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# Analytical Formulas for the Coverage of Tunable Matching Networks for Reconfigurable Applications 

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#### Abstract

Tunable matching networks are essential components for agile radio frequency systems. To optimally design such networks, the total area they cover on the Smith chart needs to be determined. In this work, the coverage areas of typical matching networks have been determined analytically for the first time. It has been found that the coverage area is encompassed by up to five arcs. Analytical expressions for the centers and radii for these arcs have been derived. The theoretical analysis is provided for four typical matching networks and verified by circuit simulation and measured data. Moreover, a dynamically load-modulated power amplifier has been designed using the presented theoretical techniques, which demonstrates a measured improvement in the power added efficiency of up to $5 \%$ in the frequency range of (0.8-0.9) GHz.


Index Terms-Smith chart, tunable matching networks, power amplifiers, re-configurable, dynamic load modulation

## I. Introduction

RECONFIGURABLE wireless transceivers are becoming crucial for future systems such as long-term evolution (LTE) and LTE-advanced. Such systems are required to be frequency agile to enable optimal utilization of the congested frequency spectrum. Therefore, these systems require tunable components like antennas, filters, and matching networks (MN).

Beside frequency agile systems, tunable MN are being used heavily for applications such as tunable/wideband antennas [1], efficiency-enhanced and load-sensitive power amplifiers (PAs) [2], [3], and range-adaptive wireless power transfer [4]. One of the main parameters the RF designer needs to know about a tunable MN, is the set of all complex impedances that can be matched to a specific load (typically $50 \Omega$ ). This set defines what is known as the coverage of the MN.

The Smith chart continues to serve as an indispensable tool for the analysis and visualization of complex loads and reflection coefficients [5]. Compared to the complex impedanceplane, the Smith chart is a superior tool because multiple quantities can be read directly, such as the complex impedance and admittance and the reflection coefficient. Therefore, the coverage of MNs defined in the complex plane of the Smith chart is of great benefit. This is particularly useful for PAs because the coverage can be plotted along with other criteria such as load-pull contours, noise circles, gain circles, etc.

In [6]-[11] the boundaries have been presented using simulations, which are less legible, do not indicate the limits of

[^0]the tuning elements, and do not provide any physical insight about the networks. In [12], theoretical formulas have been presented; however, they provide the coverage for discrete impedance points (states) and do not give the continuous coverage, and they are not directly related to the tunable capacitors commonly used in matching networks. Even though the formulas in [13]-[15] can produce the coverage for the full dynamic range, they are presented for the complex rectangular plane, not the Smith chart and for $\Pi$ networks only. In this work, the coverage areas of four of the commonly used networks have been thoroughly investigated. Theoretical formulas have been derived for the loci of the closed boundaries of the coverage areas. These formulas are derived for the complex space of the reflection coefficient rather than the impedance and can, therefore, be plotted directly on the Smith chart. It is assumed that the MN is connected to a resistive load at one end, and the reflection coefficient seen looking towards the other end defines the coverage area as illustrated in Fig. 1 (a). This configuration is different from the one in [6]-[10], [12] where a resistive load is connected at one end with the other end conjugately matched to a known impedance. The configuration used here is particularly suitable for PAs where conjugate matching is not necessarily required. The theoretical analysis has been verified by circuit simulation as well as measured data, which agrees well with the theory.

The theoretical formulas presented here are compact; therefore, very practical for use in CAD tools and provide a useful instrument for the analysis of tunable MN.


Fig. 1. Illustration of the boundary of the coverage area. (a) Problem description. (b) Networks with only one tunable element. (c) Networks with two tunable elements.

## II. Theory

The coverage of a tunable MN can be defined as the set of all complex impedances that can be matched to a specified load at a particular frequency. If the MN has only one tunable
component, its coverage will be a linear path as shown in Fig. 1(a). If the MN has more than one tunable component, then the coverage is typically an area on the Smith chart as shown in Fig. 1(b). The boundary of such area will be referred to here as the coverage boundary and is always a closed path. In this work, MNs with two tunable elements ( $C_{1}$ and $C_{2}$ ) are analyzed. Nevertheless, the theoretical methods introduced here can be applied to networks with a higher number of tunable elements, but the formulas become large and less practical. The two elements are not necessarily identical and can take any values between $C_{\min }$ and $C_{\max }$. The load is assumed to be pure resistive taking a value of $Y_{0} \Omega^{-1}$.

The first step of the analysis is to sweep both capacitors within their limits and observe the area covered by the matching network. It has been found that the coverage area is bounded by four arcs. The first two arcs are plotted by letting $C_{1} \in\left\{C_{\min }, C_{\max }\right\}$ and $C_{2}$ takes all the real numbers between $C_{\min }$ and $C_{\max }$. The remaining two arcs are plotted by letting $C_{2} \in\left\{C_{\min }, C_{\max }\right\}$ and $C_{1}$ takes all the real numbers between $C_{\min }$ and $C_{\max }$. These four arcs are parts of circles which can be completely plotted by extending the limits of the capacitors to $-\infty$ and $\infty$. For lossless networks, all of these circles are tangent to the $|\Gamma|=1$ circle. The value of the capacitor at the tangent point can be calculated by solving for $Y_{i n}$ that falls on the $|\Gamma|=1$ circle according to:

$$
\begin{array}{r}
|\Gamma|=\left|\frac{Y_{0}-Y_{i n}}{Y_{0}+Y_{i n}}\right|=1 \\
\left|Y_{0}-Y_{i n}\right|=\left|Y_{0}+Y_{i n}\right|, \tag{1b}
\end{array}
$$

which is satisfied only when the real part of $Y_{i n}$ is either zero or $\infty$ corresponding to either an open or a short circuit, respectively. A short circuit can be achieved by a shunt branch with zero impedance (if the branch has only a capacitor the capacitance can be set to $\infty$ ). An open circuit, on the other hand, can be achieved by a series branch with infinitely large impedance (if the branch has only a capacitor, its value can be set to zero).

The centers and radii of the four circles can be calculated from the tangent point and any other point. If the tangent point is denoted $\mathrm{A}\left(x_{A}, y_{A}\right)$ and the other point is denoted $\mathrm{B}\left(x_{B}, y_{B}\right)$, the center $\left(x_{c}, y_{c}\right)$ and radius $\left(R_{c}\right)$ of the circle can be calculated, as derived in Appendix B, by:

$$
\begin{align*}
x_{c} & =\frac{x_{A}\left(x_{B}^{2}-x_{A}^{2}+y_{B}^{2}-y_{A}^{2}\right)}{2\left(-x_{A}^{2}-y_{A}^{2}+x_{A} x_{B}+y_{A} y_{B}\right)}  \tag{2a}\\
y_{c} & =\frac{y_{A}\left(x_{B}^{2}-x_{A}^{2}+y_{B}^{2}-y_{A}^{2}\right)}{2\left(-x_{A}^{2}-y_{A}^{2}+x_{A} x_{B}+y_{A} y_{B}\right)} \quad \text { and }  \tag{2b}\\
R_{c} & =\sqrt{\left(x_{c}-x_{A}\right)^{2}+\left(y_{c}-y_{B}\right)^{2}} \tag{2c}
\end{align*}
$$

respectively.
The four circles for a hybrid network are plotted in Fig. 2 together with the coverage area using the theoretical formulas presented here and verified by a commercial simulator. It can be observed that a fifth circle (referred to in the figure as
$\left.C_{2}^{\prime}\right)$ is also needed to complete the boundary. This circle is a function of the inductors and transmission lines as well as the frequency and is referred to here as the auxiliary circle. For given values of these parameters, impedances inside the auxiliary circle can not be matched even if the values of the tunable capacitors extend from $-\infty$ to $\infty$ [16]. As illustrated in Fig. 2, the auxiliary circle is traced when $C_{1}$ is swept while $C_{2}$ is assigned a critical value $\left(C_{2}^{\prime}\right)$, which resonates with the transmission line when $C_{2}=Y_{0} / \omega \tan (\theta)$. Formulas for $C_{2}^{\prime}$ are derived for various network topologies in the following section. The auxiliary circle becomes part of the boundary only if $C_{2}^{\prime}$ falls within the tuning range of $\mathrm{C}_{2}$. Since the auxiliary circle defines a forbidden region, the constant circles of $C_{1, \min }$ and $C_{1, \text { max }}$ can not intersect with this circle, but only shares a single point with it and thus the three circles must be tangent.
In the following sections, four different matching networks are analyzed and the derived formulas to calculate the five circles are presented.


Fig. 2. Illustration of the four main circles and the auxiliary circle that form the boundary of the matching network. The coverage is verified by a commercial simulator.

## A. T-Matching Network

A lumped T-type MN is illustrated in Fig. 3(a). To analyze this network as mentioned in the previous section, the coordinates of the points of the $C_{1}$ and $C_{2}$ circles need to be calculated.

1) $C_{1}$ variable and $C_{2}$ constant: For the first case, $C_{2} \in$ $\left\{C_{\min }, C_{\max }\right\}$ while $C_{1}$ can take any real number between these two limits. To plot the complete circle, however, we will let $C_{1} \in \mathbb{R}_{\geq 0}$. The first point to be calculated is the point at which this circle is to the outer circle $(|\Gamma|=1)$. This point is referred to as point A and corresponds to $C_{1}=0$. Using this value, the input admittance $\left(Y_{i n}\right)$ becomes infinite, and the real and imaginary parts of the input reflection coefficient $\left(\Gamma_{i n}\right)$ are

$$
\begin{align*}
x_{A 1} & =1 \quad \text { and }  \tag{3a}\\
y_{A 1} & =0 . \tag{3b}
\end{align*}
$$

For the second point (point B ), $C_{1}$ is assumed to take an infinitely large value, which corresponds to an RF short circuit. Using this assumption, the real and imaginary values of $Y_{i n}$ can be calculated as:

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{Y_{0}\left(\omega^{2} L C_{2}\right)^{2}}{\left(\omega^{2} L C_{2}\right)^{2}+\left(Y_{0} \omega L\right)^{2}} \tag{4a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Im\left\{Y_{i n}\right\}=\frac{\omega L\left(-\omega^{2} C_{2}^{2}-Y_{0}^{2}+Y_{0}^{2} \omega^{2} L C_{2}\right)}{\left(\omega^{2} L C_{2}\right)^{2}+\left(Y_{0} \omega L\right)^{2}} \tag{4b}
\end{equation*}
$$

from which the real and imaginary values of $\Gamma_{i n}$ are:

$$
\begin{align*}
& x_{B 1}=\frac{-\left(\Re\left\{Y_{i n}\right\}\right)^{2}+Y_{0}^{2}-\left(\Im\left\{Y_{i n}\right\}\right)^{2}}{\left(\Re\left\{Y_{i n}\right\}+Y_{0}\right)^{2}+\left(\Im\left\{Y_{i n}\right\}\right)^{2}} \quad \text { and }  \tag{5a}\\
& y_{B 1}=\frac{-2 Y_{0} \Im\left\{Y_{i n}\right\}}{\left(\Re\left\{Y_{i n}\right\}+Y_{0}\right)^{2}+\left(\Im\left\{Y_{i n}\right\}\right)^{2}} \tag{5b}
\end{align*}
$$

respectively.
The coordinates of points A and B can be directly used in equation (2) to calculate the centers and radii of the circles. Two circles are obtained: one for $C_{2}=C_{\min }$ and another for $C_{2}=C_{\text {max }}$.


Fig. 3. Schematics of the four topologies analyzed in this work. (a) T-type. (b) $\Pi$-type. (c) Ladder-type. (d) Hybrid $\Pi$. $\Gamma_{i n}=x+j y$ is the input reflection coefficient and $Y_{i n}$ is the input admittance.
2) $C_{1}$ constant and $C_{2}$ variable: For this case $C_{1} \in$ $\left\{C_{\min }, C_{\max }\right\}$ while $C_{2} \in \mathbb{R}_{\geq 0}$. The first point in this case corresponds to $C_{2}=0$, which gives the following values of the real and imaginary of $\Gamma_{i n}$

$$
\begin{gather*}
x_{A 2}=\frac{Y_{0}^{2}\left(1-\omega^{2} L C_{1}\right)^{2}-\left(\omega C_{1}\right)^{2}}{Y_{0}^{2}\left(1-\omega^{2} L C_{1}\right)^{2}+\omega^{2} C_{1}^{2}}  \tag{6a}\\
y_{A 2}=\frac{-2 \omega Y_{0} C_{1}\left(1-\omega^{2} L C_{1}\right)}{Y_{0}^{2}\left(1-\omega^{2} L C_{1}\right)^{2}+\omega^{2} C_{1}^{2}} \tag{6b}
\end{gather*}
$$

This point has a unity magnitude regardless of the value of $C_{1}$ and, therefore, always lies on the outer circle. For the second point, the assignment: $C_{2}=\infty$ is used and the real and imaginary values of $\Gamma_{i n}$ are:

$$
\begin{equation*}
x_{B 2}=\frac{Y_{0}^{2}\left(1-2 \omega^{2} L C_{1}\right)+\omega^{2}\left(Y_{0}^{2} L^{2}-C_{1}^{2}\right)}{Y_{0}^{2}\left(1-2 \omega^{2} L C_{1}\right)^{2}+\omega^{2}\left(Y_{0}^{2} L+C_{1}\right)^{2}} \tag{7a}
\end{equation*}
$$

and
$y_{B 2}=\frac{\omega Y_{0}\left(Y_{0}^{2} L-C_{1}\right)\left(1-2 \omega^{2} L C_{1}\right)-\omega Y_{0}\left(Y_{0}^{2} L+C_{1}\right)}{Y_{0}^{2}\left(1-2 \omega^{2} L C_{1}\right)^{2}+\omega^{2}\left(Y_{0}^{2} L+C_{1}\right)^{2}}$,
respectively. These two points can be used in equation (2) to calculate the centers and radii of the circles of $C_{1}=C_{\min }$ and $C_{1}=C_{\max }$.
3) Auxiliary Circle: The T-type network can match impedances up to a maximum resistance. The first derivative of the real part of the input impedance with respect to $C_{2}$ can be used to determine the value of $\mathrm{C}_{2}$ that provides this resistance as derived in Appendix B-B, and given by:

$$
\begin{equation*}
C_{2}^{\prime}=\frac{1}{\omega^{2} L} \tag{8}
\end{equation*}
$$

The necessary condition for the auxiliary circle to be part of the boundary is for this value to fall within the limits of $\mathrm{C}_{2}$ :

$$
\begin{equation*}
C_{\min }<C_{2}^{\prime}<C_{\max } \tag{9}
\end{equation*}
$$

The auxiliary circle itself can be plotted by using the value of $\mathrm{C}_{2}$ defined in (8) with the formulas of section II-A1

In Fig. 4-(a), an illustration of the boundary of a T-type MN is illustrated for $L=10 \mathrm{nH}$ and frequencies of $0.5 \mathrm{GHz}, 1.2$ GHz , and 2.5 GHz . The complete circles are also included for the case of 1.2 GHz .


Fig. 4. Illustration of the boundary at three different frequencies with the values: $C_{\min }=0.5 \mathrm{pF}, C_{\max }=15 \mathrm{pF}, \theta=50, Z_{0}=50 \Omega, L_{T}=10 \mathrm{nH}, L_{\Pi}=6.2$ nH and $L_{1}=L_{2}=13 \mathrm{nH}$. (a) T-Network. (b) $\Pi$-Network. (c) Ladder Network. (d) Hybrid-П Network.

## B. П-Type Matching Network

A typical $\Pi$-Type MN is illustrated in Fig. 3(b). It consists of an inductor between two shunt capacitors. The analysis of this network can be performed by the same method used in the previous section.

1) $C_{1}$ variable and $C_{2}$ constant: For this case, the coordinates of the first point are calculated by putting $C_{1}=\infty$. At this case the input of the MN appears to have a zero impedance and, therefore, the real and imaginary values of $\Gamma_{i n}$ are

$$
\begin{align*}
& x_{A 1}=-1 \quad \text { and }  \tag{10a}\\
& y_{A 1}=0 \tag{10b}
\end{align*}
$$

respectively.
For the second point, $C_{1}$ is assigned a value of zero and the real and imaginary parts of the input admittance $\left(Y_{i n}\right)$ are

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{Y_{0}\left(1-\omega^{2} L C_{2}\right)+\omega^{2} Y_{0} L C_{2}}{\left(1-\omega^{2} L C_{2}\right)^{2}+\left(\omega Y_{0} L\right)^{2}} \tag{11a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Im\left\{Y_{i n}\right\}=\frac{\omega C_{2}\left(1-\omega^{2} L C_{2}\right)-\omega Y_{0}^{2} L}{\left(1-\omega^{2} L C_{2}\right)^{2}+\left(\omega Y_{0} L\right)^{2}} \tag{11b}
\end{equation*}
$$

respectively. The real and imaginary parts of the reflection coefficient ( $x_{B 1}$ and $y_{B 1}$ ) can be calculated from $Y_{i n}$ using (5).
2) $C_{1}$ constant and $C_{2}$ variable: For the first point of this case, $C_{2}$ is assigned a value of $\infty$ ( RF short circuit). The real and imaginary parts of $\Gamma_{i n}$ are calculated to be

$$
\begin{align*}
& x_{A 2}=\frac{\left(\omega Y_{0} L\right)^{2}-\left(1-\omega^{2} L C_{1}\right)^{2}}{\left(\omega Y_{0} L\right)^{2}+\left(1-\omega^{2} L C_{1}\right)^{2}} \quad \text { and }  \tag{12a}\\
& y_{A 2}=\frac{2 \omega Y_{0} L\left(1-\omega^{2} L C_{1}\right)}{\left(\omega Y_{0} L\right)^{2}+\left(1-\omega^{2} L C_{1}\right)^{2}} \tag{12b}
\end{align*}
$$

respectively. This point has a unity magnitude regardless of the value of $C_{1}$. For the second point, $C_{2}$ is assigned a value of zero. The real and imaginary parts of $Y_{i n}$ are:

$$
\begin{align*}
& \Re\left\{Y_{i n}\right\}=\frac{Y_{0}}{1+\left(\omega Y_{0} L\right)^{2}}, \quad \text { and }  \tag{13a}\\
& \Im\left\{Y_{i n}\right\}=\frac{\omega\left(C_{1}-Y_{0}^{2} L+C_{1}\left(\omega Y_{0} L\right)^{2}\right)}{1+\left(\omega Y_{0} L\right)^{2}} \tag{13b}
\end{align*}
$$

respectively. These values can be used to calculate the coordinates of $\Gamma_{i n}$ using (5).
3) The Auxiliary Circle: This network can match impedances up to a maximum conductance. The first derivative of the real part of the input admittance with respect to $C_{2}$ can be used to calculate the value of $\mathrm{C}_{2}$ that produces this conductance as shown in Appendix B-A and given by the following formula:

$$
\begin{equation*}
C_{2}^{\prime}=\frac{1}{\omega^{2} L} \tag{14}
\end{equation*}
$$

The condition for the auxiliary circle to be part of the boundary is the same as the one defined for the T-network in equation (9), and the auxiliary circle can be plotted by using the value of $\mathrm{C}_{2}^{\prime}$ calculated in (14) with the formulas of section II-B1.

In Fig. 4-(b), an illustration of the boundary of a $\Pi$-type MN is illustrated for $L=6.2 \mathrm{nH}$ and at frequencies of 0.5 $\mathrm{GHz}, 1.2 \mathrm{GHz}$, and 1.5 GHz . The complete boundary circles are also plotted.

## C. Ladder Matching Network

This type of MN is illustrated in Fig. 3(c). It consists of two L-sections connected in series. The analysis of this network is presented in the following sections.

1) $C_{1}$ variable and $C_{2}$ constant: For this case, $C_{1}$ is assigned the values of zero and $\infty$. When $C_{1}=\infty$ the real and imaginary parts of $\Gamma_{i n}$ are clearly given by:

$$
\begin{align*}
x_{A 1} & =-1 \quad \text { and }  \tag{15a}\\
y_{A 1} & =0 \tag{15b}
\end{align*}
$$

respectively. When $C_{1}=0$, on the other hand, the real and imaginary parts of $Y_{i n}$ are calculated as:

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{\Re\left\{Y_{2}\right\}\left(1-\omega \Im\left\{Y_{2}\right\} L_{1}\right)+\omega \Re\left\{Y_{2}\right\} \Im\left\{Y_{2}\right\} L_{1}}{\left(1-\omega \Im\left\{Y_{2}\right\} L_{1}\right)^{2}+\left(\omega \Re\left\{Y_{2}\right\} L_{1}\right)^{2}} \tag{16a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Im\left\{Y_{i n}\right\}=\frac{\Im\left\{Y_{2}\right\}\left(1-\omega \Im\left\{Y_{2}\right\} L_{1}\right)-\omega\left(\Re\left\{Y_{2}\right\}\right)^{2} L_{1}}{\left(1-\omega \Im\left\{Y_{2}\right\} L_{1}\right)^{2}+\left(\omega \Re\left\{Y_{2}\right\} L_{1}\right)^{2}} \tag{16b}
\end{equation*}
$$

respectively, where

$$
\begin{align*}
& \Re\left\{Y_{2}\right\}=\frac{Y_{0}}{1+\left(\omega Y_{0} L_{2}\right)^{2}}, \quad \text { and }  \tag{16c}\\
& \Im\left\{Y_{2}\right\}=\frac{\omega\left(C_{2}\left(1+\left(\omega Y_{0} L_{2}\right)^{2}\right)-Y_{0}^{2} L_{2}\right)}{1+\left(\omega Y_{0} L_{2}\right)^{2}} \tag{16d}
\end{align*}
$$

$\Re\left\{Y_{i n}\right\}$ and $\Im\left\{Y_{i n}\right\}$ can be used to calculate the real and imaginary parts of $\Gamma_{i n}$ using equation (5).
2) $C_{1}$ constant and $C_{2}$ variable: For this case $C_{2}$ is assigned zero and $\infty$. For $C_{2}=\infty$, which is an RF short circuit, the real and imaginary parts of $\Gamma_{i n}$ are given respectively by

$$
\begin{align*}
& x_{A 2}=\frac{\left(\omega Y_{0} L_{1}\right)^{2}-\left(1-\omega^{2} L_{1} C_{1}\right)^{2}}{\left(1-\omega^{2} L_{1} C_{1}\right)^{2}+\left(\omega Y_{0} L_{1}\right)^{2}}, \quad \text { and }  \tag{17a}\\
& y_{A 2}=\frac{2 \omega Y_{0} L_{1}\left(1-\omega^{2} L_{1} C_{1}\right)}{\left(1-\omega^{2} L_{1} C_{1}\right)^{2}+\left(\omega Y_{0} L_{1}\right)^{2}} \tag{17b}
\end{align*}
$$

This point is on the outer circle of the Smith chart because it has a magnitude of one regardless of the value of $C_{1}$. For the second point, $C_{2}=0$ and the real and imaginary parts of $Y_{i n}$ are given respectively by:

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{Y_{0}}{1+\left(\omega Y_{0}\left(L_{1}+L_{2}\right)\right)^{2}} \tag{18a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Im\left\{Y_{i n}\right\}=\frac{\omega C_{1}-\omega Y_{0}^{2}\left(L_{1}+L_{2}\right)\left(1-\omega^{2} C_{1}\left(L_{1}+L_{2}\right)\right)}{1+\left(\omega Y_{0}\left(L_{1}+L_{2}\right)\right)^{2}} . \tag{18b}
\end{equation*}
$$

3) Auxiliary Circle: This network can also match impedances up to a maximum conductance. The value of $\mathrm{C}_{2}^{\prime}$ is calculated in Appendix B-C and is given by:

$$
\begin{equation*}
C_{2}^{\prime}=\frac{1}{\omega^{2} L_{1}}+\frac{Y_{0}^{2} L_{2}}{1+\left(Y_{0} \omega L_{2}\right)^{2}} \tag{19}
\end{equation*}
$$

The condition for the auxiliary circle to be part of the boundary is given by equation (9), and the auxiliary circle can be plotted by using the value of $\mathrm{C}_{2}^{\prime}$ produced in (19) above with the formulas of section II-C1.

In Fig. 4-(c) the coverage circles together with the associated boundary are plotted for values of $L_{1}=L_{2}=13 \mathrm{nH}$, and frequencies of $0.5 \mathrm{GHz}, 1.2 \mathrm{GHz}$, and 2.5 GHz .

## D. Hybrid-П Matching Network

The last network is a hybrid- $\Pi$, which consists of a transmission line with two shunt variable capacitors at its ends as illustrated in Fig. 3-(d). This network is analyzed in the following sections.

1) $C_{1}$ variable and $C_{2}$ constant: In the first case $C_{2}$ is fixed and $C_{1}$ is a variable. The two points are obtained as in the previous sections by assigning zero and $\infty$ to $C_{1}$. For the latter the real and imaginary parts of $\Gamma_{i n}$ are:

$$
\begin{align*}
& x_{A 1}=-1 \quad \text { and }  \tag{20a}\\
& y_{A 1}=0 . \tag{20b}
\end{align*}
$$

For the second point, $C_{1}$ is assigned a value of zero. The real and imaginary parts of $Y_{i n}$ are then calculated as

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{Y_{0}^{2}\left[Y_{0}-\omega C_{2} \tan \theta+\tan \theta\left(\omega C_{2}+Y_{0} \tan \theta\right)\right]}{\left(Y_{0}-\omega C_{2} \tan \theta\right)^{2}+\left(Y_{0} \tan \theta\right)^{2}} \tag{21a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Im\left\{Y_{i n}\right\}=\frac{Y_{0}\left[\left(\omega C_{2}+Y_{0} \tan \theta\right)\left(Y_{0}-\omega C_{2} \tan \theta\right)-Y_{0}^{2} \tan \theta\right]}{\left(Y_{0}-\omega C_{2} \tan \theta\right)^{2}+\left(Y_{0} \tan \theta\right)^{2}} . \tag{21b}
\end{equation*}
$$

These values can be used in (5) to calculate the real and imaginary parts of $\Gamma_{i n}$, which can be used together with the first point to calculate the centers and radii of the circles associated with $C_{1}$.
2) $C_{1}$ constant and $C_{2}$ variable: In this case, $C_{1}$ is fixed at either $C_{1, \min }$ or $C_{1, \max }$ while $C_{2}$ is a variable. The two circle points can be calculated by assigning zero and $\infty$ to $C_{2}$. When $C_{2}=\infty$ the real and imaginary parts of $Y_{i n}$ are given by

$$
\begin{align*}
& \Re\left\{Y_{i n}\right\}=0, \quad \text { and }  \tag{22a}\\
& \Im\left\{Y_{i n}\right\}=\omega C_{1}-\frac{Y_{0}}{\tan \theta}, \tag{22b}
\end{align*}
$$

where $Y_{0}$ is the characteristic admittance of the line, which is also the same as the admittance of the load. Using these values and equation (5), the real and imaginary parts of $\Gamma_{i n}$ can be directly calculated.
In a similar way when $C_{2}=0$ the real and imaginary parts of $Y_{i n}$ are given by:

$$
\begin{align*}
& \Re\left\{Y_{i n}\right\}=Y_{0}, \quad \text { and }  \tag{23a}\\
& \Im\left\{Y_{i n}\right\}=\omega C_{1} . \tag{23b}
\end{align*}
$$

Once more, equation (5) can be used to calculate the real and imaginary parts of $\Gamma_{i n}$. From these two points, the centers and radii of the circles associated with $C_{1}$ can be directly calculated using the relations in section II.
3) Auxiliary Circle: In this case the maximum conductance that can be matched is a function of the transmission line parameters and $C_{2}$. The value of $\mathrm{C}_{2}$ that gives the maximum conductance is derived in Appendix B-D and is given by:

$$
\begin{equation*}
C_{2}^{\prime}=\frac{Y_{0}}{\omega \tan (\theta)} \tag{24}
\end{equation*}
$$

and the necessary condition for the auxiliary circle to be part of the boundary is given by (9). The auxiliary circle can be plotted by using the value of $\mathrm{C}_{2}^{\prime}$ calculated in (24) and the formulas of section II-D1. The coverage for three different frequencies are illustrated in Fig. 4-(d).

## III. Implementation of the Theoretical Results

## A. Connecting the Boundary Area

All five circles of the boundary coverage can be plotted using the formulas presented in the previous section. Since these circles intersect at multiple points, it is necessary to provide a systematic way to identify and connect the arcs of the boundary area. As illustrated in Fig. 5 (c), four points can be identified for all the combinations of the maximum and minimum values of the two capacitors. If the auxiliary circle is part of the boundary, two more points are required making a total of six points. To uniquely define any arc, the starting and ending points together with its center and radius are not sufficient, and a third point is needed. This point can be selected arbitrarily as far as it corresponds to capacitances between the minimum and maximum values. For convenience, however, the median of the capacitances is chosen. Given this information, the arcs can be plotted as detailed in Appendix C.

## B. Design Guidelines

The design and optimization of tunable MNs depends highly on the application at hand. However, important general rules can be derived based on the theoretical analysis presented in this work. The first task in the design is usually the selection of the most suitable network topology. Apart from other application-dependent requirements, the main factor in the selection is the range of capacitance required. The topology that provides the required coverage using capacitors with the smallest range should be favored. Also, the network that offers smaller capacitances should be selected because the quality factors of variable capacitors is highest at their lowest values. Once a suitable network is selected it can be optimized using the tool presented in this work with the following steps:

1) Use the formulas in Appendix B to calculate the values of L or $\theta$ to guarantee coverage of the maximum conductance (resistance for the case of the T-NW).
2) Calculate $\mathrm{C}_{2}^{\prime}$ using L or $\theta$ from the previous step.
3) Ensure that $C_{2, \text { min }}<C_{2}^{\prime}<C_{2, \text { max }}$ by re-selecting $L$ and $\theta$. This will guarantee wide boundary coverage because $\mathrm{C}_{2}^{\prime}$ defines a critical limit.
4) Since the auxiliary circle is included in the boundary by design, select the value of $\mathrm{C}_{2, \min }\left(\mathrm{C}_{2, \max }\right.$ for the case of the T-NW) close to $\mathrm{C}_{2}^{\prime}$ because values of $\mathrm{C}_{2, \text { min }}$ below and above $\mathrm{C}_{2}^{\prime}$ result in identical impedances and therefore duplicate coverage as shown in Fig. 5 (a) and (b). This condition can be derived for the case of the $\Pi$ NW, without loosing generality, by assigning $\mathrm{C}_{2}=$ $\mathrm{C}_{2}^{\prime} \pm C_{\Delta}$ in equation (33) to give:

$$
\begin{equation*}
\Re\left\{Y_{i n}\right\}=\frac{Y_{0}}{(\omega L)^{2}\left[Y_{0}^{2}+\left( \pm \omega C_{\Delta}\right)^{2}\right]} \tag{25}
\end{equation*}
$$

which clearly does not depend on the sign of $\mathrm{C}_{\Delta}$. Therefore, as far as $\mathrm{C}_{2, \text { min }}<C_{2}^{\prime}$, a region on the coverage area is covered twice as illustrated in Fig. 5 (b) (this area can also be observed in Fig. 2 as the densely covered region).
5) After selecting $\mathrm{L}, \theta$ and $\mathrm{C}_{2, \min }$ (or $\mathrm{C}_{2, \max }$ for the case of the T ), the rest of the parameters can be selected according to the required coverage.


Fig. 5. (a) Coverage area with $C_{2, \min }=C_{2}^{\prime}$. (b) Coverage area with $C_{2, \text { min }}<C_{2}^{\prime}$. The duplicate area is covered twice for the case of (b), while the uncovered area is dropped for the case of (a). (c) Illustration of the five arcs and 11 points of the boundary. Each point is defined by the values of $C_{1}$ and $C_{2} \in\{\min$, max, med, aux $\}$, which refer to the minimum, maximum, median, and $C_{2}^{\prime}$, respectively.

## IV. Measurements and Discussions

To verify the theoretical formulas presented in the previous sections, prototypes for all four matching networks have been fabricated and measured. The RT/Duroid ${ }^{\circledR} 5880$ substrate has been used with a thickness of 0.508 mm . The printed circuit boards were fabricated using Laser etching. For the tunable capacitor, the varactor BB388 from Infineon has been used. The fabricated circuits are shown in Fig. 7.

## A. Measurements

First, the C-V characteristics of the varactor have been measured at the frequencies of interest. To take the measurement, a circuit board has been fabricated for the varactor with one end shorted to the ground. The capacitance has been extracted from the $S_{11}$ measurement. In Fig. 6 the capacitance is plotted against the reverse voltage for various frequency points. A model provided by the manufacturer has been used to generate the simulated results presented in the figure. Since the used varactor is packaged, its capacitance changes with frequency due to the effects of the package parasitics. These parasitics originate from the wire-bonds and are mostly inductive resulting in a self-resonant-frequency, which can be clearly observed in the figure.


Fig. 6. Measured and simulated capacitance of the varactor against the bias voltage at multiple frequencies.

Lumped components in the theoretical analysis presented in the previous sections are assumed to be ideal, and the entire networks are assumed to be infinitesimally small. These assumptions can not be realized in practice as surface mounted components (SMT) are not ideal, and the circuits have to be large enough for fabrication and measurement reasons. To obtain results comparable with the ideal theory, the circuits have been made as small as possible. Moreover, the input/output connectors and feed lines have been de-embedded from the measured S-parameters. The parasitic effects of the solder and small conductive patterns connecting the SMT components have not been de-embedded. However, these parasitic effects are proportional to the frequency and can be reduced considerably by lowering the test frequency. For this reason, a relatively low frequency of 500 MHz has been chosen to
verify the theory. From the good match between the theory and measurement, it can be concluded that if the circuits are small enough, and the parasitics reduced (integrated designs), the theory can also be verified for higher frequencies.


Fig. 7. Photographs of the fabricated prototypes.
The measurements of the matching networks have been performed using a digital power supply (TTi MX100TP) and a network analyzer (Keysight N5242A) both controlled by a PC. The DC voltages of the two varactors have been swept from 2 to 28 V in steps of 0.5 V . The measurement has been automated, and a total of 2809 points have been measured for each frequency. The results are compared to the theory for all matching networks in Fig. 8.

## B. Analysis and Discussion

From Fig. 8 (a) a good agreement between theory and measurements can be observed with the four arcs clearly identifiable from the measurements. The minor discrepancy is mainly due to the ideal components assumed in the theoretical analysis. It can also be observed that the points at the edge of the chart, which are predicted by the theory, are not obtained in the measurement. As proofed by equation (1), when the matched impedance is on or close to the outer circle, its real part should be either zero or $\infty$. This condition can only be satisfied if the matching network is purely reactive, which is not the case in the fabricated circuits. A slight phase shift can also be observed which is mainly due to the parasitic effects of the large, manually-soldered circuit.

For the case of the $\Pi$ network (Fig. 8-(b)), the same discussion above applies with a slightly more evident phase shift. The parasitics on each circuit are different because each circuit has a different layout. Therefore, the phase change, which is mainly due to the parasitics, vary from one design to the other. It can also be observed that the circle of $\mathrm{C}_{2, \max }$ is slightly distant from the measured boundary, which indicates that the value of $\mathrm{C}_{2, \max }$ has not been estimated correctly. The maximum value of the capacitance occurs when the bias voltage is low. In such case, the C-V curve experiences an infinite slope as shown in Fig. 6 resulting in an uncertainty in the estimation of the maximum values. Another consequence of this uncertainty is the difference between the maximum limits of the two capacitors even if they are identical due to the effects of the parasitics.
For the case of the Ladder circuit, a very good match between theory and measurements is achieved as shown in Fig. 8 (c). The phase shift in this case is very minor.


Fig. 8. Measurements of the fabricated prototypes compared with the theory. (a) T-NW. (b) П-NW. (c) Ladder NW. (d) Hybrid П NW.

For the hybrid circuit, the comparison has been made at 1 GHz instead of 500 MHz (Fig. 8 (d)) because the size of the lumped circuit is small.
On all the measurements, one or two points can be observed to fall outside of the boundary. These points are found to occur at the lowest bias voltage where the varactor operates on or near its self resonant frequency (SRF) as clear in Fig. 6, which explains the unexpected behaviour.

## V. Design Example: Load Modulated PA

As an example of the presented design tool, a PA has been designed with load modulation capability. One of the challenges of PAs is the deterioration of the Power Added Efficiency (PAE) as the power is reduced (backed-off) from its maximum value. This limitation is more serious for applications with high Peak-to-Average Power Ratio (PAPR), where the input signal is considerably backed-off for most of the time. If a tunable MN is used at the output of the PA, the efficiency at power back-off can be increased by varying the output impedance presented to the transistor according to the magnitude of the input signal.
The first step in the design of load-modulated PAs is to identify the optimal output impedances for various input powers. If an accurate transistor-model is available, this task can be achieved by load and source pull simulations. In this work the GaN HEMT transistor CGH40010F from Cree has been used, which has a good simulation model. Load pull simulations have been performed for input power levels from 30 dBm down to 20 dBm ( 10 dB back-off) at a frequency of 0.8 GHz . The transistor is biased for a class B operation and the second and third harmonics have been optimized for the
maximum power level and kept constant for the other powervalues.

The second step is to properly terminate the harmonics. Since they are kept constant in the previous step, a static network between the transistor and the tunable network has been used (Fig. 10). After designing this network, the load-pull data at the transistor plane has been transformed to the tunable network plane as shown in Fig. 10. The transformed trajectory is plotted in Fig. 9 against the coverage of all four network topologies, where the techniques discussed in section III-B have been used. As expected, all four networks can provide the required coverage, but the T-network is chosen due to its lower capacitance-range (Fig. 9-e) and wider coverage.


Fig. 9. The coverage of all the networks and the load-pull trajectory (beyond the static NW) of the OMN for input power of $18-30 \mathrm{dBm}$. (a) T-NW. (b) $\Pi$ NW. (c) Ladder NW, (d) Hybrid-П NW. (e) Cost comparison.


Fig. 10. Schematic of the load-modulated power amplifier with varactors in anti-series configurations.


Fig. 11. Photographs of the fabricated power amplifier.

## A. Results and Discussion

A prototype amplifier has been fabricated as shown in Fig. 11. The MTV4090 tuning varactor has been used due to its high power handling capability. To increase the power handling even more, the tunable capacitors have been implemented as multiple varactors connected in an anti-series configuration as shown in Fig. 10 and 11.

The PAE is measured at 0.8 and 0.9 GHz for the full dynamic range of the tunable network ( $\mathrm{V}_{1}$ and $\mathrm{V}_{2}$ between 5 and 70 V ). All the measurement has been taken below the 2 dB compression point to maintain a constant gain between 13 and 15 dB . Any measurement resulting in a gain below 13 has been dropped. The results are shown in Fig. 12 where it can be observed that the amplifier achieves a maximum efficiency of $60 \%$ and $5 \%$ improvement in the PAE at 10 dBs of power back-off for the case of 0.9 GHz and $2 \%$ for the case of 0.8 GHz.
The presented design demonstrates relatively low efficiency improvement, which is due to the losses of the varactors, the package parasitics, and the manual soldering. However, it demonstrates the benefit of the presented theoretical analysis and design guidelines in the design and optimization of tunable PA.


Fig. 12. Measured power added efficiency of the dynamically load modulated power amplifier against the output power at 0.9 GHz and 0.8 GHz . The input power is swept from 20 to 28 dBm and the measured gain is steady at (13-15) dB.

## VI. Conclusion

In this work, analytical formulas for the coverage area for four typical matching networks have been derived for the Smith chart for the first time. These theoretical formulas have been validated by circuit simulation and measured results, which agree very well with the theory. The MNs analyzed in this work all have two tuning components (capacitors) and are lossless. Also, only perfect matching is considered. The findings of this work can be extended further by including
the effects of imperfect matching as well as the losses. Nevertheless, the formulas presented here are compact, therefore, suitable for circuit simulators and can be plotted directly on the Smith chart, which enables rapid design of tunable systems. As an example, a load-modulated power amplifier has been designed and tested using the theoretical tools presented in this work.

## Appendix A

DERIVATION OF THE CENTER AND RADIUS OF AN INNER TANGENTIAL CIRCLE

(a)

(b)

Fig. 13. (a) Illustration of a circle tangential with the $\Gamma=1$ circle. (b) Illustration of an arc defined by three points together with the center and radius.

In this section the center and radius of a circle tangent to the $|\Gamma|=1$ circle are derived. The circles are illustrated in Fig. 13-(a), where the tangent point is denoted $\mathrm{A}\left(x_{A}, y_{A}\right)$ and the center is denoted $\mathrm{C}\left(x_{C}, y_{C}\right)$. Point $\mathrm{B}\left(x_{B}, y_{B}\right)$ is any other point on the circle. Triangle 1 and 2 on Fig. 13-(a) are similar, therefore, the ratio of their corresponding sides are in proportion, thus

$$
\begin{gather*}
\frac{x_{C}}{x_{A}-x_{C}}=\frac{y_{C}}{y_{A}-y_{C}} \\
x_{C}=\frac{x_{A}}{y_{A}} y_{C} \tag{26}
\end{gather*}
$$

Evaluating the value of the radius $\mathrm{R}_{C}$ from triangles 1 and 3 and equating them gives:

$$
\begin{gather*}
\left(x_{A}-x_{C}\right)^{2}+\left(y_{A}-y_{C}\right)^{2}=\left(x_{B}-x_{C}\right)^{2}+\left(y_{B}-y_{C}\right)^{2} \\
\left(2 x_{B}-2 x_{A}\right) x_{C}+\left(2 y_{B}-2 y_{A}\right) y_{C} \\
=x_{B}^{2}+y_{B}^{2}-x_{A}^{2}-y_{A}^{2} \tag{27}
\end{gather*}
$$

Substituting form (26) in (27) gives

$$
\begin{array}{r}
\left(\frac{2 x_{A} x_{B}}{y_{A}}-\frac{2 x_{A}^{2}}{y_{A}}+2 y_{B}-2 y_{A}\right) y_{C}  \tag{28}\\
=x_{B}^{2}+y_{B}^{2}-x_{A}^{2}-y_{A}^{2}
\end{array}
$$

Multiplying through by $y_{A}$ and re-arranging to get $y_{C}$ as

$$
\begin{equation*}
y_{C}=\frac{y_{A}\left(x_{B}^{2}+y_{B}^{2}-x_{A}^{2}-y_{A}^{2}\right)}{2\left(-x_{A}^{2}-y_{A}^{2}+2 x_{A} x_{B}+2 y_{A} y_{B}\right)}, \tag{29}
\end{equation*}
$$

and substituting in (26) to get $x_{C}$ as

$$
\begin{equation*}
x_{C}=\frac{x_{A}\left(x_{B}^{2}+y_{B}^{2}-x_{A}^{2}-y_{A}^{2}\right)}{2\left(-x_{A}^{2}-y_{A}^{2}+2 x_{A} x_{B}+2 y_{A} y_{B}\right)} . \tag{30}
\end{equation*}
$$

The radius $\mathrm{R}_{C}$ can be calculated directly from triangle 1 as

$$
\begin{equation*}
R_{C}=\sqrt{\left(x_{A}-x_{C}\right)^{2}+\left(y_{A}-y_{C}\right)^{2}} \tag{31}
\end{equation*}
$$

## Appendix B

DERIVATION OF THE CENTERS AND RADII OF THE AUXILIARY CIRCLES
In this section the value of the capacitor $\left(\mathrm{C}_{2}\right)$ that produces the auxiliary circle is derived for all the topologies presented in Fig. 3.

## A. П Matching Network

The value of $\mathrm{C}_{2}$ that gives maximum conductance is referred to as $\mathrm{C}_{2}^{\prime}$ and can be calculated by first evaluating the real part of the admittance just before the last branch as

$$
\begin{equation*}
Y_{2}=\frac{\left(Y_{0}+j \omega C_{2}\right)\left(\frac{1}{j \omega L}\right)}{Y_{0}+j\left(\omega C_{2}-\frac{1}{\omega L}\right)} \tag{32}
\end{equation*}
$$

The real part of which (conductance) can be calculated as:

$$
\begin{equation*}
\Re\left\{Y_{2}\right\}=\frac{\frac{K_{Q} C_{2}}{L}+\frac{Y_{0}}{(\omega L)^{2}}-\frac{K_{Q} C_{2}}{L}}{Y_{0}^{2}+\left(\omega C_{2}-\frac{1}{\omega L}\right)^{2}} \tag{33}
\end{equation*}
$$

To evaluate the maximum conductance the derivative of $\Re\left\{Y_{2}\right\}$ with respect to $C_{2}$ must be zero, which gives:

$$
\begin{equation*}
\frac{2 Y_{0}\left(1-\omega^{2} L C_{2}^{\prime}\right)\left(\omega^{2} L\right)}{\left(\left(Y_{0} \omega L\right)^{2}+\left(\omega^{2} L C_{2}^{\prime}-1\right)^{2}\right)^{2}}=0 \tag{34}
\end{equation*}
$$

Therefore,

$$
\begin{gather*}
1-\omega^{2} L C_{2}^{\prime}=0, \quad \text { and }  \tag{35}\\
C_{2}^{\prime}=\frac{1}{\omega^{2} L} \tag{36}
\end{gather*}
$$

## B. T-Matching Network

In this case, the last branch is a series capacitor $\left(\mathrm{C}_{1}\right)$, the forbidden area can be defined by the maximum resistance that can be matched before $\mathrm{C}_{1}$. The value of $\mathrm{C}_{2}^{\prime}$ can be calculated by evaluating the real part of the impedance just before the last branch as

$$
\begin{equation*}
\Re\left\{Z_{2}\right\}=\Re\left\{Z_{i n}\right\}=\frac{Z_{0} \omega^{2} L^{2}}{Z_{0}^{2}+\left(\omega L-\frac{1}{\omega C_{2}}\right)^{2}} \tag{37}
\end{equation*}
$$

Equating the first derivative with respect to $C_{2}$ to zero gives:

$$
\begin{equation*}
\frac{\frac{2 Z_{0} \omega L^{2}}{C_{2}^{\prime 2}}\left(\omega L-\frac{1}{\omega C_{2}^{\prime \prime}}\right)}{\left(Z_{0}^{2}+\left(\omega L-\frac{1}{\omega C_{2}^{\prime 2}}\right)^{2}\right)^{2}}=0 \tag{38}
\end{equation*}
$$

Since $C_{2}^{\prime}=\infty$ can be excluded,

$$
\begin{equation*}
C_{2}^{\prime}=\frac{1}{\omega^{2} L} \tag{39}
\end{equation*}
$$

## C. Ladder Matching Network

For this case, a simple transformation can be used to transform $Y_{0}$ and $L_{2}$ from a series to a parallel configuration. This network can then be analysed in the same way as the $\Pi$ network. The parallel conductance and susceptance can be calculated to be

$$
\begin{gather*}
Y_{0}^{\prime}=\frac{Y_{0}}{1+\left(Y_{0} \omega L_{2}\right)^{2}} \text { and }  \tag{40}\\
B_{0}=\frac{-\omega Y_{0}^{2} L_{2}}{1+\left(Y_{0} \omega L_{2}\right)^{2}} \tag{41}
\end{gather*}
$$

respectively. From the results of the $\Pi$ network in (equation (36)) and assuming that $B_{0}$ is capacitive

$$
\begin{gather*}
C_{2}^{\prime}-\frac{-Y_{0}^{2} L_{2}}{1+\left(Y_{0} \omega L_{2}\right)^{2}}=\frac{1}{\omega^{2} L_{1}}, \quad \text { and }  \tag{42}\\
C_{2}^{\prime}=\frac{1}{\omega^{2} L_{1}}+\frac{Y_{0}^{2} L_{2}}{1+\left(Y_{0} \omega L_{2}\right)^{2}} \tag{43}
\end{gather*}
$$

## D. Hybrid П Matching Network

This network can be analyzed in the same way as the $\Pi$ network.

$$
\begin{equation*}
Y_{2}=Y_{0} \frac{Y_{0}+j\left(\omega C_{2}+Y_{0} \tan (\theta)\right)}{Y_{0}-\omega C_{2} \tan (\theta)+j Y_{0} \tan (\theta)} \tag{44}
\end{equation*}
$$

from which the real part can be evaluated as

$$
\begin{equation*}
\Re\left\{Y_{2}\right\}=\Re\left\{Y_{\text {in }}\right\}=\frac{Y_{0}^{3}\left(1+\tan ^{2}(\theta)\right)}{\left(Y_{0}-\omega C_{2} \tan (\theta)\right)^{2}+\left(Y_{0} \tan (\theta)\right)^{2}} \tag{45}
\end{equation*}
$$

Applying the maximum conductance condition results in

$$
\begin{equation*}
\frac{2 Y_{0}^{3} \omega \tan (\theta)[1+\tan (\theta)]\left[Y_{0}-\omega C_{2} \tan (\theta)\right]}{\left[\left(Y_{0}-\omega C_{2} \tan (\theta)\right)^{2}+\left(Y_{0} \tan (\theta)\right)^{2}\right]^{2}}=0 \tag{46}
\end{equation*}
$$

which gives

$$
\begin{equation*}
C_{2}^{\prime}=\frac{Y_{0}}{\omega \tan (\theta)} \tag{47}
\end{equation*}
$$

## Appendix C <br> Plotting an arc Knowing its center radius and THREE POINTS

In this section plotting an arc with a radius of $R_{C}$ and center $\mathrm{C}\left(x_{C}, y_{C}\right)$ and three points $\mathrm{A}, \mathrm{B}$ and D is illustrated as shown in Fig. (13(b)). First the angles of the three points are calculated as

$$
\begin{equation*}
\theta_{i}=\tan ^{-1}\left(\frac{y_{i}-y_{C}}{x_{i}-x_{C}}\right), i \in\{A, B, D\} \tag{48}
\end{equation*}
$$

Next, a vector of the ordered (ascending or descending) angles can be formulated as

$$
\begin{equation*}
\theta=\left[\theta_{A}, \theta_{2}, \theta_{3}, \ldots, \theta_{k}, \theta_{D}, \theta_{k+2}, \ldots, \theta_{B}\right] \tag{49}
\end{equation*}
$$

The angles in this vector always represent the right arc because they include $\theta_{D}$. From this vector, the x and y coordinates of any point in the arc can be simply calculated using:

$$
\begin{equation*}
x+j y=R_{C}\left(\cos (\theta)+x_{C}+j\left[\sin (\theta)+y_{C}\right]\right) \tag{50}
\end{equation*}
$$

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