ANALYTICAL FORMULAE AND ALGORITHMS FOR CONSTRUCTING MAGIC SQUARES FROM AN ARBITRARY SET OF 16 NUMBERS

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In this paper we seek for an answer on Smarandache type question: may one create the theory of Magic squares 4×4 in size without using properties of some concrete numerical sequences? As a main result of this theoretical investigation we adduce the solution of the problem on decomposing the general algebraic formula of Magic squares 4×4 into two complete sets of structured and four-component analytical formulae.

1 Introduction

In the general case *Magic squares* represent by themselves numerical or analytical square tables, whose elements satisfy a set of definite basic and additional relations. The basic relations therewith assign some constant property for the elements located in the rows, columns and two main diagonals of a square table, and additional relations, assign additional characteristics for some other sets of its elements.

Judging by the given general definition of Magic squares, there is no difficulty in understanding that, in terms of mathematics, the problem on Magic squares consists of the three interrelated problems

a) elucidate the possibility of choosing such a set of elements which would satisfy both the basic and all the additional characteristics of the relations;

b) determine how many Magic squares can be constructed from the chosen set of elements;

c) elaborate the practical methods for constructing these Magic squares.

It is a traditional way to solve all mentioned problems with taking into account concrete properties of the numerical sequences from which the Magic square numbers are generated. For instance, by using this way problems was solved on constructing different Magic squares of natural numbers[1 - 5], prime numbers[6, 7], Smarandache numbers of the 1st kind[8] and so on. Smarandache type question[9] arises: whether a possibility exists to construct the theory of Magic squares without using properties of concrete numerical sequences. The main goal of this paper is finding an answer on this question with respect to problems of constructing Magic squares 4×4 in size. In particular, in this investigation we

a) describe a simple way of obtaining a general algebraic formulae of Magic squares 4×4, required no use of algebraic methods, and explain why in the general case this formula does not simplify the solution of problems on constructing Magic square 4×4 (Sect. 2);

b) give a description of a set of invariant transformations of Magic squares 4×4 (Sect. 3);

c) adduce a general algorithm, suitable for constructing Magic squares from an arbitrarily given set of 16 numbers (Sect. 4);

d) discuss the problems of constructing Magic squares from the structured set of 16 elements (Sect. 5);

e) solve the problem of decomposing the general algebraic formula of Magic squares 4×4 into a complete set of the four-component formulae (Sect. 6).

2 Constructing the general algebraic formula of a Magic square 4×4

A table, presented in Fig. 1(2), consists of two orthogonal diagonal Latin squares, contained symbols A, B, C, D (L_1) and a, b, c, d (L_2). Remind[10, 11] that two Latin squares of order n are called

a) *orthogonal* if being superimposed these Latin squares form a table whose all n^2 elements are various;

b) *diagonal* if n different elements are located not only in its rows and columns, but also in its two main diagonals.

It is evident that the table 1(2) is transformed in the analytical formula of a Magic square 4×4 when its parameter $b = 0$. By using Fig. 1(2) we reveal the law governing the numbers of any Magic square 4×4 decomposed in two orthogonal diagonal Latin squares. For this aim we rearrange the sets of the symbols in the two-component algebraic formula 1(2) so as it is shown in Fig. 1(6). Further, a table 1(6) will be called *additional* one. Such name of the table is justified by the following:

a) the table 1(6), containing the same set of elements as the table 1(2), has more simple structure than the formula 1(2);

b) there exists a simple way of passing from this table to a Magic square 4x4: really, if one considers that Fig. 1(1) represents the enumeration of the cells in the table 1(6), then, for passing from this table to a Magic square it will be sufficient to arrange numbers in the new table 4x4 in the order corresponding to one in the *classical* square 1(5) {the Magic square of natural numbers from 1 to 16 }.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(1)

$A+c$	$B+b$	$C+d$	$D+a$
$D+d$	$C+a$	$B+c$	$A+b$
$B+a$	$A+d$	$D+b$	$C+c$
$C+b$	$D+c$	$A+a$	$B+d$

$(2 - L_1 + L_2)$

-	-	-	-
$-w$	-	-	$+w$
$+w$	-	-	$-w$
-	-	-	-

$(3 - W)$

$A+c$	B	$C+d$	$D+a$
$D+d-w$	$C+a$	$B+c$	$A+w$
$B+a+w$	$A+d$	D	$C+c-w$
C	$D+c$	$A+a$	$B+d$

$(4 - L_1 + L_2 + W)$

3	5	12	14
16	10	7	1
6	4	13	11
9	15	2	8

(5)

A	$A+a$	$A+c$	$A+d$
B	$B+a$	$B+c$	$B+d$
C	$C+a$	$C+c$	$C+d$
D	$D+a$	$D+c$	$D+d$

(6)

$A+w$	$A+a$	$A+c$	$A+d$
B	$B+a+w$	$B+c$	$B+d$
C	$C+a$	$C+c-w$	$C+d$
D	$D+a$	$D+c$	$D+d-w$

(7)

Fig. 1. Constructing the general algebraic formula of a Magic square 4x4.

The more simple construction of the additional table in comparison with the formula 1(2) and the possibility of passing from the additional table to a Magic square suggest solving the analogous problems on constructing the corresponding additional tables instead of solving the problems on constructing Magic squares. Further we shall always perform this replacement of one problem by another.

It is easy to establish by algebraic methods[12, 13] that the general algebraic formula of Magic square of order 4 contains 8 parameters. Thus it has one parameter less than the two-component algebraic formula, presented in Fig. 1(2) with $b = 0$. If one takes it into account, then there appears a natural possibility to seek a form for the general algebraic formula of a Magic square 4×4 basing, namely, on this two-component algebraic formula. It seems[7] that for introducing one more parameter in the algebraic formula 1(2) one may add cell-wise this formula to the Magic square, shown in Fig. 1(3) {it can be easily counted that the Magic constant of this square equals zero}. Thus, the general algebraic formula of a Magic square 4×4 {see Fig. 1(4)} is obtained as a result of the mentioned operation. Therefore it may be written in the simple analytical form

$$L_1 + L_2 + W. \tag{1}$$

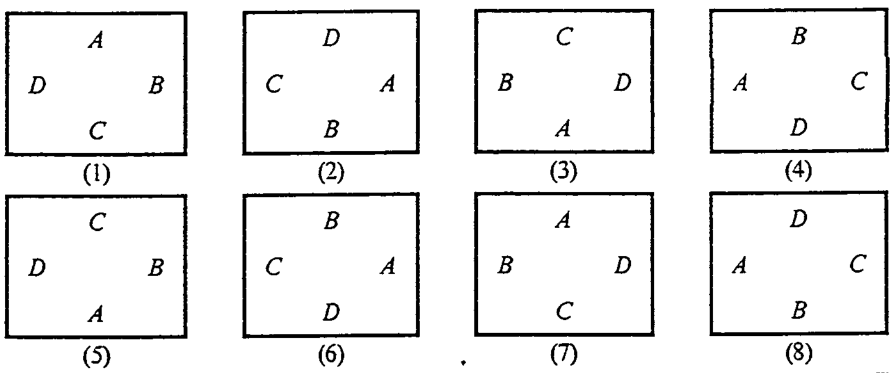
By analysing Fig. 1(7), in which the general formula of Magic square 4×4 is presented as the additional table, one may conclude, that the availability of eight but not of seven parameters results in a substantial violation of the simple regularity existing for the elements of the additional table 1(6) and by this reason, changing the problem on constructing a Magic square 4×4 by that on constructing the corresponding additional table, will not result in a facilitation of its solution in the general case {passing from the additional table 1(7) to the general algebraic formula of the Magic square 4×4 1(4) one may realise by means of the classical square 1(5) in the way mentioned above for the additional table 1(6)}.

3 A set of invariant transformations of a Magic square 4×4

By means of rotations by 90 degrees and mappings relative to the sides one can obtain from any Magic square 4×4 seven more new ones {see Fig. 2, from which one can judge on changes of a spatial orientation of a Magic square on the basis of the changes in arrangement of the symbols A, B, C and D}. Besides for $n \geq 4$ there exist such internal transformations (*M-transformations*) of a Magic square n×n (permutations of its rows and columns) by which the assigned set of

$[n/2]\{(2[n/2]-2)!!\}$ Magic squares $n\times n$ can be obtained[7] from one square with regard for rotations and mappings, where the symbol $a!!$ means the product of all natural numbers which, firstly, are not exceeding a, and, secondly, coincide with it in an evenness; $[a]$ means the integer part of a. In particular, if the cells of any Magic square 4x4 are enumerated so, as it is shown in Fig. 2(9), and under M-transformations the specific permutations of the cells of the initial square are meant, then, in this case the all 4 possible M-transformations of a square 4x4 can be represented in the form of four tables, depicted in Fig. 2(9 - 12).

It is evident, that when studying Magic squares, constructed from the same set of elements, it is worthwhile, to consider the only squares which can not be obtained from each other by rotations, mappings and M-transformations. It is usually said about such a family of Magic squares, that it is assigned with regard for rotations, mappings and M-transformations.



1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$(9-M_1)$

1	3	2	4
9	11	10	12
5	7	6	8
13	15	14	16

$(10-M_2)$

6	5	8	7
2	1	4	3
14	13	16	15
10	9	12	11

$(11-M_3)$

6	8	5	7
14	16	13	15
2	4	1	3
10	12	9	11

$(12-M_4)$

Fig. 2. A set of invariant transformations of a Magic square 4x4.

4 The algorithm for constructing Magic squares from an arbitrary 16 numbers

A complete set of Magic squares 4×4 from an arbitrarily given set of 16 numbers with regard for rotations, mappings and M-transformations one may obtain by the following algorithm[7]:

1. Calculate the sum of all 16 numbers of the given set and, having divided it into 4, obtain the value of the Magic constant S of the future Magic square 4×4;
2. Find all possible presentations of the number S in four different terms each of them belonging to the given set of the numbers;
3. If the number of various partitionings is not smaller than 14, then, using the obtained list of partitionings, form all possible various sets of four Magic rows, containing jointly 16 numbers of the given set;
4. Among the sets of four rows, of the obtained list, find such pairs of the sets which satisfy the following condition: each row of the set has one number from various rows of the other set;
5. It is possible to construct Magic squares 4×4 from the above mentioned pairs, if among the earlier found Magic rows (partitionings of the number S) one succeeds in finding the two rows such that
   - these rows do not contain identical numbers;
   - each row contains one by one number from various rows both of the first and the second set of the pair.

When constructing Magic squares 4×4 from the obtained pairs of the sets consisting of four rows and the sets of the pairs of the rows corresponding to these pairs one should bear in mind that:

- a four-row pair of sets (see point 4) gives a set of Magic rows and columns for a Magic square 4×4;
- the found pairs of the rows (see point 5) are used for forming the Magic square diagonals;
- if it is necessary to seek for Magic squares with regard for rotations, mappings and M-transformations, then each differing pair of rows, found for the given pair of sets consisting of four rows, can be utilised for construction of only one Magic square 4×4;
- the algorithm can be easy realised as a computer program.

5 Constructing Magic squares from the structured set of 16 elements

We shall say that a Magic square of order 4 possesses *the structure* (contains a structured set of elements) if it is possible to construct from its elements the eight various pairs of elements with the sum equal to 1/2 of the Magic square constant. For obtaining *the structural pattern* of a Magic square, it is sufficient to connect by lines each pair of the elements, forming this structure, directly in the Magic square. The other (*implicit*) way of representing the structural pattern of a Magic square 4×4 consists of the following: having chosen 8 various symbols we substitute each pair of numbers, forming the Magic square, by any symbol. As it has been proved by analytical methods[5], with account for rotations, reflections and *M*-transformations none Magic squares 4×4 exist, which contains in its cells 8 even and 8 odd numbers and has structure patterns another than ones shown in the implicit form in Fig. 3(1 – 6). In reality[7, 14] this statement is incorrect because for such Magic squares with respect of invariant transformations there exist 6 more new structure plots, depicted in Fig. 3(7 – 12).

1	2	3	4
5	6	7	8
5	6	7	8
1	2	3	4

(1)

1	2	3	4
5	6	7	8
8	7	6	5
4	3	2	1

(2)

1	2	3	4
5	6	7	8
3	4	1	2
7	8	5	6

(3)

1	2	3	4
1	2	3	4
5	6	7	8
5	6	7	8

(4)

1	2	1	2
3	4	3	4
5	6	7	8
7	8	5	6

(5)

1	2	3	4
1	5	6	4
7	5	6	8
7	2	3	8

(6)

1	2	3	4
3	5	6	2
7	6	5	8
4	7	8	1

(7)

1	2	3	4
5	6	1	7
7	2	8	5
8	6	4	3

(8)

1	1	2	2
3	4	3	4
5	6	7	8
8	7	6	5

(9)

1	2	3	4
3	5	6	1
7	7	8	8
5	6	4	2

(10)

1	2	3	4
5	5	6	6
4	7	8	3
2	8	1	7

(11)

1	2	3	4
5	6	7	8
6	4	8	1
7	3	2	5

(12)

Fig. 3. A complete set of possible structural patterns in a Magic square 4×4, depicted in the implicit form.

Basing on Fig. 3, for all structural patterns we shall construct a complete set of general structural analytical formulae. Thus, in this section we shall solve the problem *on decomposing* the general algebraic formula 1(4) in the structured ones.

I. Here we present a simple method suitable for constructing general algebraic formulae of Magic squares possessing the structural pattern 3(1 - 4). Besides, we point out some singularities of these four general structured analytical formulae.

As it has been established in Sect. 2 the general algebraic formula of a Magic square 4×4 may be represented, as the sum of the two diagonal Latin squares, formed by capital and small Latin letters {see Fig. 1(2)}, and the Magic square {Fig. 1(3)}, having a zero Magic constant. It turns out[7] that general structured algebraic formulae, having structural patterns 3(1 - 4), can be obtained if the required conditions of a structuredness at the fixed structural pattern are written out separately for each of the 3 tables, forming the general algebraic formula 1(4). In particular, diagonal Latin squares 1(2) and the Magic square 1(3) will have structural patterns 3(1 - 4) at the following correlations between their parameters {for convenience, the numbers of the written systems of equations are chosen so that they are identical with the numbers of structural patterns, shown in Fig. 3, by which these equations have been derived}:

$$\begin{array}{llll} 1.\ A+C=B+D, & 2.\ A+B=C+D, & 3.\ A+D=B+C, & 4.\ A+D=B+C, \\ \quad c=a+d. & \quad a=c+d, & \quad c=a+d, & \quad a=c+d, \\ & \quad e=0. & \quad e=0. & \quad e=0. \end{array} \qquad (2)$$

Starting from the extracted system of equations (2) one can easily prove that:

1) The cells of an algebraic formula having the structural pattern 3(1) contain two sequences involving elements of the following form:

$$\begin{aligned} &\text{a)}\ \ a_1+e,\ \ a_1+a,\ \ a_1+a+d,\ \ a_1+d,\ \ a_1+b,\ \ a_1+a+b+e,\\ &\qquad a_1+a+b+d,\ \ a_1+b+d;\\ &\text{b)}\ \ a_2,\ \ a_2+a,\ \ a_2+a+d,\ \ a_2+d-e,\ \ a_2+b,\ \ a_2+a+b,\\ &\qquad a_2+a+b+d-e,\ \ a_2+b+d. \end{aligned} \qquad (3)$$

One can see from a set of sequences (3) that the regularity existing between the symbols of an general algebraic formula, having structural pattern 3(1), is complicated due to the presence of the four elements containing the symbol e

$a_1 + b + 2c$	$a_1 + b$	$a_2 + c$	$a_2 + 2b + 3c$
$a_2 + b$	$a_2 + b + 2c$	$a_1 + c$	$a_1 + 2b + 3c$
$a_2 + b + 3c$	$a_2 + b + c$	$a_1 + 2b + 2c$	a_1
$a_1 + b + c$	$a_1 + b + 3c$	$a_2 + 2b + 2c$	a_2

(5)

a_2	$a_1 + 2b + c + d$	$a_1 + c$	$a_2 + 2b + 2c + d$
$a_2 + b$	$a_1 + b + 2c + d$	$a_1 + b$	$a_2 + 2c + d$
$a_1 + 2b + 2c + d$	$a_2 + b$	$a_2 + b + 2c + d$	a_1
$a_1 + 2c + d$	$a_2 + c$	$a_2 + 2b + c + d$	$a_1 + 2b$

(6)

$a_2 + b + 2c$	$a_1 + b$	a_2	$a_1 + 2c$
$a_1 + b + 2c$	$a_2 + c$	$a_1 - b + c$	$a_2 + 2b$
$a_1 - b$	$a_2 + 2b + c$	$a_1 + b + c$	$a_2 + 2c$
$a_2 + b$	$a_1 - b + 2c$	$a_2 + 2b + 2c$	a_1

(7)

a_1+2b	a_2+10b	a_1+4b	a_2+4b
$a_2 + b$	a_1+10b	a_2+8b	$a_1 + b$
a_2+9b	a_1	a_2+2b	a_1+9b
a_1+8b	a_2	a_1+6b	a_2+6b

(8)

$a_1 + 4b$	a_1+12b	a_1+10b	a_1+16b
a_1+11b	$a_1 + 8b$	$a_1 + 6b$	a_1+17b
a_1+14b	$a_1 + 7b$	a_1+21b	a_1
a_1+13b	a_1+15b	$a_1 + 5b$	$a_1 + 9b$

(10)

a_1	a_2+8b	a_2	a_1+8b
a_2+6b	a_1+6b	a_1+2b	a_2+2b
a_1+5b	$a_2 + b$	a_2+7b	a_1+3b
a_2+5b	$a_1 + b$	a_1+7b	a_2+3b

(9)

a_1+12b	a_1+16b	$a_1 + 4b$	a_1+10b
a_1+14b	$a_1 + 7b$	a_1+21b	a_1
a_1+11b	$a_1 + 6b$	$a_1 + 8b$	a_1+17b
$a_1 + 5b$	a_1+13b	$a_1 + 9b$	a_1+15b

(11)

$a_1 + 3b$	$(a_1 + a_2)/2 + 3b$	$(a_1 + a_2)/2 - b$	$a_2 + 5b$
$a_2 + 3b$	$a_2 + b$	$a_1 + 5b$	$a_1 + b$
$a_1 + 4b$	a_1	$a_2 + 4b$	$a_2 + 2b$
a_2	$(a_1 + a_2)/2 + 6b$	$(a_1 + a_2)/2 + 2b$	$a_1 + 2b$

(12)

Fig. 4. General algebraic formulae of a Magic square 4×4
with structural patterns 3(5 - 12).

{as well as in the general algebraic formula of a Magic square 4×4 shown in Fig. 1(4)}. Consequently, the knowledge of the regularity existing between the elements of the general algebraic formula with structural pattern 3(1) can not be of help in creating a convenient and practical algorithm for constructing corresponding Magic squares {as well as for the general algebraic formula 1(4)}.

2) The general algebraic formulae having structural patterns 3(2 - 4) are decomposable in sums of two diagonal Latin squares {parameters b and e are equal to zero). Hence, there is the simple regularity for the elements of additional tables of general algebraic formulae with structural patterns 3(2 - 4) and, consequently, the problem on constructing such Magic squares 4×4 from a given structured set of 16 elements is easy to solve by means of these three formulae.

II. Taking into account that for structural patterns 3(1 - 4) there exists a simple method for constructing the general algebraic formulae (see point I) we present in Fig. 4 a set of 8 general algebraic formulae which possess only structural patterns of 3(5 - 12) {the form of representing these formulae is chosen so that it reveals the regularity existing between their elements}. Analysing the analytical formulae presented in Fig. 4 we may come to the following conclusions:

1) among the all above formulae, the formulae 10 and 11 have the most simple structure: the set, consisting of their 16 elements, is completely defined by the first element of the sequence a_1 and the value of the parameter b;

2) the sets of the symbols, contained in the formulae 5, 6, 7, 8 and 9, may be represented in the form of the two identically constructed sequences consisting of 8 elements {the reader can himself get assured that the same holds true also for general algebraic formulae possessing structural patterns 3(2 - 4)};

3) there are two arithmetical sequences, each containing 6 terms and having the same progression difference in the formula 12. Thus, the complication of the regularity, governing the symbols forming the algebraic formula 12, is caused only by four of its elements {compare with the above information concerning the general algebraic formula possessing structural pattern 3(1)}.

The main conclusion which may be drawn from the above written implies that for constructing Magic squares having the structural patterns 3(2 - 12) it is preferable to use the general algebraic formulae of Magic squares 4×4, corresponding to these structural patterns.

a_1	p_1	a_1	p_1
p_1	a_1	p_1	a_1
p_1	a_1	p_1	a_1
a_1	p_1	a_1	p_1

(1) – A_1

a_2	a_2	p_2	p_2
p_2	p_2	a_2	a_2
p_2	p_2	a_2	a_2
a_2	a_2	p_2	p_2

(2) – A_2

a_3	a_3	p_3	p_3
p_3	p_3	a_3	a_3
a_3	a_3	p_3	p_3
p_3	p_3	a_3	a_3

(3) – A_3

a_4	p_4	a_4	p_4
p_4	p_4	a_4	a_4
a_4	a_4	p_4	p_4
p_4	a_4	p_4	a_4

(4) – A_4

a_5	p_5	p_5	a_5
p_5	a_5	a_5	p_5
a_5	p_5	p_5	a_5
p_5	a_5	a_5	p_5

(5) – A_5

a_6	p_6	p_6	a_6
a_6	p_6	p_6	a_6
p_6	a_6	a_6	p_6
p_6	a_6	a_6	p_6

(6) – A_6

a_7	p_7	a_7	p_7
a_7	p_7	a_7	p_7
p_7	a_7	p_7	a_7
p_7	a_7	p_7	a_7

(7) – A_7

a_8	a_8	p_8	p_8
a_8	p_8	a_8	p_8
p_8	a_8	p_8	a_8
p_8	p_8	a_8	a_8

(8) – A_8

c_1	c_1	t_1	t_1
c_1	c_1	t_1	t_1
t_1	t_1	c_1	c_1
t_1	t_1	c_1	c_1

(9) – C_1

c_2	t_2	c_2	t_2
c_2	c_2	t_2	t_2
t_2	t_2	c_2	c_2
t_2	c_2	t_2	c_2

(10) – C_2

c_3	t_3	c_3	t_3
t_3	c_3	t_3	c_3
c_3	t_3	c_3	t_3
t_3	c_3	t_3	c_3

(11) – C_3

c_4	c_4	t_4	t_4
t_4	c_4	t_4	c_4
c_4	t_4	c_4	t_4
t_4	t_4	c_4	c_4

(12) – C_4

b_1	h_1	h_1	b_1
h_1	b_1	h_1	b_1
b_1	h_1	b_1	h_1
h_1	b_1	b_1	h_1

(13) – B_1

b_2	h_2	h_2	b_2
b_2	h_2	b_2	h_2
h_2	b_2	h_2	b_2
h_2	b_2	b_2	h_2

(14) – B_2

b_3	h_3	b_3	h_3
h_3	b_3	b_3	h_3
b_3	h_3	h_3	b_3
h_3	b_3	h_3	b_3

(15) – B_3

b_4	h_4	b_4	h_4
b_4	h_4	h_4	b_4
h_4	b_4	b_4	h_4
h_4	b_4	h_4	b_4

(16) – B_4

b_5	b_5	h_5	h_5
h_5	b_5	b_5	h_5
b_5	h_5	h_5	b_5
h_5	h_5	b_5	b_5

(17) – B_5

b_6	b_6	h_6	h_6
b_6	h_6	h_6	b_6
h_6	b_6	b_6	h_6
h_6	h_6	b_6	b_6

(18) – B_6

Fig. 5. A set of A-, B-, C-forms, suitable for constructing Magic squares 4×4.

6 Four-component algebraic formulae of Magic squares 4×4

1. Four-component algebraic formulae of the classical Magic squares 4×4. Since a classical (Magic) square contains in its cells 16 different natural numbers N ($1 \leq N \leq 16$) then one may write[4, 12] the formula for decomposing the number N in 5 terms:

$$N = 8a + 4b + 2c + d + 1, \tag{4}$$

where the parameters a, b, c and d can assume only two values: either 0 or 1. By means of (4) any classical square 4×4 may be identically decomposed in 4 tables (a-, b-, c-, d-components) each of them containing 8 zeros and 8 units. From theoretical point of view[4] there exist the only three groups of Magic squares:

1) *correct squares* — all the decomposition tables are by themselves Magic squares: they have in all the rows, columns and in the two main diagonals by 2 zeros and 2 units. Further such decomposition tables we shall denote as *A-form.*

2) *regular squares* — at least one of the decomposition tables differs from correct one by existing at least one of the components of the formula, which is necessarily a regular one: each of its rows and columns contains by two zeros and two units, but this condition being not preserved for the main diagonal. Further such decomposition tables we shall denote as *B-form* if its both main diagonals contain 4 or 0 zeros (units) and *C-form* if its the main diagonals contain 1 or 3 zeros (units).

3) *irregular squares* — at least one of the decomposition tables differs from correct and regular one by existing at least one of the components of the formula has one row or one column where the number of the same symbols of one kind is distinct from two.

As it can be proved by analytical methods

a) by using A-forms one may construct[4, 12] the only 11 different algebraic formulae of correct Magic squares and with account for rotations and reflections[7] the only 7 following

$$A_1 A_2 A_5 A_6,\quad A_1 A_2 A_3 A_5,\quad A_1 A_3 A_5 A_7,\quad A_1 A_2 A_6 A_7, \tag{5}$$
$$A_2 A_3 A_6 A_7,\quad A_3 A_5 A_6 A_7,\quad A_4 A_5 A_6 A_8,$$

will be different among them, where $A_1 - A_8$ forms are presented in Fig. 5(1 – 8);

b) by using B- and C-forms one may construct[4] with account for rotations, reflections and M-transformations the only 15 different algebraic formulae of regular Magic squares

$$\begin{array}{llllll} BCAA — & B_1C_1A_2A_3, & B_1C_2A_1A_4; & & & (6)\\ BCBA — & B_1C_1B_2A_2, & B_1C_1B_3A_2, & B_1C_1B_3A_3, & B_1C_1B_4A_3, & \\ & B_1C_2B_2A_1, & B_1C_2B_5A_4, & B_1C_2B_6A_4; & & \\ BCBB — & B_1C_1B_2B_3, & B_1C_1B_2B_4, & B_1C_1B_3B_4, & B_1C_2B_2B_5, & \\ & B_1C_2B_2B_6, & B_1C_2B_5B_6, & & & \end{array}$$

where $C_1 - C_4$ and $B_1 - B_6$ forms are presented in Fig. 5(9 – 18).

c) for classical squares 4×4 the complete set of four-component algebraic formulae consists of algebraic formulae of the only correct and regular Magic squares[7] {see sets of formulae (5) and (6)}.

2. Four-component algebraic formulae of generalised Magic squares. Denote, first, A-components of a correct Magic square 4×4 by the symbols F_1, F_2, F_3 and F_4; second, the trivial Magic square, whose 16 cells are filled with units, by the symbol E. As it follows from point 1, any correct classical square 4×4 can be represented as the sum of 5 tables (the first 3 tables should be multiplied by 8, 4 and 2):

$$8F_1 + 4F_2 + 2F_3 + F_4 + E. \qquad (7)$$

An algebraic generalisation of this notation is the expression

$$\alpha F_1 + \beta F_2 + \sigma F_3 + \delta F_4 + \varepsilon E, \qquad (8)$$

which represents the general recording form of a Magic square 4×4 decomposable in the sum of the 4-th A-components. Since the numbers of a classical square 4×4 may be calculated from the formula (4), the formula (8) obviously permits to find the symbols contained in the cells of *the generalised correct* Magic square 4×4. In particular, there exist the following relations

$$\begin{array}{lllll} 1 — \varepsilon, & 5 — \varepsilon + \beta, & 9 — \varepsilon + \alpha, & 13 — \varepsilon + \alpha + \beta, & (9)\\ 2 — \varepsilon + \delta, & 6 — \varepsilon + \beta + \delta, & 10 — \varepsilon + \alpha + \delta, & 14 — \varepsilon + \alpha + \beta + \delta, & \end{array}$$

$3 — \varepsilon + \sigma$, $7 — \varepsilon + \beta + \sigma$, $11 — \varepsilon + \alpha + \sigma$, $15 — \varepsilon + \alpha + \beta + \sigma$,

$4 — \varepsilon + \sigma + \delta$, $8 — \varepsilon + \beta + \sigma + \delta$, $12 — \varepsilon + \alpha + \sigma + \delta$, $16 — \varepsilon + \alpha + \beta + \sigma + \delta$.

between natural numbers from 1 to 16 and the symbols of the generalised correct Magic square 4x4.

Note that the cells of the table, shown in Fig. 6(1), contain a complete set of the symbols of the generalised correct Magic square 4x4. These symbols are arranged so that the first cell of the table contains the symbol ε, the second one contains the symbols $\varepsilon + \delta$ and so on. Thus, the mentioned table is *additional* by the definition and permits to construct various algebraic formulae of the generalised correct Magic squares of the fourth order for the assigned correct classical squares 4x4.

Change the form of recording the table 6(1) by introducing the new symbols g, h and f with the correlations $g = \varepsilon + \beta$, $h = \varepsilon + \alpha$, $f = \varepsilon + \alpha + \beta$. The new form of the table is shown in Fig. 6(3). The table 6(3) makes it clear that the rows of the initial additional table of the generalised correct Magic square 4x4 contain the sequences of four numbers formed by the same regularity. Let it be also noted, that the new table (as it may be easily verified) completely corresponds to the initial one only if between its parameters ε, g, h and f the correlation $\varepsilon + f = g + h$ is fulfilled. Thus, for constructing concrete examples of the generalised correct Magic squares it is necessary to continue the search for the indicated sequences involving four numbers until one finds among their first terms the two pairs of numbers having the same sum.

For example, the generalised correct Magic square 4x4 may be formed from the following eight pairs of prime numbers "the twins": 29, 31; 59, 61; 71, 73; 101, 103; 197, 199; 227, 229; 239, 241; 269, 271.

The additional table, shown in Fig. 6(1) one also may use for constructing the algebraic formulae of *the generalised regular* Magic squares on the basis of the given classical squares 4x4. However, due to the fact that the condition of Magicity is not fulfilled on the diagonals of regular tables (see point 1) for obtaining algebraic formulae of Magic squares in this case one has to assign additional correlations between the parameters of the additional table. For the set (6) of regular formulae of the Magic square 4x4 these necessary correlations between the parameters of the additional table 6(1) have the form, depicted in Fig. 6(6).

e	$e+d$	$e+c$	$e+c+d$
$e+b$	$e+b+d$	$e+b+c$	$e+b+c+d$
$e+a$	$e+a+d$	$e+a+c$	$e+a+c+d$
$e+a+b$	$e+a+b+d$	$e+a+b+c$	$e+a+b+c+d$

(1)

17	29	41	53
47	59	71	83
227	239	251	263
257	269	281	293

(2)

e	$e+d$	$e+c$	$e+c+d$
g	$g+d$	$g+c$	$g+c+d$
h	$h+d$	$h+c$	$h+c+d$
f	$f+d$	$f+c$	$f+c+d$

(3)

1	7	67	73
37	43	103	109
157	163	223	229
193	197	257	263

(4)

83	113	293	503
41	71	251	461
281	311	491	701
239	269	449	659

(5)

Components of formula				Correlations
a	b	c	d	between the parameters
A	A	B	C	$c = 2d,$
B	C	A	A	$a = 2b,$
A	B	C	A	$b = 2c,$
A	B	B	C	$b = c + 2d,$
B	B	C	A	$a = b + 2c,$
B	B	B	C	$a = c + b + 2d.$

(6)

Fig. 6. Examples of constructing additional tables for the generalised correct (1–5) and regular (2 – $AABC$, 4 – $ABBC$, 5 – $BBBC$) Magic squares 4×4.

It is noteworthy, that for the given type of a four-component regular formula the set of the symbols, positioned in the cells of additional tables of the generalised regular Magic squares 4×4, does not depend upon the form of additional correlations between the parameters of the additional table 6(1), in other words, it does not depend on the position of the C-form in the a-, b-, c-, d-decompositions of regular formulae. One can be immediately convinced in this by constructing on the basis of Fig. 6(1) all six additional tables for various algebraic formulae of the generalised regular Magic squares 4×4. Thus, if it is possible to construct one additional table for algebraic formulae of the type

AABC or *ABBC* from the given set involving arbitrary 16 numbers, then it is also possible to construct the other additional tables of Magic squares of the given type, distinct from the above constructed one by the form of additional conditions for the parameters of the table 6(1). With regard for the above stated, only 3 additional tables, filled with prime numbers, for which the reader is referred to Fig. 6(2, 4, 5), suffice for constructing a complete family of different regular Magic squares 4x4.

1367	1468	2358	2457
1457	1458	2368	2367
1368	1467	2357	2458
1358	2467	1357	2468

(1)

2368	1467	2357	1458
2367	1457	1358	2468
2467	1357	1368	2458
1367	1468	2358	2457

(2)

1367	1458	2368	2457
1457	1468	2358	2367
1358	1467	2357	2468
1368	2467	1357	2458

(3)

1367	2368	1458	2457
2367	2358	1468	1457
2468	2357	1467	1358
1368	1357	2467	2458

(4)

1367	1468	2358	2457
1467	1458	2368	2357
1358	1457	2367	2468
1368	2467	1357	2458

(5)

1368	2367	2458	1457
1367	2358	2467	1458
2468	1358	1467	2357
2368	1357	1468	2457

(6)

Fig. 7. Examples of irregular four-component algebraic formulae of Magic squares 4x4.

In conclusion of this section we would like to draw attention that with regard for rotations, mappings and *M*-transformations there exist[7] 81 irregular four-component algebraic formulae of Magic squares 4x4. For instance, 6 formulae of such type are presented in Fig. 7 {for splitting the formulae, shown in Fig. 7, in four components, it suffices to retain in the formulae, at first, only the digits 1 and 2 (1st component), and then, only the digits 3 and 4 (2nd component), etc.}. Hence, the solution of the problem *on decomposing* the general algebraic formula 1(4) into a complete set of the four-component ones has following form: there are 7 formulae for correct Magic squares 4×4 {with account for rotations and reflections}, 15 and 81 formulae correspondingly for regular and irregular Magic squares 4×4 {with account for rotations, mappings and M-transformations}. Thus, it is *the main conclusion* of this section that the complete set of four-component analytical formulae of Magic squares 4×4 can not simplify the solution of the problem on constructing Magic squares 4×4 from an arbitrary given set of 16 numbers but it can make so for constructing the generalised correct and regular Magic squares 4×4.

7 Summary

As it have been demonstrated in this paper discussed Smarandache type question – whether a possibility exists to construct the theory of Magic squares without using properties of concrete numerical sequences – has the positive answer. However, to construct this theory for Magic squares 4×4 in size, the new type of mathematical problems was necessary to introduce. Indeed, in terms of algebra, any problems on constructing Magic squares without using properties of concrete numerical sequences may be formulated as ones on composing and solving the corresponding systems of algebraic equations. Thus, algebraic methods can be applied for

a) constructing the algebraic formulae of Magic squares;

b) finding the transformations translating an algebraic formula of a Magic square from one form into another one;

c) elucidating the general regularities existing between the elements of Magic squares;

d) finding for an algebraic formula of a Magic square, containing m freely chosen parameters, the equivalent set consisting of L algebraic formulae each containing the number of freely chosen parameters less than m.

The new for algebra the type of mathematical problems is presented in points (b) – (d). It is evident that without introducing these problems the algebraic methods are not effective themselves. For instance, in the common case (see Sect. 2) the general formula of Magic square 4×4 can not simplify the solution of problems on constructing Magic squares 4×4 from an arbitrary given set of 16 numbers. In particular, even when solution of discussed problems is sought by means of a computer, in calculating respect it is more preferable for obtaining the solution to use algorithm, described in Sect. 4, than one, elaborated on the base of the general formula of Magic square 4×4. But by means of decomposing the general algebraic formula of Magic squares 4×4 into complete sets of a defined type of analytical formulae one may decrease the common amount of freely chosen parameters in every such formula and, consequently, substantially simplify the regularity existing for the elements of every formula. In other words, for constructing Magic squares 4×4 from an arbitrary given set of 16 numbers there appears a peculiar possibility of using the set algebraic formulae with more simple structure instead of use one complex algebraic formula Magic square 4×4.

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