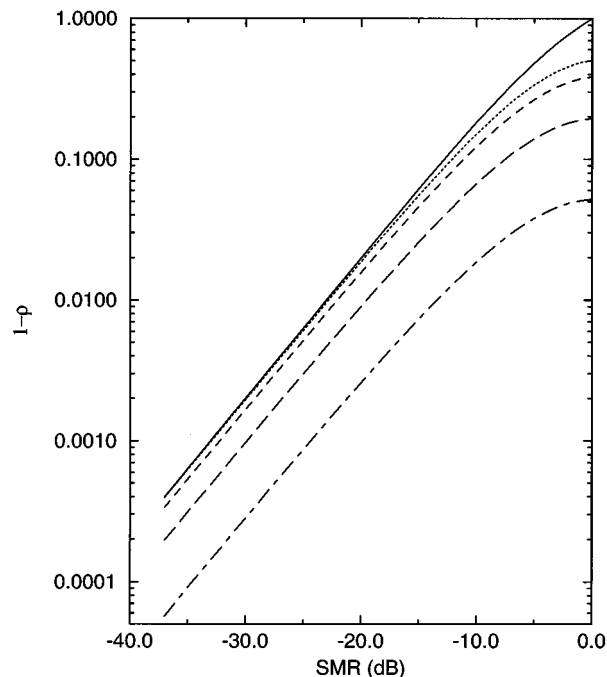

FIG. 6. Change in the normalized cross correlation as a function of the SMR for the waveform for $M_0 S_\pi$ (solid line), the envelope for multiplied noise for $M_0 S_\pi$ with M a multiplied noise (low-pass Gaussian noise multiplied by a sinusoid) and S a sinusoid, with $\vartheta=0$ (dotted line), $\vartheta=\frac{1}{8}\pi$ (dashed line), $\vartheta=\frac{1}{4}\pi$ (long-dashed line), and $\vartheta=\frac{3}{8}\pi$ (dashed–dotted line).

V. MULTIPLIED NOISE

A special type of noise to which the derivations in the previous sections are not generally applicable is multiplied noise. Multiplied noise is generated by multiplying a low-pass noise by a sinusoid. It is often used in experiments if a narrow-band noise with variable center frequency is needed (as, e.g., in psychophysical tuning curves; Johnson-Davies and Patterson, 1979) or if comodulated noise bands at different center frequencies must be generated (e.g., Moore *et al.*, 1990).[4]

A special case arises when the signal $S(t)$ is a sinusoid (with some phase offset of ϑ) with the same frequency as the sinusoid that was used to obtain the multiplied noise. In this situation, the phase difference $\phi(t)$, as defined in Eq. (19), is not uniformly distributed, but is either ϑ or $\vartheta+\pi$.[5]

For the pure waveform correlation, this nonuniform phase is of no importance and thus Eq. (13) is also valid for multiplied noise. For the envelope correlation, however, the $\cos^{2n}\phi$ factor in Eq. (22) cannot be reduced to $(2n-1)!!/(2n)!!$.

We can fix $\phi(t)$ at a value ϑ if we use the low-pass noise itself instead of the Hilbert envelope as the amplitude function $m(t)$. The distribution of such an amplitude function allows for both positive and negative function values. Assuming this fixed phase value ϑ, we will find $\cos\vartheta$ back in the final expression for the normalized cross correlation of the envelopes:

$$\rho_{E_L E_R} = 1 + \frac{1}{\langle m^2 \rangle + \langle s^2 \rangle} \sum_{n=1}^{\infty} c_n (-4)^n$$
$$\times \cos^{2n}\vartheta \left\langle \frac{m^{2n} s^{2n}}{(m^2+s^2)^{2n-1}} \right\rangle. \tag{52}$$

In the derivation, we needed either $\langle m \rangle$ or $\langle s \rangle$ to be zero in order for Eq. (25) to be valid since here $\langle \cos\phi \rangle \neq 0$.

In Fig. 5, the cross correlation of the envelope for an $M_0 S_\pi$ condition with multiplied noise as masker and a sinusoidal signal is plotted as a function of the SMR with the phase angle ϑ as parameter. In addition the waveform correlation is shown by the solid line. The phase angle ϑ has a marked effect on the envelope cross correlation. For $\vartheta=0$, the values for the cross correlation are smallest. They are also smaller than those for the conditions with a Gaussian noise masker (cf. Fig. 1). This allows a test of the hypothesis that signal detection is based on envelope correlation cues, because $M_0 S_\pi$ thresholds should be lower for a multiplied-noise masker, with $\vartheta=0$, than for a Gaussian noise masker.

As in the previous section, it is possible to find an approximation for small SMRs. When $\langle m^2 \rangle \gg \langle s^2 \rangle$, the first term of the series in Eq. (52) is sufficient for a good approximation. Using Eqs. (8) and (9), the cross correlation of the envelopes can be written as

$$\rho_{E_L E_R} \approx \frac{\langle M^2 \rangle - \langle S^2 \rangle + 2 \langle S^2 \rangle \sin^2 \vartheta}{\langle M^2 \rangle + \langle S^2 \rangle}. \tag{53}$$

FIG. 7. The normalized cross covariance as a function of the SMR for the waveform for M_0S_π (solid line), the envelope for multiplied noise for M_0S_π with M a multiplied noise (low-pass Gaussian noise multiplied by a sinusoid) and S a sinusoid, with $\vartheta=0$ (dotted line), $\vartheta=\frac{1}{8}\pi$ (dashed line), $\vartheta=\frac{1}{4}\pi$ (long-dashed line), and $\vartheta=\frac{3}{8}\pi$ (dashed–dotted line).
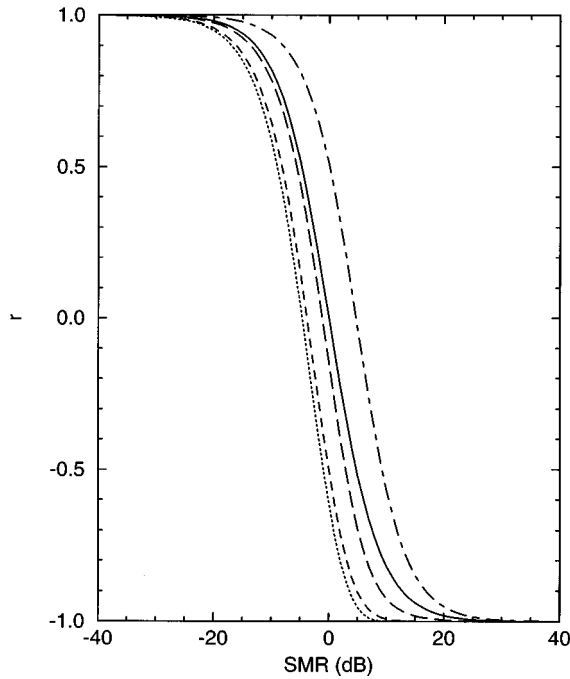
FIG. 8. Change in the normalized cross covariance as a function of the SMR for the waveform for M_0S_π (solid line), the envelope for multiplied noise for M_0S_π with M a multiplied noise (low-pass Gaussian noise multiplied by a sinusoid) and S a sinusoid, with $\vartheta=0$ (dotted line), $\vartheta=\frac{1}{8}\pi$ (dashed line), $\vartheta=\frac{1}{4}\pi$ (long-dashed line), and $\vartheta=\frac{3}{8}\pi$ (dashed–dotted line).

This leads to the interesting result that for $\vartheta=0$, the envelope correlation is approximately equal to the pure waveform correlation. On the other hand, for $\vartheta=\frac{1}{2}\pi$, the envelope correlation is equal to unity for all SMRs [cf. Eq. (52)]. The change in cross correlation of the envelopes is shown in Fig. 6 for several values of ϑ. The waveform decorrelation (solid line) is larger than any of the envelope decorrelations. For small SMRs and $\vartheta=0$, the difference between the two disappears.

For the cross covariance, apart from the expression for $\langle E_L E_R \rangle$, the expression for $\langle E_L \rangle$ as shown in Eq. (36) is also slightly different. The reason for this is again that ϕ is not uniformly distributed, as was discussed above for the cross correlation.

This results in an expression for $\langle E_L \rangle$ which depends on $\cos \vartheta$:

$$\langle E_L \rangle = \left\langle \sqrt{m^2+s^2} \right\rangle$$
$$+ \sum_{n=1}^{\infty} c_{2n} 4^n \left\langle \frac{m^{2n} s^{2n}}{(m^2+s^2)^{2n-1/2}} \right\rangle \cos^{2n} \vartheta. \quad (54)$$

In Fig. 7 the cross covariance of the envelope for multiplied noise is plotted as a function of the SMR with the
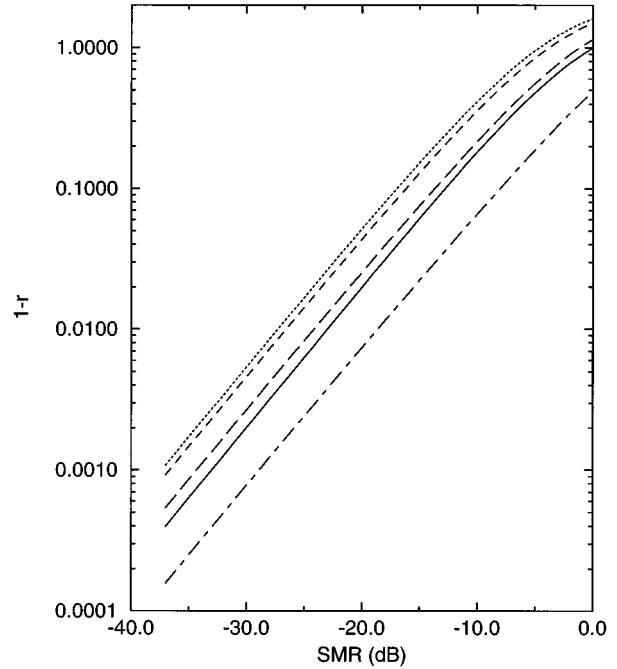
phase angle ϑ as parameter. In the range from 0 to $\frac{1}{2}\pi$, with increasing ϑ, the curves shift to the right at all SMRs.

When $\langle m^2 \rangle \gg \langle s^2 \rangle$, the following approximation can be made for the cross covariance:

$$r_{E_L E_R} \approx \frac{\langle m^2 \rangle - \langle |m| \rangle^2 - 2\langle s^2 \rangle \cos^2 \vartheta}{\langle m^2 \rangle - \langle |m| \rangle^2}. \quad (55)$$

The change in cross covariance of the envelope is shown in Fig. 8. In contrast to what we found for the cross correlation (cf. Fig. 6), the change in cross covariance for the waveform is not always larger than for the envelope.

When the masker M has a flat envelope (e.g., a sinusoid) and when the level of M is much higher than that of S, the approximation in Eq. (55) does not hold. In this case, approximations for the cross covariance are relatively difficult to derive because both the expected value of the product of the envelopes and the product of the expected value of the squared envelopes are approximately equal to the product of the means of the envelopes. Such an approximation is presented here for a signal that is a multiplied noise. Because the basic steps in the derivation of this equation are exactly parallel to that of Appendix A, only the result is presented:

$$r_{E_L E_R} \approx \frac{-\langle s^2 \rangle \cos^2 \vartheta + (\langle s^4 \rangle - \langle s^2 \rangle^2)(1/4m^2 - (1/m^2)\cos^2 \vartheta + (1/2m^2)\cos^4 \vartheta)}{\langle s^2 \rangle \cos^2 \vartheta + (\langle s^4 \rangle - \langle s^2 \rangle^2)1/4m^2}. \quad (56)$$

VI. OVERVIEW OF THE RESULTS AND DISCUSSION

In the previous sections the envelope correlation for various conditions was examined. The results can be summarized as follows.

The decisive factors determining the envelope correlation are

(1) the kind of correlation measure that is used: normalized crosscorrelation or normalized crosscovariance;
(2) the statistics of the Hilbert envelopes, m and s, of the homophasic masker M and the antiphasic signal S, respectively [in the case of multiplied noise, the relevant statistics is not that of the Hilbert envelope but it is the statistics of the temporal waveform of the low-pass noise (cf. Sec. V)]; and
(3) the statistics of the phase difference between M and S. This can be either uniform (Gaussian noise) or constant (e.g., for multiplied noise).

The approximations that are derived in the previous sections indicate a principal difference between the cross correlation and the cross covariance of the envelope at small SMRs, a level region most interesting for dichotic detection experiments. The cross correlation does *not* depend on the statistics of the masker and the signal while the cross covariance does depend on the masker statistics. When the masker has a flat envelope, the cross covariance depends on the statistics of the signal.

Given the amount of equations that were derived in this paper, an overview of the results is bound to be complicated. In Appendix B we have combined all relevant equations into one scheme.

At the end of this rather mathematical paper, we would like to mention some psychoacoustic implications and questions for future research that evolve from the derivations. Throughout this paper, we only considered the condition $M_0 S_\pi$. All statements about the relation between envelope correlation (and statistics) and the SMR are equally valid for the condition $M_\pi S_0$. Conditions that are not currently covered are those with a monaural signal ($M_0 S_m, M_\pi S_m$). We hope to be able to extend the mathematical derivations in the future to these conditions also, since they are of particular interest for the topic of comodulation masking release.

Models of binaural hearing that are based on the interaural correlation of the waveform will in general predict no influence of masker and signal statistics. From the derivations in this paper it is evident, however, that signal statistics do influence the interaural correlation of the envelopes. This provides a way to reexamine the assumption that an interaural correlator is a central building block of binaural processing.

One direct application of the analytical solutions of the present paper is the comparison of binaural masking level differences (BMLDs) at low and at high signal frequencies. It has long been known that the BMLD decreases significantly for signal frequencies above 500 Hz. One of the possible sources for this decrease is the loss of phase locking in the neural responses. On the basis of the expressions derived in the previous sections we can predict how much the BMLD should change due to a loss of fine-structure information.

Using the cross-correlation measure [Eqs. (13) and (30)], we find a 3-dB lower BMLD when only the envelope of the stimuli is available for calculating the interaural correlation.[6] Since the decrease in BMLD toward high frequencies is much larger than this value (cf. Zurek and Durlach, 1987; Kohlrausch *et al.*, 1995) additional sources such as a reduced sensitivity for interaural differences at high frequencies have to be assumed (Koehnke *et al.*, 1986).

We shortly want to address the question whether waveforms and their Hilbert envelopes are a good approximation for the internal representation of low- and high-frequency stimuli. If we reduce the transformation properties of the inner hair cell to a half-wave rectification, followed by a low-pass filter, we can calculate the properties at the hair cell output, assuming narrow-band input stimuli. At signal frequencies that are high compared to the cutoff frequency of the low-pass filter, the hair cell output approximates the Hilbert envelope of the input. At signal frequencies that are low compared to the cutoff frequency, we just get the rectified version of the input signal. The spectrum of this rectified stimulus contains the original input spectrum, the spectrum of the Hilbert envelope, plus higher frequency mirror components.

We thus can conclude that the Hilbert envelope is a reasonable description of high-frequency stimuli as they are coded at the level of the inner hair cell. For low-frequency stimuli, on the other hand, it is not generally correct to derive predictions on the interaural correlation from the (nonrectified) waveforms. Numerical simulations reveal that for rectified waveforms, the normalized cross correlation and the normalized cross covariance are not generally the same and that, in addition, the statistics of the signal influence the correlation value. Because this issue lies outside the main focus of this paper, it is discussed in Appendix C.

From Sec. III it can be deduced that a sinusoidal masker which is higher in level than a Gaussian signal gives negative values for the cross covariance of the envelope [cf. Eq. (45) and Fig. 3].[7] The cross covariance will approach -1 for low levels of the signal (increasing SMR in Fig. 3). If we assume that the diotic sinusoid alone has a cross covariance[8] of 1 there is a large change in the cross covariance for the sinusoid combined with a low-level noise. According to this line of reasoning the change in cross covariance increases with decreasing noise level. Such a result is against expectation because the noise must eventually become undetectable. When applying the cross-covariance measure to a flat envelope (sinusoidal) masker, it is therefore essential that some kind of system noise is introduced. For the masker-alone condition, the interaural cross covariance would be identical to the cross covariance of the system noise. When a signal is added to the masker at decreasing SMRs, the system noise will eventually start to dominate and determine the interaural cross covariance.

The use of the interaural correlation as a decision parameter in a detection process not only requires estimates of the mean values of correlation and covariance, but also of the standard deviations of these estimates. A detailed knowledge of the influence of stimulus parameters like bandwidth and duration on the standard deviation of correlation estimates

would allow a prediction of correlation jnd's and BMLDs. So far, it has been assumed in BMLD experiments that the parameter's bandwidth and duration influence the relevant decision statistics in the same way in diotic and dichotic conditions. This, however, has not been tested for the statistics of the interaural cross correlation in comparison to, e.g., the (diotic) energy increment. Further investigations have to show whether the slight increase of the BMLD with decreasing signal duration (e.g., Blodgett *et al.*, 1958; Kohlrausch, 1986) and the still puzzling observation of a wider "effective critical band" in dichotic masking conditions (e.g., Zurek and Durlach, 1987) indicate a particular property of our hearing system or whether they are just a consequence of the statistics of the stimulus.

ACKNOWLEDGMENTS

APPENDIX A

When a sinusoid or a frequency modulated sinusoid is used as a masker and when this masker has a much higher level than the signal, Eq. (40) cannot be used as an approximation, since $\langle m^2 \rangle = \langle m \rangle^2$. Consequently, it is not appropriate to neglect the residual terms ϵ_1, ϵ_2, and ϵ_3, since they are not small with respect to $\langle m^2 \rangle - \langle m \rangle^2$.

Therefore, even though we assumed $\langle m^2 \rangle \gg \langle s^2 \rangle$, we need to take more terms into account in our approximation, in particular, terms with s^4 factors. The first three terms of Eq. (24) can be evaluated to

$$\langle E_L E_R \rangle = \langle m^2 \rangle + \langle s^2 \rangle - \left\langle \frac{m^2 s^2}{m^2 + s^2} \right\rangle - \frac{3}{4} \left\langle \frac{s^4}{m^2} \right\rangle. \qquad (A1)$$

We need to expand the third expected value of this equation since s^4 factors will be created in this way. The following Taylor expansion can be used:

$$\frac{1}{(1+x)^k} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} k(k+1)\cdots(k+n-1)(-x)^n. \qquad (A2)$$

We can write

$$\left\langle \frac{m^2 s^2}{m^2 + s^2} \right\rangle = \left\langle \frac{s^2}{1 + s^2/m^2} \right\rangle \approx \langle s^2 \rangle - \langle s^4 \rangle \left\langle \frac{1}{m^2} \right\rangle. \qquad (A3)$$

This results in

$$\langle E_L E_R \rangle \approx m^2 + \frac{1}{4m^2} \langle s^4 \rangle. \qquad (A4)$$

When the expected value of the mean of the envelope [Eq. (36)] is written with the square-root term expanded in a Taylor series, we find

$$\langle E_L \rangle = \langle m \rangle + \sum_{n=1}^{\infty} \left(c_n \left\langle \frac{s^{2n}}{m^{2n-1}} \right\rangle \right.$$
$$\left. + c_{2n} 4^n \frac{(2n-1)!!}{2n!!} \left\langle \frac{m^{2n} s^{2n}}{(m^2+s^2)^{2n-1/2}} \right\rangle \right). \qquad (A5)$$

When Eq. (A5) is evaluated up to the fourth order in s, this yields

$$\langle E_L \rangle \approx m + \frac{1}{4m} \langle s^2 \rangle + \frac{1}{64m^3} \langle s^4 \rangle. \qquad (A6)$$

Omitting factors with a power of s higher than 4, the square of this equation can be written as

$$\langle E_L \rangle^2 \approx m^2 + \frac{1}{2} \langle s^2 \rangle + \frac{1}{16m^2} \langle s^2 \rangle^2 + \frac{1}{32m^2} \langle s^4 \rangle. \qquad (A7)$$

When Eqs. (A4) and (A7) are inserted in Eq. (33) we get a general approximation which is valid when $\langle m^2 \rangle \gg \langle s^2 \rangle$:

$$r_{E_L E_R} \approx \frac{-(1/2)\langle s^2 \rangle + (7/32m^2)\langle s^4 \rangle - (1/16m^2)\langle s^2 \rangle^2}{(1/2)\langle s^2 \rangle - (1/32m^2)\langle s^4 \rangle - (1/16m^2)\langle s^2 \rangle^2}. \qquad (A8)$$

In this last equation, we can see that in the case of a flat envelope masker that is large with respect to the signal, the statistics of the signal become important.

We can evaluate the case that the signal is a Gaussian noise. Using the symbol l for the SMR [cf. Eq. (10)], we know that

$$\frac{\langle s^4 \rangle}{m^4} = 2l^2, \qquad (A9)$$

based on general rules for Gaussian noise. After some manipulations of Eq. (A8), this yields

$$r_{E_L E_R} \approx \frac{-1 + \frac{3}{4} l}{1 - \frac{1}{4} l}. \qquad (A10)$$

This results in a relation between pure waveform correlation and envelope correlation; that is,

$$\tfrac{1}{4} \Delta r_{LR} \approx 1 + r_{E_L E_R}. \qquad (A11)$$

For the case in which we have a sinusoidal masker and a sinusoidal signal with different frequencies we get

$$\tfrac{1}{16} \Delta r_{LR} \approx 1 + r_{E_L E_R}. \qquad (A12)$$

APPENDIX B

In this appendix a concise overview of all analytical expressions is presented. First, the equations for the exact solutions for the envelope correlation are given. In these equations, $P_m(m)$ and $P_s(s)$ denote the probability distribution functions of the masker and signal amplitude functions, respectively. The probability distribution function of the phase difference between the masker and the signal is denoted as $P_\phi(\phi)$. In the second part of the appendix, the results of the approximations of the envelope correlation are presented in Table BI.

The exact solutions for cross correlation and the cross covariance of the envelope are

TABLE BI. An overview of approximations for the cross correlation and the cross covariance of the envelope for the case that $\langle m^2\rangle\gg\langle s^2\rangle$.

m	s	ϕ	$\Delta\rho_{LR}$ $=\Delta r_{LR}$	$\Delta\rho_{E_LE_R}$	$r_{E_LE_R}$	Description
P_m	P_s	unif.	$2l$	l	$1-\dfrac{\langle m^2\rangle}{\langle m^2\rangle-\langle m\rangle^2}l$	noise masker, noise signal
P_m	const.	unif.	$2l$	l	$1-\dfrac{\langle m^2\rangle}{\langle m^2\rangle-\langle m\rangle^2}l$	noise masker, sinusoidal signal
const.	P_s	unif.	$2l$	l	$\left(\dfrac{3}{8}\dfrac{\langle s^4\rangle}{\langle s^2\rangle^2}-\dfrac{1}{4}\right)l-1$	sinusoidal masker, noise signal
const.	const.	unif.	$2l$	l	$\dfrac{1}{8}l-1$	sinusoidal masker, sinusoidal signal with different freq.
Rayl.	Rayl.	unif.	$2l$	l	$1-\dfrac{1}{1-\pi/4}l$	Gaussian masker and signal
Rayl.	const.	unif.	$2l$	l	$1-\dfrac{1}{1-\pi/4}l$	Gaussian masker, sinusoidal signal
const.	Rayl.	unif.	$2l$	l	$\dfrac{1}{2}l-1$	sinusoidal masker, Gaussian signal
P_m	P_s	const.	$2l$	$2l\cos^2\vartheta$	$1-\dfrac{2\langle m^2\rangle\cos^2\vartheta}{\langle m^2\rangle-\langle|m|\rangle^2}l$	multiplied noise masker and signal
P_m	const.	const.	$2l$	$2l\cos^2\vartheta$	$1-\dfrac{2\langle m^2\rangle\cos^2\vartheta}{\langle m^2\rangle-\langle|m|\rangle^2}l$	multiplied noise masker, sinusoidal signal
const.	P_m	const.	$2l$	$2l\cos^2\vartheta$	$\dfrac{-m^2\cos^2\vartheta+(\langle s^4\rangle/\langle s^2\rangle-\langle s^2\rangle)(\frac{1}{4}-\cos^2\vartheta+\frac{1}{2}\cos^4\vartheta)}{m^2\cos^2\vartheta+\frac{1}{4}(\langle s^4\rangle/\langle s^2\rangle-\langle s^2\rangle)}l-1$	sinusoidal masker, multiplied noise signal

cross correlation:
$$\rho=\frac{A}{B} \qquad (\text{B1})$$

and

cross covariance:
$$r=\frac{A-C^2}{B-C^2}, \qquad (\text{B2})$$

with

$$A=\int m^2P_m(m)dm+\int s^2P_s(s)ds$$
$$+\sum_{n=1}^{\infty}\frac{(-4)^n}{n!}\lambda_n\left[\prod_{k=1}^{n}\left(\frac{3}{2}-k\right)\right]$$
$$\times\int\int\frac{m^{2n}s^{2n}}{(m^2+s^2)^{2n-1}}P_m(m)P_s(s)dm\,ds \qquad (\text{B3})$$

[cf. Eq. (24)],

$$B=\int m^2P_m(m)dm+\int s^2P_s(s)ds \qquad (\text{B4})$$

[cf. Eq. (36)], and

$$C=\int\int\sqrt{m^2+s^2}P_m(m)P_s(s)dm\,ds$$
$$+\sum_{n=1}^{\infty}\frac{4^n}{(2n)!}\lambda_n\left[\prod_{k=1}^{2n}\left(\frac{3}{2}-k\right)\right]$$
$$\times\int\int\frac{m^{2n}s^{2n}}{(m^2+s^2)^{2n-1/2}}P_m(m)P_s(s)dm\,ds, \qquad (\text{B5})$$

[cf. Eqs. (36) and (54)] where the value of λ_n depends on $P_\phi(\phi)$. When this distribution is uniform,

$\lambda_n=(2n-1)!!/(2n)!!$. When ϕ is a constant equal to ϑ, $\lambda_n=\cos^{2n}\vartheta$. In the latter case, Eqs. (B1) and (B2) are only valid when one of the functions m or s has a mean equal to zero.

The approximations which can be made for the situation where $\langle m^2\rangle\gg\langle s^2\rangle$ are summarized in Table BI. In this table, l is equal to $\langle s^2\rangle/\langle m^2\rangle=$SMR. In the first two columns the properties of m and s are given. The symbols P_m or P_s are used symbolically to express that m or s results from a random process and that, consequently, m or s is not constant.

We remark here that all entries in Table BI for which the envelope distribution of the masker is constant (first column) have a cross covariance $r_{E_LE_R}$ that is negative. All other entries have a positive cross covariance $r_{E_LE_R}$.

APPENDIX C

In this appendix, we investigate the influence of half-wave rectification (without subsequent low-pass filtering) on the values of the cross correlation and the cross covariance. Because this problem could not be solved analytically, we present the results of numerical simulations. We considered conditions of the type M_0S_π with a Gaussian noise masker and a sinusoidal or a Gaussian noise signal. The noises had a bandwidth of 50 Hz. Both masker and signal where centered at 250 Hz and had a duration of 4000 ms. The correlations of the half-wave rectified waveform are the averages of ten repeated simulations.

Table CI shows, as a function of the SMR (first column), the cross correlation (identical to the cross covariance) for the full waveform (second column) and in columns three to six values for half-wave rectified waveforms. These are the cross correlation and the cross covariance for a sinusoidal

TABLE CI. The correlation of the waveform ($\rho_W = r_W$), the cross correlation of the half-wave rectified waveform (ρ_{HW}), and the cross covariance of the half-wave rectified waveform (r_{HW}) are shown for an $M_0 S_\pi$ condition with a Gaussian noise masker and either a sinusoidal or a Gaussian noise signal at several SMRs. The values are the averages of ten simulations using stimuli with a duration of 4000 ms.

SMR	$\rho_W = r_w$	Sinusoidal signal		Noise signal	
		ρ_{HW}	r_{HW}	ρ_{HW}	r_{HW}
0 dB	0	0.273	−0.086	0.320	−0.002
−5 dB	0.519	0.602	0.413	0.624	0.448
−10 dB	0.818	0.837	0.761	0.842	0.768
−15 dB	0.939	0.942	0.915	0.943	0.917
−20 dB	0.980	0.981	0.972	0.981	0.972
−25 dB	0.994	0.994	0.991	0.994	0.991
−30 dB	0.998	0.998	0.997	0.998	0.997

signal (columns 3 and 4), and the cross correlation and the cross covariance for a noise signal (columns 5 and 6). For almost all SMRs, columns 2–6 show different values for the correlation. This indicates that (1) the correlation of the waveform differs from the correlation of the half-wave rectified waveform. (2) The cross correlation and the cross covariance are no longer identical if applied to the half-wave rectified waveform. (3) In the case of the half-wave rectified waveform, the statistics of the signal (sinusoid versus Gaussian noise) influence the correlation value. Differences between the correlation estimates of the full waveform and the half-wave rectified waveform are only negligible in the case of the cross correlation for SMRs of −20 dB or less. These results show that the formula for the relation between SMR and correlation [Eq. (13)] is not generally correct if applied to half-wave rectified waveforms.

[1] This condition is usually referred to as $N_0 S_\pi$ indicating an in-phase noise and an out-of-phase signal. However, in the course of this paper we will also consider the combination of a *sinusoidal* masker and a *noise* signal, for which the notation $N_0 S_\pi$ is ambiguous. For the purpose of consistency, we therefore use the more general notation $M_0 S_\pi$ throughout the paper. For the same reason, the term signal-to-masker ratio (SMR) will be used instead of the more common signal-to-noise ratio (SNR).

[2] Richard's statement with respect to $r=0$ is correct for uncorrelated noise bands (this was also the context of the statement), but not for the combination of an in-phase noise masker and an out-of-phase sinusoidal signal (see Sec. III and Figs. 3 and 4).

[3] We want to emphasize that in this paper the term "envelope" is used for the linear envelope as defined in Eqs. (14) and (15). In the terminology of Lawson and Uhlenbeck (1950), it is the waveform obtained from a linear detector. This waveform has to be distinguished from the squared envelope that results from a square-law detector. In the psychoacoustic literature, the term envelope with no further qualifications has also been used for the latter waveform (e.g., Fantini, 1991).

[4] We want to emphasize that there are several other ways to generate comodulated noise bands, besides using multiplied noise as defined at the beginning of this section.

[5] When the low-pass noise $M_{low}(t)$ is positive, the multiplied noise, $M(t) = M_{low}(t) \sin \omega t$, will have a phase that lags by ϑ as compared to $S(t) = \sin(\omega t + \vartheta)$. When $M_{low}(t)$ is negative, however, the phase lag will be $\phi(t) = \vartheta + \pi$.

[6] A similar calculation based on the normalized cross covariance results in an *increase* of the BMLD due to the loss in fine structure.

[7] In order to see this from Fig. 3, one has to follow the dotted curve (noise masker plus sinusoidal signal) to positive values of the SMR. Here, we have a high level out-of-phase sinusoid combined with a low level in-phase

noise, thus a condition of the type $M_\pi S_0$ (with the sinusoid as the masker). The envelopes of this condition are the same as those for $M_0 S_\pi$.

[8] Strictly speaking, its cross covariance is undefined because the masker envelope, after subtracting the mean, is zero. This observation indicates a principal problem in applying the cross-covariance measure of the envelope to sinusoidal signals. In the mathematical derivations we have always ignored on- and offset ramps for the stimulus; this is, however, not correct for stimuli used in psychoacoustic experiments. If the ramps are taken into account the cross covariance for a diotic sinusoidal masker will be one.

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