# Analytical Methods and Perturbation Theory for the Elliptic Restricted Three-Body Problem of Astrodynamics

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# Dedication

For my parents who launched me on my way,

for my teachers who provided guidance and navigation,

and for Susan who propelled me to the finish.

# Acknowledgments

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to fully appreciate Dr. Chichka's teaching style and expertise in the field. I wish my professor the best of luck in all his endeavors and will surely seek his advice whenever I stumble across truly interesting (read: difficult) problems of dynamics and control.

Likewise, I'd like to thank all my professors at GW and in particular those serving on my committee: Professors James D. Lee and Taeyoung Lee of the Mechanical and Aerospace Engineering department and Professor Helmut Haberzettl of the Physics department. In addition, I'd like to thank Dr. Christopher Scott and Dr. Martin Ozimek, both of the Johns Hopkins University Applied Physics Laboratory in Laurel, MD and both of whom graciously agreed to serve on my committee as experts in the areas of spacecraft dynamics and control and in particular, the three-body problem of astrodynamics. All six of my committee members, including my advocate, Dr. Chichka, are knowledgeable experts in their fields and I am grateful for their time and effort in reviewing my dissertation, attending my defense presentation, and providing valuable feedback.

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# Abstract of Dissertation

Analytical Methods and Perturbation Theory
for the Elliptic Restricted Three-Body Problem of Astrodynamics

The distinguishing characteristic of the elliptic restricted three-body problem is a pulsating potential field resulting in non-autonomous and non-integrable spacecraft dynamics, which are difficult to model using classical methods of analysis. The purpose of this study is to harness modern methods of analytical perturbation theory to normalize the system dynamics about the circular restricted three-body problem and about one of the triangular Lagrange points. The normalization is achieved through a canonical transformation of the system Hamiltonian function based on the Lie transform method introduced by Hori and Deprit in the 1960s. The classic method derives a near-identity transformation of a Hamiltonian function expanded about a single parameter such that the transformed system possesses ideal properties of integrability. One of the major contributions of this study is to extend the normalization method to two-parameter expansions and to non-autonomous Hamiltonian systems. The two-parameter extension is used to normalize the system dynamics of the elliptic restricted three-body problem such that the stability of the triangular Lagrange points may be determined using the Kolmogorov-Arnold-Moser theorem. Further dynamical analysis is performed in the transformed phase space in terms of local integrals of motion

akin to Jacobi's integral of the circular restricted three-body problem. The local phase space around the Lagrange point is foliated by invariant tori that effectively separate the planar dynamics into qualitative regions of motion. Additional analysis is presented for the incorporation of control into the normalization routine with the goal of eliminating the non-circular secular perturbations. The control method is validated on a test case and applied to the elliptic restricted three-body problem for the purposes of stabilizing the motion around the triangular Lagrange points.

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# Chapter I

# Introduction

The problem of three bodies moving under their mutual gravitational attraction has interested mathematicians since Sir Isaac Newton first postulated the system as

> three bodies whose forces decrease in a duplicate ratio of the distances, attract each other mutually; and the accelerative attractions of any two towards the third be between themselves reciprocally as the squares of the distances.[1]

The ease of posing the problem belies the complexity of its solution. In fact, Henri Poincaré proved that no analytic integrals of motion exist for the general three-body problem other than the energy and angular momentum. This then implies that the problem may not be solved in terms of algebraic functions and integrals, but only through numerical integration or infinite series representation. However, the lack of a closed-form solution does not preclude analytical methods of study. Poincaré himself devised the method of phase space surfaces of section, which in place of describing the trajectories themselves, instead describes the qualitative regions of motion.[2] In doing so, one does not dictate the state of the bodies at any given instant in time, but rather a domain of possible states for all time. However, even this approach has its limitations due in part to nonlinear effects and the onset of chaos.

If the mass of one of the three bodies is negligible as compared to the masses of the other two bodies, than it imposes no gravitational influence over the motion of the other two. This then introduces a special form of the three-body problem referred to as the restricted three-body problem (RTBP). The larger bodies are referred to as the *primaries* and the third body as the *spacecraft*. The goal is to describe the motion of the spacecraft within the gravitational field generated by the two primaries.

Since the primaries represent a conservative two-body system, their trajectories are dictated by Kepler's laws in which the orbits assume the form of a conic section.[3;4] For orbital mechanics applications, the orbit of the primaries is assumed closed such that it is either circular or elliptical referred to as the circular restricted three-body problem (CRTBP) and the elliptic restricted three-body problem (ERTBP) respectively. Distinguishing between these two cases represents a pivotal bifurcation in the behavior of the system and in the history of its study within the astrodynamics community. As discussed in the next section, the vast majority of previous research has focused on the circular case with comparatively little attention paid to the elliptic case. This is due in part to the convenient conservative properties of the CRTBP, its utility as a starting point for more accurate models and the fact that most two-body systems in our solar system are very nearly circular. Nonetheless, for the interest of greater generality and higher accuracy, the non-circular effects encompassed in the ERTBP serve as the major focus of this study.

The following section presents first a historical context for the study of the ERTBP while highlighting major sources of references for the theoretical background and known results. Upon reviewing the historical background, the motivation and contributions of the current study are described for comparisons sake. The final section provides a chapter by chapter description of the document.

## 1.1 Historical Background

As mentioned previously, the study of the three-body problem started with Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* first published in 1697. Since then, it has been the study of many great scientists whose work spawned many other fields of mathematical analysis. Other notable mathematicians of the 18th and 19th centuries who studied the three-body problem include Euler, Lagrange, Laplace, Poisson, Jacobi, Poincaré and many many others. The CRTBP received its first formal introduction in the memoirs of Leonhard Euler in 1760 and is often named in his honor. Euler and Lagrange are further credited with formally defining the system equilibrium points, which are usually referred to as the Lagrange points. Meanwhile, Jacobi is credited for discovering the energy-type integral of motion from which Hill further introduced the notion of curves of zero velocity, which qualitatively describe the motion of the system based on its energy content. The lectures of Jacobi and the subsequent work of Poincaré, Von Zeipel and Hill are all of particular relevance to this study as they introduce and develop the influential field of canonical perturbation theory. Building on the pioneering work of Lagrange, Hamilton and Liouville, Jacobi described the general notion of a perturbed system that is normalized through a canonical transformation. His successors, Poincaré and Von Zeipel developed the theory leading to the widely applied method of transformation called Von Zeipel's method.[1;2;5;6;7;8;9;10]

Of the many modern published works regarding the CRTBP, none is as comprehensive or oft-cited as Szebehely's *Theory of Orbits*. Besides describing the general methods of formulation and transformation of the RTBP, Szebehely also discusses at length the stability of the system and the qualitative behavior as described in terms of Jacobi's integral and Hill's curves of zero velocity. He also provides a brief but concise introduction to the ERTBP including a derivation of the Nechvile transformation.[11] There have been far too many other studies regarding the CRTBP to attempt any

sort of comprehensive list here. However, those particularly relevant to this study include the early papers of Breakwell and Pringle who derive a useful transformation of the linearized CRTBP and Deprit and Deprit-Bartholomé who treat the nonlinear stability of the circular triangular Lagrange points.[12;13] Of the more recent studies of the CRTBP, those that incorporate elements of perturbation theory include the works of Meyer et al., particularly in the comprehensive book *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, the books by Boccaletti et al., and the series of numerical and semi-analytical studies conducted by the Barcelona group of Gómez et al.[14;15;16;17;18]

While the dynamics of the ERTBP has received far less study than the CRTBP, there are still quite a few sources of reference dating back to the 1960s including the general dynamical treatments of Szebehely, Kopal and Lyttleton, Broucke, Contopoulos and the stability analyses conducted by Danby, Alfriend and Rand, Deprit and Rom, and Nayfeh et al. In the first group, the studies discuss the basic dynamics of the system, the transformation to a pulsating, rotating reference frame, and the implications of losing Jacobi's integral.[11;19;20;21;22;23;24] The latter group analyzes the linear stability of the elliptic triangular Lagrange points starting with a numerical treatment by Danby and its subsequent verification by the others.[25;26;27;28;29;30] Since then, a few studies have been conducted including those by Cheng and Schmidt and a series of recent works by Astakhov and Farrelly, Lhotka et al., and Erdi et al. that apply modern dynamical systems theory to the RTBP including the noncircular case.[31;32;33;34;35] While there are a few other references available regarding the ERTBP, the cited works roughly represent the extent of study conducted thus far. Much of the content included in the present study builds off these previous works, but to the author's knowledge, the theorems and analyses presented herein represent a novel and original treatment of the full nonlinear ERTBP.

With regard to the method of canonical transformation of an expanded Hamilto-

nian function, the Lie transform method was introduced at roughly the same time by Hori and Deprit. While the presentations from both authors are certainly unique, the general methodology is the same. The comprehensive treatment conducted in Deprit's work serves as the primary reference used in this study due to its discussion regarding Hamiltonian systems and the derivation of the non-autonomous remainder function. Nonetheless, the classic method is referred to as the Deprit-Hori Lie transform method or simply the DH method.[36,37] The methodology presented in these two ground-breaking studies has since been developed further in subsequent papers by Hori and Cary and extended to systems expanded about two, three, or the general case of $N$ parameters by Varadi, Ahmed, and Andrade respectively.[38,39,40,41,42] It has also been applied for certain simplified forms of the ERTBP in the aforementioned papers by Deprit, Meyer, and Schmidt.[15,16,28,32] One of the major contributions of the present study is to extend the work included in these previous studies, first to a general non-autonomous Hamiltonian function expanded about two parameters and then specifically as applied to the ERTBP. In addition, a novel control scheme is introduced through the incorporation of input terms in the DH transformation. To the author's knowledge, the extension of the DH method to non-autonomous two-parameter systems and the incorporation of control terms represent novel developments to the Deprit-Hori Lie transform normalization theory.

Finally, topics from general dynamical systems theory and celestial mechanics is referenced from the books by Wintner, Arnold, Tabor, Battin, Goldstein et al., Danby, Kappeler and Pöschel, and Kibble and Berkshire as well as the aforementioned works of Boccaletti and Meyer.[3,4,16,18,43,44,45,46,47,48] The so-called KAM theory was first developed by Kolomogorov, Moser, and Arnold in the 1960s.[49,50,51] It has since been celebrated and discussed in countless works including many of the general dynamics texts cited previously.

## 1.2  Motivation

The original motivation for studying the three-body problem was to predict the motion of the Moon under the gravitational influence of both Earth and the Sun. Other applications include the evolution of the solar system, the dynamics of star systems, and trajectory design for interplanetary spacecraft. Most systems of interest exhibit low eccentricities such that the CRTBP is often applied as a sufficient approximation. However, for the purposes of generality and higher accuracy, this study incorporates the non-circular effects in terms of the ERTBP. The motivation is then to characterize the non-circular effects by studying the dynamics of the ERTBP and in particular, treating the ERTBP as a perturbed system about the CRTBP.

The stability of motion about the elliptic Lagrange points has been studied previously for the linearized system and the nonlinear circular system.[11;13;15;16] Of the five points, three are unstable in the sense of Lyapunov as easily demonstrated using Lyapunov's indirect method.[46;48] The remaining two Lagrange points are stable in the linearized CRTBP, but in a non-hyperbolic sense such that the nonlinear stability may not be inferred from the corresponding linearized system. Instead one may treat the nonlinear and non-circular terms as perturbations to the linearized CRTBP with the aim of extending its stability properties to the nonlinear ERTBP. This then motivates the need to apply canonical perturbation theory and KAM theory.

The DH method of perturbation theory provides a methodical means of generating canonical transformations of a Hamiltonian system as represented in expanded form about a single system parameter. For the purposes of analyzing motion about the elliptic Lagrange points, the methodology must be extended to expansions about two system parameters (that is about the circular case and linearized case). A few previous studies have considered the extension of the DH method to two or more parameters, but do so only for autonomous systems.[40;41;42] Thus, in order to apply these methods, one would need to first expand the phase space to account for the non-conservation

of the Hamiltonian function. While in principle these methods could be used to treat motion about the nonlinear elliptic Lagrange points, one of the goals of this study is to derive a novel formulation of the two-parameter DH method that treats the non-autonomous system directly in terms of the classical remainder function.[43] The motivation is primarily academic in nature: to determine whether there is a better approach to applying the two-parameter DH method and in a more intuitive manner. However, it is also of use to derive the method in terms of the remainder function in order to isolate the non-autonomous effects within the transformation.

Further contributions regard the incorporation of control within the DH method to derive a novel feedback control law that acts directly on the secular behavior of the Hamiltonian system. The motivation is again primarily academic: to explore the formulation of the method and its application for feedback stabilization of motion about the elliptic Lagrange points. No assertion is made regarding the optimality or general performance of the control method as compared to other control strategies such as linear quadratic regulators, Lyapunov control or controlled Lagrangians and Hamiltonians.[52;53] However, the groundwork is laid for future trade studies between the various control formulations.

## 1.3    Document Organization

The document is organized into five chapters as follows. Theoretical background material is provided in Chapter II beginning with a dynamical formulation of the ERTBP in a rotating reference frame and using canonical units. The energy of the spacecraft relative to the primaries is defined in terms of spherical coordinates and used to formulate the dynamical equations of motion based on Lagrangian and Hamiltonian mechanics. From the system Hamiltonian formulation, a series of canonical transformations are applied that normalize the dynamics into a pulsating reference frame with the true anomaly of the primaries serving as the independent variable.

Chapter II also presents relevant theory regarding the CRTBP including the classic Lagrange points and the Jacobi integral. Finally, the chapter concludes with a basic introduction to canonical perturbation theory and the theory of Kolmogorov, Arnold and Moser (KAM) and its implications for analyzing the stability of nearly integrable Hamiltonian systems in Birkhoff normal form.

Chapter III builds on the fundamental topics of perturbation theory introduced in Chapter II. After describing its predecessor, the Von Zeipel method, the Deprit-Hori Lie transform method (DH method) is introduced for the classic case of a Hamiltonian system expanded about a single parameter. Upon introducing the basic principles of the single parameter DH method, an extension to the method is provided for the case of a non-autonomous Hamiltonian function expanded about two parameters. The extension to the classic methodology is presented in the form of an original theorem for the transformation of the Hamiltonian function and a corollary for the derivation of the explicit state transformation equations. Proofs of the theorem and associated corollary are provided in the spirit of Deprit's original proof from 1969.[37] In addition, a preliminary investigation is conducted regarding the implementation of feedback control within the DH method transformation space. A simple means of deriving a control law is introduced that can be implemented after the uncontrolled system has already been normalized through the DH method. Chapter III concludes with a demonstration of the two-parameter DH method and feedback control scheme as applied to a single degree-of-freedom damped oscillator system. The uncontrolled system is normalized through the DH method into Birkhoff normal form after which feedback control is introduced with the goal of eliminating the higher-order secular perturbations.

In Chapter IV the theory introduced in Chapters II and III are applied to the ERTBP. The unperturbed system is the linearized CRTBP whose behavior is in the form of harmonic oscillation. The ERTBP is expanded about the unperturbed case

and a linear transformation is applied such that the system is expressed as a perturbation to an integrable system of oscillators represented using action-angle type variables. The two-parameter DH method then provides the basis for normalizing the perturbed system, the ERTBP, about the unperturbed system, the linearized CRTBP. Convergence is demonstrated by comparing the system response in the transformed phase space to a numerically generated truth solution. Upon transforming the ERTBP a series of dynamical analyses are conducted being with a study of the linear and non-linear stability of motion about the planar elliptic triangular Lagrange points. The motion is shown to be nonlinearly stable for a region of the system parameter space excepting locations of resonance. Further analysis of the local phase space around the triangular Lagrange points is provided in terms of local integrals of motion generated in the DH method. The phase portrait is described in terms of a family of level sets of the integrals of motion that foliate the local phase space and effectively define regions of stability around the Lagrange point. Chapter IV concludes with a preliminary design of a stabilizing feedback control law that eliminates non-circular perturbation terms in order to stabilize the system about the circular case.

Chapter V concludes the document with a thorough review of all the relevant results from the study and draws conclusions regarding the methodology in general and its application to the ERTBP. Unresolved problems of interest are discussed for the purpose of laying the foundation for the continued study of the three-body problem using analytical methods and perturbation theory.

# Chapter II

# Theoretical Background

In this chapter, the necessary theoretical background is established to treat the dynamics of the elliptic restricted three-body problem (ERTBP). Topics include the Hamiltonian formulation and canonical transformation of the ERTBP, the dynamics of the circular restricted three-body problem (CRTBP) including the stability of the equilibrium points and Jacobi's integral, and the fundamentals of perturbation and Kolmogorov-Arnold-Moser (KAM) theory.

Before presenting the specific theory used in this study, a few key definitions and notations are presented for clarity. A *dynamical system* is defined as a set of differential equations and initial conditions that together define the evolution of a system in the form

$$\frac{dx}{dt} = f(x(t), t) \tag{2.1}$$

$$x(t_0) = x_0 \tag{2.2}$$

where $t$ is the independent variable of integration (usually time) and $x$ is the $N$-dimensional state vector encompassing all the necessary variables to describe the instantaneous state of the system (usually the position and velocity components). The solution to Eqs. 2.1 and 2.2 is represented by a time-dependent function $x(t)$ called the *trajectory* emanating from the initial condition $x_0$ at time $t = t_0$. If the function

$f$ in Eq. 2.1 is not explicitly dependent on the independent variable, then the system is called *autonomous*. Otherwise, it is called *non-autonomous*. The implications of non-autonomous dynamics are far-reaching and represent a major factor driving the derivations used in this study.

For the dynamical system defined in Eq. 2.1, the collection of all possible values of $x$ is called the *phase space* or *state space*. The *equilibrium points* of the system $x_{\mathrm{e}}$ are the locations in the phase space at which the dynamics identically go to zero, that is, $f(x_{\mathrm{e}}, t) = 0$ for all time. The close neighborhood around an equilibrium point is called the *local phase space*. The equilibrium point is called *stable* in the sense of Lyapunov if trajectories starting in the local phase space remain within some bounded region around the equilibrium point for all time. It is unstable if trajectories diverge away from the equilibrium point over time. Finally, it is called asymptotically stable if trajectories approach the equilibrium point as $t \to \infty$ and asymptotically unstable if trajectories approach the equilibrium point in negative time as $t \to -\infty$. In general, the stability properties of an equilibrium point may be categorized as one of four non-trivial types: a node, a hyperbolic saddle, a spiral, or a center. The first three all exhibit symptoms of asymtotic stability or instability while the center represents the transitional case of neutral stability.[46;48]

For physical systems, the dynamical equations of motion are often posed in the form of Newton's second law where the time-rate of change of the system momentum is equal to the sum of all externally applied forces as represented by

$$\frac{d}{dt}(m\vec{v}) = \vec{F}(\vec{r}, \vec{v}, t) \tag{2.3}$$

where $m$ is the mass of the moving body, $\vec{r}$ and $\vec{v}$ are the inertial position and velocity vectors, and $\vec{F}$ encompasses all the externally applied forces acting on the center mass of the moving body. For constant mass systems, the left-hand side of Newton's law

reduces to the mass times the acceleration vector, $m\dfrac{d\vec{v}}{dt}$. For energy-based methods, Newton's law is re-formulated in terms of the scalar *potential function*, which is defined such that

$$\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} = -\nabla U(\vec{r}, t) \tag{2.4}$$

where $\nabla$ represents the gradient operator acting on the potential function $U$. Note that the scalar potential function depends on the position vector $\vec{r}$ (and possibly $t$), but not on the velocity. The scalar potential function is directly related to the system potential energy. The specific *kinetic energy* is defined by the square of the magnitude of the inertial velocity

$$T = \frac{||\vec{v}||^2}{2} \tag{2.5}$$

where $|| \cdot ||$ represents the Euclidean norm. The kinetic energy and potential energy collectively define the total energy of the system. The system dynamics in Eq. 2.1 can be completely defined in terms of these energies through the generalized methods of Lagrangian and Hamiltonian mechanics.[46;48] These will be introduced and discussed in the sequel for the analysis of the ERTBP.

## 2.1 Elliptic Restricted Three-Body Problem

Before formally defining the ERTBP, first consider it's precedessor: the problem of two bodies. Two bodies with constant mass move freely in three-dimensional space under the sole influence of their mutual gravitation attraction. By defining a frame of reference with its origin affixed to the center of mass of one of the two bodies, the motion of the second body is described relative to the first. In the absence any perturbation, the relative dynamics yield Keplerian motion in which the second body orbits the first within a conic section: any of a circle, ellipse, parabola or hyperbola as shown in Fig. 2.1. Thanks to the pioneering work of Kepler and Newton, the
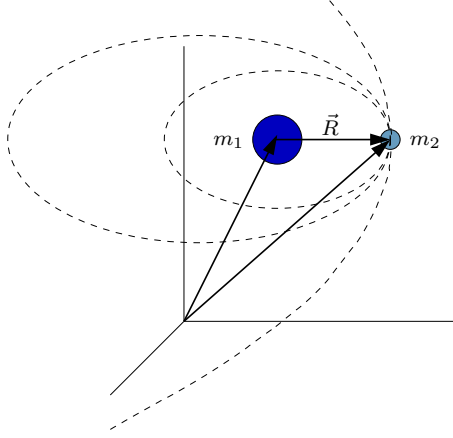
Figure 2.1: Two Body Keplerian System.

two-body problem has a closed-form solution given by the equation of a conic section

$$R(\nu) = \frac{p}{1 + e \cos \nu} \tag{2.6}$$

where $R$, $p$ and $e$ are the instantaneous distance between the two bodies, the orbit semi-latus rectum, and the orbit eccentricity respectively. The term $\nu$ is the classic true anomaly defined by the angle swept relative to the position of closest approach (periapsis) as shown in Fig. 2.1. The time rate of change of the true anomaly with respect to time is

$$\frac{d\nu}{dt} = \frac{\sqrt{G(m_1 + m_2)}}{p}(1 + e \cos \nu)^2 \tag{2.7}$$

where $G$ is Newton's gravitational constant and $m_1$ and $m_2$ are the masses of the two bodies. For the circular case, $e = 0$ and the distance between the primaries and the time rate of change of $\nu$ are constant. Otherwise, for non-circular orbits with $e > 0$, these values vary periodically with time. Rather than incorporating Kepler's transcendental equation, this study will generally apply the true anomaly as the independent variable of integration.[3]

Having described the two-body problem and its solution, the restricted three-body problem is defined formally as the motion of a massless particle within the

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gravitational potential field generated by two massive bodies (called the primaries) who move about their mutual center of mass in Keplerian orbits. Further, the orbit of the primaries is assumed closed such that their trajectories are either circular or elliptical.

### 2.1.1   Physical Description

#### 2.1.1.1   Orbit of the Primaries

Since the primaries move within a two-body, Keplerian orbit, their mutual center of mass will move along a straight line in inertial space and may serve as the origin of an inertial reference frame. This is referred to as the inertial barycentric reference frame as shown in Fig. 2.2. The $X$-axis is defined along the fixed two-body Laplace
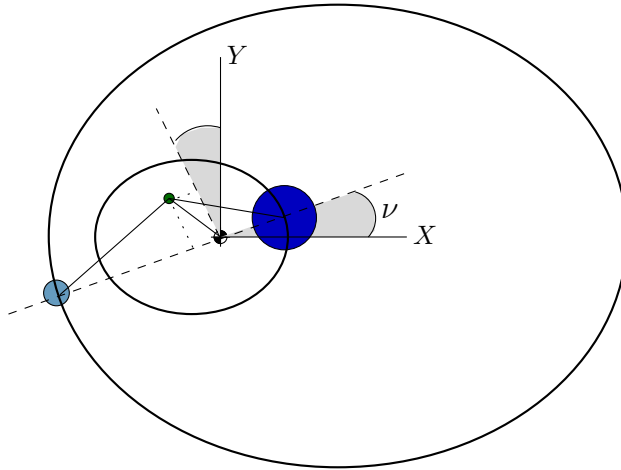


Figure 2.2: Inertial Barycentric Reference Frame

vector and the $Z$-axis is defined in the direction of the primaries' angular velocity vector, that is, normal to the orbital plane. The $Y$-axis completes the orthogonal triad and lies parallel to the semi-minor axis of the orbit ellipse. Within this orbit-fixed reference frame, the positions of the primaries are defined by the vectors $\vec{R}_1$ and $\vec{R}_2$ or in terms of their planar orbital elements, which include the semi-latus rectum $p$, the eccentricity $e$, and time-varying true anomaly $\nu$. Note that the usual out-of-plane

orbital elements are all zero by definition. Further, the eccentricity is limited to the range $0 \le e < 1$ in order to maintain a closed, periodic orbit for the primaries.

The parameter $\mu$ is defined as the ratio of the smaller primary mass $m_2$ to the sum of both primary masses

$$\mu \triangleq \frac{m_2}{m_1 + m_2} \qquad \Longrightarrow \qquad 1 - \mu = \frac{m_1}{m_1 + m_2} \qquad (2.8)$$

Without loss of generality, $m_2$ is assumed to be the less massive of the two primaries such that $0 < \mu \le 0.5$. Defining the position of the smaller primary relative to the larger primary as the vector $\vec{R} \triangleq \vec{R_2} - \vec{R_1}$, Newton's law of gravitation dictates the classic two-body equation of motion

$$\frac{d^2\vec{R}}{dt^2} = -\frac{G(m_1 + m_2)}{R^3}\vec{R} \qquad (2.9)$$

where $G$ is Newton's gravitational constant and $R$ is the magnitude of $\vec{R}$. The solution is in the form of a conic section described previously in Eq. 2.6 wherein $R(\nu) = p/(1 + e\cos\nu)$ varies periodically with $\nu$. Canonical units are applied wherein $G$ and $m_1 + m_2$ have values of unity and the primaries' orbital period is $2\pi$. The masses of the primaries are then $m_1 = 1 - \mu$ and $m_2 = \mu$ and the distances to the center of mass are $R_1 = \mu R$ and $R_2 = (1 - \mu)R$ as shown in Fig. 2.3.
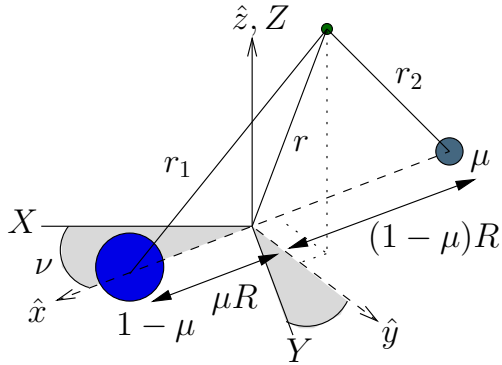


Figure 2.3: Synodic Barycentric Reference Frame

Define the $\hat{x}$-axis along the line connecting the two primaries with positive direction toward the larger primary as shown in Fig. 2.3. This defines a synodic reference frame in which the position of the primaries are always along the $\hat{x}$-axis,

$$\vec{R}_1 = \mu R\,\hat{x} \qquad \text{and} \qquad \vec{R}_2 = -\left(1-\mu\right)R\,\hat{x} \tag{2.10}$$

but their magnitudes fluctuate over a single orbit, as depicted in Fig. 2.4. As such, the



(a) $0 < \nu < \pi/2$        (b) $\pi < \nu < 3\pi/2$
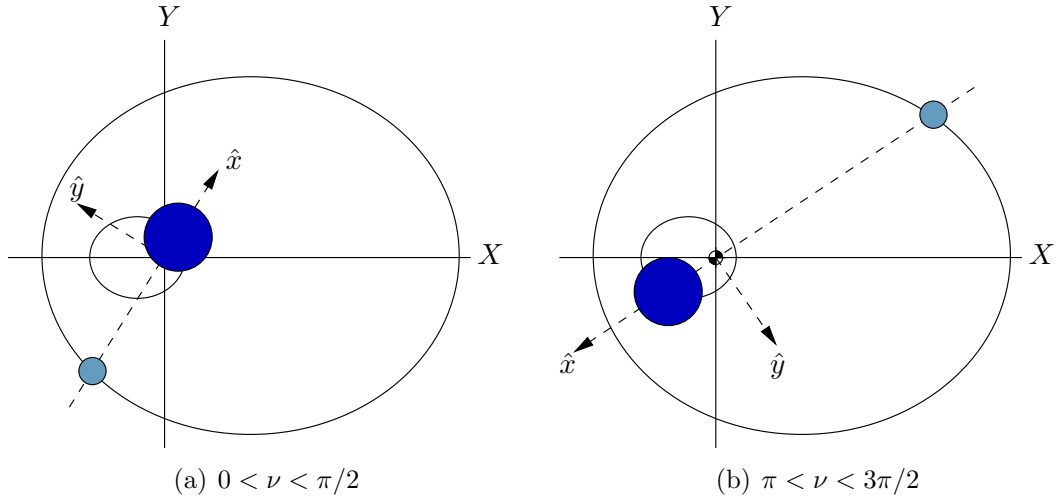
Figure 2.4: Pulsating Synodic Barycentric Reference Frame

inertial reference frame and synodic reference frame are related by a rotation about the $\hat{z} = Z$ axis by the angle $\nu$. The $\hat{x}$-axis revolves at the rate given in Eq 2.7, which in canonical units defines an angular velocity and angular acceleration as shown in Eqs. 2.11 and 2.12

$$\dot{\nu} = \frac{d\nu}{dt} = \frac{\sqrt{p}}{R^2} = \frac{(1+e\cos\nu)^2}{p^{3/2}} \tag{2.11}$$

$$\ddot{\nu} = \frac{d^2\nu}{dt^2} = \frac{-2e\sin\nu}{R^3} = \frac{2e\sin\nu}{p^3}\left(1+e\cos\nu\right)^3 \tag{2.12}$$

where the dot superscript represents the derivative with respect to time.

### 2.1.1.2   Spacecraft Dynamics

The position of the spacecraft relative to the center of mass of the two primaries is represented by the vector $\vec{r}$ as shown in Fig. 2.3. In addition, define the position of the spacecraft relative to the larger primary as $\vec{r}_1$ and relative to the smaller primary as $\vec{r}_2$ such that the vector $\vec{r}$ satisfies

$$\vec{r} = \vec{r}_1 + \vec{R}_1 = \vec{r}_2 + \vec{R}_2 \tag{2.13}$$

The spacecraft position vector is expressed using either Cartesian coordinates as in Eq. 2.14 or spherical coordinates as in Eq. 2.15.

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \tag{2.14}$$

$$= r\left(\cos\theta\sin\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\phi\,\hat{z}\right) \tag{2.15}$$

The use of spherical coordinates is depicted in Fig. 2.5. The terms $x$, $y$, and $z$ are the components of the position along each of the synodic reference axes while $r$ is the magnitude of the position, the angle $\theta$ is measured from the vector $\vec{R}_1$ and the angle $\phi$ is measured from the $\hat{z} = Z$ verical axis. Whenever possible, the system dynamics are presented in both sets of standard coordinate systems, however, on occasion only one of the two may be used for the sake of simplicity or consistency.

Incorporating the rotational effects, the spacecraft inertial velocity vector is expressed in either Cartesian or spherical coordinates in Eqs. 2.16 and 2.17

$$\vec{v} = (\dot{x} - \nu y)\hat{x} + (\dot{y} + \nu x)\hat{y} + \dot{z}\,\hat{z} \tag{2.16}$$

$$= \dot{r}\,\hat{e}_r + r\dot{\phi}\,\hat{e}_\phi + r\left(\nu + \dot{\theta}\right)\sin\phi\,\hat{e}_\theta \tag{2.17}$$

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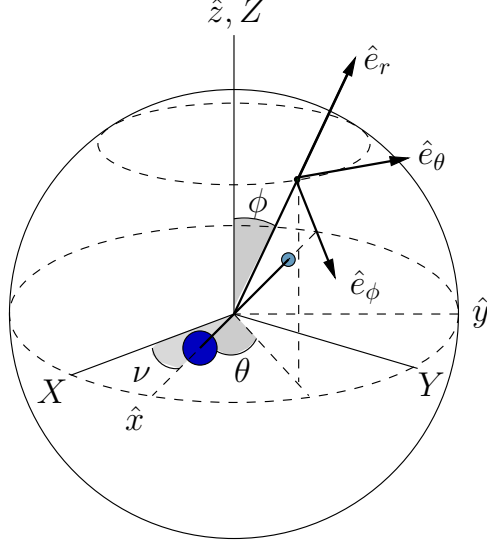
Figure 2.5: Cartesian and Spherical Coordinates

The spacecraft inertial kinetic energy is represented by

$$2T = v^2 = (\dot{x} - \dot{\nu}y)^2 + (\ddot{y} + \dot{\nu}x)^2 + \dot{z}^2$$
$$= \dot{r}^2 + r^2\dot{\phi}^2 + r^2\left(\dot{\nu} + \dot{\theta}\right)^2 \sin^2\phi \tag{2.18}$$

The motion of the spacecraft is dictated by the gravitational potential field of the two primary bodies represented by the potential function

$$U = -\frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{2.19}$$

The distances between the spacecraft and primaries are given by Eq. 2.20 and Eq. 2.21.

$$r_1^2 = (x - \mu R)^2 + y^2 + z^2$$
$$= r^2 + \mu^2 R^2 - 2\mu r R \sin\phi \cos\theta \tag{2.20}$$
$$r_2^2 = (x + (1 - \mu)R)^2 + y^2 + z^2$$
$$= r^2 + (1 - \mu)^2 R^2 + 2(1 - \mu)rR\sin\phi\cos\theta \tag{2.21}$$

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The gravitational potential field generated by the two primaries in Eq. 2.19 is represented in Fig. 2.6 for motion within the 2-D orbital plane. Note that this is just a snapshot of the potential field, which rotates and fluctuates as the two primaries orbit about their mutual center of mass and periodically approach and recede from one another. Note that in reference to spacecraft trajectory design, the primaries are frequently referred to as the planet-moon system and the massless particle as the spacecraft.
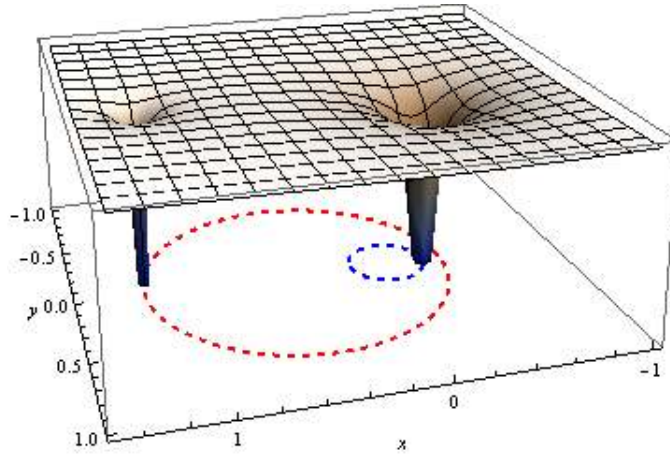


Figure 2.6: Two-Body Potential Field

### 2.1.2 Hamiltonian Dynamics

As an alternative and more generalized approach to solving the ERTBP, one may now introduce the theory of Hamiltonian mechanics where the equations of motion are derived according to Hamilton's Principle and the calculus of variations.[46;48] Hamilton's Principle states that the motion of a system over a given time interval is such that the so-called action functional

$$I = \int_{t_0}^{t} \mathcal{L} \, dt \tag{2.22}$$

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has a stationary value along the path of motion. The integrand function is called the Lagrangian function and defined as the difference between the specific kinetic energy and the potential, $\mathcal{L} = T - U$. According to the calculus of variations, the action functional will have a stationary value when its variation vanishes

$$\delta I = \delta \int_{t_0}^{t} \mathcal{L} \, dt = 0 \tag{2.23}$$

This is the first-order necessary condition for the standard calculus of variations problem with fixed end points, i.e. the minimization of some functional with a set of prescribed boundary conditions. The solution for such a problem is the classic Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{2.24}$$

where $q_i$ represents the system coordinates and $i$ ranges from 1 to $n$, the number of system degrees of freedom. For any conservative system, using any set of reference coordinates, one may construct the Lagrangian function in terms of the kinetic and potential energy and apply the Euler-Lagrange equations to derive the corresponding equations of motion. This represents a generalized and robust approach to analyzing dynamical systems as compared to the traditional application of Newton's laws of motion.

The Hamiltonian formulation is an extension of the Lagrangian formulation expressed in terms of the generalized momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{2.25}$$

instead of the generalized velocities $\dot{q}_i$. In place of the Lagrangian function, $\mathcal{L} =$

$\mathcal{L}(q, \dot{q}, t)$, the so-called Hamiltonian function is defined as

$$\mathcal{H}(q, p, t) = \sum_{i=1}^{n} (p_i \, \dot{q}_i) - \mathcal{L}(q, \dot{q}, t) \tag{2.26}$$

where the generalized velocities $\dot{q}_i$ are implicit functions of $p_i$ through Eq. 2.25. To derive the Hamiltonian equations of motion, consider the total differential of the Lagrangian function

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q_i} \, dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \, d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \, dt \tag{2.27}$$

In light of Eq. 2.24, the time rate of change of the generalized momenta satisfies

$$\dot{p}_i = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \tag{2.28}$$

such that the total differential of the Lagrangian reduces to

$$\begin{aligned} d\mathcal{L} &= \dot{p}_i \, dq_i + p_i \, d\dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \, dt \\ &= \dot{p}_i \, dq_i + d(p_i \, \dot{q}_i) - \dot{q}_i \, dp_i + \frac{\partial \mathcal{L}}{\partial t} \, dt \\ d\left(\mathcal{L} - p_i \dot{q}_i\right) &= \dot{p}_i \, dq_i - \dot{q}_i \, dp_i + \frac{\partial \mathcal{L}}{\partial t} \, dt \end{aligned} \tag{2.29}$$

Substituting the Hamiltonian function into the left-hand side of Eq. 2.29 yields

$$d\mathcal{H} = -\dot{p}_i \, dq_i + \dot{q}_i \, dp_i - \frac{\partial \mathcal{L}}{\partial t} \, dt \tag{2.30}$$

which must be equivalent to the total differential of the Hamiltonian function given by

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q_i} \, dq_i + \frac{\partial \mathcal{H}}{\partial p_i} \, dp_i + \frac{\partial \mathcal{H}}{\partial t} \, dt \tag{2.31}$$

Comparing like-terms yields the Hamiltonian form of the system equations of motion

$$\frac{dq_i}{dt} = \frac{\partial}{\partial p_i}\mathcal{H}(q, p, t)$$
$$\frac{dp_i}{dt} = -\frac{\partial}{\partial q_i}\mathcal{H}(q, p, t)$$
$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{2.32}$$

which includes $2n$ first-order differential equations of motion in place of the $n$ second-order differential equations contained in the Euler-Lagrange equations.

Thus, provided the potential and kinetic energies of a system, the Lagrangian function is computed by the difference between the two and expressed in terms of generalized coordinates and velocities $q_i$ and $\dot{q}_i$. The Hamiltonian function is computed from Eq. 2.26 in terms of the generalized coordinates and momenta $q_i$ and $p_i$ and the system dynamics are represented in canonical form in Eq. 2.32. The advantage of the Hamiltonian formulation is that the equations of motion are represented as first-order differential equations rather than second-order equations. In addition, the time rate of change of the Hamiltonian function is

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \mathcal{H}}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial \mathcal{H}}{\partial t} \tag{2.33}$$

wherein the first two terms in Eq. 2.33 cancel each other out under Eqs. 2.32. Thus, the Hamiltonian itself is conserved if it is not explicitly dependent on the time (i.e. if it is autonomous).

### 2.1.2.1 ERTBP as a Hamiltonian System

Returning to the ERTBP, the Lagrangian function is expressed in terms of spherical coordinates as the difference in the kinetic energy from Eq. 2.18 and the potential

function in Eq. 2.19 such that

$$\mathcal{L} = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\phi}^2 + r^2\left(\dot{\nu} + \dot{\theta}\right)^2\sin^2\phi\right] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \tag{2.34}$$

Note the inclusion of the $\dot{\nu}$ term, which appears due to the definition of $\theta$ as being measured from the rotating $\hat{x}$-axis. To formulate the dynamics using Hamiltonian mechanics, generalized momenta are introduced according to

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \dot{r}$$
$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi}$$
$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = r^2\left(\dot{\nu} + \dot{\theta}\right)\sin^2\phi \tag{2.35}$$

whose inverse equations are

$$\dot{r} = p_r$$
$$\dot{\phi} = \frac{p_\phi}{r^2}$$
$$\dot{\theta} = \frac{p_\theta}{r^2\sin^2\phi} - \dot{\nu} \tag{2.36}$$

The Hamiltonian function is defined in Eq. 2.26 with $q = (r, \phi, \theta)$ and $p = (p_r, p_\phi, p_\theta)$ representing the generalized coordinate and momentum vectors respectively. Upon substituting in Eqs. 2.35 and 2.36, the Hamiltonian function becomes

$$\mathcal{H}(q, p, t) = \dot{r}p_r + \dot{\phi}p_\phi + \dot{\theta}p_\theta - \mathcal{L}$$
$$= \frac{1}{2}\left[p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2\sin^2\phi}\right] - \dot{\nu}p_\theta - \frac{1-\mu}{r_1} - \frac{\mu}{r_2} \tag{2.37}$$

where again the distances between the spacecraft and primaries are given by Eq. 2.20 and Eq. 2.21.

Since the terms $\dot{\nu}$ and $R$ depend explicitly on the true anomaly $\nu$, which is itself a function of time, the Hamiltonian function will vary with time such that the system is non-autonomous. Nonetheless, the representation of the system dynamics may be considerably simplified through a series of canonical transformations referred to collectively as the Nechvile transformation. The details of the transformation are discussed in the sequel and in Szebehely's *Theory of Orbits*, but first dictate a brief introduction to the theory of canonical transformations in general.[11]

### 2.1.2.2 Canonical Transformations

One of the advantages to using a Hamiltonian formulation is that transformations of the phase space may be derived directly in terms of the Hamiltonian function through so-called generating functions (or sometimes, generators). Consider the general state transformation

$$q_i = Q_i\left(\hat{q}, \hat{p}, t\right)$$
$$p_i = P_i\left(\hat{q}, \hat{p}, t\right) \tag{2.38}$$

where the set $(q, p)$ corresponds to the original set of generalized coordinates and momenta and the set $(\hat{q}, \hat{p})$ represents an alternative set of generalized coordinates and momenta. The transformation is called canonical if the form of the Hamiltonian equations of motion remain intact in the transformed phase space, that is, if they take the form

$$\frac{d\hat{q}_i}{dt} = \frac{\partial}{\partial \hat{p}_i}\mathcal{K}(\hat{q}, \hat{p}, t)$$
$$\frac{d\hat{p}_i}{dt} = -\frac{\partial}{\partial \hat{q}_i}\mathcal{K}(\hat{q}, \hat{p}, t) \tag{2.39}$$

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where $\mathcal{K}(\hat{q}, \hat{p}, t)$ is the transformed Hamiltonian function. Thus, under a canonical transformation the fundamental form of the Hamiltonian dynamics is not changed, only the phase space representation and corresponding Hamiltonian function. To derive a necessary condition between the two canonical representations, recall that the Euler-Lagrange equations were derived based on Hamilton's principle and the calculus of variations whereby the variation of the functional vanishes under the condition shown in Eq. 2.23. Substituting the Hamiltonian function in place of the Lagrangian function yields the equivalent condition

$$
\delta \int_{t_0}^{t} \left[ \dot{q}_i p_i - \mathcal{H}(q, p, t) \right] dt = 0
\tag{2.40}
$$

Likewise, if the transformation from $(q, p)$ to $(\hat{q}, \hat{p})$ is canonical with a transformed Hamiltonian given by $\mathcal{K}(\hat{q}, \hat{p}, t)$, then Hamilton's principle must also hold in the transformed phase space such that

$$
\delta \int_{t_0}^{t} \left[ \dot{\hat{q}}_i \hat{p}_i - \mathcal{K}(\hat{q}, \hat{p}, t) \right] dt = 0
\tag{2.41}
$$

In order for both of these canonical formulations to represent the same physical system, the integrands in Eqs. 2.40 and 2.41 can at most differ by the total time derivative of a scalar function. Thus, the two formulations must satisfy the necessary condition

$$
\sum_{j=1}^{n} \dot{q}_i p_i - \mathcal{H}(q, p, t) = \sum_{j=1}^{n} \dot{\hat{q}}_i \hat{p}_i - \mathcal{K}(\hat{q}, \hat{p}, t) + \frac{d\mathcal{W}}{dt}
\tag{2.42}
$$

where $\mathcal{W}$ is called the generating function and may depend on time and any of the old or new variables, but must include some of both to be non-trivial. The various formulations of $\mathcal{W}$ include $\mathcal{W}(q, \hat{q}, t)$, $\mathcal{W}(q, \hat{p}, t)$, $\mathcal{W}(p, \hat{q}, t)$, and $\mathcal{W}(p, \hat{p}, t)$. Depending on which of the four possible generating functions is applied, the transformation

equations will take on different forms. For example, applying the generating function, $\mathcal{W}(q, \hat{p}, t)$, results in the transformation equations

$$
\begin{aligned}
p_i &= \frac{\partial}{\partial q_i} \mathcal{W}(q, \hat{p}, t) \\
\hat{q}_i &= \frac{\partial}{\partial \hat{p}_i} \mathcal{W}(q, \hat{p}, t)
\end{aligned}
\tag{2.43}
$$

The necessary condition for this to represent a canonical transformation is

$$
\mathcal{K}(\hat{q}, \hat{p}, t) = \mathcal{H}\Big(Q(\hat{q}, \hat{p}, t), P(\hat{q}, \hat{p}, t), t\Big) + \frac{\partial \mathcal{W}}{\partial t}
\tag{2.44}
$$

wherein the first part on the right-hand side is the original Hamiltonian function written explicitly in terms of the state transformation equations in Eqs. 2.38 and the second part is the so-called remainder function whose form and dependencies change depending on the form for $\mathcal{W}$. Note that in order to derive the explicit state transformation equations (as well as the inverse equations), one must invert Eqs. 2.43, which is often difficult for complicated systems. This foreshadows the motivation behind the Deprit-Hori Lie transform method discussed in the sequel.

### 2.1.2.3  Nechvile Transformation

Having introduced the notion of canonical transformation, one may apply the so-called Nechvile transformation to the ERTBP. Consider again the Hamiltonian function from Eq. 2.37

$$
\mathcal{H}(q, p, t) = \frac{1}{2} \left[ p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right] - \dot{\nu} p_\theta - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
\tag{2.37}
$$

Pulsating coordinates are introduced through a normalization of the spacecraft position $r$ by the distance between the primaries $R$, that is, $r \to r/R$. The transformed is described in terms of a rotating and pulsating reference frame in Szebehely and

defined in the form of a generating function in Schmidt.[11;32] In both references, the transformation is expressed starting from a set of Cartesian coordinates along either the inertial reference axes $(X, Y, Z)$ or synodic reference axes $(\hat{x}, \hat{y}, \hat{z})$. However, the generating function in Schmidt can be re-formulated into spherical coordinates with the result being

$$\mathcal{W}(q, \hat{p}, t) = \frac{r}{R}\hat{p}_r + \phi\hat{p}_\phi + \theta\hat{p}_\theta + \frac{\dot{R}}{2R}r^2 \tag{2.45}$$

where $\dot{R} = (\partial R/\partial\nu)\dot{\nu} = R^2 e \sin\nu/p$. The corresponding state transformation equations are in the form $\hat{q} = \hat{Q}(q, p, t)$ and $p = P(\hat{q}, \hat{p}, t)$ as shown in Eqs. 2.46

$$\begin{aligned}
\hat{r} &= \frac{\partial\mathcal{W}}{\partial\hat{p}_r} = \frac{r}{R} & p_r &= \frac{\partial\mathcal{W}}{\partial r} = \frac{\hat{p}_r}{R} + \dot{R}\hat{r} \\
\hat{\phi} &= \frac{\partial\mathcal{W}}{\partial\hat{p}_\phi} = \phi & p_\phi &= \frac{\partial\mathcal{W}}{\partial\phi} = \hat{p}_\phi \\
\hat{\theta} &= \frac{\partial\mathcal{W}}{\partial\hat{p}_\theta} = \theta & p_\theta &= \frac{\partial\mathcal{W}}{\partial\theta} = \hat{p}_\theta
\end{aligned} \tag{2.46}$$

with the inverse equations in the form $q = Q(\hat{q}, \hat{p}, t)$ and $\hat{p} = \hat{P}(q, p, t)$ as shown in Eqs. 2.47.

$$\begin{aligned}
r &= R\hat{r} & \hat{p}_r &= Rp_r - \dot{R}r \\
\phi &= \hat{\phi} & \hat{p}_\phi &= p_\phi \\
\theta &= \hat{\theta} & \hat{p}_\theta &= p_\theta
\end{aligned} \tag{2.47}$$

Substituting the generating function and state transformation equations in Eqs. 2.45-2.47 into the canonical transformation condition in Eq. 2.44 yields the transformed Hamiltonian function

$$\begin{aligned}
\mathcal{K}(\hat{q}, \hat{p}, t) &= \mathcal{H}(Q(\hat{q}, \hat{p}, t), P(\hat{q}, \hat{p}, t), t) + \frac{\partial}{\partial t}\mathcal{W}(\hat{p}, Q(\hat{q}, \hat{p}, t), t) \\
&= \frac{1}{2R^2}\left[\hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{r}^2} + \frac{\hat{p}_\theta^2}{\hat{r}^2\sin^2\hat{\phi}}\right] - \frac{\sqrt{p}}{R^2}\hat{p}_\theta - \frac{1}{R}\left(\frac{1-\mu}{\hat{r}_1} + \frac{\mu}{\hat{r}_2}\right) + \frac{e\cos\nu}{2R}\hat{r}^2 \tag{2.48}
\end{aligned}$$

where $\hat{r}_1 \triangleq r_1/R$ and $\hat{r}_2 \triangleq r_2/R$ are expressed in terms of the spherical coordinates by the relations

$$\hat{r}_1^2 = \hat{r}^2 + \mu^2 - 2\mu\hat{r}\sin\hat{\phi}\cos\hat{\theta} \tag{2.49}$$

$$\hat{r}_2^2 = \hat{r}^2 + (1-\mu)^2 + 2(1-\mu)\hat{r}\sin\hat{\phi}\cos\hat{\theta} \tag{2.50}$$

The distances $\hat{r}_1$ and $\hat{r}_2$ are independent of $R$ and therefore time whereby the positions of the primaries are fixed relative to the pulsating reference frame. However, the Hamiltonian function is still explicitly dependent on time through the term $R$ such that the system is still non-autonomous.

The second step in the Nechvile transformation is a change of independent variable from time to the true anomaly of the primaries. This serves to isolate the non-autonomous terms in the Hamiltonian function and to circumvent the need to solve Kepler's equation in order to relate the instantaneous true anomaly to time. Szebehely applies such a transformation directly to the equations of motion while Broucke discusses the general process of transforming the independent variable of a Hamiltonian system and includes an application to the ERTBP.[11;21] As Broucke demonstrates, the change of independent variables is facilitated by a transformation in the form

$$\mathcal{G}(\hat{q}, \hat{p}, \nu, t) = \frac{1}{\dot{\nu}}\, \mathcal{K}(\hat{q}, \hat{p}, t) \tag{2.51}$$

where in the case of the ERTBP, the multiplier is given by

$$\dot{\nu} = \frac{\sqrt{p}}{R^2} \tag{2.11}$$

In general, such a transformation would lose the canonical form of the Hamiltonian equations of motion. However, since in this case Eq. 2.11 is only dependent on $\nu$ and not on $q$ or $t$, the canonical form is preserved. Substituting Eq. 2.11 into Eq. 2.51

yields the transformed Hamiltonian function

$$
\mathcal{G}(\hat{q}, \hat{p}, \nu) = \frac{R^2}{\sqrt{p}} \mathcal{K}(\hat{q}, \hat{p}, t)
$$

$$
= \frac{1}{2\sqrt{p}} \left[ \hat{p}_r^2 + \frac{\hat{p}_\phi^2}{\hat{r}^2} + \frac{\hat{p}_\theta^2}{\hat{r}^2 \sin^2 \hat{\phi}} \right] - \hat{p}_\theta - \frac{R}{\sqrt{p}} \left( \frac{1-\mu}{\hat{r}_1} + \frac{\mu}{\hat{r}_2} \right) + \frac{Re \cos \nu}{2\sqrt{p}} \hat{r}^2 \qquad (2.52)
$$

with the canonical equations of motion

$$
\frac{d\hat{q}}{d\nu} = \frac{\partial}{\partial \hat{p}} \mathcal{G}(\hat{q}, \hat{p}, \nu)
$$

$$
\frac{d\hat{p}}{d\nu} = -\frac{\partial}{\partial \hat{q}} \mathcal{G}(\hat{q}, \hat{p}, \nu)
$$

$$
\frac{d\mathcal{G}}{d\nu} = \frac{\partial \mathcal{G}}{\partial \nu} \qquad (2.53)
$$

To further simplify the formulation, a scaling transformation is applied in Eq. 2.54 where the generalized momenta is normalized by the primary orbit's angular momentum $\sqrt{p}$.

$$
\left\{ \begin{array}{c} p_r \\ p_\phi \\ p_\theta \end{array} \right\} = \frac{1}{\sqrt{p}} \left\{ \begin{array}{c} \hat{p}_r \\ \hat{p}_\phi \\ \hat{p}_\theta \end{array} \right\} \qquad (2.54)
$$

As discussed in Goldstein, et al, such a transformation is canonical with multiplier $1/\sqrt{p}$ such that the resultant Hamiltonian function is

$$
\mathcal{H}(q, p, \nu) = \frac{1}{\sqrt{p}} \mathcal{G}(\hat{q}, \hat{p}, \nu)
$$

$$
= \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right) - p_\theta - \frac{R}{p} \left( \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) + \frac{Re \cos \nu}{2p} r^2
$$

$$
= \frac{1}{2} \left( r^2 + p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right) - p_\theta - \frac{R}{p} \left( \frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right) \qquad (2.55)
$$

where the distances to the primaries are

$$r_1^2 = r^2 + \mu^2 - 2\mu r \sin\phi \cos\theta \tag{2.56}$$

$$r_2^2 = r^2 + (1-\mu)^2 + 2(1-\mu)r\sin\phi\cos\theta \tag{2.57}$$

and the notation $(q,p)$ and $\mathcal{H}(q,p,\nu)$ is recycled to represent the new scaled, pulsating system with $\nu$ as the independent variable.[46]

This completes the so-called Nechvile transformation under which the system dynamics have been transformed from the original synodic barycentric reference frame to a scaled and pulsating synodic barycentric reference frame with $\nu$ serving as the independent variable of integration. The form of the resultant Hamiltonian function is consistent with those derived in Schmidt and in Deprit and Rom, which both use Cartesian coordinates rather than spherical coordinates.[28;32] While the Hamiltonian function is still non-autonomous, the $\nu$-dependency has been isolated within the term $R/p = 1/(1+e\cos\nu)$ pre-multiplying the amended potential function $\dfrac{r^2}{2} + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2}$.

While the transformation was presented in terms of spherical coordinates, other coordinates systems are certainly valid and in fact, most texts favor the use of Cartesian coordinates. Unfortunately, the nonlinear transformation between spherical and Cartesian coordinates is not canonical, so switch between the two, one would need to repeat the entire Nechvile transformation using the Cartesian coordinates $(q_x, q_y, q_z)$ and momenta $(p_x, p_y, p_z)$ instead resulting in the equivalent Hamiltonian function

$$\begin{aligned}
\mathcal{H}(q,p,\nu) =& \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + q_y p_x - q_x p_y + \frac{q_x^2 + q_y^2 + q_z^2}{2} \\
& - \frac{R}{p}\left(\frac{q_x^2 + q_y^2 + q_z^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right)
\end{aligned} \tag{2.58}$$

The distances from the primaries are expressed in terms of Cartesian coordinates by

the relations

$$r_1^2 = (q_x - \mu)^2 + q_y^2 + q_z^2 \tag{2.59}$$

$$r_2^2 = (q_x + 1 - \mu)^2 + q_y^2 + q_z^2 \tag{2.60}$$

The essential features of the Hamiltonian function are the same in either representation including the isolation of the non-autonomous effects within the term $R/p$ and the representation of the positions of the primaries as being fixed relative to the scaled, pulsating and rotating reference frame. One advantage to expressing the dynamics in Cartesian coordinates is that the symmetry across the $\hat{x}$-axis is readily apparent wherein the dynamics are invariant under the mapping $(q_x, q_y, q_z, p_x, p_y, p_z, \nu) \rightarrow (q_x, -q_y, q_z, -p_x, p_y, p_z, -\nu)$. As such, the time-forward dynamics are equivalent to the time-reverse dynamics with $q_y$ and $p_x$ negated. In any event, the underlying system dynamics are the same regardless of the choice of coordinate system. Both representations in spherical and Cartesian coordinates are applied at various stages of this study for the sake of simplicity, ease of visualization, and to allow comparisons to formulations from other studies.

In the circular case where $e \rightarrow 0$, the distance between the primaries is fixed even in the non-pulsating reference frame. Further, the time rate of change of $\nu$ is effectively unity such that time and true anomaly are equivalent independent variables. Substituting $R/p = 1$ into the elliptic Hamiltonian function yields the circular Hamiltonian function in spherical coordinates as

$$\mathcal{H}(q, p) = \frac{1}{2} \left( p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} \right) - p_\theta - \left( \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right) \tag{2.61}$$

where

$$r_1^2 = r^2 + \mu^2 - 2\mu r \sin\phi \cos\theta \tag{2.62}$$

$$r_2^2 = r^2 + (1-\mu)^2 + 2(1-\mu) r \sin\phi \cos\theta \tag{2.63}$$

Since the Hamiltonian function is no longer explicitly dependent on the independent variable, it is autonomous and conserved in the Hamiltonian equations of motion. Historically, this conservation property is presented in terms of the classic Jacobi integral.

## 2.2   Lagrange Equilibrium Points

The canonical state space equations of motion generated from the Hamiltonian function in Eq. 2.55 are

$$
\begin{aligned}
\frac{dr}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_r} = p_r \\
\frac{d\phi}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{r^2} \\
\frac{d\theta}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{r^2 \sin^2\phi} - 1 \\
\frac{dp_r}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\phi^2}{r^3} + \frac{p_\theta^2}{r^3 \sin^2\phi} - \frac{Re\cos\nu}{p} r \\
&\quad - \frac{R}{p}\left[ \frac{1-\mu}{r_1^3}(r - \mu\sin\phi\cos\theta) + \frac{\mu}{r_2^3}(r + (1-\mu)\sin\phi\cos\theta) \right] \\
\frac{dp_\phi}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial \phi} = \frac{p_\theta^2 \cos\phi}{r^2 \sin^3\phi} + \frac{R}{p}\mu(1-\mu) r \cos\phi\cos\theta \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right) \\
\frac{dp_\theta}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{R}{p}\mu(1-\mu) r \sin\phi\sin\theta \left( \frac{1}{r_1^3} - \frac{1}{r_2^3} \right)
\end{aligned} \tag{2.64}
$$

which can be validated against those included in Szebehely and in Broucke.[11;21] The system possesses five equilibrium points defined by setting the terms on the left-hand

side of Eq. 2.64 to zero yielding

$$\phi_e = \pi/2 \qquad p_{r,e} = 0 \qquad p_{\phi,e} = 0 \qquad p_{\theta,e} = r_e^2 \qquad (2.65)$$

and

$$0 = r_e - r_e \left( \frac{1-\mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} \right) + \mu(1-\mu) \cos\theta_e \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right)$$
$$0 = \mu \left( 1 - \mu \right) r_e \sin\theta_e \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right) \qquad (2.66)$$

wherein a factor of $R/p$ was divided out as a common multiplier. Since the equilibrium equations are independent of the true anomaly, they are fixed relative to the synodic, pulsating reference frame. Since they are further independent of the eccentricity, the form of the equations match those presented in Szebehely for the CRTBP. As such, the elliptic equilibrium points are co-located with the five classic Lagrange points depicted in Fig. 2.7. While the locations of the elliptic Lagrange points correspond to those of the classic circular Lagrange points, the distinction must be made that the former are fixed only in the pulsating synodic frame and with $\nu$ as the independent variable while the latter are fixed even in the original, non-pulsating synodic reference frame with time as the independent variable.

Three of the equilibrium points lie along the $\hat{x}$-axis and are referred to as the collinear Lagrange points, $L_1$, $L_2$ and $L_3$. Their locations are defined by $\theta_e = 0$ (or $\pi$) and

$$0 = r_e - r_e \left( \frac{1-\mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} \right) \pm \mu(1-\mu) \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right) \qquad (2.67)$$

where $r_{1,e} = |r_e \mp \mu|$ and $r_{2,e} = |r_e \pm (1-\mu)|$. The sign of the $\mp$ and $\pm$ symbols are determined based on whether the point in question is on the positive or negative side of the $\hat{x}$-axis and whether or not it is between the two primaries. For each of the
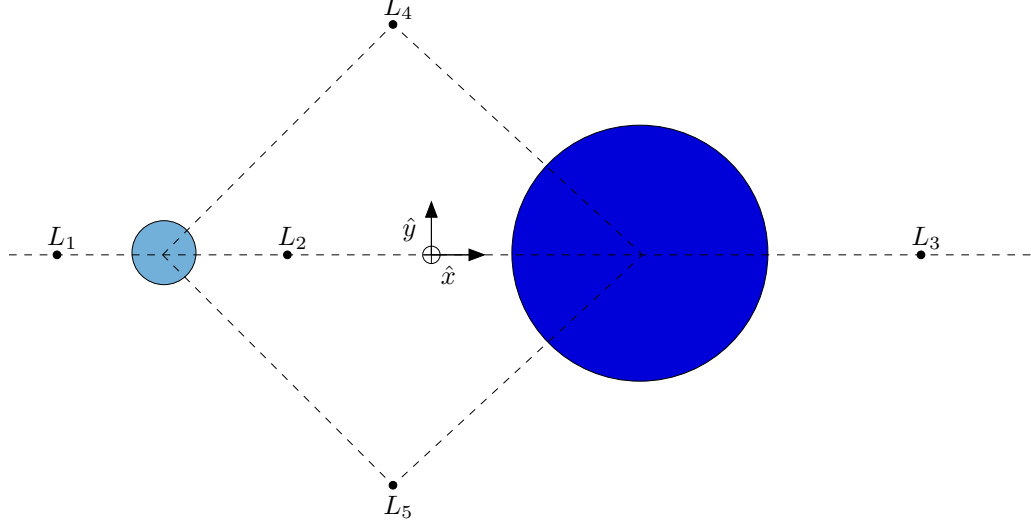
Figure 2.7: Five Lagrange Points

three possible locations dictated by the $\mp$ and $\pm$ symbols, there is one real solution such that in total there are three collinear equilibrium points corresponding to the classic $L_1$, $L_2$ and $L_3$ Lagrange points depicted in Fig. 2.7.

The remaining two Lagrange points $L_4$ and $L_5$ are located at the vertices of an equilateral triangle with sides of length $r_1 = r_2 = 1$ as depicted in Fig. 2.7. Substituting these constraints into Eqs. 2.66 defines the locations of the triangular equilibrium explicitly in terms of $\mu$ as

$$
\begin{aligned}
r_e \cos \theta_e &= \mu - \frac{1}{2} & p_{r,e} &= 0 \\
\phi_e &= \frac{\pi}{2} & p_{\phi,e} &= 0 \\
r_e \sin \theta_e &= \pm\sqrt{3/4} & p_{\theta,e} &= \left(\mu - \frac{1}{2}\right)^2 + \frac{3}{4}
\end{aligned}
\tag{2.68}
$$

where the $\pm$ changes sign for motion about either $L_4$ or $L_5$. The corresponding normalized Cartesian coordinates are $x_e = \mu - \frac{1}{2}$, $y_e = \pm\sqrt{\frac{3}{4}}$ and $z_e = 0$, which again emphasize the symmetry about the $\hat{x}$-axis. The stability of motion about the elliptic Lagrange points, particularly for the triangular points, is a major point of focus for this study and will be treated in the sequel using canonical perturbation theory and

KAM theory.

## Linearization about the Circular Lagrange Points

As the eccentricity goes to zero, the pulsation effect disappears such that the five equilibrium points correspond exactly to the five classic Lagrange points whose locations are fixed in the synodic reference frame and defined by

$$0 = r_e - r_e \left( \frac{1-\mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} \right) \pm \mu(1-\mu) \left( \frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} \right) \tag{2.67}$$

for $L_1$-$L_3$, the collinear points, and by

$$
\begin{aligned}
r_e \cos \theta_e &= \mu - \frac{1}{2} & p_{r,e} &= 0 \\
\phi_e &= \frac{\pi}{2} & p_{\phi,e} &= 0 \\
r_e \sin \theta_e &= \pm \sqrt{3/4} & p_{\theta,e} &= \left( \mu - \frac{1}{2} \right)^2 + \frac{3}{4}
\end{aligned}
\tag{2.68}
$$

for $L_4$ and $L_5$, the triangular points. As a first-cut stability analysis, the circular Hamiltonian function is linearized about any of the five Lagrange points. The corresponding linearized state space equations of motion are

$$
\frac{d}{d\nu}
\begin{pmatrix}
\delta r \\
\delta \phi \\
\delta \theta \\
\delta p_r \\
\delta p_\phi \\
\delta p_\theta
\end{pmatrix}
= J
\begin{bmatrix}
a(\mu) & 0 & d(\mu) & 0 & 0 & -2/r_e \\
0 & b(\mu) & 0 & 0 & 0 & 0 \\
d(\mu) & 0 & c(\mu) & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/r_e^2 & 0 \\
-2/r_e & 0 & 0 & 0 & 0 & 1/r_e^2
\end{bmatrix}
\begin{pmatrix}
\delta r \\
\delta \phi \\
\delta \theta \\
\delta p_r \\
\delta p_\phi \\
\delta p_\theta
\end{pmatrix}
\tag{2.69}
$$

wherein the matrix

$$
J =
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{bmatrix}
\tag{2.70}
$$

is the so-called symplectic matrix. The second matrix in Eq. 2.69 is the Hessian of the Hamiltonian function evaluated at the equlibrium point with coefficients defined by

$$
\begin{aligned}
a(\mu) &= 3 + \frac{1-\mu}{r_{1,e}^3} + \frac{\mu}{r_{2,e}^3} - \frac{3(1-\mu)(r_e - \mu\cos\theta_e)^2}{r_{1,e}^5} - \frac{3\mu(r_e + (1-\mu)\cos\theta_e)^2}{r_{2,e}^5} \\
b(\mu) &= r_e^2 + \mu(1-\mu)r_e\cos\theta_e\frac{R}{p}\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3}\right) \\
c(\mu) &= \mu(1-\mu)\left(-r_e\cos\theta_e\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3}\right) + 3r_e^2\sin^2\theta_e\left(\frac{\mu}{r_{1,e}^5} + \frac{1-\mu}{r_{2,e}^5}\right)\right) \\
d(\mu) &= \mu(1-\mu)\sin\theta_e\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} - 3r_e\left(\frac{r_e - \mu\cos\theta_e}{r_{1,e}^5} - \frac{r_e + (1-\mu)\cos\theta_e}{r_{2,e}^5}\right)\right)
\end{aligned}
$$

$$(2.71)$$

For motion about the collinear Lagrange points where $\theta_e = 0$, the Hessian coefficients are

$$
\begin{aligned}
a(\mu) &= 1 - \frac{1-\mu}{r_{1,e}^3} - \frac{\mu}{r_{2,e}^3} + \frac{3(1-\mu)(r_e - \mu\cos\theta_e)^2}{r_{1,e}^5} + \frac{3\mu(r_e + (1-\mu)\cos\theta_e)^2}{r_{2,e}^5} \\
b(\mu) &= -\mu(1-\mu)r_e\cos\theta_e\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3}\right) \\
c(\mu) &= \mu(1-\mu)\left(r_e\cos\theta_e\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3}\right) - 3r_e^2\sin^2\theta_e\left(\frac{\mu}{r_{1,e}^5} + \frac{1-\mu}{r_{2,e}^5}\right)\right) \\
d(\mu) &= -\mu(1-\mu)\sin\theta_e\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3} - 3r_e\left(\frac{r_e - \mu\cos\theta_e}{r_{1,e}^5} - \frac{r_e + (1-\mu)\cos\theta_e}{r_{2,e}^5}\right)\right)
\end{aligned}
$$

$$(2.72)$$

where the radial distance, $r_e$ is implicitly defined by the constraint

$$
0 = r_e\left(1 - \frac{1-\mu}{r_{1,e}^3} - \frac{\mu}{r_{2,e}^3}\right) + \mu(1-\mu)\left(\frac{1}{r_{1,e}^3} - \frac{1}{r_{2,e}^3}\right)
\tag{2.73}
$$

Across the range $0 < \mu < 1$, the characteristic eigenvalues of the collinear points take the form of $\lambda = \pm ia, \pm c, \pm i$ wherein one of the planar eigenvalues has a positive real part. As such, the collinear Lagrange points are unstable. While it is of interest

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to further study the stability properties of the collinear points, the remainder of the present study focuses instead on motion about the triangular Lagrange points.

For motion about the triangular Lagrange points, $r_{1,e} = r_{2,e} = 1$, $r_e \cos \theta_e = \mu - 1/2$ and $r_e \sin \theta_e = \pm\sqrt{3/4}$ such that the Hessian coefficients simplify to

$$a(\mu) = 3r_e^2 + 3\mu(1 - \mu) \cos^2 \theta_e$$

$$b(\mu) = 0$$

$$c(\mu) = 3\mu(1 - \mu)r_e^2 \sin^2 \theta_e$$

$$d(\mu) = -3\mu(1 - \mu)r_e \sin \theta_e \cos \theta_e \qquad (2.74)$$

and the linearized Hamiltonian function simplifies to

$$
\mathcal{H} = \frac{1}{2}\left[\left(\frac{1}{r_e}\delta p_\theta - 2\delta r\right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2}\delta p_\phi^2\right]
$$
$$
- \frac{3}{2}\left[r_e^2 \delta r^2 + \mu(1 - \mu)\left(\cos \theta_e \delta r - r_e \sin \theta_e \delta \theta\right)^2\right] \qquad (2.75)
$$

Note that $b(\mu)$ is zero, which also extends to the ERTBP such that the non-autonomous terms divide out of the out-of-plane dynamics in the linearized phase space. Further, the in-plane and out-of-plane linearized dynamics are de-coupled and the latter is characterized by a pair of imaginary eigenvalues $\lambda = \pm i$. Thus, in the linearized space, the out-of-plane dynamics are in the form of a center with harmonic oscillation about the Lagrange point. For the planar dynamics, the CRTBP and ERTBP are distinct and must be treated separately. In the former, the linearized system characteristic equation is

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1 - \mu) = 0 \qquad (2.76)$$

37

which possesses four complex roots and a bifurcation at Routh's critical mass ratio

$$\mu_c = \left(1 - \frac{\sqrt{69}}{9}\right)/2 \approx 0.0385$$

as shown in Fig. 2.8. The bifurcation proceeds as follows: for mass ratios less than Routh's critical value, $\mu < \mu_c$, the linearized circular system has distinct eigenvalues lying on the imaginary axis $\lambda = \pm i\omega_s$, $\pm i\omega_\ell$ where $0 < \omega_\ell < \sqrt{2}/2 < \omega_s < 1$. As the mass ratio approaches $\mu_c$, these imaginary eigenvalues converge into a pair of identical eigenvalues $\lambda^{(2)} = \pm i\frac{\sqrt{2}}{2}$ at the critical point where $\mu = \mu_c$. Finally, for mass ratios greater than Routh's critical value, $\mu > \mu_c$, the eigenvalues become complex and are symmetric about the imaginary axis such that some of the eigenvalues include a positive real part $\lambda = \pm a \pm ib$.
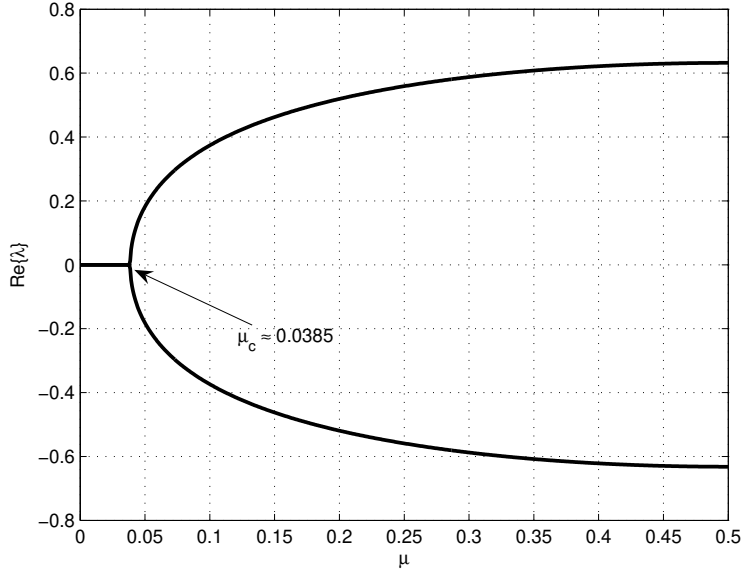


Figure 2.8: Bifurcation of the Linearized Triangular Lagrange Points

Since the circular triangular Lagrange points are linearly stable in the sense of Lyapunov, they serve as the starting point for the normalization and analysis of the non-circular and nonlinear system associated with the ERTBP. In subsequent

chapters, the nonlinear stability of the elliptic triangular Lagrange points is treated by applying canonical perturbation theory and KAM theory.

## 2.3   Jacobi Integral

A useful property in Hamiltonian mechanics is that the rate of change of the Hamiltonian function is given directly by its partial derivative

$$\frac{d\mathcal{H}}{d\nu} = \frac{\partial \mathcal{H}}{\partial \nu} \tag{2.77}$$

so that in the case of the autonomous CRTBP, where there is no explicit $\nu$-dependence, the Hamiltonian function is itself an integral of motion. In fact, it is directly related to the celebrated Jacobi's integral, which is used extensively in the dynamical analysis of the CRTBP and was first introduced by Jacobi in 1836.[8] To map the correspondence between the two, consider the classic derivation of Jacobi's integral in terms of spherical coordinates. Representing the potential function by $U(r, \phi, \theta)$ the Hamiltonian function of the ERTBP derived in Eq. 2.55 assumes the compact form

$$\mathcal{H}(q, p, \nu) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi}\right) - p_\theta + \frac{R}{p}\left(\frac{e \cos \nu}{2} r^2 + U(r, \phi, \theta)\right) \tag{2.78}$$

whose state space equations of motion are

$$
\begin{aligned}
\frac{dr}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_r} = p_r \\
\frac{d\phi}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{r^2} \\
\frac{d\theta}{d\nu} &= \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{r^2 \sin^2 \phi} - 1 \\
\frac{dp_r}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\phi^2}{r^3} + \frac{p_\theta^2}{r^3 \sin^2 \phi} - \frac{R}{p}\left(re\cos\nu + \frac{\partial U}{\partial r}\right) \\
\frac{dp_\phi}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial \phi} = \frac{p_\theta^2 \cos\phi}{r^2 \sin^3 \phi} - \frac{R}{p}\frac{\partial U}{\partial \phi} \\
\frac{dp_\theta}{d\nu} &= -\frac{\partial \mathcal{H}}{\partial \theta} = -\frac{R}{p}\frac{\partial U}{\partial \theta}
\end{aligned}
\tag{2.79}
$$

The corresponding second-order differential equations of motion are

$$
\begin{aligned}
\frac{d^2 r}{d\nu^2} - r\left(\frac{d\phi}{d\nu}\right)^2 - r\sin^2\phi\left(1 + \frac{d\theta}{d\nu}\right)^2 &= -\frac{R}{p}\left(re\cos\nu + \frac{\partial U}{\partial r}\right) \\
\frac{d^2\phi}{d\nu^2} + \frac{2}{r}\frac{dr}{d\nu}\frac{d\phi}{d\nu} - \cos\phi\sin\phi\left(1 + \frac{d\theta}{d\nu}\right)^2 &= -\frac{1}{r^2}\frac{R}{p}\frac{\partial U}{\partial \phi} \\
\frac{d^2\theta}{d\nu^2} + \frac{2}{r}\frac{dr}{d\nu}\left(1 + \frac{d\theta}{d\nu}\right) + \frac{2\cos\phi}{\sin\phi}\frac{d\phi}{d\nu}\left(1 + \frac{d\theta}{d\nu}\right) &= -\frac{1}{r^2 \sin^2 \phi}\frac{R}{p}\frac{\partial U}{\partial \theta}
\end{aligned}
\tag{2.80}
$$

which are consistent with those derived in Szebehely.[11] Multiplying each in turn by $\frac{dr}{d\nu}$, $r^2\frac{d\phi}{d\nu}$, and $r^2\sin^2\phi\frac{d\theta}{d\nu}$ and summing the three together yields

$$
\frac{1}{2}\frac{d}{d\nu}\left(\left(\frac{dr}{d\nu}\right)^2 + \left(r\frac{d\phi}{d\nu}\right)^2 + \left(r\sin\phi\left(1 + \frac{d\theta}{d\nu}\right)\right)^2\right) = -\frac{\partial U}{\partial \theta} - \frac{R}{p}\left(re\cos\nu\frac{dr}{d\nu} + \frac{dU}{d\nu}\right)
\tag{2.81}
$$

where the left-hand side represents the total $\nu$-derivative of the inertial kinetic energy. For the circular case, $e = 0$ and $R/p = 1$ and Eq. 2.81 simplifies to

$$
\frac{1}{2}\frac{d}{d\nu}\left(\left(\frac{dr}{d\nu}\right)^2 + \left(r\frac{d\phi}{d\nu}\right)^2 + \left(r\sin\phi\left(1 + \frac{d\theta}{d\nu}\right)\right)^2\right) = \frac{d}{d\nu}\left(p_\theta - U\right)
\tag{2.82}
$$

wherein the right-hand side is the total derivative of the amended potential function, $(p_\theta - U)$. Integrating Eq. 2.82 and converting back to generalized momenta yields the equation for Jacobi's integral

$$p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi} = 2p_\theta - 2U + C \tag{2.83}$$

where $C$ is the integration constant called Jacobi's constant. In fact, comparing Eq. 2.83 to Eq. 2.61 yields

$$\mathcal{H} = \frac{C}{2} \tag{2.84}$$

that is, the circular Hamiltonian function in the synodic reference frame is equivalent to half of Jacobi's integral function.

A useful application of Jacobi's integral is the generation of Hill regions to qualitatively describe the dynamical regions of motion. These are shown in Fig. 2.9 for the Earth-Moon system ($\mu = 0.0124$) where the blue regions denote areas of accessibility for a given value of $C$. The black and blue dots correspond to the primaries and Lagrange points respectively. As the value of $C$ increases, the kinetic energy of the spacecraft increases and the regions of accessibility expand. The maximum amount of energy is scaled to $C = 0$, which corresponds to the energies at the triangular Lagrange points.

Unfortunately, Eq. 2.81 does not represent a perfect first integral in the non-circular case when $e > 0$. This again reflects the non-autonomous nature of the ERTBP wherein the Hamiltonian function is not itself an integral of motion. However, one of the subsequent goals of this study is to apply canonical perturbation theory to the ERTBP in order to extend the conservative properties of the CRTBP within the local phase space of an equilibrium point. One may then derive local regions of motion in the tradition of Hill's regions.
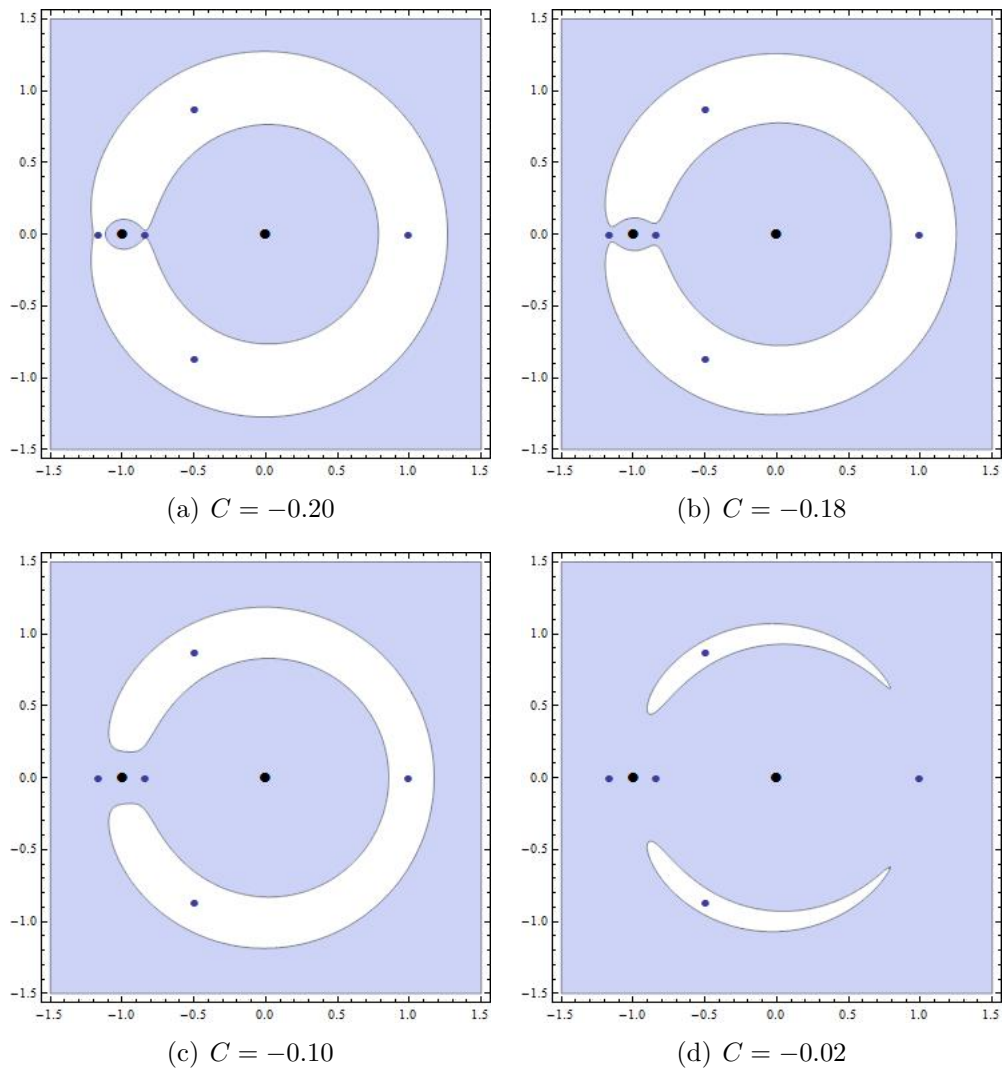
(a) $C = -0.20$　　　　　　　　　　(b) $C = -0.18$

(c) $C = -0.10$　　　　　　　　　　(d) $C = -0.02$

Figure 2.9: Hill Regions in the Earth-Moon CRTBP

## 2.4 KAM Theory

### 2.4.1 Integrability and Invariant Tori

If a Hamiltonian system with $n$ degrees of freedom is conservative, that is if the energy is conserved, the phase space will be confined to a $(2n-1)$-dimensional energy manifold. Thus, specifying the energy level of the system effectively reduces the system degrees of freedom by one. If in addition, the system possesses $n-1$ other independent functions, which all depend on the system state variables and are all conserved, then the system phase space will be further reduced onto an $n$-dimensional manifold parameterized by $n$ integrals of motion. As such, one may then apply a change of variables such that the system solution is easily defined in terms of these $n$ integrals of motion. This property of possessing $n$ independent integrals of motion is known formally as Liouville's theorem of integrability, which is stated in Boccaletti and Pucacco as

> when we have a system of $n$ (independent) first integrals in involution,
> the Hamiltonian system is completely integrable ... if the $n$ first integrals
> are in involution [all their Poisson brackets vanish indentically], then the
> corresponding momenta are also constant and we have all the possible
> $2n$ independent first integrals and therefore the complete solution of the
> problem.[18]

In more general terms, a system is integrable if its phase space is "foliated" by invariant submanifolds parameterized by the integrals of motion. The so-called Poincaré-Hopf theorem dictates that this $n$-dimensional manifold has the topology of an $n$-dimensional torus. The space of the torus is defined by $n$ initial conditions (or $n$ values for the integrals of motion) and is invariant: the solution remains on this torus for all time. Varying the set of initial conditions will then cause the solution to transition to a different torus manifold. In this manner, the $(2n-1)$-dimensional energy

manifold is said to have a foliation in $n$-dimensional invariant tori.[18;45;47]

The trajectory curves are said to "wind" the $n$-dimensional tori and are often easiest to describe in terms of action-angle variables. Action-angle variables are derived from a canonical transformation of an integrable system from the phase space variables $(q, p)$ to the set $(\theta, I)$ wherein the action variables $I$ are constant representing the $n$ integrals of motion that parameterize the invariant tori. The angular variables $\theta$ are then linear functions of the independent variable characterizing the winding of the tori by the trajectories. On occasion, the nomenclature is abused and a set of canonical variables $(\theta, I)$ is called action-angle variables even when $I$ is not constant and $\theta$ does not follow a linear curve. Since this is technically incorrect, every effort is made to use the term correctly and when necessary to label some sets as being action-angle-type variables when they are not precisely so.

In the simplest case of harmonic oscillation, the Hamiltonian expressed in terms of action-angle variables takes the form

$$\mathcal{H} = \sum_i \omega_i I_i \tag{2.85}$$

where $\omega_i$ are the system natural frequencies such that the angular histories follow the linear functions

$$\theta_i(t) = \theta_i(t_0) + \frac{\partial \mathcal{H}}{\partial I_i}(t - t_0) = \theta_i(t_0) + \omega_i(t - t_0) \tag{2.86}$$

Eq. 2.85 is the simplest example of a system in Birkhoff normal form. In general, a system is said to be in Birkhoff normal form of degree $N$ if it may be transformed into

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \ldots + \mathcal{H}_{2N} + \mathcal{H}_{2N+1} \tag{2.87}$$

where $\mathcal{H}_{2k}$ for $1 \leq k \leq N$ is a homogeneous polynomial of degree $k$ in the action-type variables $I_i$ and $\mathcal{H}_{2N+1}$ has a series expansion that starts with terms not less than degree $2N + 1$.[16] In such a system, the action-type variables are truly action variables (constants of motion) when the system is truncated to terms of order $2N$. Thus, if one can express a system in Birkhoff normal form, there must exist a set of *local* integrals of motion wherein up to the truncation of order $2N$, the action-type variables are conserved and parameterize local invariant tori in the local phase space of the expansion in Eq. 2.87. The transformation of the ERTBP into Birkhoff normal form represents the primary goal of Chapters III and IV. In deriving such a transformation, one must take care to account for (and generally avoid) any resonant modes as evident in the statement of the KAM theorem to follow.

## 2.4.2 Resonance and Degeneracy

The effects from resonance plays a pivotal role in dynamical systems particularly in the areas of perturbation and KAM theory. For an integrable system possessing an infinite family of invariant tori (as parameterized by the values of $I$), one may differentiate between resonant and nonresonant cases depending on the natural frequencies of the system: $\omega_i$. In the non-resonant case, the natural frequencies are all linearly independent satisfying

$$\vec{k} \cdot \vec{\omega} \neq 0 \quad \forall \vec{k} \in \mathbb{Z}^n - \{0\} \tag{2.88}$$

where $\vec{\omega}$ represents the vector of natural frequencies. In particular, for a 2-dimensional system with two natural frequencies, $\omega_1$ and $\omega_2$, resonance occurs if the ratio of the frequencies is a non-zero integer in the form

$$\frac{\omega_1}{\omega_2} \in \mathbb{Z} - \{0\} \tag{2.89}$$

As mentioned previously, resonant and non-resonant tori densely fill the phase space of an integrable system. Non-resonant tori exhibit quasi-periodic phase curves in which a single trajectory will move along the torus without repeating itself and is therefore dense everywhere on the torus. On the other hand, phase curves moving along resonant tori will have some degree of periodicity wherein the trajectories will eventually return to its initial state. Generally, one speaks of a set of frequencies as being strictly resonant or strictly non-resonant. However, in certain instances, one may further characterize the non-resonant frequencies as being either strongly or weakly non-resonant where essentially the latter are closer to resonance than the former. Formally, one may define the boundary for strong non-resonance in terms of the Diophantine condition, which requires the existence of a pair of constants $\kappa > 0$ and $\tau > 0$ such that

$$|\vec{k} \cdot \vec{\omega}| \geq \kappa/|\vec{k}|^\tau \qquad \forall \ \vec{k} \in \mathbb{Z} - \{0\} \tag{2.90}$$

In addition to resonance, one may also classify a Hamiltonian system based on its degeneracy properties. For a Hamiltonian system in Birkhoff normal form with natural frequencies $\omega_i = \partial \mathcal{H}_2 / \partial I_i$, the system is called non-degenerate if the integrable part satisfies the condition

$$\det\left(\frac{\partial^2 \mathcal{H}_0}{\partial \vec{I}^2}\right) = \det\left(\frac{\partial \vec{\omega}}{\partial \vec{I}}\right) \neq 0 \tag{2.91}$$

where $\vec{I}$ and $\vec{\omega}$ represent the vector of action variables and vector of natural frequencies. In non-degenerate systems, dependency on the action-type variables causes the natural frequencies to vary from torus to torus. On the other hand, degenerate systems such as the oscillator have constant natural frequencies across all possible tori. Independent from the non-degeneracy condition, one may also classify a system as

being isoenergetically non-degenerate if its integrable part satisfies the condition

$$\det \begin{pmatrix} \dfrac{\partial^2 \mathcal{H}_0}{\partial \vec{I}^2} & \dfrac{\partial \mathcal{H}_0}{\partial \vec{I}} \\[2ex] \dfrac{\partial \mathcal{H}_0}{\partial \vec{I}} & 0 \end{pmatrix} \neq 0 \tag{2.92}$$

Isoenergetically non-degenerate systems exhibit variations in the ratio of the natural frequencies on different tori even as the energy is held fixed.[18]

### 2.4.3  KAM Theorem

Most practical problems of interest do not immediately exhibit the ideal properties of integrability discussed in the previous sections. However, by expanding about small parameters, often the full system can be approximated by an integrable one under perturbation from higher-order terms. This general topic is referred to as *perturbation theory* about which the entirety of Chapter III is based. Nonetheless, it is useful to preview one of the major tenets of modern perturbation theory: namely the combined theories of Kolmogrov, Arnold, Moser, or simply KAM theory.

Consider the case of an integrable Hamiltonian system $\mathcal{H}_0$, which being integrable, possesses the properties discussed in the previous sections: $n$ integrals of motion and a phase space foliated by invariant tori. To this integrable system a series of higher-order perturbations are applied in the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_n \tag{2.93}$$

The aim of KAM theory is to answer the question of whether the invariant tori of the unperturbed system persist in the perturbed, "nearly-integrable" system of Eq. 2.93. In 1954, Kolmogorov provided a theorem addressing this issue whose proof was later derived by Moser for the case of twist maps and then by Arnold for analytic Hamiltonian systems.[49;50;51] In their honor, the theorem is referred to as the KAM

theorem and may be stated as follows.

> If the unperturbed system [$\mathcal{H}_0$] is non-degenerate or isoenergetically non-degenerate, then for a sufficiently small Hamiltonian perturbation most non-resonant invariant tori do not vanish but are only slightly deformed, so that in the phase space of the perturbed system [$\mathcal{H}_0 + \mathcal{H}_n$] there are invariant tori densely filled with [quasi-periodic] phase curves winding around them, with a number of independent frequencies equal to the number of degrees of freedom. [18]

Kolmogorov also required that the Hamiltonian function be real-analytic in the neighborhood of the origin, which is unnecessarily strong in general, but suffices for the present study.[50] It also requires periodicity in the higher-order terms with respect to the angle-type variables, differentiability to a sufficiently high order, and non-resonance as defined by the Diophantine condition in Eq. 2.90. The basic premise of the theorem is that up to a certain order of truncation and away from resonance, a nearly-integrable system behaves like an integrable one. As such, by expanding and truncating a complicated system around the unperturbed case, one may derive approximate, local integrals of motion and foliate the local space with invariant tori.

Unfortunately, while the KAM theorem dictates the preservation of invariant tori in nearly-integrable systems, it does not necessarily guarantee stability. The only time this is true is for 2-dimensional systems in which a family of invariant tori effectively separate the phase space into non-communicating parts, that is, any trajectory originating in the space between two non-resonant tori is necessarily confined to remain in this space for all time. Unfortunately, this property does not extend to systems of dimension $n > 2$ for which trajectories could escape through gaps in the tori in what is known as Arnold diffusion. Thus, application of the KAM theorem as a rigorous means to prove stability is only valid for the 2-dimensional, planar systems.[18]

Another complication with KAM theory is the requirement of non-degeneracy.

Consider the case of a perturbed harmonic oscillator with an unperturbed system defined by $\mathcal{H}_0 = \vec{\omega} \cdot \vec{I}$, which is both degenerate and isoenergetically degenerate. While it is conceivable that sufficiently small perturbations to such a system could still preserve invariant tori, the KAM theorem as stated above is invalidated by the system degeneracy. However, Arnold proved that in such a *properly degenerate* system, the first-order perturbation term can effectively "remove the degeneracy" if the perturbed Hamiltonian is of the necessary form. Arnold provides a tractable statement to this effect in the framework of a Hamiltonian system in Birkhoff normal form. Consider the 2-D Hamiltonian system

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \ldots + \mathcal{H}_{2N} + \mathcal{H}_{2N+1} \tag{2.87}$$

where $\mathcal{H}$ is real-analytic in the neighborhood of the origin, $\mathcal{H}_2$ is in the form $\omega_s I_s - \omega_\ell I_\ell$, $\mathcal{H}_{2k}$ is a homogeneous polynomial of degree $k$ in $I_s$ and $I_\ell$, and $\mathcal{H}_{2N+1}$ has a series expansion of degree not less than $2N + 1$. It is further assumed that the frequencies are sufficiently non-resonant. Under these assumptions, Arnold proved that the origin of the system is stable if for at least one $k$ in the range $1 \leq k \leq N$, the system satisfies the inequality

$$D_{2k} = \mathcal{H}_{2k}(I_s = \omega_\ell, I_\ell = \omega_s) \neq 0 \tag{2.94}$$

Often it is sufficient to demonstrate satisfaction of Eq. 2.94 only to second-order, that is

$$D_4 = (AI_s^2 + 2BI_sI_\ell + CI_\ell^2)/2 \neq 0 \tag{2.95}$$

where $A$, $B$ and $C$ are coefficients depending on the natural frequencies $\omega_i$. However, if for certain values of $\omega_i$ (including non-resonant cases) Eq. 2.95 is not satisfied, one must then demonstrate that a higher-order part is non-zero, for instance $D_6 \neq 0$. Meyer et al. provides a nice proof of the theorem by applying a canonical transformation of the Hamiltonian for small perturbations and in the neighborhood of the

origin and invoking Moser's invariant curve theorem to show that the transformed system is stable. They also include a thorough treatment of the planar CRTBP and by applying the stating formulation of the KAM theorem, demonstrate the nonlinear stability of the planar triangular Lagrange points of the CRTBP.[16;51]

# Chapter III

# Canonical Perturbation Theory

The previous Chapter introduced the basic topics and definitions associated with the perturbation theory and Kolmogrov-Arnold-Moser (KAM) theory. This Chapter further details some of the relevant methods used in perturbation theory with a focus on the Lie transform method derived by Deprit and Hori in the 1960s and herein referred to as the DH method. Motivation is provided in terms of its application to the elliptic restricted three-body problem (ERTBP) and advantages over the classic Von Zeipel method. This then leads to an introduction of the classic, single parameter DH method described in the original papers of Deprit and Hori as well as the more recent works of Meyer et al. and Boccaletti and Pucacco.[16;18;36;37] Following a description of the single parameter method, an original derivation is provided detailing an extension of the DH method to non-autonomous, two-parameter systems that may be compared to similar methods derived in Varadi, Ahmed and Andrade.[40;41;42] Validation of the two-parameter DH method is achieved through the normalization of the single degree-of-freedom damped oscillator. Finally, a preliminary investigation into incorporating control terms in the normalization is conducted with the goal of deriving control laws in the transformed phase space.

## 3.1 Motivation

The basic theory of canonical transformations was introduced in Section 2.1.2.2 for the purposes of transforming the ERTBP from the time-dependent synodic reference frame to a $\nu$-dependent pulsating, synodic reference frame. While the resultant Hamiltonian system still exhibited non-autonomous and non-integrable characteristics, the formulation was significantly simplified with the positions of the primaries and equilibrium points fixed and the non-autonomous and non-circular effects isolated within a single term $R/p$.

Thus, the primary motivation for applying a canonical transformation is to realize a set of canonical variables in which the system formulation is as simple as possible. In the ideal case, the transformed system is integrable and autonomous possessing useful properties of conservation that may be used to apply qualitative analysis of the system behavior. In perturbation theory, the system is represented as an expansion about a simplified form that exhibits the desired properties of integrability. The goal is then to derive a canonical transformation that normalizes the expanded system about the simplified integrable form. Within the expanded formulation, the canonical transformation is derived term by term yielding a series of first-order differential equations for each order in the expansion. The classic method is Von Zeipel's method, which results in a mixed variable solution. An improved direct formulation is provided in the DH method, which harnesses the unique properties of the Lie transform.

The DH method applies to a system expanded about a single parameter. For systems expanded about two parameters, the transformation could be derived as a composition of a pair of single parameter transformations. However, the differential equations associated with each application of the single parameter method is often more complicated and less solvable than those generated from a synchronous two-parameter transformation. Such an extension of the DH method to two-parameter expansions is presented in Varadi, but only for autonomous systems.[40] While in principle Varadi's

method could be used for a non-autonomous two-parameter system, it would first require an expansion of the phase space. However, with the aim of developing a more intuitive methodology, the sequel presents a non-autonomous two-parameter extension to the DH method expressed directly in terms of the original phase space and incorporating a remainder function as first described in Section 2.1.2.2.

Finally, in the interest of generating a novel and potential useful feedback control algorithm, the two-parameter DH method is presented in the formulation of control theory by appending a forcing function to the original Hamiltonian function. This provides a differential control law that is unique from any of the traditional methods of linearized control, Lyapunov theory and controlled Lagrangians.[52;53] In presenting the DH control method, the foundation is laid for a future trade study comparing the relative merits of each of these possible control strategies.

## 3.2   Perturbation Theory and Von Zeipel's Method

Consider a Hamiltonian system $\mathcal{H}(q, p, \nu)$ where $(q, p)$ are generalized state variables and $\nu$ represents the generalized independent variable. Jacobi postulated a canonical transformation under which the transformed state variables are independent integrals of motion in the form

$$
\begin{aligned}
\frac{d\hat{q}_i}{d\nu} &= \frac{\partial \mathcal{K}}{\partial \hat{p}_i} = 0 \\
\frac{d\hat{p}_i}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{q}_i} = 0
\end{aligned}
\tag{3.1}
$$

If such a set exists then the system is integrable and the transformed Hamiltonian function is identically zero

$$
\mathcal{K}(\hat{q}, \hat{p}, \nu) \equiv 0
\tag{3.2}
$$

On the other hand, for a given Hamiltonian system, one can search for integrals of motion by attempting to derive a transformation that satisfies Eq. 3.2. In his celebrated lectures from the late 19th century, Jacobi postulated such an approach wherein the transformation is defined implicitly through a generating function in the form $\mathcal{W}(q, \hat{p}, \nu)$ as introduced in section 2.1.2.2. The state transformation equations $q = Q(\hat{q}, \hat{p}, \nu)$ and $p = P(\hat{q}, \hat{p}, \nu)$ are derived from Eqs. 2.43

$$
\begin{aligned}
p_i &= \frac{\partial}{\partial q_i} \mathcal{W}(q, \hat{p}, \nu) \\
\hat{q}_i &= \frac{\partial}{\partial \hat{p}_i} \mathcal{W}(q, \hat{p}, \nu)
\end{aligned}
\tag{2.43}
$$

and the transformed Hamiltonian must satisfy the necessary condition

$$
\mathcal{K}(\hat{q}, \hat{p}, \nu) = \mathcal{H}(q, p, \nu) + \frac{\partial \mathcal{W}}{\partial \nu}
\tag{2.44}
$$

Substituting Eqs. 2.43 into Eq. 2.44 and constraining it to zero yields the condition

$$
\mathcal{H}(q, \frac{\partial \mathcal{W}}{\partial q}, \nu) + \frac{\partial \mathcal{W}}{\partial \nu} = 0
\tag{3.3}
$$

which is a first-order partial differential equation for $\mathcal{W}$ known as the Hamilton-Jacobi equation (HJE). The HJE is often very difficult to solve except for a few known solutions and cases that can be solved using separation of variables. Nonetheless, it has been successfully applied to many dynamical systems including the celestial two-body problem for which Delaunay derived a set of canonical constants of motion, which are directly related to the classical orbital elements.[4]

In addition to the celebrated HJE, Jacobi further presented the fundamental problem of perturbation theory: an integrable system perturbed by higher-order effects

as defined by the expanded Hamiltonian function

$$\mathcal{H} = \mathcal{H}_0 + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n \tag{3.4}$$

where $\epsilon$ parameterizes the order of the perturbation terms and goes to zero for the unperturbed system. The unperturbed system $\mathcal{H}_0$ is assumed autonomous and in the form of an integrable system whose phase space is foliated by invariant tori as described in Chapter II. The aim of perturbation theory is to derive a canonical transformation of the Hamiltonian system in the expanded form

$$\mathcal{H} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n(q, p, \nu) \quad \rightarrow \quad \mathcal{K} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(\hat{q}, \hat{p}, \nu) \tag{3.5}$$

where $\mathcal{K}$ is the new Hamiltonian function under the transformation $q = Q(\hat{q}, \hat{p}, \nu)$ and $p = P(\hat{q}, \hat{p}, \nu)$. The transformation is designed such that the transformed Hamiltonian function $\mathcal{K}(\hat{q}, \hat{p}, \nu)$ possesses ideal properties of integrability and autonomy.

Upon presenting the main ideas of perturbation theory, Jacobi left the development of the method of solution to his successors: Poincaré and Von Zeipel for whom the following method is named. Since the unperturbed part of Eq. 3.4 is already integrable, its form is left un-changed under the transformation corresponding to a "near-identity" transformation of the form

$$q_i = Q_i(\hat{q}, \hat{p}, \nu, \epsilon) = \hat{q}_i + \dots$$
$$p_i = P_i(\hat{q}, \hat{p}, \nu, \epsilon) = \hat{p} + \dots \tag{3.6}$$

in addition to the near-identity inverse transformation of the form

$$\hat{q}_i = \widehat{Q}_i(q, p, \nu, \epsilon) = q_i + \dots$$
$$\hat{p}_i = \widehat{P}_i(q, p, \nu, \epsilon) = p + \dots \tag{3.7}$$

Von Zeipel's method introduces a corresponding generating function that is also formulated in expanded form

$$W(q, \hat{p}, \nu) = q \cdot \hat{p} + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} W_n(q, \hat{p}, \nu) \tag{3.8}$$

where the zeroth-order part corresponds to the identity transformation. The resultant transformed Hamiltonian function is

$$\begin{aligned}
\mathcal{K}(q, \hat{p}, \nu) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(q, \hat{p}, \nu) \\
&= \sum_{n=0}^{\infty} \left[ \mathcal{H}\left(q, \frac{\partial W_n}{\partial q}, \nu\right) + \frac{\partial W_n}{\partial \nu} \right] \\
&= \mathcal{H}_0(q, \hat{p}, \nu) + \sum_{n=1}^{\infty} \left[ \mathcal{H}\left(q, \frac{\partial W_n}{\partial q}, \nu\right) + \frac{\partial W_n}{\partial \nu} \right]
\end{aligned} \tag{3.9}$$

Separating Eqs. 3.9 into terms of equal power in $\epsilon$ yields the series of ordered differential equations

$$\begin{aligned}
\mathcal{K}_1(q, \hat{p}, \nu) &= \mathcal{H}_1(q, \hat{p}, \nu) + \frac{\partial W_1}{\partial \nu} \\
\mathcal{K}_2(q, \hat{p}, \nu) &= \mathcal{H}_2(q, \hat{p}, \nu) + \frac{\partial W_1}{\partial q} \cdot \frac{\partial \mathcal{H}_1}{\partial p}(q, \hat{p}, \nu) + \frac{\partial W_2}{\partial \nu} \\
&\vdots \\
\mathcal{K}_n(q, \hat{p}, \nu) &= \mathcal{Q}_n(q, \hat{p}, \nu) + \frac{\partial W_n}{\partial \nu}
\end{aligned} \tag{3.10}$$

which term by term, relates the defined generating function $W$ to the transformed Hamiltonian $\mathcal{K}$ through the functions $\mathcal{Q}_i$ known *a priori* in increasing order of $\epsilon$ (that is, $\mathcal{Q}_i$ depends only on the original Hamiltonian function and terms of lesser order in $\epsilon$). One may then choose a desired form for $\mathcal{K}$ and solve Eqs. 3.10 for the necessary generating function to achieve this desired form. Limitations on how one may define the new Hamiltonian terms are provided by the constraint that the

expanded transformation remain analytic in the original state variables such that the series remains convergent. For many problems including those of celestial mechanics, the terms in $\mathcal{H}$ and thereby $\mathcal{Q}_i$ consist of periodic parts and secular parts. Thus, to average out the former, one may define the new Hamiltonian terms $\mathcal{K}_i$ by the secular terms $<\mathcal{Q}_i>$ and thereby insure the convergence of the transformation and normalization to Birkhoff normal form.[4;18;38]

There are two major disadvantages to Von Zeipel's method. The first is that the transformation is not canonically invariant, but instead varies for different sets of canonical variables.[38] In addition, the entire transformation is formulated based on a generating function in the form $\mathcal{W}(q, \hat{p}, \nu)$, which includes both new and old variables. As such, an important and often difficult step to solving Von Zeipel's method is the inversion of the resultant state transformation equations introduced in Eqs. 2.43. For complicated systems and high-order expansions, such inversions are costly if not impossible to implement. Due to these inherent limitations in Von Zeipel's method, both Deprit and Hori independently set out to devise a new method that avoids these issues by harnessing the elegant properties of the Lie transform.[36;37] In doing so, the so-called Deprit-Hori Lie transform method (or sometimes just Deprit's method or simply the Lie transform method) achieves the same result as Von Zeipel's method, but in a canonically invariant form and with the resultant transformation equations expressed directly in terms of the new state variables.

## 3.3  Deprit-Hori Lie Transform Method

The primary feature of the Deprit-Hori Lie transform method (DH method) is the application of the Lie transform operator to construct a canonical transformation in expanded form. For Hamiltonian systems, the Lie transform between two functions

$\mathcal{H}$ and $\mathcal{W}$ is equivalent to the classic Poission bracket

$$\mathcal{L}_\mathcal{W}\mathcal{H} \triangleq [\mathcal{H},\mathcal{W}] = \sum_{i=1}^{N} \left( \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \mathcal{W}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \mathcal{W}}{\partial q_i} \right) \tag{3.11}$$

which is generally referred to as the Lie derivative of $\mathcal{H}$ generated by $\mathcal{W}$. Further, the $n$th Lie derivative is defined as $\mathcal{L}_\mathcal{W}^n \mathcal{H} = \mathcal{L}_\mathcal{W}\left(\mathcal{L}_\mathcal{W}^{n-1}\mathcal{H}\right)$ and the zero-order derivative is the identity operator $\mathcal{L}_\mathcal{W}^0 \mathcal{H} = \mathcal{H}$. For the purposes of the DH method, the functions $\mathcal{H}$ and $\mathcal{W}$ as operated on by Eq. 3.11 are assumed real and analytic in a bounded domain of the phase space. This insures a convergent, real-valued series expansion for the resultant canonical transformation in a neighborhood of the unperturbed system (that is, for small $\epsilon$).

### 3.3.1 Single Parameter Deprit-Hori Lie Transform Method

In the original formulation of the DH method, the original Hamiltonian function is represented by the series expansion

$$\mathcal{H}(q,p,\epsilon,\nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\mathcal{H}_n^{(0)}(q,p,\nu) \tag{3.12}$$

about the single system parameter $\epsilon$. The Hamiltonian system is transformed under the near-identity state transformation $(q,p) \to (\hat{q},\hat{p})$ given explicitly by

$$q = Q(\hat{q},\hat{p},\epsilon,\nu)$$

$$p = P(\hat{q},\hat{p},\epsilon,\nu) \tag{3.13}$$

and defined implicitly through a generating function $\mathcal{W}(\hat{q},\hat{p},\nu)$, which is also represented by its series expansion

$$\mathcal{W}(\hat{q},\hat{p},\epsilon,\nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!}\mathcal{W}_{n+1}(\hat{q},\hat{p},\nu) \tag{3.14}$$

Note that the generating function is formulated directly in terms of the transformed state variables rather than the mixed variable formulation seen in Von Zeipel's method. The transformed Hamiltonian function is constructed term by term in the series

$$
\begin{aligned}
\mathcal{K}(\hat{q}, \hat{p}, \epsilon, \nu) &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{K}_n(\hat{q}, \hat{p}, \nu) \\
&= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left( \mathcal{H}_0^{(n)}(\hat{q}, \hat{p}, \nu) + \mathcal{R}_0^{(n)}(\hat{q}, \hat{p}, \nu) \right) \\
&= \widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \nu) + \mathcal{R}(\hat{q}, \hat{p}, \epsilon, \nu)
\end{aligned}
\tag{3.15}
$$

where $\widehat{\mathcal{H}}$ represents the original Hamiltonian function written explicitly in terms of the state transformation equations

$$
\widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \nu) = \mathcal{H}\left( Q(\hat{q}, \hat{p}, \epsilon, \nu), P(\hat{q}, \hat{p}, \epsilon, \nu), \epsilon, \nu \right)
\tag{3.16}
$$

and $\mathcal{R}$ is a remainder function in the manner of $\dfrac{d\mathcal{W}}{d\nu}$ of Eq. 2.44. Based on the descriptions and proofs included in the original papers of Deprit and Hori and the recent works of Boccaletti and Pucacco and Meyer et al., the DH method is stated concisely in Theorem III.1. [18;16;36;37]

**Theorem III.1.** *Consider a non-autonomous Hamiltonian function expanded about a small parameter $\epsilon$ as represented by the series*

$$
\mathcal{H}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{H}_n^{(0)}(q, p, \nu)
\tag{3.12}
$$

*A canonical transformation is generated from the function*

$$
\mathcal{W}(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \mathcal{W}_{n+1}(\hat{q}, \hat{p}, \nu)
\tag{3.14}
$$

*such that the transformed Hamiltonian function may be constructed term by term in*

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*the series*

$$\mathcal{K}(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left( \mathcal{H}_0^{(n)}(\hat{q}, \hat{p}, \nu) + \mathcal{R}_0^{(n)}(\hat{q}, \hat{p}, \nu) \right) \tag{3.15}$$

*by the recursive equations*

$$\mathcal{H}_n^{(r+1)} = \mathcal{H}_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} \mathcal{H}_{n-i}^{(r)} \tag{3.17}$$

*and*

$$\mathcal{R}_0^{(n)} = \mathcal{S}_0^{(n-1)} \tag{3.18}$$

*where*

$$\begin{aligned} \mathcal{S}_n^{(0)} &= -\frac{\partial}{\partial \nu} \mathcal{W}_{n+1} \\ \mathcal{S}_n^{(r+1)} &= \mathcal{S}_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} \mathcal{S}_{n-i}^{(r)} \end{aligned} \tag{3.19}$$

*The expansion of the explicit state transformation equations $q = Q(\hat{q}, \hat{p}, \epsilon, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \nu)$ is represented by the series*

$$q = Q(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} q_0^{(n)}(\hat{q}, \hat{p}, \nu) \tag{3.20}$$

$$p = P(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} p_0^{(n)}(\hat{q}, \hat{p}, \nu) \tag{3.21}$$

*and may be constructed using the recursive equations*

$$q_n^{(r+1)} = q_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} q_{n-i}^{(r)} \tag{3.22}$$

*and*

$$p_n^{(r+1)} = p_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} p_{n-i}^{(r)} \tag{3.23}$$

*where $q_0^{(0)} = \hat{q}$, $p_0^{(0)} = \hat{p}$, and $q_n^{(0)} = p_n^{(0)} = 0$ for $n > 0$. Moreover, the inverse*

*transformation*

$$\hat{q} = \widehat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{q}_0^{(n)}(q, p, \nu) \tag{3.24}$$

$$\hat{p} = \widehat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{p}_0^{(n)}(q, p, \nu) \tag{3.25}$$

*is obtained by defining the inverse generating function*

$$\widehat{\mathcal{W}} = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \widehat{\mathcal{W}}_0^{(n)} \tag{3.26}$$

*with $\widehat{\mathcal{W}}_n^{(0)} = -\mathcal{W}_n$ and constructed using the recursive equation*

$$\widehat{\mathcal{W}}_n^{(r+1)} = \widehat{\mathcal{W}}_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\mathcal{W}_{i+1}} \widehat{\mathcal{W}}_{n-i}^{(r)} \tag{3.27}$$

*and applying the recursive equations*

$$\hat{q}_n^{(r+1)} = \hat{q}_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\widehat{\mathcal{W}}_{i+1}} \hat{q}_{n-i}^{(r)} \tag{3.28}$$

*and*

$$\hat{p}_n^{(r+1)} = \hat{p}_{n+1}^{(r)} + \sum_{i=0}^{n} \binom{n}{i} \mathcal{L}_{\widehat{\mathcal{W}}_{i+1}} \hat{p}_{n-i}^{(r)} \tag{3.29}$$

*where $\hat{q}_0^{(0)} = q$, $\hat{p}_0^{(0)} = p$, and $\hat{q}_n^{(0)} = \hat{p}_n^{(0)} = 0$ for $n > 0$.*

For a proof of the original DH method described in Theorem III.1, please refer to the original papers of Deprit and Hori or to either of the recent books by Boccaletti and Pucacco and Meyer et al.[18;16;36;37]

Just like Von Zeipel's method, the algorithmic scheme presented in Eq. 3.17 and elsewhere in the proof is recursive in the sense that each successive term is dependent only on the original Hamiltonian function and terms of lesser order. This is visualized in Deprit's triangle in Figure 3.1 where the terms along the left-most vertical

correspond to the original Hamiltonian function under the identity transformation $\mathcal{H}(\hat{q}, \hat{p}, \epsilon, \nu)$ and the terms along the right-most diagonal correspond to the transformed Hamiltonian $\widehat{\mathcal{H}}$. Similarly, the recursive construction of the remainder function $\mathcal{R}$ through Eqs. 3.18 and 3.19 can also be visualized by Deprit's triangle, but with $-\partial \mathcal{W}/\partial \nu$ along the left-most vertical and $\mathcal{S}$ along the right-most diagonal.
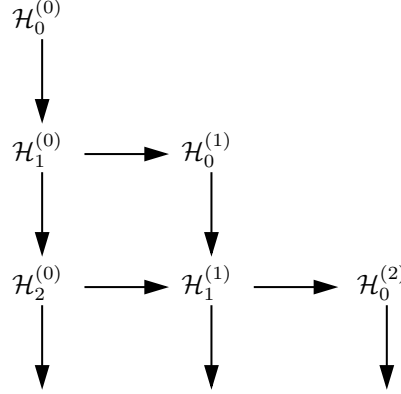


Figure 3.1: Deprit's Triangle

In deriving the ordered equations in Eq. 3.15, the DH method produces a series of first-order differential equations relating the original Hamiltonian to the transformed Hamiltonian through the generating function $\mathcal{W}$. Carrying out this operation to order 3 yields the series of first-order differential equations

$$
\begin{aligned}
\mathcal{K}_0 &= \mathcal{H}_0^{(0)} \\
\mathcal{K}_1 &= -\frac{\partial \mathcal{W}_1}{\partial \nu} + \mathcal{L}_{\mathcal{W}_1} \mathcal{H}_0^{(0)} + \mathcal{H}_1^{(0)} \\
\mathcal{K}_2 &= -\frac{\partial \mathcal{W}_2}{\partial \nu} + \mathcal{L}_{\mathcal{W}_2} \mathcal{H}_0^{(0)} + \mathcal{H}_2^{(0)} + \mathcal{L}_{\mathcal{W}_1} \left( \mathcal{H}_1^{(0)} + \mathcal{K}_1 \right) \\
\mathcal{K}_3 &= -\frac{\partial \mathcal{W}_3}{\partial \nu} + \mathcal{L}_{\mathcal{W}_3} \mathcal{H}_0^{(0)} + \mathcal{H}_3^{(0)} + \mathcal{L}_{\mathcal{W}_2} \left( 2\mathcal{H}_1^{(0)} + \mathcal{K}_1 \right) + \mathcal{L}_{\mathcal{W}_1} \left( \mathcal{H}_2^{(0)} + 2\mathcal{K}_2 - \mathcal{L}_{\mathcal{W}_1} \mathcal{K}_1 \right)
\end{aligned}
$$

(3.30)

which relate the generating function $\mathcal{W}$ to the transformed Hamiltonian $\mathcal{K}$ in the same manner as Von Zeipel's Eqs. 3.10. Unlike Von Zeipel's method, the differential equations in Eqs. 3.30 are expressed directly in terms of the transformed state vari-

ables $(\hat{q}, \hat{p})$. Further, for every order of $\epsilon$, the differential equations in Eqs. 3.30 take the form of the so-called *homological differential equation*

$$\frac{\partial \mathcal{W}_i}{\partial \nu} - \mathcal{L}_{\mathcal{W}_i} \mathcal{H}_0^{(0)} = \mathcal{Q}_i - \mathcal{K}_i \tag{3.31}$$

where $\mathcal{H}_0^{(0)}$ is the unperturbed Hamiltonian ($\epsilon = 0$) and $\mathcal{Q}_i$ is known prior to solving the homological equation based on the original Hamiltonian function and terms of lesser order. The form of $\mathcal{K}$ is also prescribed prior to solving Eq. 3.31 based on the desired form for the normalized system. Thus, provided a solution to Eq. 3.31 exists, one may derive the necessary generating function to normalize the Hamiltonian, which in turn defines the state transformation equations

$$q = Q(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} q_0^{(n)}(\hat{q}, \hat{p}, \nu) \tag{3.20}$$

$$p = P(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} p_0^{(n)}(\hat{q}, \hat{p}, \nu) \tag{3.21}$$

and the inverse equations

$$\hat{q} = \widehat{Q}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{q}_0^{(n)}(q, p, \nu) \tag{3.24}$$

$$\hat{p} = \widehat{P}(q, p, \epsilon, \nu) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \hat{p}_0^{(n)}(q, p, \nu) \tag{3.25}$$

through Eqs. 3.28 and 3.29.

### 3.3.2 Two-Parameter Deprit-Hori Lie Transform Method

The DH method provides a robust methodology for normalizing a perturbed Hamiltonian system about its unperturbed form as parameterized by the small parameter $\epsilon$. As an extension to the single parameter method, consider a doubly-expanded

Hamiltonian system

$$\mathcal{H}(q, p, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) \tag{3.32}$$

whose higher-order terms are parameterized by two quantities: $\epsilon$ and $\gamma$. This two-parameter case was treated for autonomous systems in 1985 by Varadi, whose analysis was subsequently extended to three parameters by Ahmed and to $N$ parameters by Andrade.[40;41;42] While it is possible to apply Varadi's method to non-autonomous systems, it requires an expansion of the phase space to incorporate the independent variable and Hamiltonian function as extra state variables. This renders the system autonomous and allows for the application of Varadi's formulation.

To circumvent the need to expand the phase space, an original theorem is presented in the sequel in which the two-parameter DH method is formulated directly in terms of the original non-autonomous system by applying a remainder function in the tradition of the single parameter method presented in Theorem III.1. As before, the Lie transform operator defined in Eq. 3.11 is applied to formulate the canonical transformation, but now through a pair of generating functions

$$\mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{W}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \tag{3.33}$$

$$\mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{V}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \tag{3.34}$$

The transformed Hamiltonian function is constructed term by term in the series

$$
\begin{aligned}
\mathcal{K}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{K}_{m,n}(\hat{q}, \hat{p}, \nu) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{H}_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) + \mathcal{R}_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \right) \\
&= \widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) + \mathcal{R}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)
\end{aligned}
\tag{3.35}
$$

where as before, the terms in $\widehat{\mathcal{H}}$ represent the old Hamiltonian written explicitly in terms of the transformed state variables

$$\widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \mathcal{H}\Big(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu\Big) \tag{3.36}$$

and the terms in $\mathcal{R}$ comprise the remainder function and are generated from a pair of intermediary functions $\mathcal{S}$ and $\mathcal{T}$. The two-parameter, non-autonomous theorem is presented in Theorem III.2 followed by a proof that is modeled in part on those included in Deprit and Meyer et al. for the single parameter case.[16;37] In addition, the construction of the explicit state transformation equations are treated separately in Corollary III.3.

**Theorem III.2.** *Consider a non-autonomous Hamiltonian function expanded about two small parameters as represented by the series*

$$\mathcal{H}(q, p, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) \tag{3.32}$$

*A canonical transformation is generated from the pair of functions*

$$\mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{W}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \tag{3.33}$$

$$\mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{V}_{m+1,n+1}(\hat{q}, \hat{p}, \nu) \tag{3.34}$$

*such that the transformed Hamiltonian function may be constructed term by term in the series*

$$\mathcal{K}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{H}_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) + \mathcal{R}_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \right) \tag{3.35}$$

*by the extended recursive equations*

$$\mathcal{H}_{m,n}^{(r,s+1)} = \mathcal{H}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{H}_{m,n}^{(r+1,s)} = \mathcal{H}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)} \qquad (3.37)$$

*and*

$$\mathcal{R}_{0,0}^{(m,n)} = \begin{cases} \mathcal{S}_{0,0}^{(0,n-1)} & m = 0, n \neq 0 \\[2mm] \mathcal{T}_{0,0}^{(m-1,0)} & m \neq 0, n = 0 \\[2mm] \mathcal{S}_{0,0}^{(m,n-1)} \quad = \mathcal{T}_{0,0}^{(m-1,n)} & m, n \neq 0 \end{cases} \qquad (3.38)$$

*where*

$$\mathcal{S}_{m,n}^{(0,0)} = -\frac{\partial}{\partial\nu}\mathcal{W}_{m+1,n+1}$$

$$\mathcal{S}_{m,n}^{(r,s+1)} = \mathcal{S}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{S}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{S}_{m,n}^{(r+1,s)} = \mathcal{S}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{S}_{m-j,n-i}^{(r,s)} \qquad (3.39)$$

*and*

$$\mathcal{T}_{m,n}^{(0,0)} = -\frac{\partial}{\partial\nu}\mathcal{V}_{m+1,n+1}$$

$$\mathcal{T}_{m,n}^{(r,s+1)} = \mathcal{T}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{T}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{T}_{m,n}^{(r+1,s)} = \mathcal{T}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{T}_{m-j,n-i}^{(r,s)} \qquad (3.40)$$

*Moreover, the terms appearing in the expanded transformation containing "mixed parameters" (i.e. of the form $\epsilon^n \gamma^m$ with both $m$ and $n$ non-zero) may be obtained equivalently using either of the two generating functions $\mathcal{W}$ or $\mathcal{V}$ such that these*

*generating functions satisfy the Deprit commutation condition,*

$$\frac{\partial \mathcal{V}}{\partial \epsilon} - \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L}_\mathcal{W} \mathcal{V} = 0 \tag{3.41}$$

*which term by term implies*

$$\mathcal{V}_{m+1,n+2} - \mathcal{W}_{m+2,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{V}_{m-j+1,n-i+1} = 0 \tag{3.42}$$

*Proof.* The canonical transformation from coordinates $(q, p)$ to $(\hat{q}, \hat{p})$ may be formulated in terms of the generating functions $\mathcal{W}$ and $\mathcal{V}$ through the pair of non-autonomous systems

$$\frac{\partial}{\partial \epsilon} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.43}$$

and

$$\frac{\partial}{\partial \gamma} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p} \mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q} \mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.44}$$

with initial conditions $q(\epsilon, \gamma = 0) = \hat{q}$ and $p(\epsilon, \gamma = 0) = \hat{p}$. Notice that both these systems are in Hamiltonian form where $\mathcal{W}$ and $\mathcal{V}$ take the place of the Hamiltonian function and $\epsilon$ and $\gamma$ serve as the independent variables. For well-behaved functions, the general solution for this coupled system is a pair of analytic functions defining the near-identity state transformation equations,

$$q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$$

$$p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \tag{3.45}$$

where $Q(\epsilon, \gamma = 0) = \hat{q}$ and $P(\epsilon, \gamma = 0) = \hat{p}$.

In the spirit of Deprit's original proof, one may define an extended phase space by appending the independent variable $\nu$ and the Hamiltonian function to the original state variables

$$q \to \{q, \nu\}$$
$$p \to \{p, -\mathcal{H}\} \tag{3.46}$$

To apply the Lie operator within the extended phase space but still maintain its definition within the original $(q, p)$ phase space, one must incorporate the additional terms

$$\mathcal{L}_W - \frac{\partial W}{\partial \mathcal{H}} \frac{\partial}{\partial \nu} + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{L}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}}$$
$$\mathcal{L}_V - \frac{\partial V}{\partial \mathcal{H}} \frac{\partial}{\partial \nu} + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{L}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} \tag{3.47}$$

wherein the generating functions will vary with $\nu$, but will not be explicitly dependent on the Hamiltonian itself such that $\dfrac{\partial W}{\partial \mathcal{H}} = \dfrac{\partial V}{\partial \mathcal{H}} = 0$. The Deprit operators are defined as

$$\mathcal{D}_W \triangleq \frac{\partial}{\partial \epsilon} + \mathcal{L}_W$$
$$\mathcal{D}_V \triangleq \frac{\partial}{\partial \gamma} + \mathcal{L}_V \tag{3.48}$$

and incorporating the additional terms from Eqs. 3.47 for the expanded phase space, the extended Deprit operators are defined as

$$\mathcal{E}_W \triangleq \frac{\partial}{\partial \epsilon} + \mathcal{L}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{D}_W + \frac{\partial W}{\partial \nu} \frac{\partial}{\partial \mathcal{H}}$$
$$\mathcal{E}_V \triangleq \frac{\partial}{\partial \gamma} + \mathcal{L}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} = \mathcal{D}_V + \frac{\partial V}{\partial \nu} \frac{\partial}{\partial \mathcal{H}} \tag{3.49}$$

Note that the original Deprit operators satisfy the following conditions: the zeroth-

order Deprit operators act as identity operations, $\mathcal{D}_\mathcal{W}^0 \mathcal{H} = \mathcal{D}_\mathcal{V}^0 \mathcal{H} = \mathcal{H}$ and successive applications of the Deprit operators are given by

$$\mathcal{D}_\mathcal{W}^s \mathcal{D}_\mathcal{V}^r = \left( \frac{\partial}{\partial \epsilon} + \mathcal{L}_\mathcal{W} \right)^s \left( \frac{\partial}{\partial \gamma} + \mathcal{L}_\mathcal{V} \right)^r \tag{3.50}$$

The question of commutativity will be treated in the sequel.

When applied to a function $F(q, p, \epsilon, \gamma, \nu)$ with no explicit dependence on the Hamiltonian, the right-most terms in Eqs. 3.49 are zero and the extended Deprit operators reduce to the original Deprit operators, $\mathcal{E}_\mathcal{W} F = \mathcal{D}_\mathcal{W} F$ and $\mathcal{E}_\mathcal{V} F = \mathcal{D}_\mathcal{V} F$. Furthermore, under the transformation $q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$, the partial derivative of $F$ with respect to $\epsilon$ is

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon} \left( F(q, p, \epsilon, \gamma, \nu) \Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right) &= \left( \frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial \epsilon} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial \epsilon} \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \\
&= \left( \frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial q}\frac{\partial \mathcal{W}}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial \mathcal{W}}{\partial q} \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \\
&= \left( \frac{\partial F}{\partial \epsilon} + \mathcal{L}_\mathcal{W} F \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \\
&= \left( \mathcal{D}_\mathcal{W} F \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}}
\end{aligned}
\tag{3.51}
$$

and by the same logic, the partial derivative with respect to $\gamma$ is

$$\frac{\partial}{\partial \gamma}\left( F(q,p,\epsilon,\gamma,\nu)\Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right) = \left( \mathcal{D}_\mathcal{V} F \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \tag{3.52}$$

By extension, one may also express the mixed $\epsilon$- and $\gamma$-derivatives to any order by

$$\frac{\partial^n}{\partial \epsilon^n}\frac{\partial^m}{\partial \gamma^m}\left( F(q,p,\epsilon,\gamma,\nu)\Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right) = \left( \mathcal{D}_\mathcal{W}^n \mathcal{D}_\mathcal{V}^m F \right)\Bigg|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \tag{3.53}$$

Upon examining Eq. 3.53, it is evident that the Deprit operators may be applied within an expansion series in $\epsilon$ and $\gamma$. This approach is subsequently applied to the Hamiltonian function under the transformation $(q, p) \rightarrow (\hat{q}, \hat{p})$.

Consider the canonical transformation of a non-autonomous Hamiltonian function resulting in a transformed Hamiltonian function in the form

$$
\begin{aligned}
\mathcal{K} &= \mathcal{H}\Big(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu\Big) + \mathcal{R}\,(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\
&= \widehat{\mathcal{H}}\,(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) + \mathcal{R}\,(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)
\end{aligned}
\tag{3.54}
$$

where $\widehat{\mathcal{H}}$ represents the original Hamiltonian function written explicitly in terms of the state transformation equations and $\mathcal{R}$ represents a remainder function. The expansion of $\widehat{\mathcal{H}}$ in a Taylor series about $\epsilon = 0$ and $\gamma = 0$ is

$$
\begin{aligned}
\widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)\Big|_{\epsilon, \gamma = 0} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \frac{\partial^n}{\partial \epsilon^n} \frac{\partial^m}{\partial \gamma^m} \left( \mathcal{H}(q, p, \epsilon, \gamma, \nu)\Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right)\Bigg|_{\epsilon, \gamma = 0}
\end{aligned}
\tag{3.55}
$$

For the time being, the Hamiltonian function is treated as in function $F$ used previously, that is, with no explicit dependence on the Hamiltonian as a state variable. Eq. 3.55 is represented in terms of the extended Deprit operators by substituting Eq. 3.53 in for the $\epsilon$ and $\gamma$ partial derivatives such that

$$
\widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{D}_{\mathcal{W}}^n \mathcal{D}_{\mathcal{V}}^m \mathcal{H}(q, p, \epsilon, \gamma, \nu)\Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right)\Bigg|_{\epsilon, \gamma = 0}
\tag{3.56}
$$

Further introducing the subscripted and superscripted formulation

$$\mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(0,0)}$$

$$\mathcal{D}_{\mathcal{W}}^s \mathcal{D}_{\mathcal{V}}^r \mathcal{H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(r,s)} \tag{3.57}$$

the expansion of the original Hamiltonian function is represented by the series of functions $\mathcal{H}_{m,n}^{(0,0)}$ and the expansion of $\widehat{\mathcal{H}}$ is represented by

$$\widehat{\mathcal{H}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{D}_{\mathcal{W}}^n \mathcal{D}_{\mathcal{V}}^m \mathcal{H}(q, p, \epsilon, \gamma, \nu) \Big|_{\substack{q=Q(\hat{q},\hat{p},\epsilon,\gamma,\nu) \\ p=P(\hat{q},\hat{p},\epsilon,\gamma,\nu)}} \right) \Bigg|_{\epsilon,\gamma=0}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \tag{3.58}$$

Within the subscripted and superscripted formulation, the Lie derivatives satisfy

$$\mathcal{L}_{\mathcal{W}} \mathcal{H}_{m,n}^{(r,s)} = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{L}_{\mathcal{V}} \mathcal{H}_{m,n}^{(r,s)} = \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)} \tag{3.59}$$

such that the terms included in the expansion series under the Deprit operator satisfy

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(r,s+1)} = \mathcal{D}_{\mathcal{W}} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(r,s)} \right)$$

$$= \frac{\partial}{\partial \epsilon} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(r,s)} \right) + \mathcal{L}_{\mathcal{W}} \left( \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(r,s)} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{H}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)} \right) \tag{3.60}$$

and likewise

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m!\, n!} \mathcal{H}_{m,n}^{(r+1,s)} = \mathcal{D}_{\mathcal{V}}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m!\, n!} \mathcal{H}_{m,n}^{(r,s)}\right)$$

$$= \frac{\partial}{\partial \gamma}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m!\, n!} \mathcal{H}_{m,n}^{(r,s)}\right) + \mathcal{L}_{\mathcal{V}}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m!\, n!} \mathcal{H}_{m,n}^{(r,s)}\right)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m \epsilon^n}{m!\, n!} \left(\mathcal{H}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)}\right) \quad (3.61)$$

Thus, comparing Eqs. 3.60 and 3.61 to Eq. 3.57, all the unknown functions $\mathcal{H}_{m,n}^{(r,s)}$ may be constructed term by term using recursive equations

$$\mathcal{H}_{m,n}^{(r,s+1)} = \mathcal{H}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{H}_{m,n}^{(r+1,s)} = \mathcal{H}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{H}_{m-j,n-i}^{(r,s)} \quad (3.37)$$

referred to as the extended recursive equations. The formulae in Eqs. 3.37 are recursive in the sense that each successive term is dependent only on terms preceding it, starting with the original Hamiltonian $\mathcal{H}_{m,n}^{(0,0)}$. The construction may be visualized in the tradition of Deprit's triangle, but now in the form of a pyramid as shown in Figure 3.2. The terms in the central column correspond to the original Hamiltonian function under the identity transformation while the terms along each outermost diagonal correspond to terms in $\widehat{\mathcal{H}}$ with increasing order in $\epsilon$ and $\gamma$ toward the base of the pyramid.

Thus far, the method has provided the means to construct the explicit substitution of the state transformation equations into the original Hamiltonian through the Deprit operators $\mathcal{D}_{\mathcal{W}}$ and $\mathcal{D}_{\mathcal{V}}$. While this provides the function $\widehat{\mathcal{H}}$ in Eq. 3.54, one must still account for the remainder function.
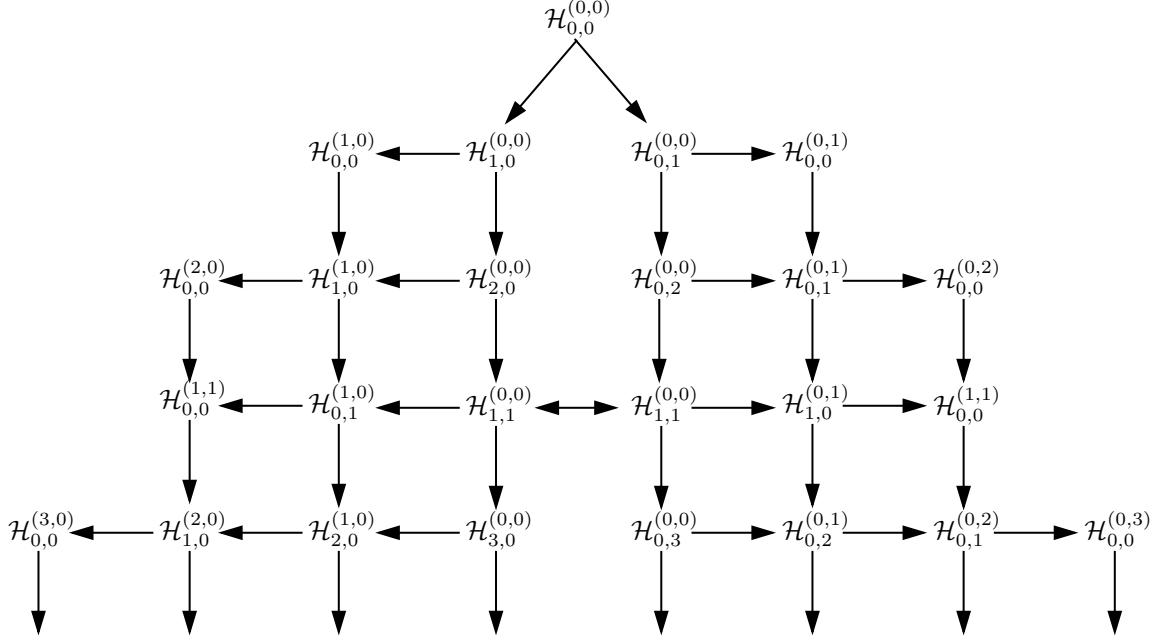
Figure 3.2: Deprit's Pyramid

Applying the extended Deprit operators to the Hamiltonian state variable yields

$$\mathcal{E}_\mathcal{W}\mathcal{H} = \mathcal{D}_\mathcal{W}\mathcal{H} - \frac{\partial \mathcal{W}}{\partial \nu}$$
$$\mathcal{E}_\mathcal{V}\mathcal{H} = \mathcal{D}_\mathcal{V}\mathcal{H} - \frac{\partial \mathcal{V}}{\partial \nu} \tag{3.62}$$

and for higher-order and mixed terms,

$$\mathcal{E}_\mathcal{W}^s\mathcal{E}_\mathcal{V}^r\mathcal{H} = \mathcal{D}_\mathcal{W}^s\mathcal{D}_\mathcal{V}^r\mathcal{H} - \mathcal{D}_\mathcal{W}^{s-1}\mathcal{D}_\mathcal{V}^r\frac{\partial \mathcal{W}}{\partial \nu} - \mathcal{D}_\mathcal{W}^s\mathcal{D}_\mathcal{V}^{r-1}\frac{\partial \mathcal{V}}{\partial \nu} \tag{3.63}$$

The first term appearing in Eq. 3.63 is equivalent to the explicit substitution of the state transformation equations into the original Hamiltonian function while treating it as being independent of the Hamiltonian state variable. This effectively generates the autonomous part of the transformation represented by the term $\widehat{\mathcal{H}}$ in Eq. 3.54. The other two terms appearing in Eq. 3.63, $-\mathcal{D}_\mathcal{W}^{s-1}\mathcal{D}_\mathcal{V}^r\dfrac{\partial \mathcal{W}}{\partial \nu}$ and $-\mathcal{D}_\mathcal{W}^s\mathcal{D}_\mathcal{V}^{r-1}\dfrac{\partial \mathcal{V}}{\partial \nu}$ comprise the remainder function and may be constructed term by term in the same manner as

the Hamiltonian function, but as acting on the functions $\dfrac{\partial \mathcal{W}}{\partial \nu}$ and $\dfrac{\partial \mathcal{V}}{\partial \nu}$. To do so in a constructive manner, define the intermediary functions $\mathcal{S}$ and $\mathcal{T}$ by

$$\mathcal{S}_{m,n}^{(0,0)} = -\frac{\partial}{\partial \nu} \mathcal{W}_{m+1,n+1}$$

$$\mathcal{T}_{m,n}^{(0,0)} = -\frac{\partial}{\partial \nu} \mathcal{V}_{m+1,n+1} \tag{3.64}$$

such that in Eq. 3.63, the remainder function $\mathcal{R}$ is constructed term by term according to

$$\mathcal{R}_{0,0}^{(m,n)} = \begin{cases} \mathcal{S}_{0,0}^{(0,n-1)} & m = 0, n \neq 0 \\ \mathcal{T}_{0,0}^{(m-1,0)} & m \neq 0, n = 0 \\ \mathcal{S}_{0,0}^{(m,n-1)} = \mathcal{T}_{0,0}^{(m-1,n)} & m, n \neq 0 \end{cases} \tag{3.65}$$

and the recursive formulae

$$\mathcal{S}_{m,n}^{(r,s+1)} = \mathcal{S}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{S}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{S}_{m,n}^{(r+1,s)} = \mathcal{S}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{S}_{m-j,n-i}^{(r,s)} \tag{3.39}$$

and

$$\mathcal{T}_{m,n}^{(r,s+1)} = \mathcal{T}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{T}_{m-j,n-i}^{(r,s)}$$

$$\mathcal{T}_{m,n}^{(r+1,s)} = \mathcal{T}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \mathcal{T}_{m-j,n-i}^{(r,s)} \tag{3.40}$$

Together with Eqs. 3.37, Eqs. 3.39 and 3.40 define the complete transformation of the Hamiltonian function into $\mathcal{K}$ as formulated in Eq. 3.35.

To complete the proof, one must address the issue of commutativity in the Deprit operators. Based on the preceding discussion, one may derive either of the following

representations

$$\mathcal{D_W D_V H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \, \epsilon^n}{m! \, n!} \mathcal{H}_{m,n}^{(1,1)}$$

$$\mathcal{D_V D_W H} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \, \epsilon^n}{m! \, n!} \mathcal{H}_{m,n}^{(1,1)} \tag{3.66}$$

For consistency's sake, the terms on the right-hand side should be the same in either case, which implies that the Deprit operators are commutative, that is,

$$\mathcal{D_W D_V} = \mathcal{D_V D_W} \tag{3.67}$$

The Lie derivatives are not themselves commutative, but instead satisfy the condition

$$\mathcal{L_V L_W} = \mathcal{L_W L_V} - \mathcal{L_{\mathcal{L_W} V}} \tag{3.68}$$

wherein the last term represents the Lie derivative generated by $\mathcal{L_W V}$. Expanding the Deprit commutative condition in Eq. 3.67 based on the definition of the Deprit operator in Eq. 3.48 yields the constraint

$$\mathcal{D_W D_V} = \frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \epsilon} \mathcal{L_V} + \mathcal{L_W} \frac{\partial}{\partial \gamma} + \mathcal{L_W L_V}$$

$$= \mathcal{D_V D_W} = \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \epsilon} + \frac{\partial}{\partial \gamma} \mathcal{L_W} + \mathcal{L_V} \frac{\partial}{\partial \epsilon} + \mathcal{L_V L_W} \tag{3.69}$$

wherein the first two terms

$$\frac{\partial}{\partial \epsilon} \frac{\partial}{\partial \gamma} = \frac{\partial}{\partial \gamma} \frac{\partial}{\partial \epsilon} \tag{3.70}$$

are equivalent operations according to the symmetry of second-order derivatives (Clairaut's theorem). Substituting the Lie commutation condition in Eq. 3.68 into

Eq. 3.69 and re-arranging terms yields

$$0 = \frac{\partial}{\partial \epsilon} \mathcal{L}_{\mathcal{V}} + \mathcal{L}_{\mathcal{W}} \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial \gamma} \mathcal{L}_{\mathcal{W}} - \mathcal{L}_{\mathcal{V}} \frac{\partial}{\partial \epsilon} + \mathcal{L}_{\mathcal{L}_{\mathcal{W}} \mathcal{V}} \tag{3.71}$$

Moreover, the partial derivatives and Lie operators satisfy the conditions

$$\frac{\partial}{\partial \epsilon} \mathcal{L}_{\mathcal{V}} - \mathcal{L}_{\mathcal{V}} \frac{\partial}{\partial \epsilon} = \mathcal{L}_{\partial \mathcal{V}/\partial \epsilon}$$
$$\frac{\partial}{\partial \gamma} \mathcal{L}_{\mathcal{W}} - \mathcal{L}_{\mathcal{W}} \frac{\partial}{\partial \gamma} = \mathcal{L}_{\partial \mathcal{W}/\partial \gamma} \tag{3.72}$$

where the last terms are the Lie derivatives generated by $\dfrac{\partial \mathcal{V}}{\partial \epsilon}$ and $\dfrac{\partial \mathcal{W}}{\partial \gamma}$ respectively.
Substituting Eqs. 3.72 into Eq. 3.71 yields

$$0 = \mathcal{L}_{\partial \mathcal{V}/\partial \epsilon} - \mathcal{L}_{\partial \mathcal{W}/\partial \gamma} + \mathcal{L}_{\mathcal{L}_{\mathcal{W}} \mathcal{V}}$$
$$= \mathcal{L}_{\partial \mathcal{V}/\partial \epsilon - \partial \mathcal{W}/\partial \gamma + \mathcal{L}_{\mathcal{W}} \mathcal{V}} \tag{3.73}$$

which implies the condition

$$0 = \frac{\partial \mathcal{V}}{\partial \epsilon} - \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L}_{\mathcal{W}} \mathcal{V} \tag{3.74}$$

This condition is herein referred to as the Deprit commutation condition in reference to its derivation from Eq. 3.67. By applying the recursive algorithms used previously, the Deprit commutation condition can also be expressed term by term as

$$0 = \mathcal{V}_{m+1,n+2} - \mathcal{W}_{m+2,n+1} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \mathcal{V}_{m-j+1,n-i+1} \tag{3.75}$$

**QED**

Having presented the transformation of the Hamiltonian function in Theorem III.2, one may further derive the corresponding state transformation equations through

Corollary III.3.

**Corollary III.3.** *The expansion of the explicit state transformation equations $q = Q(\hat{q}, \hat{p}, \epsilon, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \nu)$ are represented by the series*

$$q = Q(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} q_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \tag{3.76}$$

$$p = P(\hat{q}, \hat{p}, \epsilon, \nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} p_{0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \tag{3.77}$$

*and may be constructed using the recursive equations*

$$q_{m,n}^{(r,s+1)} = q_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} q_{m-j,n-i}^{(r,s)}$$

$$q_{m,n}^{(r+1,s)} = q_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} q_{m-j,n-i}^{(r,s)} \tag{3.78}$$

*and*

$$p_{m,n}^{(r,s+1)} = p_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} p_{m-j,n-i}^{(r,s)}$$

$$p_{m,n}^{(r+1,s)} = p_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} p_{m-j,n-i}^{(r,s)} \tag{3.79}$$

*where $q_{0,0}^{(0,0)} = \hat{q}$, $p_{0,0}^{(0,0)} = \hat{p}$, and $q_{m,n}^{(0,0)} = p_{m,n}^{(0,0)} = 0$ for $m + n > 0$. The inverse transformation is constructed in terms of the inverse generating functions*

$$\widehat{\mathcal{W}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \widehat{\mathcal{W}}_{0,0}^{(m,n)}(q, p, \nu) \tag{3.80}$$

$$\widehat{\mathcal{V}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \widehat{\mathcal{V}}_{0,0}^{(m,n)}(q, p, \nu) \tag{3.81}$$

*derived from*

$$\widehat{\mathcal{W}}_{m,n}^{(0,0)} = -\mathcal{W}_{m,n}$$

$$\widehat{\mathcal{V}}_{m,n}^{(0,0)} = -\mathcal{V}_{m,n} \tag{3.82}$$

*and the recursive equations*

$$\widehat{\mathcal{W}}_{m,n}^{(r,s+1)} = \widehat{\mathcal{W}}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \widehat{\mathcal{W}}_{m-j,n-i}^{(r,s)}$$

$$\widehat{\mathcal{W}}_{m,n}^{(r+1,s)} = \widehat{\mathcal{W}}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \widehat{\mathcal{W}}_{m-j,n-i}^{(r,s)} \tag{3.83}$$

*The explicit inverse state transformation equations are represented by the series*

$$\hat{q} = \widehat{Q}(q,p,\epsilon,\nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!} \hat{q}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.84}$$

$$\hat{p} = \widehat{P}(q,p,\epsilon,\nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!} \hat{p}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.85}$$

*and generated in terms of the inverse generating functions through the recursive equations*

$$\hat{q}_{m,n}^{(r,s+1)} = \hat{q}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{W}}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)}$$

$$\hat{q}_{m,n}^{(r+1,s)} = \hat{q}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{V}}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)} \tag{3.86}$$

*and*

$$\hat{p}_{m,n}^{(r,s+1)} = \hat{p}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{W}}_{j+1,i+1}} \hat{p}_{m-j,n-i}^{(r,s)}$$

$$\hat{p}_{m,n}^{(r+1,s)} = \hat{p}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{V}}_{j+1,i+1}} \hat{p}_{m-j,n-i}^{(r,s)} \tag{3.87}$$

*where* $\hat{q}_{0,0}^{(0,0)} = q$, $\hat{p}_{0,0}^{(0,0)} = p$, *and* $\hat{q}_{m,n}^{(0,0)} = \hat{p}_{m,n}^{(0,0)} = 0$ *for* $m + n > 0$.

*Proof.* Consider the explicit state transformation equations $q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$ expanded about $\epsilon = 0$ and $\gamma = 0$ as represented by the Taylor series

$$q = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}\frac{\partial^n}{\partial\epsilon^n}\frac{\partial^m}{\partial\gamma^m}\Big(Q(\hat{q},\hat{p},\epsilon,\gamma,\nu)\Big)\Big|_{\epsilon,\gamma=0}$$
$$p = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}\frac{\partial^n}{\partial\epsilon^n}\frac{\partial^m}{\partial\gamma^m}\Big(P(\hat{q},\hat{p},\epsilon,\gamma,\nu)\Big)\Big|_{\epsilon,\gamma=0} \qquad (3.88)$$

The expansion series are represented in terms of the Deprit operators by substituting in Eq. 3.53 yielding

$$q = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}\Big(\mathcal{D}_{\mathcal{W}}^n\mathcal{D}_{\mathcal{V}}^m Q(\hat{q},\hat{p},\epsilon,\gamma,\nu)\Big)\Big|_{\epsilon,\gamma=0}$$
$$p = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}\Big(\mathcal{D}_{\mathcal{W}}^n\mathcal{D}_{\mathcal{V}}^m P(\hat{q},\hat{p},\epsilon,\gamma,\nu)\Big)\Big|_{\epsilon,\gamma=0} \qquad (3.89)$$

The subscripted and superscripted formulation is defined as before such that the expanded state transformation equations are

$$q = Q(\hat{q},\hat{p},\epsilon,\nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}q_{0,0}^{(m,n)}(\hat{q},\hat{p},\nu) \qquad (3.76)$$

$$p = P(\hat{q},\hat{p},\epsilon,\nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!}p_{0,0}^{(m,n)}(\hat{q},\hat{p},\nu) \qquad (3.77)$$

and the zero superscript terms correspond to the identity transformation,

$$q_{m,n}^{(0,0)} = \begin{cases} \hat{q} & m = n = 0 \\ 0 & m + n > 0 \end{cases} \qquad (3.90)$$

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and

$$p_{m,n}^{(0,0)} = \begin{cases} \hat{p} & m = n = 0 \\ 0 & m + n > 0 \end{cases} \tag{3.91}$$

Since the state transformation equations are not explicitly dependent on the Hamiltonian function as a state variable, they transform in the same manner as $\widehat{\mathcal{H}}$ from Theorem III.2. Therefore, by the same logic used in the proof of Theorem III.2, Eqs. 3.76 and 3.77 may be generated using the recursive equations

$$q_{m,n}^{(r,s+1)} = q_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} q_{m-j,n-i}^{(r,s)}$$

$$q_{m,n}^{(r+1,s)} = q_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} q_{m-j,n-i}^{(r,s)} \tag{3.78}$$

and

$$p_{m,n}^{(r,s+1)} = p_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} p_{m-j,n-i}^{(r,s)}$$

$$p_{m,n}^{(r+1,s)} = p_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} p_{m-j,n-i}^{(r,s)} \tag{3.79}$$

Thus, after deriving the generating functions from Theorem III.2, the explicit state transformation equations are generated directly in terms of the transformed state variables through the recursive equations in Eqs. 3.78 and 3.79.

Recall that the canonical transformation $q = Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$ and $p = P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$ is defined by the coupled pair of systems

$$\frac{\partial}{\partial \epsilon} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q} \mathcal{W}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.43}$$

and

$$\frac{\partial}{\partial \gamma}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial p}\mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial q}\mathcal{V}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.44}$$

wherein the generating functions serve as the system Hamiltonian functions and the parameters $\epsilon$ and $\gamma$ act as independent variables. The inversion of the canonical transformation, defined as

$$\hat{q} = \widehat{Q}(q, p, \epsilon, \gamma, \nu) \tag{3.92}$$

$$\hat{p} = \widehat{P}(q, p, \epsilon, \gamma, \nu) \tag{3.93}$$

satisfies the complementary system equations

$$\frac{\partial}{\partial \epsilon}\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \hat{p}}\widehat{\mathcal{W}}(q, p, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial \hat{q}}\widehat{\mathcal{W}}(q, p, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.94}$$

and

$$\frac{\partial}{\partial \gamma}\begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \hat{p}}\widehat{\mathcal{V}}(q, p, \epsilon, \gamma, \nu) \\ -\frac{\partial}{\partial \hat{q}}\widehat{\mathcal{V}}(q, p, \epsilon, \gamma, \nu) \end{pmatrix} \tag{3.95}$$

in terms of the inverse generating functions $\widehat{\mathcal{W}}(q, p, \epsilon, \gamma, \nu)$ and $\widehat{\mathcal{V}}(q, p, \epsilon, \gamma, \nu)$. Wintner demonstrated that for the canonical transformation $(q, p) \to (\hat{q}, \hat{p})$ with remainder function $\mathcal{R}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)$, the remainder function for the inverse transformation $(q, p) \leftarrow (\hat{q}, \hat{p})$ is[43]

$$\widehat{\mathcal{R}} = -\mathcal{R}\Big(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu\Big) \tag{3.96}$$

implying that the inverse generating functions in Eqs. 3.94 and 3.95 are

$$\widehat{\mathcal{W}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = -\mathcal{W}\Big(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu\Big)$$
$$\widehat{\mathcal{V}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) = -\mathcal{V}\Big(Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu\Big) \tag{3.97}$$

Thus, to construct the inverse generating functions under the canonical transformation as described in Eqs. 3.97, one need only apply the DH method in the same manner as in the transformation of the Hamiltonian function. The subscript and superscript formulation is introduced such that

$$\widehat{\mathcal{W}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \widehat{\mathcal{W}}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.80}$$

$$\widehat{\mathcal{V}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \widehat{\mathcal{V}}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.81}$$

and the zeroth-order terms correspond to the negative identity transformation

$$\widehat{\mathcal{W}}_{m,n}^{(0,0)} = -\mathcal{W}_{m,n}(\hat{q},\hat{p},\nu)$$

$$\widehat{\mathcal{V}}_{m,n}^{(0,0)} = -\mathcal{V}_{m,n}(\hat{q},\hat{p},\nu) \tag{3.82}$$

The transformed inverse generating functions are then constructed through the DH recursive equations

$$\widehat{\mathcal{W}}_{m,n}^{(r,s+1)} = \widehat{\mathcal{W}}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{j+1,i+1}} \widehat{\mathcal{W}}_{m-j,n-i}^{(r,s)}$$

$$\widehat{\mathcal{W}}_{m,n}^{(r+1,s)} = \widehat{\mathcal{W}}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{j+1,i+1}} \widehat{\mathcal{W}}_{m-j,n-i}^{(r,s)} \tag{3.83}$$

The inverse state transformation equations are constructed using the DH method in the same manner as the forward state transformation equations, but starting from Eqs. 3.94 and 3.95 instead of Eqs. 3.43 and 3.44. Thus, the inverse transformation equations are represented by the expansion series

$$\hat{q} = \widehat{Q}(q,p,\epsilon,\nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \hat{q}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.84}$$

$$\hat{p} = \widehat{P}(q,p,\epsilon,\nu) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \hat{p}_{0,0}^{(m,n)}(q,p,\nu) \tag{3.85}$$

with $\hat{q}_{0,0}^{(0,0)} = q$, $\hat{p}_{0,0}^{(0,0)} = p$, and $\hat{q}_{m,n}^{(0,0)} = \hat{p}_{m,n}^{(0,0)} = 0$ for $m + n > 0$ and are generated through the recursive equations

$$\hat{q}_{m,n}^{(r,s+1)} = \hat{q}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{W}}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)}$$

$$\hat{q}_{m,n}^{(r+1,s)} = \hat{q}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{V}}_{j+1,i+1}} \hat{q}_{m-j,n-i}^{(r,s)} \qquad (3.98)$$

$$\hat{p}_{m,n}^{(r,s+1)} = \hat{p}_{m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{W}}_{j+1,i+1}} \hat{p}_{m-j,n-i}^{(r,s)}$$

$$\hat{p}_{m,n}^{(r+1,s)} = \hat{p}_{m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\widehat{\mathcal{V}}_{j+1,i+1}} \hat{p}_{m-j,n-i}^{(r,s)} \qquad (3.99)$$

**QED**

### 3.3.3   Homological Equations and Averaging

Having presented and proved the two-parameter DH method in Theorem III.2 and the state transformation equations in Corollary III.3, the practical application of the method to perturbed Hamiltonian systems is presented here. Starting with an expanded Hamiltonian function in the form

$$\mathcal{H}(q, p, \epsilon, \gamma, \nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\gamma^m}{m!}\frac{\epsilon^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) \qquad (3.32)$$

the application of Eq. 3.35 and Eqs. 3.37 through 3.40 in the two-parameter DH method yields a series of first-order partial differential equations - one for each order in the expansion. Through order 3, the un-mixed ($m = 0$ and/or $n = 0$) equations

are

$$\mathcal{K}_{0,0} = \mathcal{H}_{0,0}^{(0,0)}$$

$$\mathcal{K}_{0,1} = \mathcal{H}_{0,0}^{(0,1)} + \mathcal{S}_{0,0}^{(0,0)} = -\frac{\partial \mathcal{W}_{1,1}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{0,1}^{(0,0)}$$

$$\mathcal{K}_{1,0} = \mathcal{H}_{0,0}^{(1,0)} + \mathcal{T}_{0,0}^{(0,0)} = -\frac{\partial \mathcal{V}_{1,1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,0}^{(0,0)}$$

$$\mathcal{K}_{0,2} = \mathcal{H}_{0,0}^{(0,2)} + \mathcal{S}_{0,0}^{(0,1)} = -\frac{\partial \mathcal{W}_{1,2}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{1,2}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{0,2}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}} \left( \mathcal{H}_{0,1}^{(0,0)} + \mathcal{K}_{0,1} \right)$$

$$\mathcal{K}_{2,0} = \mathcal{H}_{0,0}^{(2,0)} + \mathcal{T}_{0,0}^{(1,0)} = -\frac{\partial \mathcal{V}_{2,1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{2,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{2,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}} \left( \mathcal{H}_{1,0}^{(0,0)} + \mathcal{K}_{1,0} \right)$$

$$\mathcal{K}_{0,3} = \mathcal{H}_{0,0}^{(0,3)} + \mathcal{S}_{0,0}^{(0,2)} = -\frac{\partial \mathcal{W}_{1,3}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{1,3}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{0,3}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,2}} \left( 2\mathcal{H}_{0,1}^{(0,0)} + \mathcal{K}_{0,1} \right)$$
$$+ \mathcal{L}_{\mathcal{W}_{1,1}} \left( \mathcal{H}_{0,2}^{(0,0)} + 2\mathcal{K}_{0,2} - \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{K}_{0,1} \right)$$

$$\mathcal{K}_{3,0} = \mathcal{H}_{0,0}^{(3,0)} + \mathcal{T}_{0,0}^{(2,0)} = -\frac{\partial \mathcal{V}_{3,1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{3,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{3,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{2,1}} \left( 2\mathcal{H}_{1,0}^{(0,0)} + \mathcal{K}_{1,0} \right)$$
$$+ \mathcal{L}_{\mathcal{V}_{1,1}} \left( \mathcal{H}_{2,0}^{(0,0)} + 2\mathcal{K}_{2,0} - \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{K}_{1,0} \right) \tag{3.100}$$

which are consistent with the single-parameter equations presented in the original papers of Deprit and Hori.[36;37] In addition, the mixed-variable terms are those appearing in the transformed Hamiltonian with non-zero powers of both $\epsilon$ and $\gamma$. This is a departure from the single-parameter case, which obviously has no such mixing of multiple parameters. The lowest order mixed term is $\mathcal{K}_{1,1}$, which is derived from either of the two formulations

$$\mathcal{K}_{1,1} = \mathcal{H}_{0,0}^{(1,1)} + \mathcal{S}_{0,0}^{(1,0)} = -\frac{\partial \mathcal{W}_{2,1}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{2,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{H}_{1,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{K}_{0,1}$$
$$= \mathcal{H}_{0,0}^{(1,1)} + \mathcal{T}_{0,0}^{(0,1)} = -\frac{\partial \mathcal{V}_{1,2}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{1,2}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,1}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{H}_{0,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{K}_{1,0}$$
$$\tag{3.101}$$

Likewise, the terms $\mathcal{K}_{1,2}$ and $\mathcal{K}_{2,1}$ are derived from

$$
\begin{aligned}
\mathcal{K}_{1,2} = \mathcal{H}_{0,0}^{(1,2)} + \mathcal{S}_{0,0}^{(1,1)} &= -\frac{\partial \mathcal{W}_{2,2}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{2,2}}\mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,2}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,2}}\mathcal{H}_{1,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}}\mathcal{K}_{0,2} \\
&\quad + \mathcal{L}_{\mathcal{W}_{2,1}}\left(\mathcal{K}_{0,1} + \mathcal{H}_{0,1}^{(0,0)}\right) + \mathcal{L}_{\mathcal{W}_{1,1}}\left(\mathcal{H}_{1,1}^{(0,0)} + \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{V}_{1,1}}\mathcal{K}_{0,1}\right) \\
= \mathcal{H}_{0,0}^{(1,2)} + \mathcal{T}_{0,0}^{(0,2)} &= -\frac{\partial \mathcal{V}_{1,3}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{1,3}}\mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{1,2}^{(0,0)} + 2\mathcal{L}_{\mathcal{V}_{1,2}}\mathcal{H}_{0,1}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}}\mathcal{H}_{0,2}^{(0,0)} \\
&\quad + \mathcal{L}_{\mathcal{W}_{1,2}}\mathcal{K}_{1,0} + \mathcal{L}_{\mathcal{W}_{1,1}}\left(2\mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{K}_{1,0}\right) \\
\mathcal{K}_{2,1} = \mathcal{H}_{0,0}^{(2,1)} + \mathcal{S}_{0,0}^{(2,0)} &= -\frac{\partial \mathcal{V}_{2,2}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{2,2}}\mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{2,1}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{2,1}}\mathcal{H}_{0,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{K}_{2,0} \\
&\quad + \mathcal{L}_{\mathcal{V}_{1,2}}\left(\mathcal{K}_{1,0} + \mathcal{H}_{1,0}^{(0,0)}\right) + \mathcal{L}_{\mathcal{V}_{1,1}}\left(\mathcal{H}_{1,1}^{(0,0)} + \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{K}_{1,0}\right) \\
= \mathcal{H}_{0,0}^{(2,1)} + \mathcal{T}_{0,0}^{(1,1)} &= -\frac{\partial \mathcal{W}_{3,1}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{3,1}}\mathcal{H}_{0,0}^{(0,0)} + \mathcal{H}_{2,1}^{(0,0)} + 2\mathcal{L}_{\mathcal{W}_{2,1}}\mathcal{H}_{1,0}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{H}_{2,0}^{(0,0)} \\
&\quad + \mathcal{L}_{\mathcal{V}_{2,1}}\mathcal{K}_{0,1} + \mathcal{L}_{\mathcal{V}_{1,1}}\left(2\mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{V}_{1,1}}\mathcal{K}_{0,1}\right)
\end{aligned}
\tag{3.102}
$$

This pattern continues for higher orders wherein each mixed term appearing in the original Hamiltonian has two available avenues for deriving the corresponding transformed term. The pair of formulations are distinguished by a pair of complementary generating functions that are related according to the Deprit commutation condition

$$
0 = \frac{\partial \mathcal{V}}{\partial \epsilon} - \frac{\partial \mathcal{W}}{\partial \gamma} + \mathcal{L}_{\mathcal{W}}\mathcal{V}
\tag{3.74}
$$

where at each order in the expansion of $\mathcal{W}$ and $\mathcal{V}$, the complementary generating functions satisfy Eq. 3.74 in the expanded form

$$
0 = \mathcal{V}_{m+1,n+2} - \mathcal{W}_{m+2,n+1} + \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\mathcal{L}_{\mathcal{W}_{j+1,i+1}}\mathcal{V}_{m-j+1,n-i+1}
\tag{3.75}
$$

From Eqs. 3.75, one may derive the ordered Deprit commutation conditions

$$\mathcal{V}_{1,2} - \mathcal{W}_{2,1} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{1,1} = 0$$

$$\mathcal{V}_{2,2} - \mathcal{W}_{3,1} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{2,1} + \mathcal{L}_{\mathcal{W}_{2,1}}\mathcal{V}_{1,1} = 0$$

$$\mathcal{V}_{1,3} - \mathcal{W}_{2,2} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{1,2} + \mathcal{L}_{\mathcal{W}_{1,2}}\mathcal{V}_{1,1} = 0$$

$$\mathcal{V}_{1,4} - \mathcal{W}_{2,3} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{1,3} + 2\mathcal{L}_{\mathcal{W}_{1,2}}\mathcal{V}_{1,2} + \mathcal{L}_{\mathcal{W}_{1,3}}\mathcal{V}_{1,1} = 0$$

$$\mathcal{V}_{3,2} - \mathcal{W}_{4,1} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{3,1} + 2\mathcal{L}_{\mathcal{W}_{2,1}}\mathcal{V}_{2,1} + \mathcal{L}_{\mathcal{W}_{3,1}}\mathcal{V}_{1,1} = 0$$

$$\mathcal{V}_{2,3} - \mathcal{W}_{3,2} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{2,2} + \mathcal{L}_{\mathcal{W}_{2,1}}\mathcal{V}_{1,2} + \mathcal{L}_{\mathcal{W}_{1,2}}\mathcal{V}_{2,1} + \mathcal{L}_{\mathcal{W}_{2,2}}\mathcal{V}_{1,1} = 0 \qquad (3.103)$$

and likewise for higher-orders. Thus, one need only derive a solution for one of the two complementary mixed-parameter generating functions and use Eqs. 3.103 to solve for the other (for example, solve the homological differential equation for $\mathcal{W}_{2,1}$ then apply the first of Eqs. 3.103 to determine $\mathcal{V}_{1,2}$ or vice versa).

Whether one is applying the DH method about a single parameter or multiple parameters, the key operation is to solve the homological equation appearing in Eqs. 3.100, 3.101 and 3.102 in either of the forms

$$\frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} - \mathcal{L}_{\mathcal{W}_{i+1,j}}\mathcal{H}_{0,0} = \mathcal{Q}_{i,j} - \mathcal{K}_{i,j} \qquad (3.104)$$

or

$$\frac{\partial \mathcal{V}_{i,j+1}}{\partial \nu} - \mathcal{L}_{\mathcal{V}_{i,j+1}}\mathcal{H}_{0,0} = \mathcal{P}_{i,j} - \mathcal{K}_{i,j} \qquad (3.105)$$

All of the terms included in $\mathcal{H}_{0,0}$, $\mathcal{Q}_{i,j}$, and $\mathcal{P}_{i,j}$ are known prior to solving the homological equation, either from the expansion of the original Hamiltonian function or from previously derived terms of lesser order. The goal is then to prescribe a desired form for $\mathcal{K}$ and solve the homological equations for each of the corresponding terms in the expanded generating function. However, one must take some care in choosing the form of $\mathcal{K}$ in order to insure a realizable transformation. Since the transformation

is formulated in terms of expansion series the DH method is convergent to the real solution if the state transformation equations are everywhere analytic in the original, real-valued coordinate system. This insures that the expanded solution converges to the true solution as more and more terms are included in the expansion. To remain analytic and locally convergent, the secular variations in the system response must remain intact within the transformed phase space. However, one may still average out the periodic variations by defining the transformed Hamiltonian function in terms of the functional average

$$\mathcal{K}_{i,j} = < \mathcal{Q}_{i,j} > \text{ or } < \mathcal{P}_{i,j} > \tag{3.106}$$

where $\mathcal{Q}_{i,j}$ and $\mathcal{P}_{i,j}$ are the terms appearing in the homological equations shown in Eqs. 3.104 and 3.105. When expressed in action-angle variables, the averaging normalization in Eq. 3.106 eliminates periodic variations in the independent variable $\nu$ and fast variables $\theta_i$, leaving only secular variations in the slow variables $I_i$. Thus, while one speaks of prescribing a desired form for $\mathcal{K}$, in fact the desired form is itself dictated by the conditions of the averaging method.

In any event, having prescribed the form for $\mathcal{K}$, the solution to the ordered series of homological equations defines the necessary ordered generating functions, $\mathcal{W}_i$ and $\mathcal{V}_i$, that achieve the desired transformation. The form of the homological equations is essentially the same at all orders and depends primarily on the form of the unperturbed system $\mathcal{H}_{0,0}^{(0,0)}$. Therefore, one may immediately determine the practicality of applying the DH method based on the form of the unperturbed system and whether it permits a solution to the homological equation.

## 3.4 Incorporation of Control

To incorporate control authority in the perturbed Hamiltonian system, one may append the original Hamiltonian function with a non-autonomous forcing function $\mathcal{U}(q, p, \epsilon, \gamma, \nu)$. Expanding about the two parameters $\epsilon$ and $\gamma$ results in the controlled Hamiltonian function

$$
\begin{aligned}
\mathcal{H}_c(q, p, \epsilon, \gamma, \nu) &= \mathcal{H}(q, p, \epsilon, \gamma, \nu) + \mathcal{U}(q, p, \epsilon, \gamma, \nu) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m \, \epsilon^n}{m! \, n!} \left( \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) + \mathcal{U}_{m,n}^{(0,0)}(q, p, \nu) \right)
\end{aligned}
\tag{3.107}
$$

which may be normalized using the DH method in the same manner as the uncontrolled system, but in place of $\mathcal{H}$ one inserts $\mathcal{H}_c$. Since the forcing function is non-autonomous, it will also contribute to the remainder function $\mathcal{R} \rightarrow \mathcal{R}_c$ and the generating functions $\mathcal{W} \rightarrow \mathcal{W}_c$ and $\mathcal{V} \rightarrow \mathcal{V}_c$. Nonetheless, under the same general assumptions stated previously, the controlled Hamiltonian function in Eq. 3.107 may be normalized using the DH method in the same manner as the uncontrolled case resulting in a transformed controlled Hamiltonian function in the form

$$
\begin{aligned}
\mathcal{K}_c &= \mathcal{H}_c \Big( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \Big) + \mathcal{R}_c(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\
&= \mathcal{H} \Big( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \Big) \\
&\quad + \mathcal{U} \Big( Q(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), P(\hat{q}, \hat{p}, \epsilon, \gamma, \nu), \epsilon, \gamma, \nu \Big) + \mathcal{R}_c(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) \\
&= \widehat{\mathcal{H}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) + \widehat{\mathcal{U}}(\hat{q}, \hat{p}, \epsilon, \gamma, \nu) + \mathcal{R}_c(\hat{q}, \hat{p}, \epsilon, \gamma, \nu)
\end{aligned}
\tag{3.108}
$$

where $\widehat{\mathcal{H}}$ and $\widehat{\mathcal{U}}$ are the uncontrolled Hamiltonian function and forcing function written explicitly in terms of the state transformation equations and $\mathcal{R}_c$ is the controlled remainder function. The question persists as to how the state transformation $(q, p) \rightarrow (\hat{q}, \hat{p})$ is itself affected by the additional forcing function. Consider, the

homologic equation in the controlled case

$$\frac{\partial \mathcal{W}_{c,i+1,j}}{\partial \nu} - \mathcal{L}_{\mathcal{W}_{c,i+1,j}} \mathcal{H}_{c,0,0}^{(0,0)} = \mathcal{Q}_{c,i,j} - \mathcal{K}_{c,i,j} \tag{3.109}$$

where $\mathcal{Q}_{c,i,j}$ is comprised of terms known *a priori* and $\mathcal{K}_{c,i,j}$ is the portion of Eq. 3.108 at order $\gamma^i$ and $\epsilon^j$. The controlled generating function $\mathcal{W}_{c,i+1,j}$ is then defined by the solution to the homological equation in Eq. 3.109, which will depend in large part on the form of the controlled unperturbed system

$$\mathcal{H}_{c,0,0}^{(0,0)} = \mathcal{H}_{0,0}^{(0,0)} + \mathcal{U}_{0,0}^{(0,0)} \tag{3.110}$$

However, for the purposes of this study, it is assumed that the unperturbed system already possesses ideal properties of integrability and stability and thereby represents the desired solution. Therefore, setting the zeroth order forcing function term to zero

$$\mathcal{U}_{0,0}^{(0,0)} \equiv 0 \tag{3.111}$$

the unperturbed system is left in its uncontrolled form. As such, the controller is in the form of a perturbation-damping controller. In other applications, it may be desired to incorporate control at the zeroth-order for the purposes of affecting the unperturbed system as well. However, in this case, Eq. 3.110 implies that control terms only appear on the right-hand side of Eq. 3.109 within $\mathcal{Q}_{c,i,j}$. Therefore, the conjecture is made that if the forcing function is in the same form as the original Hamiltonian function then the solution to Eq. 3.109 may be de-coupled into a sum of independent generating functions

$$\mathcal{W}_c = \mathcal{W} + \mathcal{W}_u \tag{3.112}$$

and likewise

$$\mathcal{V}_c = \mathcal{V} + \mathcal{V}_u \tag{3.113}$$

The resultant controlled homological equation then takes either the form

$$
\begin{aligned}
\mathcal{K}_{c,i,j} = &-\frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{i,j} \\
&- \frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{u,i,j} \\
&+ \mathcal{U}_{i,j}^{(0,0)} + \mathcal{Q}_{c,i,j}
\end{aligned}
\tag{3.114}
$$

or

$$
\begin{aligned}
\mathcal{K}_{c,i,j} = &-\frac{\partial \mathcal{V}_{i,j+1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{i,j+1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{P}_{i,j} \\
&- \frac{\partial \mathcal{V}_{u,i,j+1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{u,i,j+1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{P}_{u,i,j} \\
&+ \mathcal{U}_{i,j}^{(0,0)} + \mathcal{P}_{c,i,j}
\end{aligned}
\tag{3.115}
$$

where $\mathcal{Q}_{u,i,j}$, $\mathcal{Q}_{c,i,j}$, $\mathcal{P}_{u,i,j}$, and $\mathcal{P}_{c,i,j}$ are known *a priori* to solving the homological equation. The first line in Eq. 3.114 corresponds to the uncontrolled result

$$\mathcal{K}_{i,j} = -\frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{i,j} \tag{3.116}$$

while the remaining terms $\mathcal{U}_{i,j}^{(0,0)}$ and $\mathcal{W}_{u,i+1,j}$ are designed as to modify $\mathcal{K}_{i,j}$ into the desired form for $\mathcal{K}_{c,i,j}$ through the control law

$$
\begin{aligned}
\mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} = &-\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{u,i,j} \\
&+ \mathcal{U}_{i,j}^{(0,0)} + \mathcal{Q}_{c,i,j}
\end{aligned}
\tag{3.117}
$$

and likewise for $\mathcal{V}_{u,i,j+1}$.

There are effectively two control functions introduced in the control law in Eq. 3.117:

the input function $\mathcal{U}_{i,j}^{(0,0)}$ and its corresponding generating function $\mathcal{W}_{u,i+1,j}$. As such, there is some flexibility in how one may design the control scheme in order to achieve some desired form for the controlled Hamiltonian function $\mathcal{K}_c$. One approach is to define the control input by

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \tag{3.118}$$

such that Eq. 3.117 dictates the condition

$$-\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{u,i,j} = 0 \tag{3.119}$$

which effectively sets the control generating functions $\mathcal{W}$ and $\mathcal{V}$ to zero. Thus, this control formulation acts on the secular terms in the transformed Hamiltonian function without affecting the underlying state transformation equations. An alternative approach is to define the controlled generating function by

$$\begin{aligned} 0 = {} & -\frac{\partial \mathcal{W}_{i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{i+1,j}} \mathcal{H}_{0,0}^{(0,0)} \\ & -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} \end{aligned} \tag{3.120}$$

such that $\mathcal{W}_{u,i+1,j} = -\mathcal{W}_{i+1,j}$ and the differential terms are removed from the control law. Eq. 3.117 then dictates a control input function

$$\begin{aligned} \mathcal{U}_{i,j}^{(0,0)} &= \mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} + \frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} - \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} - (\mathcal{Q}_{u,i,j} + \mathcal{Q}_{c,i,j}) \\ &= \mathcal{K}_{c,i,j} - \mathcal{H}_{i,j}^{(0,0)} \end{aligned} \tag{3.121}$$

wherein $(\mathcal{Q}_{i,j} + \mathcal{Q}_{u,i,j} + \mathcal{Q}_{c,i,j}) \rightarrow \mathcal{H}_{i,j}^{(0,0)}$ at all orders. Thus, the control law in Eq. 3.121 is expressed directly in terms of the higher-order terms in the original Hamiltonian function.

In either control strategy, one may first derive the complete transformation of the uncontrolled system in terms of the generating functions $\mathcal{W}$ and $\mathcal{V}$ yielding the transformed Hamiltonian $\mathcal{K}$. Upon deriving the uncontrolled response, Eq. 3.117 is imposed in terms of a prescribed solution for $\mathcal{K}_c$. The corresponding controlled state transformation equations are defined implicitly through the controlled generating functions $\mathcal{W}_c = \mathcal{W} + \mathcal{W}_u$ and $\mathcal{V}_c = \mathcal{V} + \mathcal{V}_u$ in the same manner as the uncontrolled system presented in Corollary III.3. The controlled state transformation equations are represented by the series

$$q = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} q_{c,0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \tag{3.122}$$

$$p = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} p_{c,0,0}^{(m,n)}(\hat{q}, \hat{p}, \nu) \tag{3.123}$$

where the zero superscript terms correspond to the identity transformation,

$$q_{c,m,n}^{(0,0)} = \begin{cases} \hat{q} & m = n = 0 \\ 0 & m + n > 0 \end{cases} \tag{3.124}$$

and

$$p_{c,m,n}^{(0,0)} = \begin{cases} \hat{p} & m = n = 0 \\ 0 & m + n > 0 \end{cases} \tag{3.125}$$

and are otherwise generated from

$$q_{c,m,n}^{(r,s+1)} = q_{c,m,n+1}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{W}_{c,j+1,i+1}} q_{c,m-j,n-i}^{(r,s)}$$

$$q_{c,m,n}^{(r+1,s)} = q_{c,m+1,n}^{(r,s)} + \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{n}{i} \binom{m}{j} \mathcal{L}_{\mathcal{V}_{c,j+1,i+1}} q_{c,m-j,n-i}^{(r,s)} \tag{3.78}$$

and

$$p_{c,m,n}^{(r,s+1)} = p_{c,m,n+1}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{W}_{c,j+1,i+1}} p_{c,m-j,n-i}^{(r,s)}$$

$$p_{c,m,n}^{(r+1,s)} = p_{c,m+1,n}^{(r,s)} + \sum_{i=0}^{n}\sum_{j=0}^{m} \binom{n}{i}\binom{m}{j} \mathcal{L}_{\mathcal{V}_{c,j+1,i+1}} p_{c,m-j,n-i}^{(r,s)} \tag{3.79}$$

To see this control scheme in effect, consider the zeroth and first-order controlled homological equations

$$
\begin{aligned}
\mathcal{K}_{c,0,0} &= \mathcal{H}_{c,0,0}^{(0,0)} &&= \mathcal{K}_{0,0}\\
\mathcal{K}_{c,0,1} &= \mathcal{H}_{c,0,0}^{(0,1)} + \mathcal{S}_{c,0,0}^{(0,0)} &&= \mathcal{K}_{0,1} + \mathcal{U}_{0,1}^{(0,0)} - \frac{\partial \mathcal{W}_{u,1,1}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,1,1}} \mathcal{H}_{0,0}^{(0,0)}\\
\mathcal{K}_{c,1,0} &= \mathcal{H}_{c,0,0}^{(1,0)} + \mathcal{T}_{c,0,0}^{(0,0)} &&= \mathcal{K}_{1,0} + \mathcal{U}_{1,0}^{(0,0)} - \frac{\partial \mathcal{V}_{u,1,1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{u,1,1}} \mathcal{H}_{0,0}^{(0,0)}
\end{aligned}
\tag{3.126}
$$

and the second-order equations

$$
\begin{aligned}
\mathcal{K}_{c,0,2} &= \mathcal{H}_{c,0,0}^{(0,2)} + \mathcal{S}_{c,0,0}^{(0,1)} = \mathcal{K}_{0,2} + \mathcal{U}_{0,2}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}}\left(\mathcal{U}_{0,1}^{(0,0)} + \mathcal{U}_{0,0}^{(0,1)} + \mathcal{S}_{u,0,0}^{(0,0)}\right)\\
&\quad - \frac{\partial \mathcal{W}_{u,1,2}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,1,2}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{u,1,1}}\left(\mathcal{H}_{c,0,1}^{(0,0)} + \mathcal{H}_{c,0,0}^{(0,1)} + \mathcal{S}_{c,0,0}^{(0,0)}\right)\\
\mathcal{K}_{c,2,0} &= \mathcal{H}_{c,0,0}^{(2,0)} + \mathcal{T}_{c,0,0}^{(1,0)} = \mathcal{K}_{2,0} + \mathcal{U}_{2,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}}\left(\mathcal{U}_{1,0}^{(0,0)} + \mathcal{U}_{0,0}^{(1,0)} + \mathcal{T}_{u,0,0}^{(0,0)}\right)\\
&\quad - \frac{\partial \mathcal{V}_{u,2,1}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{u,2,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{u,1,1}}\left(\mathcal{H}_{c,1,0}^{(0,0)} + \mathcal{H}_{c,0,0}^{(1,0)} + \mathcal{T}_{c,0,0}^{(0,0)}\right)\\
\mathcal{K}_{c,1,1} &= \mathcal{H}_{c,0,0}^{(1,1)} + \mathcal{S}_{c,0,0}^{(1,0)} = \mathcal{K}_{1,1} + \mathcal{U}_{1,1}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}}\mathcal{U}_{0,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}}\left(\mathcal{U}_{0,0}^{(1,0)} + \mathcal{T}_{u,0,0}^{(0,0)}\right)\\
&\quad - \frac{\partial \mathcal{V}_{u,1,2}}{\partial \nu} + \mathcal{L}_{\mathcal{V}_{u,1,2}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{u,1,1}} \mathcal{H}_{c,0,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{u,1,1}}\left(\mathcal{H}_{c,0,0}^{(1,0)} + \mathcal{T}_{c,0,0}^{(0,0)}\right)\\
&= \mathcal{H}_{c,0,0}^{(1,1)} + \mathcal{T}_{c,0,0}^{(0,1)} = \mathcal{K}_{1,1} + \mathcal{U}_{1,1}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{U}_{1,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{1,1}}\left(\mathcal{U}_{0,0}^{(0,1)} + \mathcal{S}_{u,0,0}^{(0,0)}\right)\\
&\quad - \frac{\partial \mathcal{W}_{u,2,1}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,2,1}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{L}_{\mathcal{W}_{u,1,1}} \mathcal{H}_{c,1,0}^{(0,0)} + \mathcal{L}_{\mathcal{V}_{u,1,1}}\left(\mathcal{H}_{c,0,0}^{(0,1)} + \mathcal{S}_{c,0,0}^{(0,0)}\right)
\end{aligned}
$$

$$\tag{3.127}$$

Within these controlled homological equations, the control laws are designed using

Eqs. 3.118 and 3.119, Eqs. 3.120 and 3.121, or some other formulation of Eq. 3.117. In the first case, the control generating functions are $\mathcal{W}_{u,i,j} = \mathcal{V}_{u,i,j} = 0$ and the control laws to first and second-order are

$$
\begin{aligned}
\mathcal{U}_{0,1}^{(0,0)} &= \mathcal{K}_{c,0,1} - \mathcal{K}_{0,1} \\
\mathcal{U}_{1,0}^{(0,0)} &= \mathcal{K}_{c,1,0} - \mathcal{K}_{1,0}
\end{aligned}
\tag{3.128}
$$

and

$$
\begin{aligned}
\mathcal{U}_{0,2}^{(0,0)} &= \mathcal{K}_{c,0,2} - \mathcal{K}_{0,2} - \mathcal{L}_{\mathcal{W}_{1,1}} \left( \mathcal{U}_{0,1}^{(0,0)} + \mathcal{U}_{0,0}^{(0,1)} \right) \\
\mathcal{U}_{2,0}^{(0,0)} &= \mathcal{K}_{c,2,0} - \mathcal{K}_{2,0} - \mathcal{L}_{\mathcal{V}_{1,1}} \left( \mathcal{U}_{1,0}^{(0,0)} + \mathcal{U}_{0,0}^{(1,0)} \right) \\
\mathcal{U}_{1,1}^{(0,0)} &= \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{U}_{0,1}^{(0,0)} - \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{U}_{0,0}^{(1,0)} \\
&= \mathcal{K}_{c,1,1} - \mathcal{K}_{1,1} - \mathcal{L}_{\mathcal{W}_{1,1}} \mathcal{U}_{1,0}^{(0,0)} - \mathcal{L}_{\mathcal{V}_{1,1}} \mathcal{U}_{0,0}^{(0,1)}
\end{aligned}
\tag{3.129}
$$

Alternatively, one could impose the control generating functions $\mathcal{W}_{u,i,j} = -\mathcal{W}_{i,j}$ and $\mathcal{V}_{u,i,j} = -\mathcal{V}_{i,j}$ such that the control laws are expressed directly in terms of the higher-order terms in the original Hamiltonian function

$$
\begin{aligned}
\mathcal{U}_{0,1}^{(0,0)} &= \mathcal{K}_{c,0,1} - \mathcal{H}_{0,1}^{(0,0)} \\
\mathcal{U}_{1,0}^{(0,0)} &= \mathcal{K}_{c,1,0} - \mathcal{H}_{1,0}^{(0,0)} \\
\mathcal{U}_{0,2}^{(0,0)} &= \mathcal{K}_{c,0,2} - \mathcal{H}_{0,2}^{(0,0)} \\
\mathcal{U}_{2,0}^{(0,0)} &= \mathcal{K}_{c,2,0} - \mathcal{H}_{2,0}^{(0,0)} \\
\mathcal{U}_{1,1}^{(0,0)} &= \mathcal{K}_{c,1,1} - \mathcal{H}_{1,1}^{(0,0)}
\end{aligned}
\tag{3.130}
$$

No matter how the control laws are formulated, the aim is to prescribe a desired form for the controlled Hamiltonian function $\mathcal{K}_c$ and derive the corresponding series of control functions that achieve the desired form within the DH transformation.

## 3.5  Nonlinear Oscillator with Damping

To validate the two-parameter DH method and controlled DH method, a test case is presented in the form of a nonlinear oscillator with external damping. The standard equation of motion for the nonlinear oscillator with damping is

$$\frac{dq^2}{dt^2} + \alpha \frac{dq}{dt} + \omega_0^2 \sin q = 0 \tag{3.131}$$

where $q$ is the angular displacement, $\omega_0$ is the natural, undamped frequency, and $\alpha$ is the damping coefficient. The dynamics may be converted into a Hamiltonian formulation by applying the method discussed in Huang and Lin with the resultant Hamiltonian function

$$\mathcal{H}(q, p, t) = e^{-\alpha t} \left( \frac{p^2}{2} - e^{2\alpha t} \omega_0^2 \cos q \right) \tag{3.132}$$

where $p = e^{\alpha t} \dot{q}$ is the generalized momenta.[54] The origin is an equilibrium point at which the Hamiltonian function assumes the time-dependent function

$$\mathcal{H}(0, 0, t) = -\omega_0^2 e^{\alpha t} \tag{3.133}$$

The function in Eq. 3.133 may be subtracted from the general Hamiltonian function in Eq. 3.132 without affecting the underlying dynamics such that the Hamiltonian function assumes a value of zero at the equilibrium point. Expanding the resultant Hamiltonian function about the origin and the zero-damping case yields the series

representation

$$\mathcal{H}(q, p, t) = \frac{p^2}{2} \left( 1 - \alpha t + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right) \right)$$
$$+ \omega_0^2 \left( 1 + \alpha t + \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right) \right)$$
$$\left( \frac{q^2}{2} - \frac{\gamma^2 q^4}{4!} + \frac{\gamma^4 q^6}{6!} - \frac{\gamma^6 q^8}{8!} + \mathrm{O}\left(\gamma^8\right) \right) \qquad (3.134)$$

where $\gamma$ parameterizes the scale of the even-powered nonlinear terms. Due to the form of Eq. 3.132, only even-powers of $q$ and $\gamma$ appear in the expansion. Action-angle type variables are introduced in terms of the canonical transformation generating function

$$\mathcal{S}(q, \theta) = \frac{\omega_0}{2} q^2 \tan^{-1} \theta \qquad (3.135)$$

or directly in terms of the state transformation equations

$$q = \sqrt{\frac{2I}{\omega_0}} \sin \theta$$
$$p = \sqrt{2I\omega_0} \cos \theta \qquad (3.136)$$

Expressed in terms of action-angle type variables, the expanded Hamiltonian function is

$$\mathcal{H}(\theta, I, t) = \omega_0 I \left( 1 - \alpha t \cos 2\theta + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{3!} \cos 2\theta + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right) \right)$$
$$+ \left( 1 + \alpha t + \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right) \right)$$
$$\left( -\frac{\gamma^2 I^2}{6} \sin^4 \theta + \frac{\gamma^4 I^3}{90\omega_0} \sin^6 - \frac{\gamma^6 I^4}{2520\omega_0^2} \sin^8 \theta + \mathrm{O}\left(\gamma^8\right) \right) \qquad (3.137)$$

In the spirit of the DH method, the expanded Hamiltonian function in Eq. 3.137 is represented by the series

$$\mathcal{H}(\theta, I, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma^m}{m!} \frac{\alpha^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(\theta, I, t) \tag{3.138}$$

where $\mathcal{H}_{m,n}^{(0,0)}$ are non-autonomous functions of the action-angle variables and the time.

The unperturbed system is the linearized oscillator with zero damping

$$\mathcal{H}_{0,0}^{(0,0)} = \omega_0 I = \frac{1}{2} \left( p^2 + \omega_0 q^2 \right) \tag{3.139}$$

which possesses an integral of motion in the action variable $I$ such that the phase space is foliated by invariant tori parameterized by the constant value $I_0 = I(t_0)$ and wound linearly by $\theta(t) = \theta(t_0) + \omega_0(t - t_0)$. Being a single degree of freedom system, the invariant tori correspond to 2-dimensional circles in the planar phase space as shown in Figure 3.3. Points farther from the origin correspond to larger values of $I_0$ and thereby larger maximum displacements in $q$. A typical response for the unperturbed
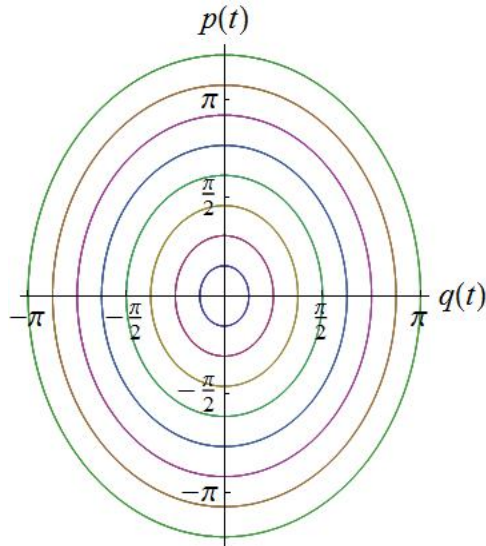


Figure 3.3: Linearized Oscillator with Zero Damping

system is shown in Fig. 3.4 in terms of the state trajectories as functions of time.
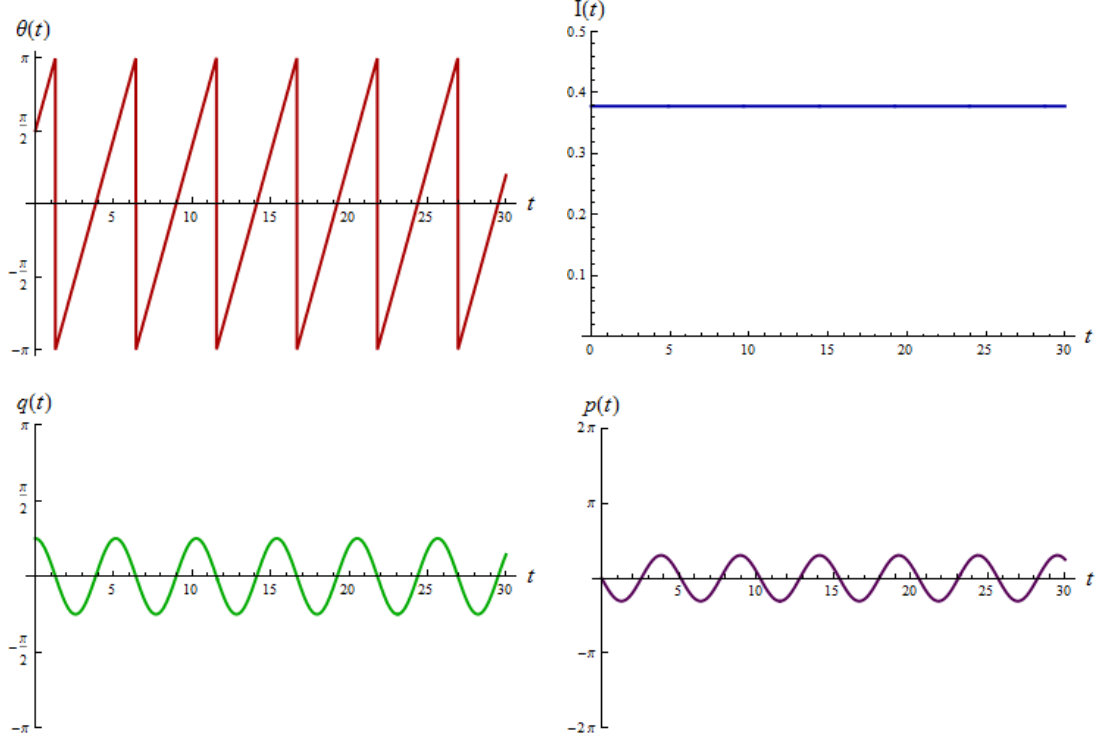
Figure 3.4: Linearized Oscillator Response

The perturbed system incorporates the nonlinear terms and time-dependent terms in Eqs. 3.132 and 3.138 and is treated using the non-autonomous two-parameter DH method presented in Theorem III.2 and Corollary III.3. The transformation $\mathcal{H}(\theta, I, t) \rightarrow \mathcal{K}(\hat{\theta}, \hat{I}, t)$ is defined implicitly through the pair of generating functions $\mathcal{W}(\hat{\theta}, \hat{I}, t)$ and $\mathcal{V}(\hat{\theta}, \hat{I}, t)$ which are themselves derived from the solution to the homological differential equations in either of the two forms

$$\frac{\partial \mathcal{W}_{m,n}}{\partial t} + \omega_0 \frac{\partial \mathcal{W}_{m,n}}{\partial \theta} = \mathcal{Q}_{m,n} - \mathcal{K}_{m,n}$$
$$\frac{\partial \mathcal{V}_{m,n}}{\partial t} + \omega_0 \frac{\partial \mathcal{V}_{m,n}}{\partial \theta} = \mathcal{P}_{m,n} - \mathcal{K}_{m,n} \qquad (3.140)$$

where the terms $\mathcal{K}_{m,n}$ constitute the transformed Hamiltonian function and the terms $\mathcal{Q}_{m,n}$ and $\mathcal{P}_{m,n}$ are known prior to solving the homological equations. Many of these terms are zero due to the form of Eq. 3.132 including $\mathcal{P}_{1,0} = \mathcal{Q}_{0,2} = \mathcal{Q}_{1,1} = 0$ among

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others. Some of the lower-order cases that are not zero are listed in Eqs. 3.141

$$\mathcal{Q}_{0,1} = -\omega_0 t I \cos 2\theta$$

$$\mathcal{P}_{2,0} = -\frac{I^2}{3} \sin^4 \theta$$

$$\mathcal{Q}_{0,3} = -t^2 I \cos \theta \sin \theta$$

$$\mathcal{P}_{2,1} = \frac{I^2}{8\omega_0} \left( \sin 2\theta + \omega_0 t \left( 1 - \cos 2\theta - \cos 4\theta \right) \right)$$

$$\mathcal{Q}_{0,4} = -\frac{3It^2}{2\omega_0}$$

$$\mathcal{P}_{4,0} = -\frac{I^3}{160\omega_0} \left( 15 - 35 \cos 2\theta + 2 \cos 4\theta + 3 \cos 6\theta \right)$$

$$\mathcal{Q}_{2,2} = -\frac{I^2 t}{32\omega_0} \left( 19 \sin 2\theta + 4 \sin 4\theta + 4\omega_0 t (1 - \cos 2\theta - \cos 4\theta) \right) \qquad (3.141)$$

Defining the transformed Hamiltonian by the secular part of the original Hamiltonian yields

$$\mathcal{K}(\hat{\theta}, \hat{I}, t) = \omega_0 \hat{I} \left( 1 - \frac{\alpha^4 t^2}{16\omega_0^2} + \frac{\alpha^6 t^2}{576\omega_0^4} (-3 + 16\omega_0^2 t^2) + \dots \right)$$

$$+ \frac{\hat{I}^2}{16} \left( -1 + \alpha t - \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t(-67 + 8\omega_0^2 t^2)}{48\omega_0^2} - \frac{\alpha^4 t^2 (9 + 8\omega_0^2 t^2)}{192\omega_0^2} + \dots \right)$$

$$+ \frac{\hat{I}^3}{128\omega_0} \left( -\frac{1}{2} + \alpha t + \frac{\alpha^2 (221 - 96\omega_0^2 t^2)}{96\omega_0^2} + \dots \right) + \dots \qquad (3.142)$$

which is integrable up to the order of truncation. The generating functions that achieve this transformation are computed from the homological equations at each

order of $\alpha$ and $\gamma$ with the first few non-zero terms being

$$\mathcal{W}_{1,1} = -\frac{\hat{I}}{4\omega_0}\left(\cos 2\hat{\theta} + 2\omega_0 t \sin 2\hat{\theta}\right)$$

$$\mathcal{W}_{1,3} = \frac{\hat{I}}{8\omega_0^3}\left[\left(-1 + 2\omega_0^2 t^2\right)\cos 2\hat{\theta} - 2\omega_0 t \sin 2\hat{\theta}\right]$$

$$\mathcal{W}_{3,1} = \frac{\hat{I}^2}{384\omega_0^2}\left[-48 - 32\cos 2\hat{\theta} + 5\cos 4\hat{\theta} + 4\omega_0 t\left(-8\sin 2\hat{\theta} + \sin 4\hat{\theta}\right)\right]$$

$$\mathcal{V}_{2,1} = -\frac{\hat{I}^2}{96\omega_0}\left(-8\sin 2\hat{\theta} + \sin 4\hat{\theta}\right)$$

$$\mathcal{V}_{2,2} = -\frac{\hat{I}^2}{128\omega_0^2}\left[8 + 12\cos 2\hat{\theta} + \cos 4\hat{\theta} + 4\omega_0 t\left(2\sin 2\hat{\theta} + \sin 4\hat{\theta}\right)\right] \qquad (3.143)$$

The mixed-parameter term $\mathcal{K}_{2,1} = \hat{I}^2 t/16$ is derived from either of the two complementary generating functions $\mathcal{W}_{3,1}$ and $\mathcal{V}_{2,2}$, which are related by the Deprit commutation condition,

$$\mathcal{V}_{2,2} - \mathcal{W}_{3,1} + \mathcal{L}_{\mathcal{W}_{2,1}}\mathcal{V}_{1,1} + \mathcal{L}_{\mathcal{W}_{1,1}}\mathcal{V}_{2,1} = 0 \qquad (3.144)$$

Truncating to sufficiently high-order, the transformed Hamiltonian function in Eq. 3.142 is in Birkhoff normal form possessing the action-type variable $\hat{I}$ as an integral of motion. The equations of motion are

$$\frac{d\theta}{dt} = \frac{\partial \mathcal{K}}{\partial I} = \omega_0\left(1 - \frac{\alpha^4 t^2}{16\omega_0^2} + \frac{\alpha^6 t^2}{576\omega_0^4}(-3 + 16\omega_0^2 t^2)\right)$$
$$+ \frac{\hat{I}}{8}\left(-1 + \alpha t - \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t(-67 + 8\omega_0^2 t^2)}{48\omega_0^2} - \frac{\alpha^4 t^2(9 + 8\omega_0^2 t^2)}{192\omega_0^2}\right)$$
$$+ \frac{3\hat{I}^2}{128\omega_0}\left(-\frac{1}{2} + \alpha t + \frac{\alpha^2(221 - 96\omega_0^2 t^2)}{96\omega_0^2}\right) - \frac{5\hat{I}^3}{2048\omega_0^2} + \ldots$$
$$\frac{dI}{dt} = -\frac{\partial \mathcal{K}}{\partial \theta} = 0 \qquad (3.145)$$

wherein the angle-type variable is no longer a linear function of time, but rather varies nonlinearly.

To numerically validate the transformation, set the natural frequency to $\omega_0 =$

$\sqrt{1.5}$ rads/s, the damping coefficient to $\alpha = 0.05$, and consider an initial angular displacement of $q(0) = \pi/4$ radians released from rest. The equations of motion associated with the full nonlinear Hamiltonian function in Eq. 3.132 are integrated with respect to time in the original state variables $(q, p)$ resulting in the expected damped oscillatory behavior seen in Figure 3.5.
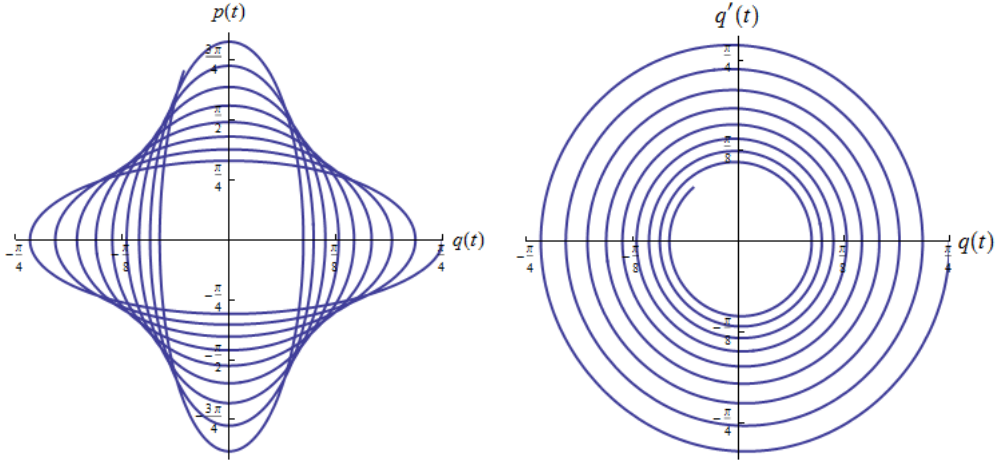


Figure 3.5: Phase Portrait for the Damped Oscillator

To simplify the system formulation, one may now apply the two-parameter DH method about the linearized, undamped oscillator. Using the Deprit inverse state transformation equations, the initial conditions of the $(q, p)$ and $(\theta, I)$ domain are converted into initial conditions in the transformed $(\hat{\theta}, \hat{I})$ domain. The trajectories in the transformed phase space are computed in closed-form from Eq. 3.145 and converted back into the original $(q, p)$ and $(\theta, I)$ phase space. This provides an approximate, local solution to the original system that converges to the true solution as more terms are included in the expansion. The resultant state trajectory is plotted in Figs. 3.6 and 3.7 for increasing orders in the expansion (dashed lines) as compared to true solution (solid lines). As the truncation order increases, the solution converges to the true solution. Note that the amplitude of the generalized momenta increases exponentially as $t \to \infty$ while the angular displacement and angular velocity converge on the origin.

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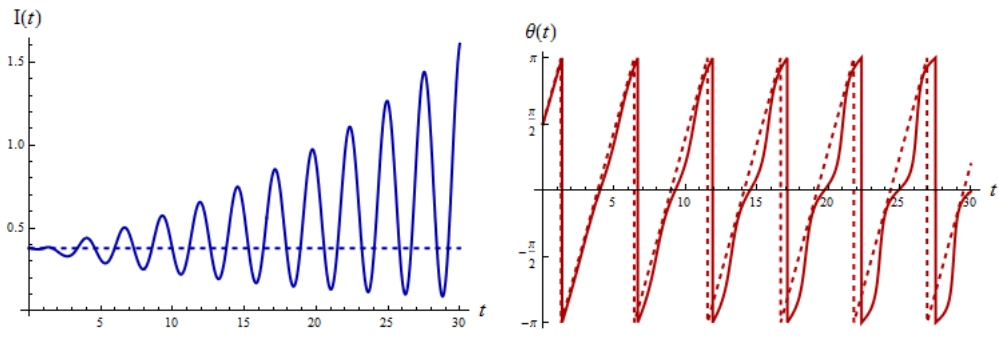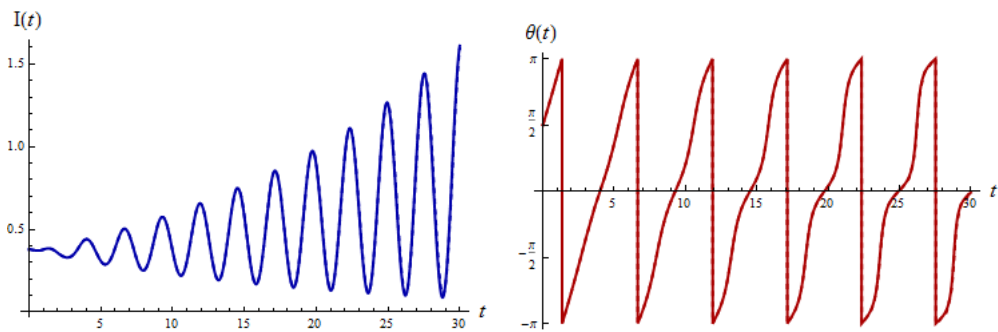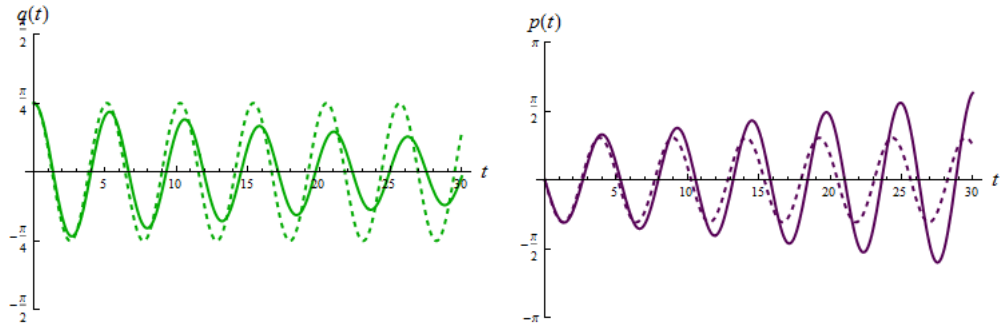
(a) 0th-Order Solution
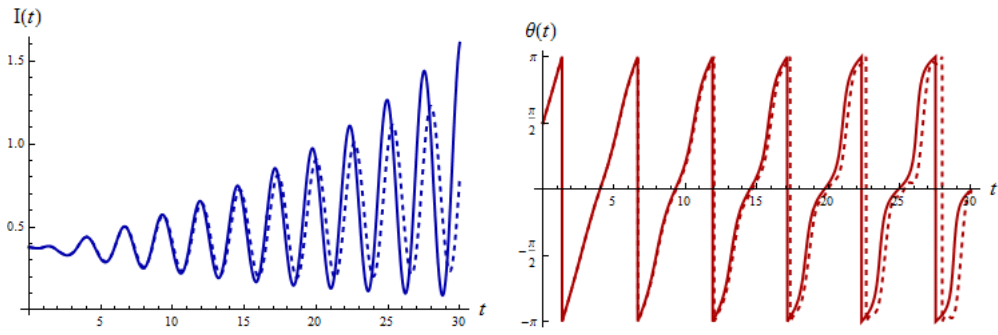
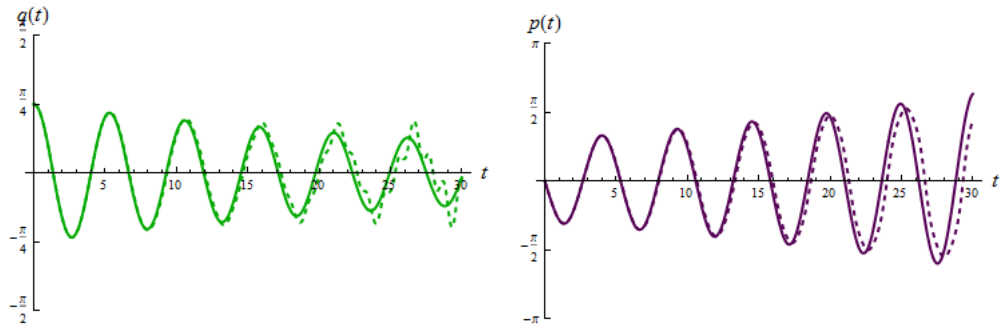(b) 2nd-Order Solution
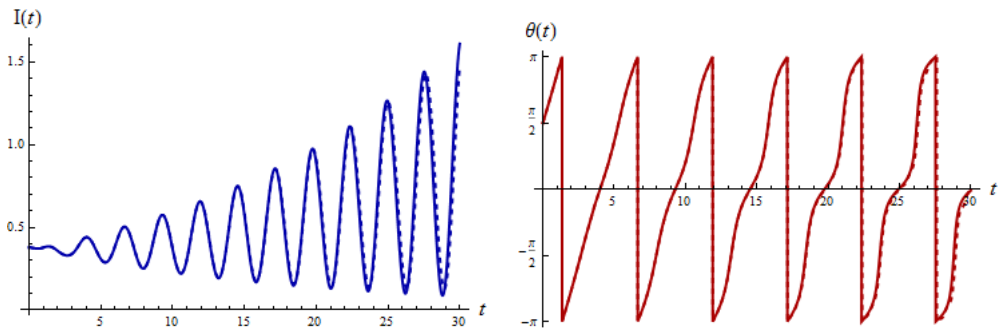
(c) 4th-Order Solution

(d) 6th-Order Solution
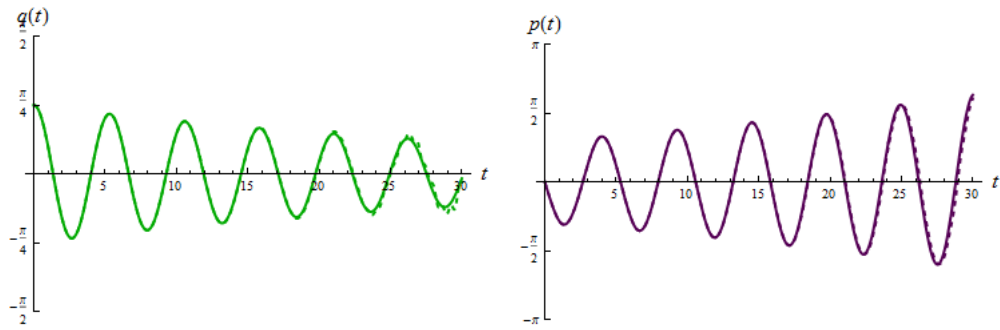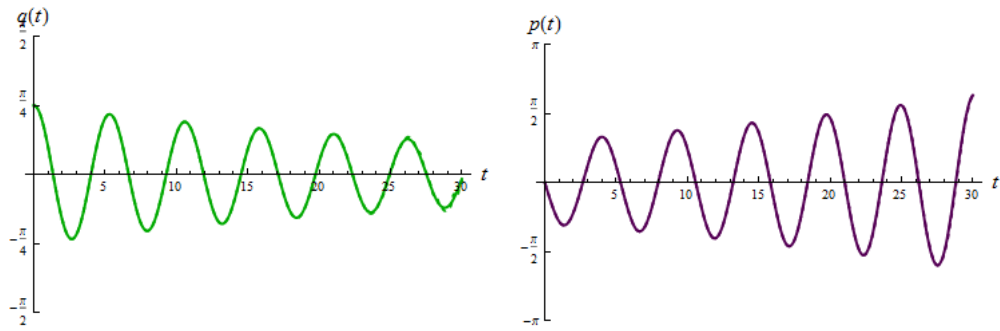
Figure 3.6: DH Convergence for the Uncontrolled Damped Oscillator in $(\theta, I)$

(a) $0^{\text{th}}$-Order Solution

(b) $2^{\text{nd}}$-Order Solution

(c) $4^{\text{th}}$-Order Solution

(d) $6^{\text{th}}$-Order Solution

Figure 3.7: DH Convergence for the Uncontrolled Damped Oscillator in $(q, p)$

Having analyzed the natural dynamics of the damped oscillator, one may now consider the implementation of control using the method detailed in Section 3.4. Since the natural motion of the damped oscillator converges toward the origin of the $(q, \dot{q})$ phase space, one may consider the control problem of eliminating or at least slowing down the decay rate. A practical example for such a problem could be the active control of a perpetual oscillator subject to damping.

A conservative approach to achieving this controlled behavior is to eliminate all the secular variations in the system response by prescribing a controlled transformed Hamiltonian function in the form

$$\mathcal{K}_c = \mathcal{K}_{0,0} = \omega_0 \hat{I} \tag{3.146}$$

This effectively forces the system to the unperturbed form, that is, the linearized oscillator. The control law is defined at each order in the expansion by Eq. 3.117

$$\mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} = -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{u,i,j}$$
$$+ \mathcal{U}_{i,j}^{(0,0)} + \mathcal{Q}_{c,i,j} \tag{3.117}$$

wherein $\mathcal{K}_{c,i,j} = 0$ for $i + j > 0$ and $\mathcal{H}_{0,0}^{(0,0)} = \omega_0 I$. Two control strategies were discussed in Section 3.4 and are applied in the sequel for the case of the damped oscillator. First, consider the case where the control generating functions are zero, $\mathcal{W}_u = \mathcal{V}_u = 0$ and the control law reduces to

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \qquad i + j \neq 0 \tag{3.147}$$

Based on the transformed Hamiltonian function given in Eq. 3.142 the control law

expressed in the transformed domain takes the explicit form

$$
\begin{aligned}
\mathcal{U}(\hat{\theta}, \hat{I}, t) = &-\bigg[ \omega_0 \hat{I} \left( -\frac{\alpha^4 t^2}{16\omega_0^2} + \frac{\alpha^6 t^2}{576\omega_0^4}(-3 + 16\omega_0^2 t^2) \right) \\
&+ \frac{\hat{I}^2}{16} \left( -1 + \alpha t - \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t(-67 + 8\omega_0^2 t^2)}{48\omega_0^2} - \frac{\alpha^4 t^2 (9 + 8\omega_0^2 t^2)}{192\omega_0^2} \right) \\
&+ \frac{\hat{I}^3}{128\omega_0} \left( -\frac{1}{2} + \alpha t + \frac{\alpha^2(221 - 96\omega_0^2 t^2)}{96\omega_0^2} \right) - \frac{5\hat{I}^4}{8192\omega_0^2} + \dots \bigg] \qquad (3.148)
\end{aligned}
$$

The corresponding control law in the original phase space is

$$
\begin{aligned}
\mathcal{U}(\theta, I, t) = &\frac{I\alpha^4 t^2}{16\omega_0} + \frac{I^2}{384\omega_0^2} \bigg( 9\alpha^2 + 24\omega_0^2(1 - \alpha t + 2\alpha^2 t^2 - 2\alpha t(1 - \alpha t)\cos 2\theta) \\
&- \alpha^2(3 - 12\omega_0^2 t^2)\cos 4\theta + 24\alpha\omega_0(1 - \alpha t)\sin 2\theta - 12\alpha^2 \omega_0 t \sin 4\theta \bigg) \\
&+ \frac{I^3}{768\omega_0^2} \left( 3\omega_0 + 8\omega_0 \cos 2\theta - 2\omega_0 \cos 4\theta \right) + \dots \qquad (3.149)
\end{aligned}
$$

Enforcing the control law for the numerical example introduced previously ($\omega_0 = \sqrt{1.5}$ rads/s, $\alpha = 0.05$, $q(0) = \pi/4$) results in the controlled response shown in Figs. 3.8 and 3.9 for increasing orders in the expansion represented by dashed lines while the un-controlled, damped response is represented by solid lines. The corresponding control function profiles are shown in Fig. 3.10. Note that the zeroth-order case corresponds to uncontrolled natural dynamics with $\mathcal{U} = 0$. Otherwise, as the truncation order and magnitude of the control input increase, the effect from the control law becomes more pronounced and the controlled response converges toward the linearized oscil-lator solution presented previously in Figs. 3.3 and 3.4. The controlled action-type variable converges toward a fixed value while the state trajectories approach harmonic oscillation. For further convergence, one need only incorporate more terms in the DH normalization and resultant control law.

105
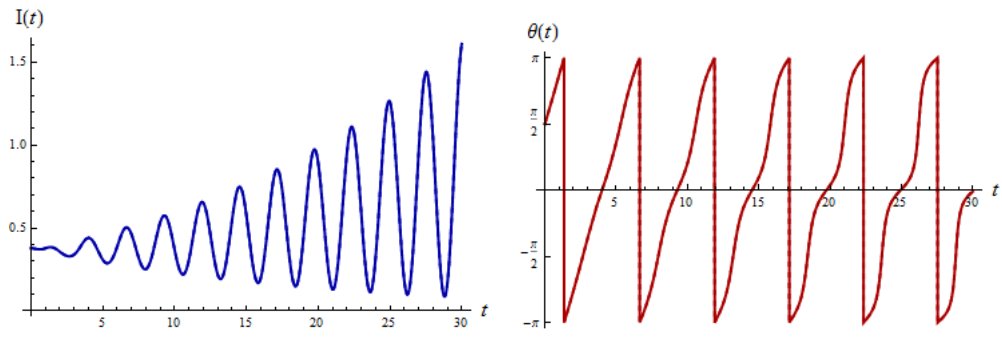
(a) $0^{\text{th}}$-Order Solution
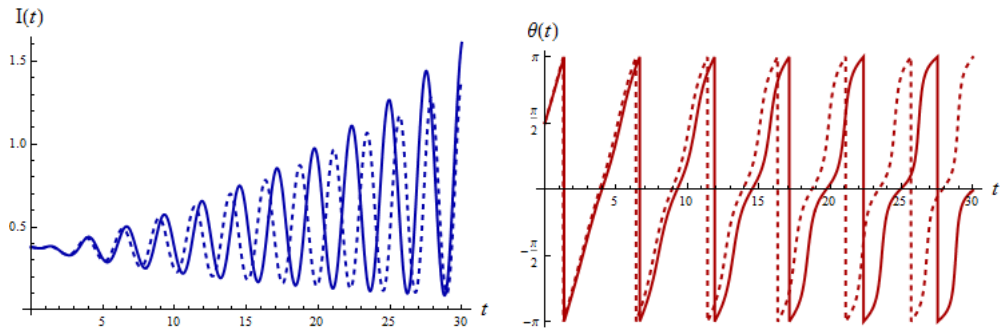
(b) $2^{\text{nd}}$-Order Solution
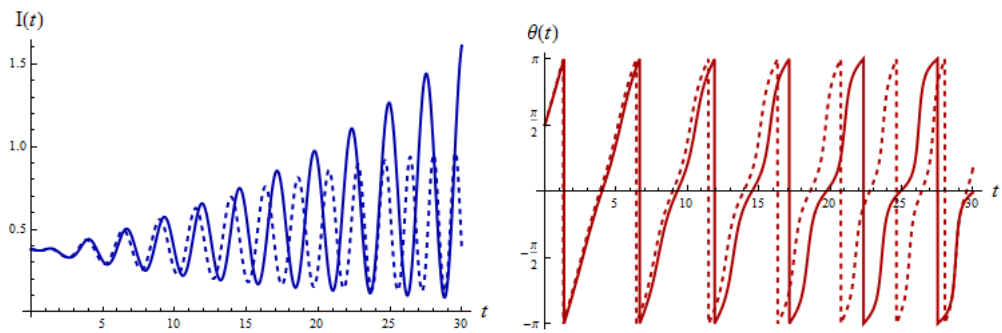
(c) $4^{\text{th}}$-Order Solution

(d) $6^{\text{th}}$-Order Solution

Figure 3.8: Controlled Response for the Damped Oscillator in $(\theta, I)$
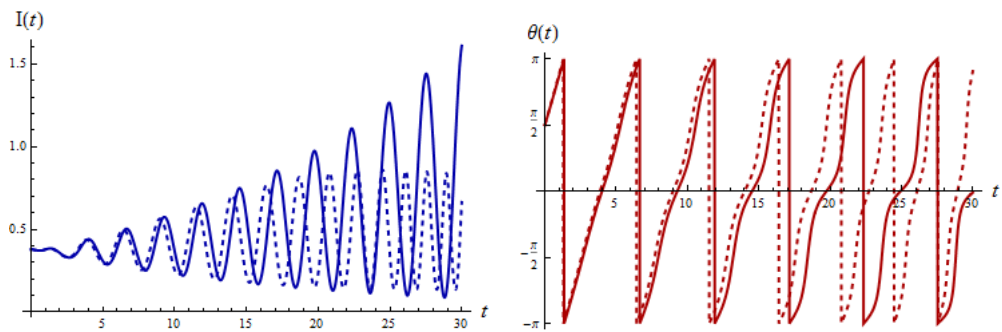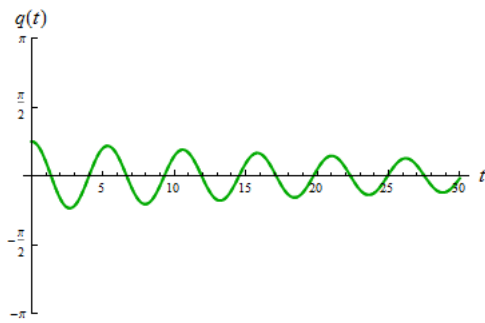
106

(a) 0th-Order Solution



(b) 2nd-Order Solution
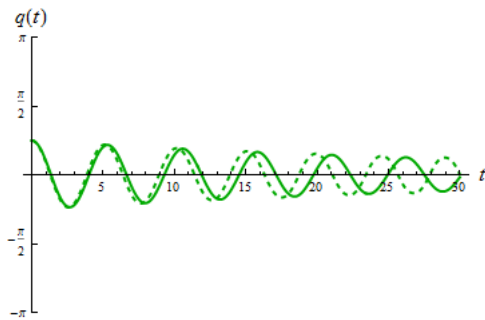


(c) 4th-Order Solution



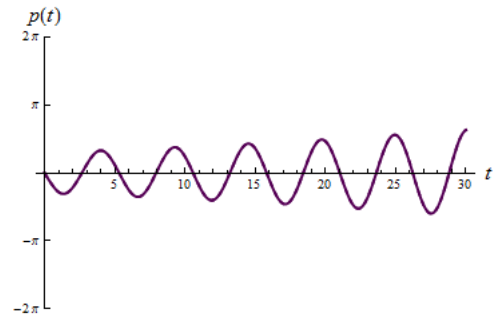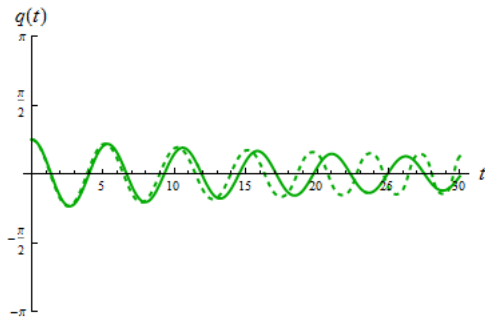(d) 6th-Order Solution

Figure 3.9: Controlled Response for the Damped Oscillator in $(q, p)$

(a) 0$^{\text{th}}$-Order Solution

(b) 2$^{\text{nd}}$-Order Solution

(c) 4$^{\text{th}}$-Order Solution

(d) 6$^{\text{th}}$-Order Solution
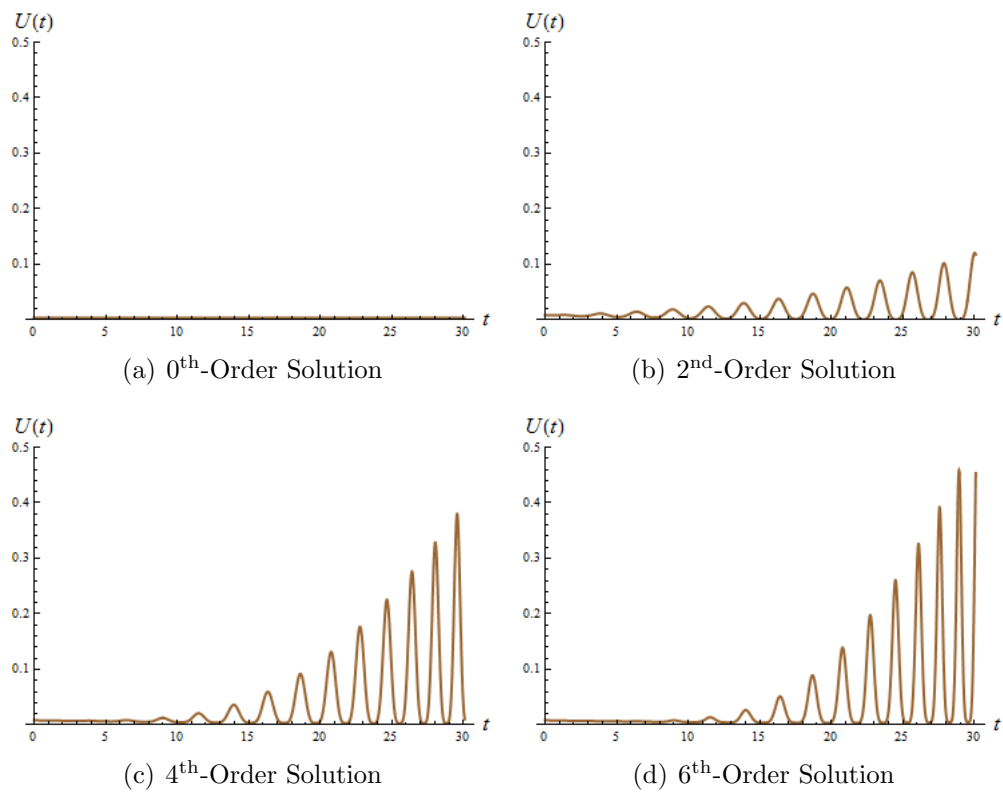
Figure 3.10: Control Function Profile

As an alternative to the previous approach, consider a control law defined by $\mathcal{W}_u = -\mathcal{W}$ and $\mathcal{V}_u = -\mathcal{V}$ and the control law

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{H}_{i,j}^{(0,0)} = -\mathcal{H}_{i,j}^{(0,0)} \qquad i + j \neq 0 \tag{3.150}$$

Since the control is designed to eliminate the higher-order perturbations, the control law takes the form of direct negative feedback on the original Hamiltonian function. This is precisely the type of control one would achieve by simply eliminating the higher-order perturbations directly from the original Hamiltonian function in Eq. 3.137. In this case, the result is somewhat trivial since it can be derived without ever applying the DH method transformation. Nonetheless, for illustration purposes, the resultant control law is

$$\begin{aligned}
\mathcal{U}(\theta, I, t) = &-\Bigg[\omega_0 I \left(-\alpha t \cos 2\theta + \frac{\alpha^2 t^2}{2} - \frac{\alpha^3 t^3}{3!}\cos 2\theta + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right)\right) \\
&+ \left(1 + \alpha t + \frac{\alpha^2 t^2}{2} + \frac{\alpha^3 t^3}{3!} + \frac{\alpha^4 t^4}{4!} + \mathrm{O}\left(\alpha^5\right)\right) \\
&\left(-\frac{\gamma^2 I^2}{6}\sin^4\theta + \frac{\gamma^4 I^3}{90\omega_0}\sin^6 - \frac{\gamma^6 I^4}{2520\omega_0^2}\sin^8\theta + \mathrm{O}\left(\gamma^8\right)\right)\Bigg]
\end{aligned} \tag{3.151}$$

and as one might expect the controlled response converges much faster as shown in Figs. 3.11 through 3.13 for the previous numerical example ($\omega_0 = \sqrt{1.5}$ rads/s, $\alpha = 0.05$, $q(0) = \pi/4$). As before, the system response converges to the linearized oscillator solution for which the action variable is constant and the angle variable varies linearly with time.

(a) $0^{\text{th}}$-Order Solution
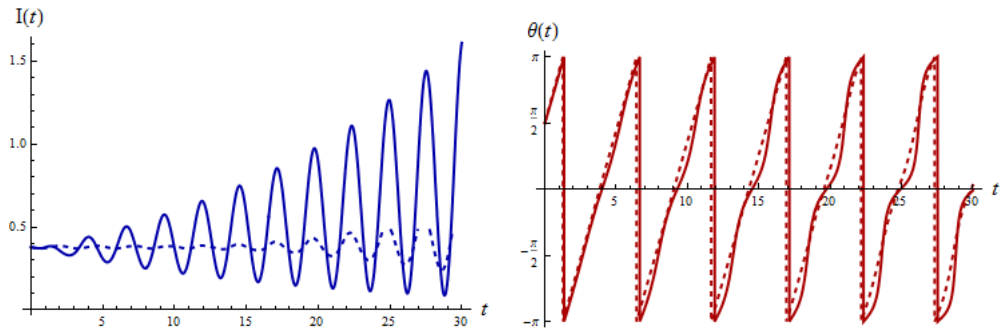
(b) $2^{\text{nd}}$-Order Solution
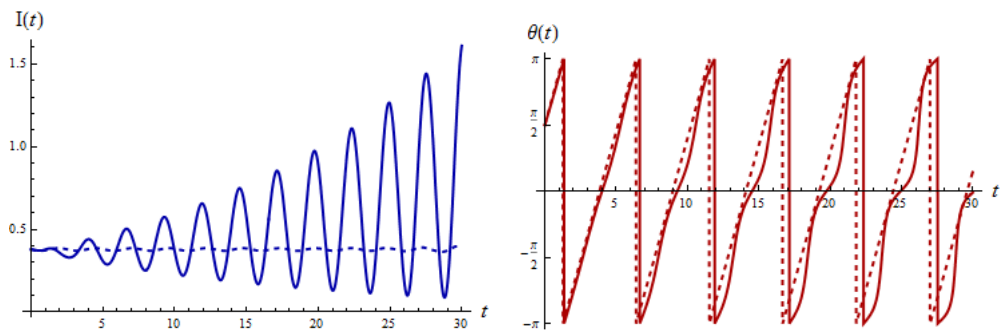
(c) $4^{\text{th}}$-Order Solution
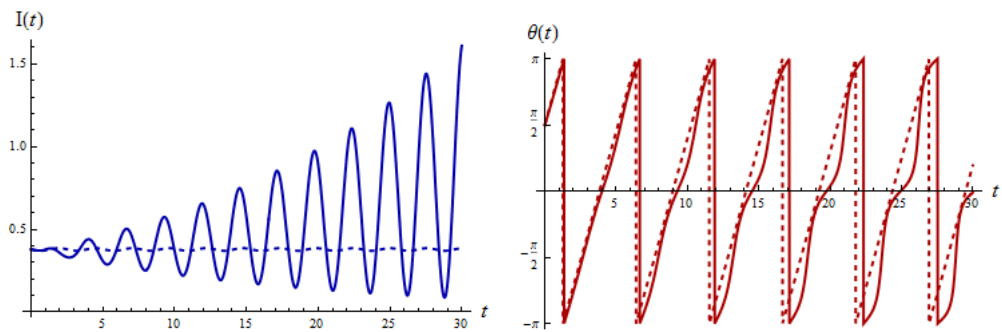
(d) $6^{\text{th}}$-Order Solution
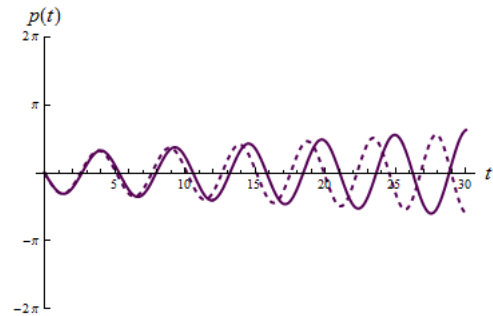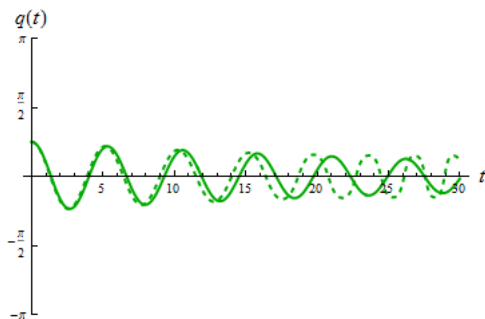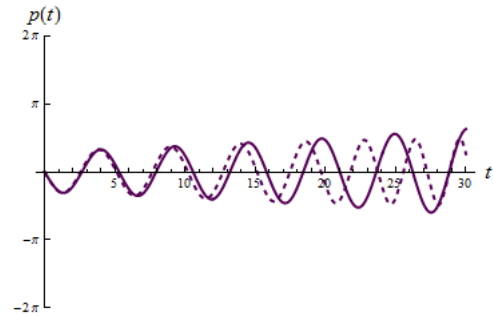
Figure 3.11: Controlled Response for the Damped Oscillator in $(\theta, I)$
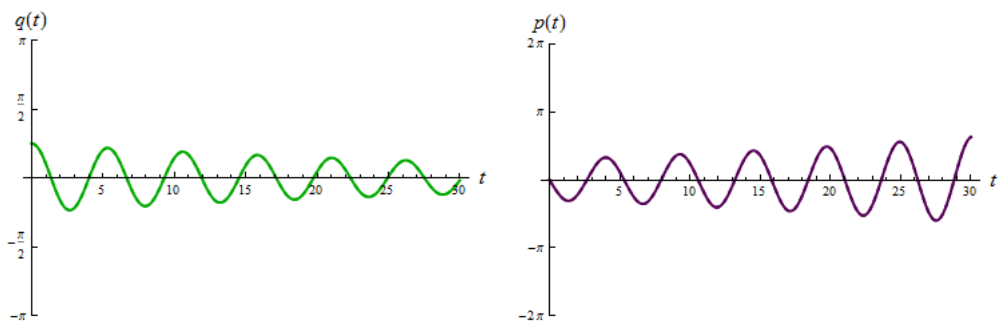
(a) 0<sup>th</sup>-Order Solution
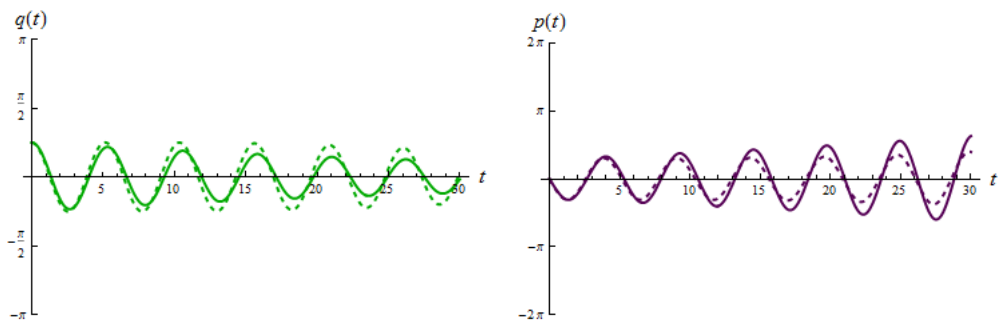


(b) 2<sup>nd</sup>-Order Solution



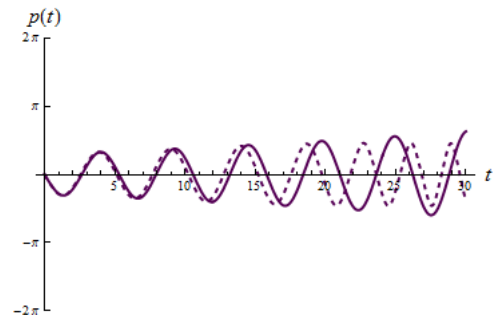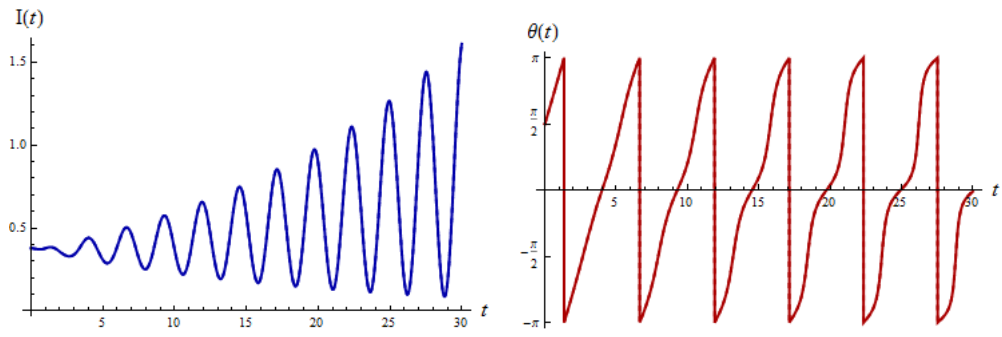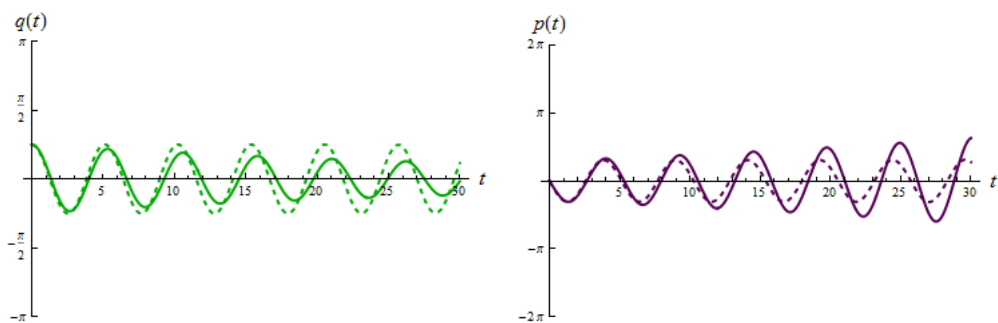(c) 4<sup>th</sup>-Order Solution
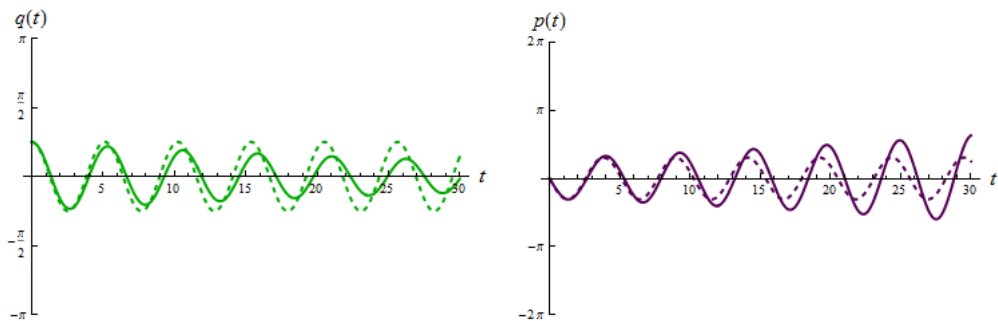


(d) 6<sup>th</sup>-Order Solution

Figure 3.12: Controlled Response for the Damped Oscillator in $(q, p)$
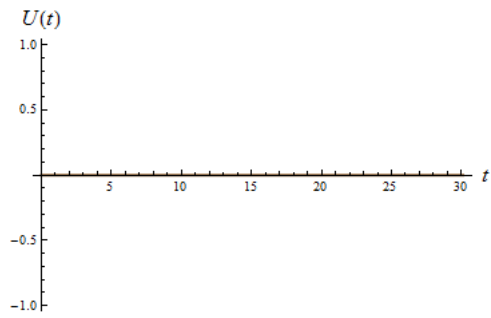
(a) $0^{\text{th}}$-Order Solution
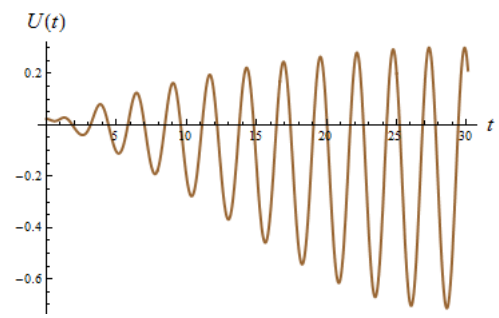
(b) $2^{\text{nd}}$-Order Solution
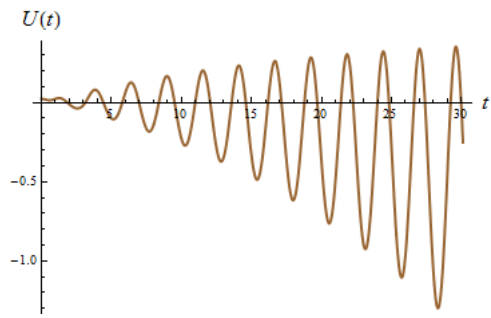
(c) $4^{\text{th}}$-Order Solution
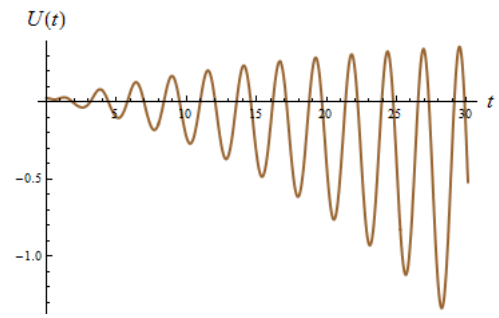
(d) $6^{\text{th}}$-Order Solution

Figure 3.13: Control Function Profile

# Chapter IV

# Dynamical Analysis

The previous chapter outlined the major facets of the classic Deprit-Hori Lie transform method (DH method) including a novel extension to two-parameter systems with dependence on the independent variable and the implementation of control within the DH method. This chapter applies the DH method specifically to the elliptic restricted-three body problem (ERTBP) and analyzes the system dynamics within the transformed phase space.

## 4.1   System Normalization

One of the most powerful methods of analytical mechanics is the process of "normalization" for dynamical systems. The term normalization or normal form is widely used with varying connotations. However, the general idea is to apply a canonical transformation to a complicated dynamical system in order to simplify its representation to something more tractable. For example, in the case of a linear system, the Jordan normal form simplifies the equations of motion by describing them along the system eigenvectors. Perturbation theory normalizes a system by decomposing it into an unperturbed, integrable part plus a series of higher-order perturbation terms. The fully perturbed system is then normalized about the unperturbed system using the classic Von Zeipel method or more rigorously through the DH method. The system is

then expressed in the so-called Birkhoff normal form, which simplifies the equations of motion by describing them along a set of local integrals of motion and angular variables winding a local invariant torus.

For the ERTBP, the system Hamiltonian function is expanded about the circular case in the hope of harnessing Jacobi's integral as an approximate integral in the non-circular case. The ERTBP Hamiltonian function was derived in Chapter II and presented in Eq. 2.55

$$\mathcal{H}(q, p, \nu) = \frac{1}{2}\left(r^2 + p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi}\right) - p_\theta - \frac{R}{p}\left(\frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) \tag{2.55}$$

where

$$r_1^2 = r^2 + \mu^2 - 2\mu r \sin \phi \cos \theta \tag{2.56}$$

$$r_2^2 = r^2 + (1 - \mu)^2 + 2(1 - \mu) r \sin \phi \cos \theta \tag{2.57}$$

The non-circular and non-autonomous effects are isolated within the multiplier $R/p = 1/(1 + e \cos \nu)$, which is expanded about the circular case in the series

$$\frac{R}{p} = \sum_{n=0}^{\infty} (-e \cos \nu)^n \tag{4.1}$$

such that the expanded Hamiltonian function is represented by

$$\mathcal{H}(q, p, \nu) = \frac{1}{2}\left(r^2 + p_r^2 + \frac{p_\phi^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \phi}\right) - p_\theta$$
$$- \left(\frac{r^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\right) \sum_{n=0}^{\infty} (-e \cos \nu)^n \tag{4.2}$$

At this point, one could attempt a normalization of Eq. 4.2 about $e = 0$ using the single parameter DH method. Doing so would result in a series of homologic equations

in the form

$$\frac{\partial \mathcal{W}_n}{\partial \nu} + \frac{\partial \mathcal{W}_n}{\partial r} p_r + \frac{\partial \mathcal{W}_n}{\partial \phi} \frac{p_\phi}{r^2} + \frac{\partial \mathcal{W}_n}{\partial \theta} \left( \frac{p_\theta}{r^2 \sin^2 \phi} - 1 \right) + \frac{\partial \mathcal{W}_n}{\partial p_r} \left( \frac{p_\phi^2}{r^3} + \frac{p_\theta^2}{r^3 \sin^2 \phi} + \frac{\partial U}{\partial r} \right)$$
$$+ \frac{\partial \mathcal{W}_n}{\partial p_\phi} \left( \frac{p_\theta^2 \cos \phi}{r^2 \sin^3 \phi} + \frac{\partial U}{\partial \phi} \right) + \frac{\partial \mathcal{W}_n}{\partial p_\theta} \frac{\partial U}{\partial \theta} = \mathcal{Q}_n - \mathcal{K}_n \tag{4.3}$$

where

$$U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \tag{4.4}$$

Eq. 4.3 is a first-order partial differential equation with respect to the generating function $\mathcal{W}$, but is highly nonlinear with respect to the state variables. A solution may exist for such a complicated nonlinear partial differential equation, in either analytical or at least numerical form, but is outside the scope of this study and is relegated to the auspices of future work.

In lieu of attempting to solve the complicated partial differential equation in Eq. 4.3, one may instead limit motion to the neighborhood of one of the triangular Lagrange points (or possibly some other point such as a collinear Lagrange point or one of the primaries). Expansion about either of the triangular Lagrange points yields the series representation

$$\mathcal{H}(\delta q, \delta p, \nu) = \frac{1}{2} \left[ \left( \frac{1}{r_e} \delta p_\theta - 2 \delta r \right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2} \delta p_\phi^2 \right]$$
$$- \frac{3}{2} \sum_{n=0}^{\infty} (-e \cos \nu)^n \left[ r_e^2 \delta r^2 + \mu (1 - \mu) \left( \cos \theta_e \delta r - r_e \sin \theta_e \delta \theta \right)^2 \right]$$
$$+ O \left( ||(\delta q, \delta p)||^3 \right) \tag{4.5}$$

where the generalized coordinates and momenta $(\delta q, \delta p)$ are defined relative to the

Lagrange point coordinates and momenta

$$r_e \cos\theta_e = \mu - \frac{1}{2} \qquad\qquad p_{r,\mathrm{e}} = 0$$

$$\phi_e = \frac{\pi}{2} \qquad\qquad p_{\phi,\mathrm{e}} = 0$$

$$r_e \sin\theta_e = \pm\sqrt{3/4} \qquad\qquad p_{\theta,\mathrm{e}} = \left(\mu - \frac{1}{2}\right)^2 + \frac{3}{4} \qquad (2.68)$$

where the $\pm$ changes sign for motion about either $L_4$ or $L_5$.

While the expansion about the circular case is easily parameterized by the eccentricity, it is less obvious how to parameterize the expansion about the Lagrange point. In order to study the motion in the vicinity of the Lagrange point, the magnitude of the nonlinear terms must be small as compared to the linear terms. One way to parameterize this constraint is to scale the state variables by $(\delta q, \delta p) \to \gamma(\delta q, \delta p)$ and taking $\gamma < 1$ such that higher-order terms remain small. Upon normalizing the system, the scaling transformation is inverted $(\delta q, \delta p) \to \gamma^{-1}(\delta q, \delta p)$ to return to the original phase space. This formally justifies the use of $\gamma$ as a parameterization of the expansion about the Lagrange point. In practice, one may simply skip the scaling transformation by setting $\gamma = 1$ and confining the solution space to the neighborhood of the Lagrange point. Nonetheless, to mirror the formulation used in Chapter III, $\gamma$ is incorporated as a small parameter of the system associated with the expansion about the Lagrange point.

In the spirit of the DH method, the doubly-expanded Hamiltonian function in Eq. 4.5 is represented as the perturbed system

$$\mathcal{H}(\delta q, \delta p, e, \gamma, \nu) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma^m}{m!} \frac{e^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(\delta q, \delta p, \nu) \qquad (4.6)$$

where $\gamma$ parameterizes the magnitude of the nonlinear terms and $\mathcal{H}_{m,n}^{(0,0)}$ corresponds to terms of order $||(\delta q, \delta p)||^{m+2}$ and $e^n$. The function $\mathcal{H}_{0,0}^{(0,0)}$ corresponds to the linearized

CRTBP, which represents the unperturbed system about which the DH method normalizes the perturbed system, that is, the nonlinear ERTBP. The motiviation for doing so lies in the ideal properties of autonomy and integrability in the unperturbed system.

### 4.1.1 Unperturbed System

The simplest approximation of the full ERTBP system is the linearized, circular case represented by the term $\mathcal{H}_{0,0}^{(0,0)}$ and referred to as the unperturbed system. It is represented through either of Eqs. 4.7 and 4.8 using spherical and Cartesian coordinates respectively.

$$\mathcal{H}_{0,0}^{(0,0)}(\delta q, \delta p) = \frac{1}{2} \left[ \left( \frac{1}{r_e} \delta p_\theta - 2\delta r \right)^2 + \delta p_r^2 + r_e^2 \delta \phi^2 + \frac{1}{r_e^2} \delta p_\phi^2 \right] \\ - \frac{3}{2} \left[ r_e^2 \delta r^2 + \mu(1-\mu) \left( \cos\theta_e \delta r - r_e \sin\theta_e \delta\theta \right)^2 \right] \quad (4.7)$$

$$\mathcal{H}_{0,0}^{(0,0)}(\delta q, \delta p) = \frac{1}{2} \left[ (\delta p_x + \delta q_y)^2 + (\delta p_y - \delta q_x)^2 + \delta p_z^2 + \delta q_z^2 \right] \\ - \frac{3}{8} \left[ \delta q_x^2 + 3\delta q_y^2 + 2\sqrt{3}(1-2\mu)\delta q_x \delta q_y \right] \quad (4.8)$$

As discussed in Chapter II, the dynamics of the linearized CRTBP are characterized by a set of purely imaginary eigenvalues within the range $0 < \mu < \mu_c = (1-\sqrt{69}/9)/2$. The ensuing motion is in the form of harmonic oscillation about the Lagrange point with natural frequencies determined by the mass ratio $\mu$ through the equations

$$\omega_s = \sqrt{\frac{1 + \sqrt{1 - 27\mu(1-\mu)}}{2}} \qquad \omega_\ell = \sqrt{\frac{1 - \sqrt{1 - 27\mu(1-\mu)}}{2}} \qquad \omega_z = 1 \quad (4.9)$$

and satisfying the inequality

$$0 < \omega_\ell < \frac{\sqrt{2}}{2} < \omega_s < 1 \tag{4.10}$$

with the unity eigenvalue identified with the linearized out-of-plane dynamics. The frequency $\omega_\ell$ corresponds to slow oscillation about the Lagrange point with period $2\pi/\omega_\ell$ while $\omega_s$ corresponds to fast, epicyclic oscillation with period $2\pi/\omega_s$. The combination of both modes yields coupled harmonic oscillation in the planar dynamics and circular oscillation in the out-of-plane.

To elucidate the oscillatory behavior of the unperturbed system, Breakwell and Pringle provide a linear transformation of the planar CRTBP within the stable regime that expresses the linearized system directly in the form of coupled harmonic oscillators.[12] Starting from the Cartesian representation in Eq. 4.8 the elements of the Breakwell and Pringle transformation are given explicitly in Breakwell and Pringle, Deprit and Rom, and Meyer and Schmidt.[12;15;28] Extending the transformation to the three-dimensional phase space and introducing the parameters

$$\alpha \triangleq 3\sqrt{3}(1 - 2\mu) \quad \text{and} \quad \beta \triangleq 1 + (\alpha^2 - 27)/4, \tag{4.11}$$

satisfying $\sqrt{23} < \alpha < 3\sqrt{3}$ and $0 < \beta < 1$ yields the linear transformation

$$\begin{pmatrix} \delta q_x \\ \delta q_y \\ \delta q_z \\ \delta p_x \\ \delta p_y \\ \delta p_z \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 9+4\omega_s^2 & -9-\omega_\ell^2 & 0 \\ -8 & -8 & 0 & -\alpha & \alpha & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1-4\omega_s^2 & -1-4\omega_\ell^2 & 0 & \alpha & -\alpha & 0 \\ \alpha & \alpha & 0 & 9-4\omega_s^2 & -9+4\omega_\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} q_s \; \omega_s/\sqrt{\sqrt{\beta}\sqrt{11+2\sqrt{\beta}}} \\ q_\ell \; \omega_\ell/\sqrt{\sqrt{\beta}\sqrt{11-2\sqrt{\beta}}} \\ q_z \\ p_s \; /\omega_s/\sqrt{\sqrt{\beta}\sqrt{11+2\sqrt{\beta}}} \\ p_\ell \; /\omega_\ell/\sqrt{\sqrt{\beta}\sqrt{11-2\sqrt{\beta}}} \\ p_z \end{pmatrix} \tag{4.12}$$

where $(q_s, q_\ell, q_z, p_s, p_\ell, p_z)$ are new generalized state variables identified with the short and long-period (SLP) modes of the system. Note that the sign conventions used in Eq. 4.12 correspond to the linear normalization about the $L_5$ Lagrange point, but

could easily be adapted for motion about $L_4$, particularly in light of the symmetry between the two. The resultant transformed circular Hamiltonian function is expressed in terms of the SLP variables as

$$\mathcal{H}_{0,0}^{(0,0)} = \frac{1}{2}\left(\omega_s^2 q_s^2 + p_s^2\right) - \frac{1}{2}\left(\omega_\ell^2 q_\ell^2 + p_\ell^2\right) + \frac{1}{2}\left(\omega_z^2 q_z^2 + p_z^2\right) \tag{4.13}$$

By expanding the non-canonical transformation from spherical coordinates to Cartesian coordinates, one may equally derive the linear transformation starting from Eq. 4.7 as

$$
\begin{pmatrix} \delta r \\ \delta \phi \\ \delta \theta \\ \delta p_r \\ \delta p_\phi \\ \delta p_\theta \end{pmatrix} = \frac{1}{2}
\begin{bmatrix}
\cos\theta_e & 0 & -r_e \sin\theta_e & 0 & 0 & 0 \\
\sin\theta_e & 0 & r_e \cos\theta_e & 0 & 0 & 0 \\
0 & -r_e & 0 & 0 & 0 & 0 \\
\sin\theta_e & 0 & -r_e \cos\theta_e & \cos\theta_e & 0 & -\sin\theta_e/r_e \\
-\cos\theta_e & 0 & -r_e \sin\theta_e & \sin\theta_e & 0 & \cos\theta_e/r_e \\
0 & 0 & 0 & 0 & -1/r_e & 0
\end{bmatrix}^{-1}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 9+4\omega_s^2 & -9-\omega_\ell^2 & 0 \\
-8 & -8 & 0 & -\alpha & \alpha & 0 \\
0 & 0 & 2 & 0 & 0 & 0 \\
-1-4\omega_s^2 & -1-4\omega_\ell^2 & 0 & \alpha & -\alpha & 0 \\
\alpha & \alpha & 0 & 9-4\omega_s^2 & -9+4\omega_\ell^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\begin{pmatrix}
q_s \ \omega_s/\sqrt{\sqrt{\beta}\sqrt{11+2\sqrt{\beta}}} \\
q_\ell \ \omega_\ell/\sqrt{\sqrt{\beta}\sqrt{11-2\sqrt{\beta}}} \\
q_z \\
p_s \ /\omega_s/\sqrt{\sqrt{\beta}\sqrt{11+2\sqrt{\beta}}} \\
p_\ell \ /\omega_\ell/\sqrt{\sqrt{\beta}\sqrt{11-2\sqrt{\beta}}} \\
p_z
\end{pmatrix}
\tag{4.14}
$$

which provides the same normalized form for the transformed Hamiltonian function

$$\mathcal{H}_{0,0}^{(0,0)} = \frac{1}{2}\left(\omega_s^2 q_s^2 + p_s^2\right) - \frac{1}{2}\left(\omega_\ell^2 q_\ell^2 + p_\ell^2\right) + \frac{1}{2}\left(\omega_z^2 q_z^2 + p_z^2\right) \tag{4.13}$$

The Hamiltonian state-space equations of motion take the convenient form

$$
\begin{aligned}
\frac{dq_i}{d\nu} &= \frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial p_i} = \pm p_i \\
\frac{dp_i}{d\nu} &= -\frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial q_i} = \mp \omega_i^2 q_i
\end{aligned}
\tag{4.15}
$$

and the corresponding orbit solutions are periodic in the form

$$
\left\{ \begin{array}{c} q_i(\nu) \\ p_i(\nu) \end{array} \right\} = \left[ \begin{array}{cc} \cos\left(\omega_i\left(\nu - \nu_0\right)\right) & \omega_i^{-1}\sin\left(\omega_i\left(\nu - \nu_0\right)\right) \\ -\omega_i\sin\left(\omega_i\left(\nu - \nu_0\right)\right) & \cos\left(\omega_i\left(\nu - \nu_0\right)\right) \end{array} \right] \left\{ \begin{array}{c} q_{i,0} \\ p_{i,0} \end{array} \right\}
\tag{4.16}
$$

for $i = s$, $l$, and $z$ and with $q_{i,0}$ and $p_{i,0}$ representing the initial conditions at $\nu = \nu_0$.

For the purposes of numerical analysis, it is often necessary to work within the framework of complex coordinates. To this end, one may introduce the transformation

$$
\begin{pmatrix} \delta q_x \\ \delta q_y \\ \delta q_z \\ \delta p_x \\ \delta p_y \\ \delta p_z \end{pmatrix} = \frac{1}{2} \left[ \begin{array}{cccccc} 0 & 0 & 0 & 9+4\omega_s^2 & -9-4\omega_\ell^2 & 0 \\ -8 & -8 & 0 & -\alpha & \alpha & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ -1-4\omega_s^2 & -1-4\omega_\ell^2 & 0 & \alpha & -\alpha & 0 \\ \alpha & \alpha & 0 & 9-4\omega_s^2 & -9+4\omega_\ell^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array} \right]
$$
$$
\left( \mathrm{Id} + i \left[ \begin{array}{cccccc} 0 & 0 & 0 & \omega_s & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_\ell & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1/\omega_s & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\omega_\ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right] \right) \begin{pmatrix} \xi_s \\ \xi_\ell \\ \xi_z \\ \eta_s \;/\left(11\sqrt{\beta}+2\beta\right)/2 \\ \eta_\ell \;/\left(11\sqrt{\beta}-2\beta\right)/2 \\ \eta_z \qquad /2 \end{pmatrix}
\tag{4.17}
$$

where $(\xi_s, \xi_\ell, \xi_z, \eta_s, \eta_\ell, \eta_z)$ are complex-valued SLP state variables and Id is the identity matrix.[15] Applying this transformation to the original unperturbed Hamiltonian funciton in Eq. 4.8 yields

$$
\mathcal{H}_{0,0}^{(0,0)} = i\omega_s\xi_s\eta_s - i\omega_\ell\xi_\ell\eta_\ell + i\xi_z\eta_z
\tag{4.18}
$$

such that the linearized equations of motion assume the decoupled form

$$
\frac{d\xi_i}{d\nu} = \frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial \eta_i} = \pm i\omega_i\xi_i
$$
$$
\frac{d\eta_i}{d\nu} = -\frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial \xi_i} = \mp i\omega_i\eta_i
\tag{4.19}
$$

and the corresponding orbit solutions decouple into

$$\xi_i(\nu) = \xi_{i,0} \left[ \cos\left( \pm\omega_i \left( \nu - \nu_0 \right) \right) + i \sin\left( \pm\omega_i \left( \nu - \nu_0 \right) \right) \right]$$

$$\eta_i(\nu) = \eta_{i,0} \left[ \cos\left( \pm\omega_i \left( \nu - \nu_0 \right) \right) + i \sin\left( \pm\omega_i \left( \nu - \nu_0 \right) \right) \right] \tag{4.20}$$

for $i = s$, $l$, and $z$ and with $\xi_{i,0}$ and $\eta_{i,0}$ representing the complex initial conditions at $\nu = \nu_0$.

Being integrable, the unperturbed system may be formulated using action-angle variables through the explicit transformation equations

$$
\begin{aligned}
I_s &\triangleq \frac{\omega_s^2 q_s^2 + p_s^2}{2\omega_s} & \tan\theta_s &\triangleq \omega_s \frac{q_s}{p_s} \\
I_\ell &\triangleq \frac{\omega_\ell^2 q_\ell^2 + p_\ell^2}{2\omega_\ell} & \tan\theta_\ell &\triangleq \omega_\ell \frac{q_\ell}{p_\ell} \\
I_z &\triangleq \frac{\omega_z^2 q_z^2 + p_z^2}{2\omega_z} & \tan\theta_z &\triangleq \omega_z \frac{q_z}{p_z}
\end{aligned}
\tag{4.21}
$$

or equivalently

$$
\begin{aligned}
I_s &\triangleq i\xi_s\eta_s & \tan\theta_s &\triangleq \frac{\xi_s + i\omega_s\eta_s/2/\alpha}{i\xi_s + \omega_s\eta_s/2/\alpha} \\
I_\ell &\triangleq i\xi_\ell\eta_\ell & \tan\theta_\ell &\triangleq \frac{\xi_\ell + i\omega_\ell\eta_\ell/2/\alpha}{i\xi_\ell + \omega_\ell\eta_\ell/2/\alpha} \\
I_z &\triangleq i\xi_z\eta_z & \tan\theta_z &\triangleq \frac{\xi_s + i\omega_z\eta_z/2}{i\xi_z + \omega_z\eta_z/2}
\end{aligned}
\tag{4.22}
$$

with $\alpha = -9 + 14\omega_s^2 + 8\omega_s^4 = 13 - 30\omega_\ell^2 + 8\omega_\ell^4$. The resultant Hamiltonian function is expressed in Birkhoff normal form

$$\mathcal{H}_{0,0}^{(0,0)} = \omega_s\, I_s - \omega_\ell\, I_\ell + \omega_z I_z \tag{4.23}$$

with the corresponding equations of motion are

$$\frac{d\theta_i}{d\nu} = \frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial I_i} = \pm\omega_i$$

$$\frac{dI_i}{d\nu} = -\frac{\partial \mathcal{H}_{0,0}^{(0,0)}}{\partial \theta_i} = 0 \tag{4.24}$$

and the orbit solution is

$$\theta_i(\nu) = \theta_{i,0} \pm \omega_i(\nu - \nu_0) \pmod{2\pi}$$

$$I_i(\nu) = I_{i,0} \tag{4.25}$$

for $i = s$, $l$, and $z$ and with $\theta_{i,0}$ and $I_{i,0}$ representing the action-angle initial conditions at $\nu = \nu_0$. The integrability of the unperturbed system is evident directly in terms of the conserved action variables $I_i$, which together define $n$ integrals of motion parameterizing an $n$-dimensional torus on which the trajectories are wound by the linearly $\nu$-varying angular variables.

For the purposes of applying KAM theory in the planar dynamics, the unperturbed system must either satisfy the non-degeneracy conditions in Eqs. 2.91 and/or 2.92, or be in the proper Birkhoff normal form presented at the end of Chapter II. In this case, since the natural frequencies of the unperturbed system are constant, the system is properly degenerate such that the classic KAM theorem does not apply. Therefore, one must resort to the second approach, that of normalizing the perturbed system into Birkhoff normal form. Further, one must take care to avoid conditions of resonance in the frequencies, which can de-stabilize the system regardless of its degenerate properties. The first example of resonance was seen in Figure 2.8 at the bifurcation point $\mu = \mu_c$. Resonance also occurs at mass ratios whose ratio of natural frequencies $\omega_s/\omega_\ell$ approaches an integer value. For example, the 2:1 resonance occurs

at

$$\frac{\omega_s}{\omega_\ell} = 2 \quad \implies \quad \mu = \mu_2 = \frac{45 - \sqrt{1833}}{90} \approx 0.0242939 \tag{4.26}$$

and the 3:1 resonance occurs at

$$\frac{\omega_s}{\omega_\ell} = 3 \quad \implies \quad \mu = \mu_3 = \frac{15 - \sqrt{213}}{30} \approx 0.013516 \tag{4.27}$$

While it is possible to achieve stability in a resonant case, the conditions are more strenuous. In the case of the CRTBP, Meyer et al demonstrated that motion about the triangular Lagrange points in either of the 2:1 resonant case $\mu = \mu_2$ or the 3:1 resonant case $\mu = \mu_3$ is unstable.[16] Elsewhere in the range $0 < \mu < \mu_c$, motion about the triangular points is nonlinearly stable as demonstrated in the sequel.

### 4.1.2 Perturbed System

The unperturbed system $\mathcal{H}_{0,0}^{(0,0)}$ is autonomous and integrable with natural frequencies $\omega_s$, $\omega_\ell$ and $\omega_z = 1$ while the perturbed system

$$\mathcal{H}(\delta q, \delta p, e, \gamma, \nu) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma^m}{m!} \frac{e^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(\delta q, \delta p, \nu) \tag{4.6}$$

is non-autonomous and non-integrable. The unperturbed system corresponds to the linearized CRTBP while the perturbed system is the nonlinear ERTBP. The Breakwell and Pringle linear transformation of the unperturbed, linearized CRTBP is applied to the nonlinear and non-circular ERTBP such that the perturbed system is expressed in terms of action-angle-type variables $(\theta, I)$ in the form

$$\mathcal{H}(\theta, I, \nu) = \omega_s I_s - \omega_\ell I_\ell + \omega_z I_z + \sum_{m+n>0}^{\infty} \frac{\gamma^m}{m!} \frac{e^n}{n!} \mathcal{H}_{m,n}^{(0,0)}(\theta, I, \nu) \tag{4.28}$$

The higher-order perturbations are still explicitly dependent on $\nu$ and the angular variables $\theta_i$ such that the normalized perturbed system is non-autonomous and non-

integrable. However, since the system is represented in expanded form both about the circular case and the linearized case, the true anomaly and angular variables only appear in the guise of periodic functions, that is, $\mathcal{H}_{m,n}^{(0,0)} = \mathcal{H}_{m,n}^{(0,0)}(I_i, \cos\theta_i, \sin\theta_i, \cos\nu, \sin\nu)$ for $m + n > 0$. Therefore, the DH method may be applied to average out these periodic terms with the goal of normalizing the perturbed Hamiltonian function into a locally integrable form reminiscent of the unperturbed case.

Recall that the pivotal step to implementing the DH method is solving the homological equation in Eq. 3.31. Substituting the unperturbed Hamiltonian function $\mathcal{H}_{0,0}^{(0,0)} = \omega_s I_s - \omega_\ell I_\ell + \omega_z I_z$ into Eqs. 3.31 yields a first-order partial differential equation in the form

$$\left( \frac{\partial}{\partial\nu} + \omega_s\frac{\partial}{\partial\theta_s} - \omega_\ell\frac{\partial}{\partial\theta_\ell} + \omega_z\frac{\partial}{\partial\theta_z} \right) \mathcal{W}_{i+1,j} = \mathcal{Q}_{i,j} - \mathcal{K}_{i,j} \qquad (4.29)$$

where the subscripts $i, j$ denote a particular order in $\gamma$ and $\epsilon$ respectively, $\mathcal{W}_{i+1,j}$ represents the corresponding part of the generating function and $\mathcal{Q}_{i,j}$ encompasses all the functions defined *a priori* through the original Hamiltonian function or from solutions at lesser orders. Finally, $\mathcal{K}_{i,j}$ represents the corresponding part of the transformed Hamiltonian function, which is defined by the periodic average

$$\mathcal{K}_{i,j} \triangleq <\mathcal{Q}_{i,j}> = \frac{1}{4(2\pi)} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \mathcal{Q}_{i,j}\, d\nu\, d\theta_s\, d\theta_\ell\, d\theta_z \qquad (4.30)$$

such that the true anomaly and angular variables are effectively eliminated from the Hamiltonian function leaving only the action-type variables, that is, $\mathcal{K} = \mathcal{K}(I)$.

Since the true anomaly and angular-type variables only appear in the form of periodic functions in the original Hamiltonian function, they will likewise only appear within periodic functions in the generating functions, transformed Hamiltonian, and

terms appearing in $\mathcal{Q}_{i,j}$. Thus, the solution to Eq. 4.29 as given by the inversion

$$\mathcal{W}_{i+1,j} = \left( \frac{\partial}{\partial \nu} + \omega_s \frac{\partial}{\partial \theta_s} - \omega_\ell \frac{\partial}{\partial \theta_\ell} + \omega_z \frac{\partial}{\partial \theta_z} \right)^{-1} \left( \mathcal{Q}_{i,j} - \mathcal{K}_{i,j} \right), \qquad (4.31)$$

is equivalent to the explicit substitution defined by

$$
\begin{aligned}
\cos \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right) &\rightarrow \frac{\sin \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right)}{i_1 + i_2 \omega_s - i_3 \omega_\ell + i_4 \omega_z} \\
\sin \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right) &\rightarrow \frac{- \cos \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right)}{i_1 + i_2 \omega_s - i_3 \omega_\ell + i_4 \omega_z}
\end{aligned}
\qquad (4.32)
$$

The term in the denominator goes to zero under resonant conditions further illustrating the pitfalls that arise when dealing with cases of resonance. However, even if $\omega_s$ and $\omega_\ell$ are not in resonance, one must still account for resonance occurring between the out-of-plane angular variable $\theta_z$ and the true anomaly (as in $\cos(\nu - \theta_z)$ for which the denominator in Eq. 4.32 goes to zero). One option is to attempt to scale the natural frequencies of the system away from $\omega_z = 1$. On the other hand, one can also apply an alternative substitution solution in the form

$$\cos \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right) \rightarrow \nu \cos \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right) \qquad (4.33)$$

$$\sin \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right) \rightarrow \nu \sin \left( i_1 \nu + i_2 \theta_s + i_3 \theta_\ell + i_4 \theta_z \right). \qquad (4.34)$$

Since this substitution introduces secular terms with respect to $\nu$, the corresponding state transformation will also exhibit small secular drift within the out-of-plane dynamics. However, the scale of the drift is relatively small and the period of oscillation is extremely large such that these secular effects are negligible. Note that previous studies usually avoid this issue by simply limiting analyses to the planar system.

### 4.1.3 Transformed Hamiltonian Function

Upon applying the DH method using the averaging operation discussed in the previous section, the ERTBP Hamiltonian function is transformed into the Birkhoff normal form

$$
\begin{aligned}
\mathcal{K} = & \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \hat{I}_s \quad - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \hat{I}_\ell + \omega_z \hat{I}_z \\
& + \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s^2 \quad + \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \hat{I}_\ell^2 \\
& + \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_z^2 \quad + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_s \hat{I}_\ell \\
& + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_s \hat{I}_z + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_\ell \hat{I}_z + \mathrm{O} \left( \hat{I}_i^3 \right) \quad (4.35)
\end{aligned}
$$

where the coefficients of the action-type variables are infinite series in the mass ratio and even-powers of the eccentricity. The ordered perturbation coefficients $\tilde{\omega}_{i,j}$, $\alpha_i$, and $\tilde{\alpha}_{i,j}$ are fixed functions of the mass ratio and natural frequencies and due to the form of Eq. 2.55, all odd-powers of the eccentricity are eliminated under the averaging operation shown in Eq. 4.30. The resultant transformed Hamiltonian function as shown in Eq. 4.35 is autonomous in $\nu$ and integrable with respect to the action-type variables.

The ordered perturbation coefficients in Eq. 4.35 are comprised of the natural frequencies $\omega_i$, the linearized non-circular coefficents $\tilde{\omega}_{i,j}$, and the nonlinear circular and non-circular coefficients $\alpha_i$ and $\tilde{\alpha}_{i,j}$. While all the coefficients are fixed functions of the mass ratio, most of these functions are too large and complicated to display

symbolically, with the exception of some of the lower-ordered terms listed in Eq. 4.36

$$\tilde{\omega}_{s,1} = \frac{\omega_s(1-\omega_s^2)(7-6\omega_s^2)}{2(1-2\omega_s^2)(1-4\omega_s^2)} \qquad \alpha_{s\ell} = -\frac{\omega_s\omega_\ell(43+64\omega_s^2\omega_\ell^2)}{3(1-2\omega_s^2)(1-2\omega_\ell^2)(1-5\omega_s^2)(1-5\omega_\ell^2)}$$

$$\tilde{\omega}_{\ell,1} = \frac{\omega_\ell(1-\omega_\ell^2)(7-6\omega_\ell^2)}{2(1-2\omega_\ell^2)(1-4\omega_\ell^2)} \qquad \alpha_{zz} = -\frac{2\omega_s^2\omega_\ell^2}{3(12+\omega_s^2\omega_\ell^2)}$$

$$\alpha_{ss} = \frac{\omega_\ell^2(81-696\omega_s^2+124\omega_s^4)}{72(1-2\omega_s^2)^2(1-5\omega_s^2)} \qquad \alpha_{sz} = -\frac{16\omega_s\omega_\ell^2}{3(4-9\omega_s^2+2\omega_s^4)}$$

$$\alpha_{\ell\ell} = \frac{\omega_s^2(81-696\omega_\ell^2+124\omega_\ell^4)}{72(1-2\omega_\ell^2)^2(1-5\omega_\ell^2)} \qquad \alpha_{\ell z} = \frac{16\omega_s^2\omega_\ell}{3(4-9\omega_\ell^2+2\omega_\ell^4)} \qquad (4.36)$$

Eqs. 4.36 are consistent with those derived in previous studies that focused on either the planar CRTBP or the planar linearized ERTBP.[13;28;15]

To validate the transformation, a numerical scenario is implemented for the Earth-Moon system of mass ratio $\mu = 0.0124$, which is sufficiently non-resonant per the Diophantine condition in Eq. 2.90. For arbitrary eccentricity, its transformed Hamiltonian function is

$$\begin{aligned}
\mathcal{K} = {} & \left(0.9535 + 0.0155e^2 - 0.0355e^4\right)\hat{I}_s - \left(0.3015 + 0.8497e^2 + 2.4324e^4\right)\hat{I}_\ell + \hat{I}_z \\
& + \left(0.1195 - 0.5126e^2 + 2.1422e^4\right)\hat{I}_s^2 + \left(0.3241 - 69.7551e^2 - 3732.5768e^4\right)\hat{I}_\ell^2 \\
& + \left(-0.0023 + 0.0009e^2 + 0.0005e^4\right)\hat{I}_z^2 - \left(1.7880 + 11.2013e^2 + 148.1181e^4\right)\hat{I}_s\hat{I}_\ell \\
& + \left(0.0914 - 0.0707e^2 + 0.1499e^4\right)\hat{I}_s\hat{I}_z + \left(0.2286 + 1.6415e^2 + 9.8214e^4\right)\hat{I}_\ell\hat{I}_z \\
& + O\left(I^3\right)
\end{aligned} \qquad (4.37)$$

Two eccentricities are presented: the true Earth-Moon eccentricity of $e = 0.0549$ and a hypothetical eccentricity of $e = 0.2$ to exaggerate the non-circular effects.

A Poincaré surface of section is generated by simulating a series of state trajectories that each start at a different set of initial conditions. The surface of section is then composed of all the points of intersection between the simulated state trajectories and a given plane (in this case, the $q_x - q_y$ plane). For the nearly circular

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Earth-Moon system of eccentricity $e = 0.0549$, the equations of motion for the full, un-expanded Hamiltonian function in Eq. 2.55 are integrated over 1000 orbits to generate the Poincaré surface of sections shown in Fig. 4.1. In generating the surfaces of section, any local trajectory that cross the $q_x$ axis and/or $q_y$ axis is deemed unstable and excluded from the simulation. As such, the surfaces of section provide a rough estimation of the region of stability about the Lagrange points.

Two cases are included in Fig. 4.1. The lower-left hand quadrant shows the stable surface of section near the ERTBP $L_5$ Lagrange point while the upper-left hand quadrant shows the surface of section near the CRTBP $L_4$ Lagrange point. Since the dynamics are symmetric about the $q_x$ axis, the two can be compared in order to characterize the non-circular effects. In addition, the small and large primaries are represented as blue dots in Fig. 4.1 located on the $\hat{x}$-axis at $-\mu$ and $1-\mu$ respectively.
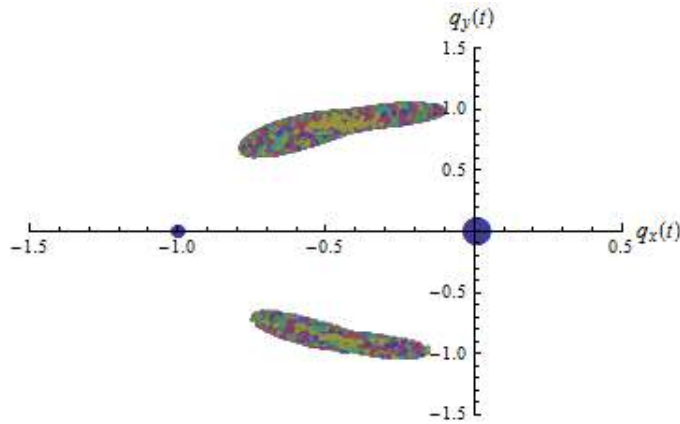


Figure 4.1: Poincaré Surface of Section near Earth-Moon $L_5$

The surfaces of section in Fig. 4.1 give a rough numerical approximation of the region of stability about the Lagrange points in both the circular and non-circular systems. In comparing the two results, the non-circular effects appear to shrink the size of the surface of section such that the region of stability in the ERTBP is smaller than the corresponding region of stability for the CRTBP. Thus, while not entirely de-stabilizing the system, the non-circular effects appear to reduce the region of stability

about a triangular Lagrange point.

A typical state trajectory around the Earth-Moon $L_5$ Lagrange point is shown in Fig. 4.2. The initial conditions are defined as $I_{s,0} = 0.0001$, $I_{\ell,0} = 0.0002$, $I_{z,0} = 0.0001$ and $\theta_{s,0} = \theta_{\ell,0} = \theta_{z,0} = 0$ and the trajectory is integrated in terms of the full ERTBP Hamiltonian function for a span of twice the long-period, $4\pi/\omega_\ell$, which is equivalent to approximately 6.3 short-period orbits. The corresponding phase portrait is rep-



(a) In-Plane Trajectory

(b) Out-of-Plane Trajectory

(c) Phase Portrait in Cartesian Phase Space
(Red = $x$, Green = $y$, Yellow = $z$)

(d) Phase Portrait in SLP Phase Space
(Cyan = $s$, Brown = $\ell$, Yellow = $z$)
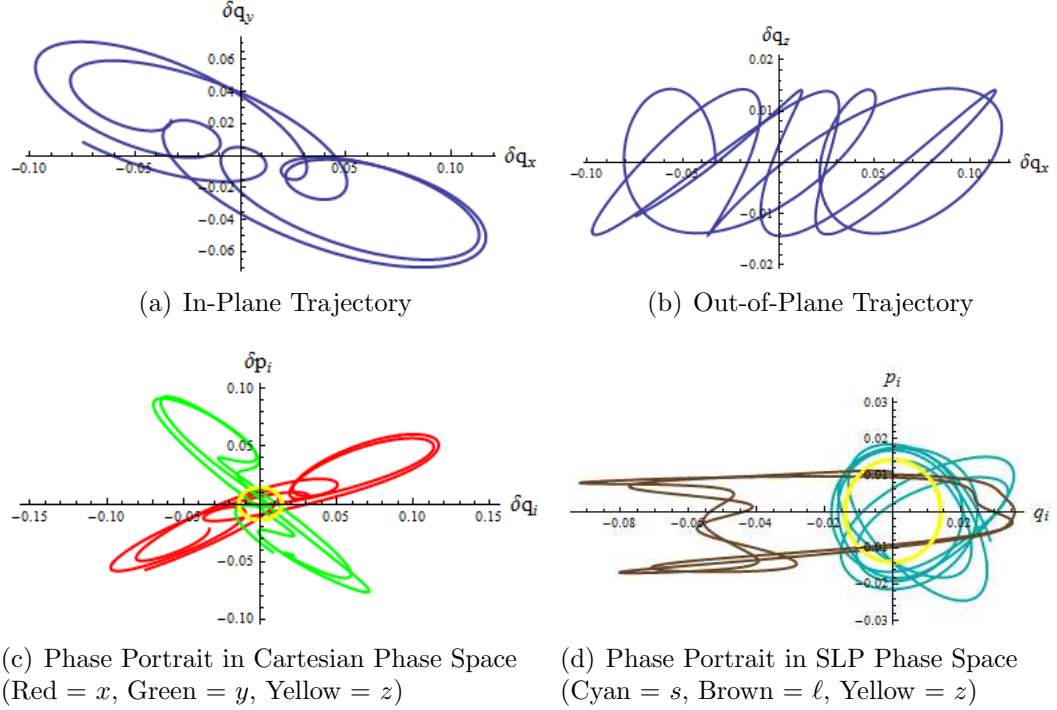
Figure 4.2: State Trajectories relative to Earth-Moon $L_5$

resented in Fig. 4.2(c) in terms of the Cartesian variables $(\delta q_x, \delta q_y, \delta q_z, \delta p_x, \delta p_y, \delta p_z)$ and in Fig. 4.2(d) in terms of the real-valued short and long-period (SLP) variables $(q_s, q_\ell, q_z, p_s, p_\ell, p_z)$ introduced in Eq. 4.12. The phase portraits are represented along each reference direction with red, green and yellow corresponding to the $x$, $y$ and $z$ dynamics in the Cartesian phase space and cyan, brown and yellow corresponding to the $s$, $\ell$ and $z$ dynamics in the SLP phase space. Fig. 4.2(d) particularly emphasizes the oscillatory behavior of the system as the phase trajectories continuously wind the origin.

Rather than numerically integrating the full nonlinear dynamics, the solutions may be approximated in the transformed phase space under the DH transformation. Since the transformed Hamiltonian function is in integrable Birkhoff normal form, the normalized equations of motion are expressed within the transformed local phase space by

$$
\begin{aligned}
\frac{d\hat{\theta}_s}{d\nu} = \frac{\partial \mathcal{K}}{\partial \hat{I}_s} = {} & \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) + 2 \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_\ell + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_z + \mathrm{O}\left( \hat{I}_i^2 \right) \\
\frac{d\hat{\theta}_\ell}{d\nu} = \frac{\partial \mathcal{K}}{\partial \hat{I}_\ell} = {} & -\left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) + 2 \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \hat{I}_\ell \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_s + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_z + \mathrm{O}\left( \hat{I}_i^2 \right) \\
\frac{d\hat{\theta}_z}{d\nu} = \frac{\partial \mathcal{K}}{\partial \hat{I}_z} = {} & \, \omega_z + 2 \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_z \\
& + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_s + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_\ell + \mathrm{O}\left( \hat{I}_i^2 \right)
\end{aligned}
\tag{4.38}
$$

and

$$
\begin{aligned}
\frac{d\hat{I}_s}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_s} = 0 \\
\frac{d\hat{I}_\ell}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_\ell} = 0 \\
\frac{d\hat{I}_z}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_z} = 0
\end{aligned}
\tag{4.39}
$$

For the Earth-Moon system, the truncated equations of motion are

$$\frac{d\hat{\theta}_s}{d\nu} = \left(0.9535 + 0.0155e^2 - 0.0355e^4\right) + 2\left(0.1195 - 0.5126e^2 + 2.1422e^4\right)\hat{I}_s$$

$$- \left(1.7880 + 11.2013e^2 + 148.1181e^4\right)\hat{I}_\ell + \left(0.0914 - 0.0707e^2 + 0.1499e^4\right)\hat{I}_z + \ldots$$

$$\frac{d\hat{\theta}_\ell}{d\nu} = -\left(0.3015 + 0.8497e^2 + 2.4324e^4\right) + 2\left(0.3241 - 69.7551e^2 - 3732.5768e^4\right)\hat{I}_\ell$$

$$- \left(1.7880 + 11.2013e^2 + 148.1181e^4\right)\hat{I}_s + \left(0.2286 + 1.6415e^2 + 9.8214e^4\right)\hat{I}_z + \ldots$$

$$\frac{d\hat{\theta}_z}{d\nu} = 1 + 2\left(-0.0023 + 0.0009e^2 + 0.0005e^4\right)\hat{I}_z + \left(0.0914 - 0.0707e^2 + 0.1499e^4\right)\hat{I}_s$$

$$+ \left(0.2286 + 1.6415e^2 + 9.8214e^4\right)\hat{I}_\ell + \ldots$$

$$\frac{d\hat{I}_s}{d\nu} = \frac{d\hat{I}_\ell}{d\nu} = \frac{d\hat{I}_z}{d\nu} = 0 \tag{4.40}$$

The solutions are then transformed back into the original set of action-angle type variables using the explicit state transformation mapping $(\hat{\theta}, \hat{I}) \rightarrow (\theta, I)$ as derived in Corollary III.3. This provides the solutions within the original phase space, which are expressed using any of the real-valued set of variables $(\theta, I)$, $(q_i, p_i)$ or $(\delta q, \delta p)$ as related to each other by the linear transformations presented in Section 4.1.1. For the numerical Earth-Moon system, the validity of the approximation is demonstrated in Figs. 4.3 through 4.6, which show the numerically-derived truth solution and the analytically-derived DH solutions at increasing orders of truncation in the expansion. Note that the zeroth-order DH solution corresponds to the unperturbed system exhibiting constant action-type variables and linearly varying angle-type variables. Due to the inherent ambiguity associated with the inverse tangent function, noise appears in the convergence of the solution within the SLP phase space, particularly at higher-orders in the DH transformation. Nonetheless, the solution derived in the transformed phase space converges quickly to the truth solution.

(a) $0^{\text{th}}$-Order DH Solution

(b) $2^{\text{nd}}$-Order DH Solution

(c) $4^{\text{th}}$-Order DH Solution
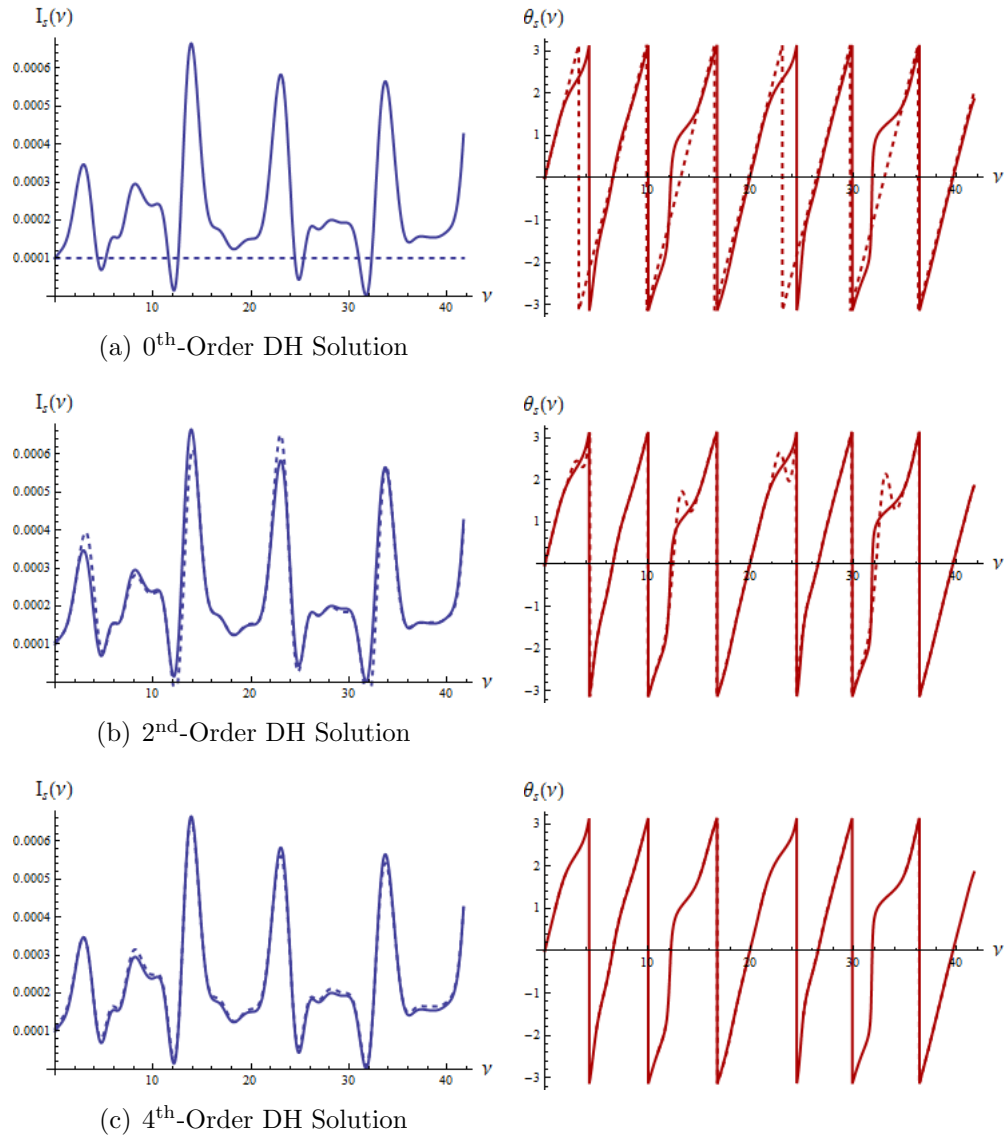
Figure 4.3: DH Convergence for Earth-Moon along $I_s$ and $\theta_s$

(a) $0^{\text{th}}$-Order DH Solution

(b) $2^{\text{nd}}$-Order DH Solution

(c) $4^{\text{th}}$-Order DH Solution
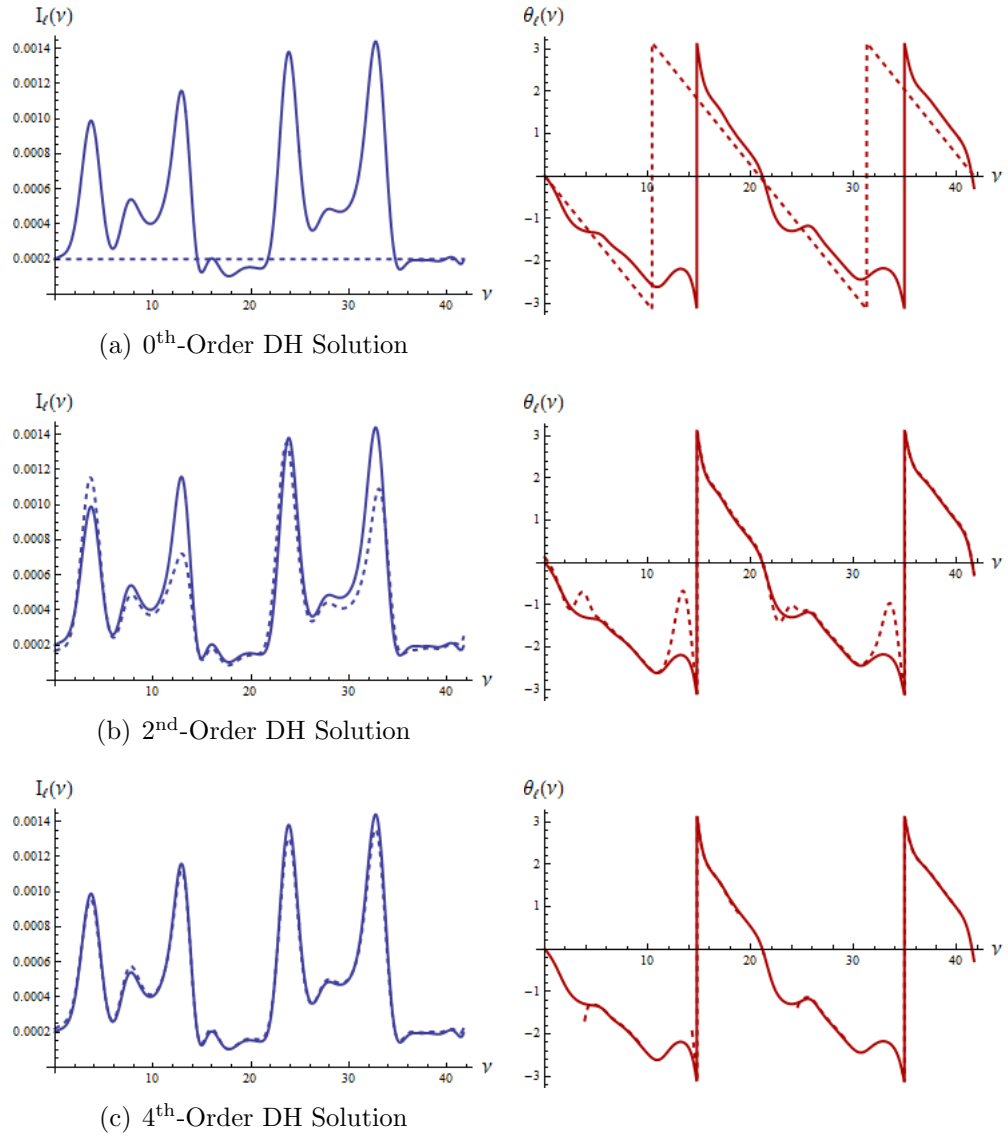
Figure 4.4: DH Convergence for Earth-Moon along $I_\ell$ and $\theta_\ell$

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(a) $0^{\text{th}}$-Order DH Solution



(b) $2^{\text{nd}}$-Order DH Solution



(c) $4^{\text{th}}$-Order DH Solution
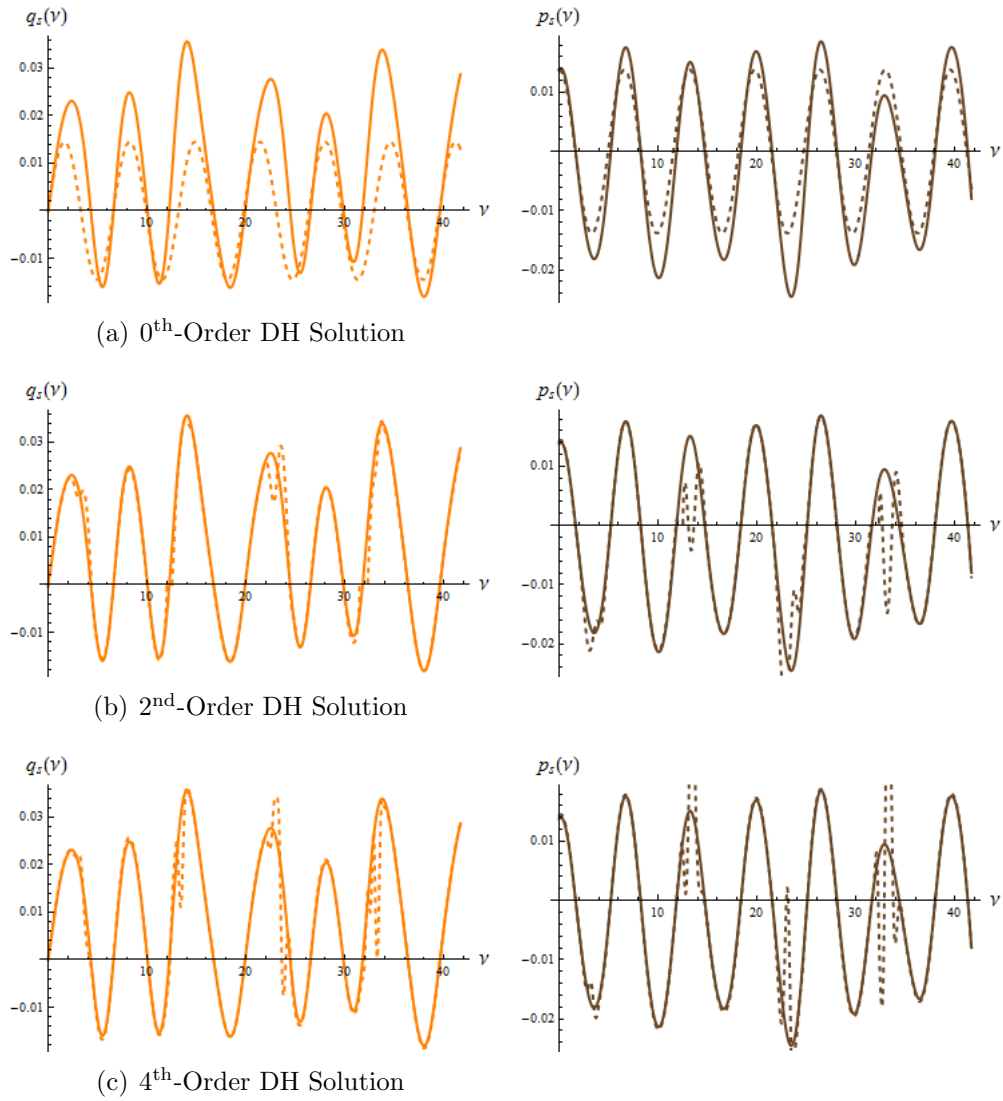
Figure 4.5: DH Convergence for Earth-Moon along $q_s$ and $p_s$

(a) 0$^{\text{th}}$-Order DH Solution



(b) 2$^{\text{nd}}$-Order DH Solution



(c) 4$^{\text{th}}$-Order DH Solution

Figure 4.6: DH Convergence for Earth-Moon along $q_\ell$ and $p_\ell$

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To exaggerate the non-circular effects in the transformation, consider the Earth-Moon system $\mu = 0.0124$, but with an increased eccentricity of $e = 0.2$. The equations of motion for the full Hamiltonian function in Eq. 2.55 are numerically integrated over 1000 orbits to generate the Poinc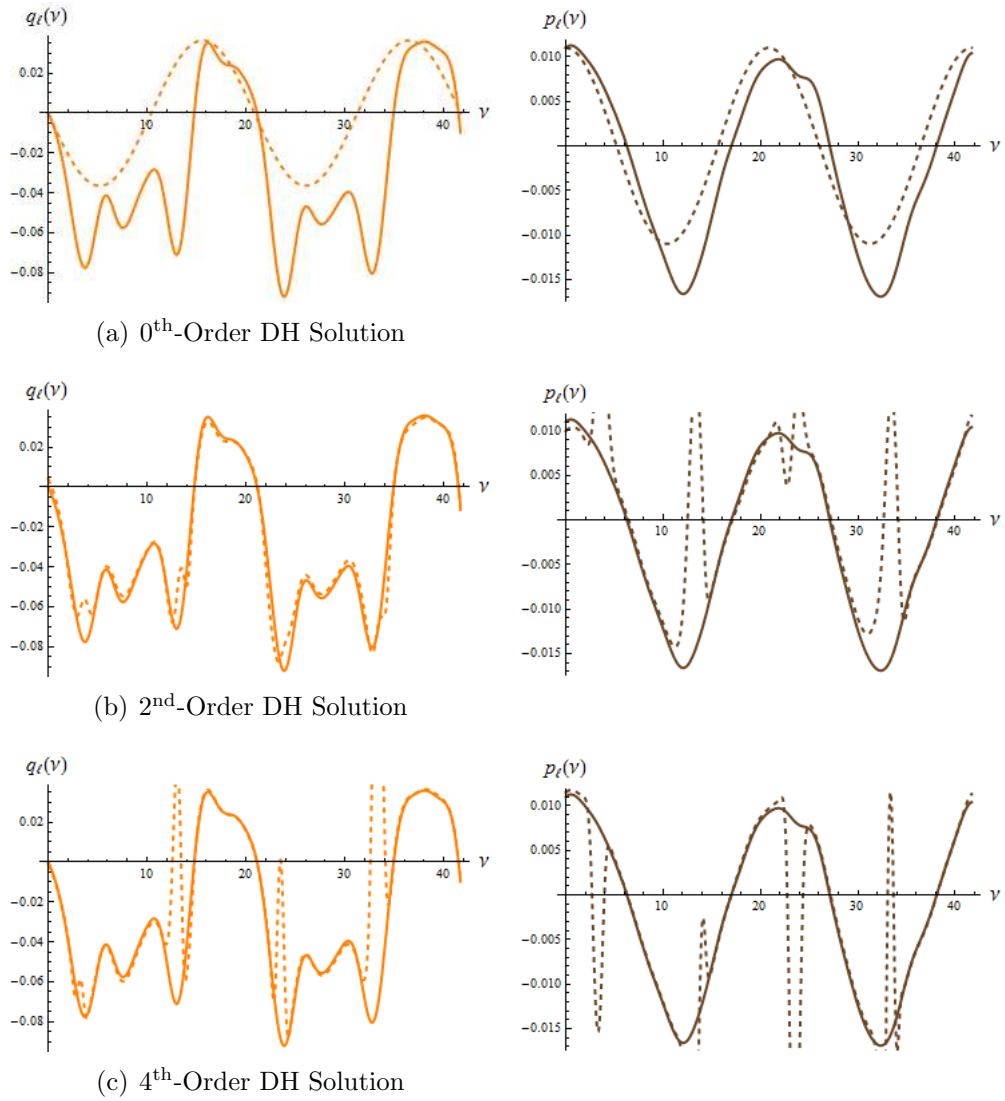aré surfaces of section shown in Fig. 4.7. As before, the surfaces of section include the circular case near $L_4$ and the non-circular case near $L_5$ and only include stable solutions as roughly defined by whether the state trajectories cross either of the reference axes in the span of 1000 orbits. As such, the surfaces of section provide a rough numerical approximation for the regions of stability, which may be compared for the two symmetric cases of the CRTBP and ERTBP included in Fig. 4.7.
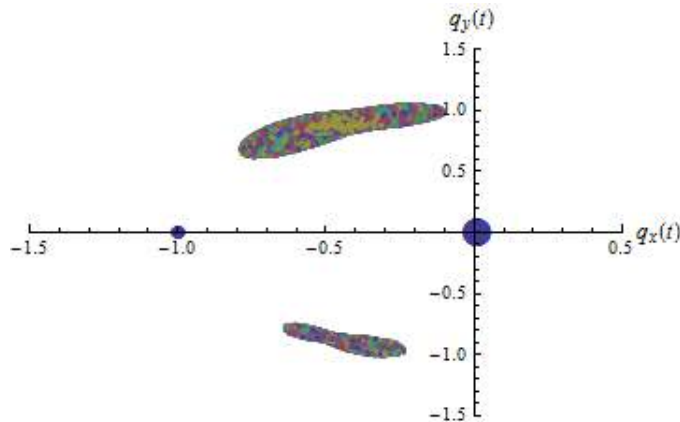


Figure 4.7: Poincaré Surface of Section near Eccentric Earth-Moon $L_5$

In this case, the higher eccentricity effectively shrinks the region of stability around the ERTBP $L_5$ Lagrange even further than seen previously in Fig. 4.1. The effects are particularly strong on the Moon-side of the stability region, which shows a greater reduction in Fig. 4.7 than the Earth-side. Based on these numerical results, one would expect higher eccentricities to exhibit increasingly smaller regions of stability as compared to the CRTBP. Eventually, the region of stability will shrink to zero corresponding to an unstable system.

A typical state trajectory of the Earth-Moon system with $e = 0.2$ is shown in Fig. 4.8 over a period of $4\pi/\omega_\ell$. As before, the corresponding phase portrait is rep-

(a) In-Plane Trajectory

(b) Out-of-Plane Trajectory

(c) Phase Portrait
(Red = $x$, Green = $y$, Blue = $z$)

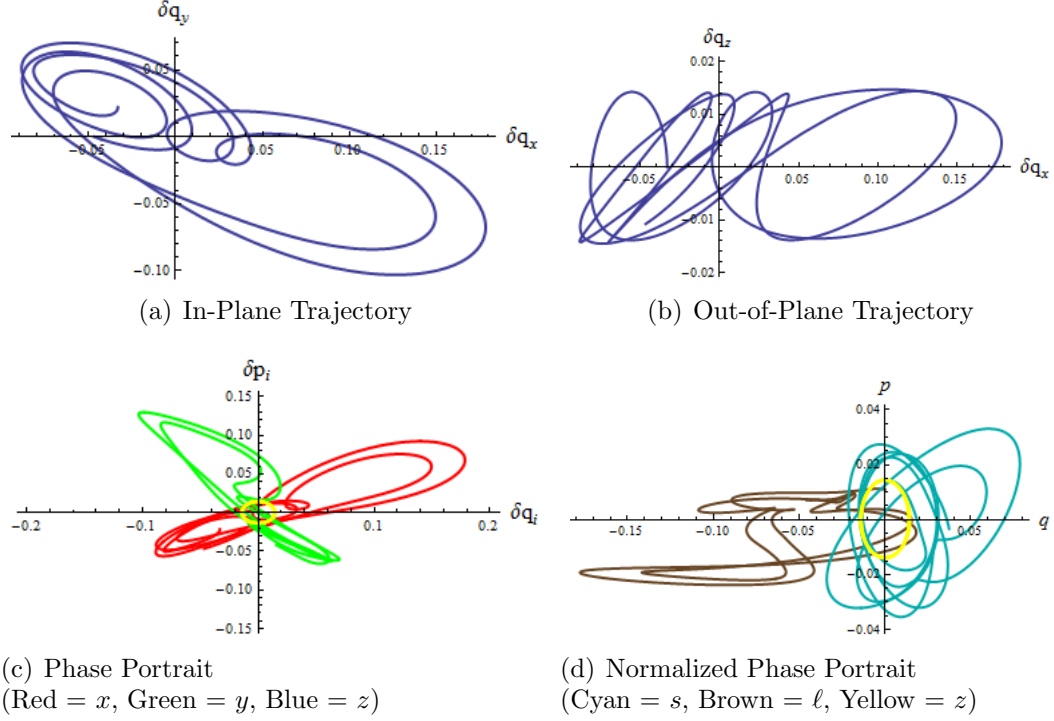(d) Normalized Phase Portrait
(Cyan = $s$, Brown = $\ell$, Yellow = $z$)

Figure 4.8: State Trajectories relative to Earth-Moon $L_5$

resented in Fig. 4.8(c) in terms of the Cartesian variables $(\delta q_x, \delta q_y, \delta q_z, \delta p_x, \delta p_y, \delta p_z)$ and in Fig. 4.8(d) in terms of the real-valued SLP variables $(q_s, q_\ell, q_z, p_s, p_\ell, p_z)$ introduced in Eq. 4.12. The phase portraits are represented along each reference direction with red, green and yellow corresponding to the $x$, $y$ and $z$ dynamics in the Cartesian phase space and cyan, brown and yellow corresponding to the $s$, $\ell$ and $z$ dynamics in the SLP phase space. The state trajectories in Fig. 4.8 may be compared to those of the nearly-circular case shown in Fig. 4.2. Larger eccentricities exhibit a more pronounced asymmetry in the state trajectories with a shift toward the positive $\delta p_s$ and $\delta p_\ell$ planes corresponding to a shift toward the positive $\delta q_x$ and negative $\delta q_y$ planes. This is most likely a reflection of the asymmetric properties of the two-body ellipse about its minor axis, that is, the speeding up and slowing down of the two-body rotation when passing from periapsis to apoapsis and vice versa.

In any event, the convergence of the DH transformation for the higher eccentricity case is demonstrated in Figs. 4.9 through 4.12, which show the numerically-derived

truth solution and the analytically-derived DH solutions at increasing orders of truncation in the expansion. As before, the DH solutions are computed in closed-form from Eqs. 4.40 and then transformed back to the original phase space using the state transformation equations derived in Corollary III.3. The zeroth-order DH solution corresponds to the unperturbed system exhibiting constant action-type variables and linearly varying angle-type variables. As evident in Figs. 4.9 through 4.12, the DH transformation converges toward the truth solution as the order of the expansion is increased, but also exhibits more noise than the previous case due to the more complicated, high-eccentricity state transformation equations.

(a) $0^{th}$-Order DH Solution



(b) $2^{nd}$-Order DH Solution



(c) $4^{th}$-Order DH Solution
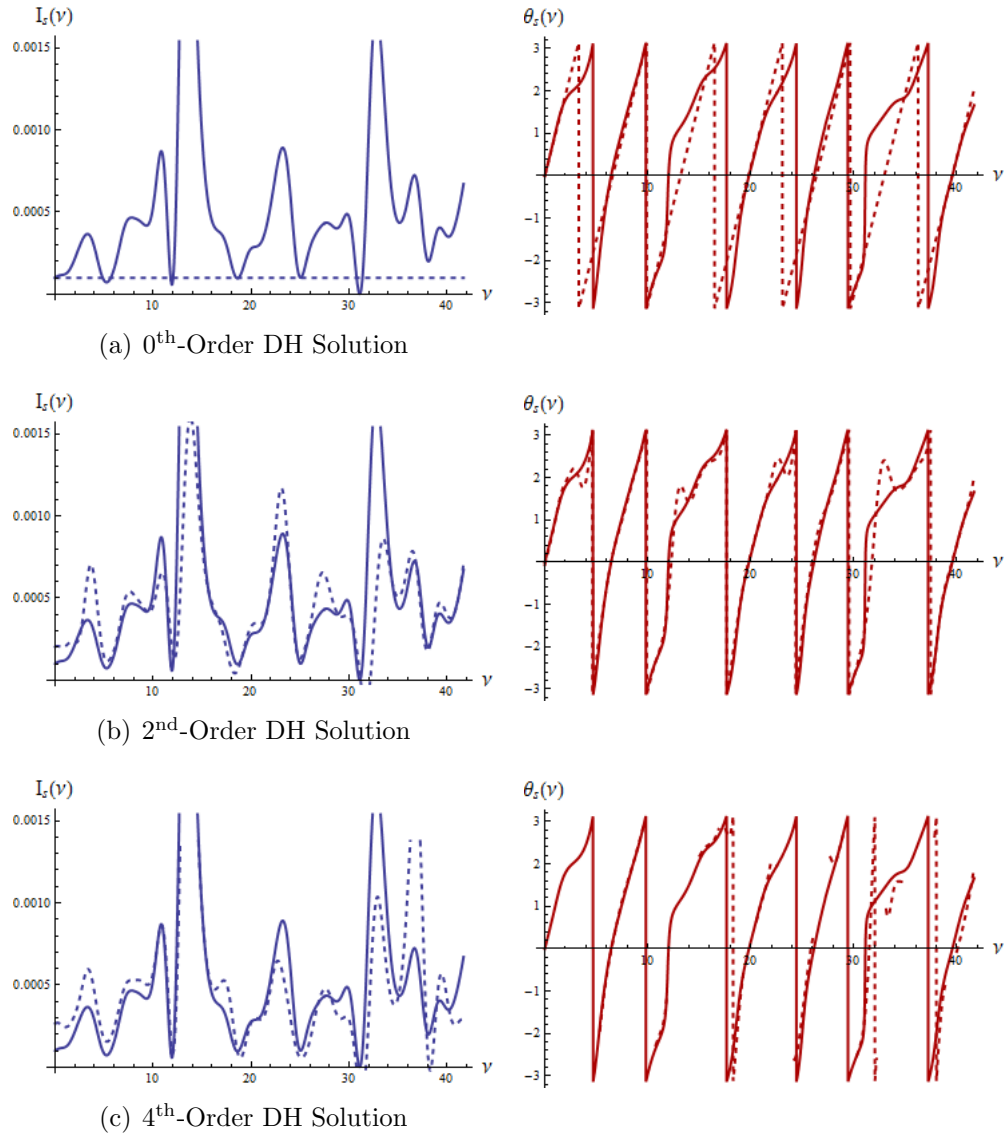
Figure 4.9: DH Convergence for Eccentric Case along $I_s$ and $\theta_s$

(a) $0^{\text{th}}$-Order DH Solution



(b) $2^{\text{nd}}$-Order DH Solution



(c) $4^{\text{th}}$-Order DH Solution
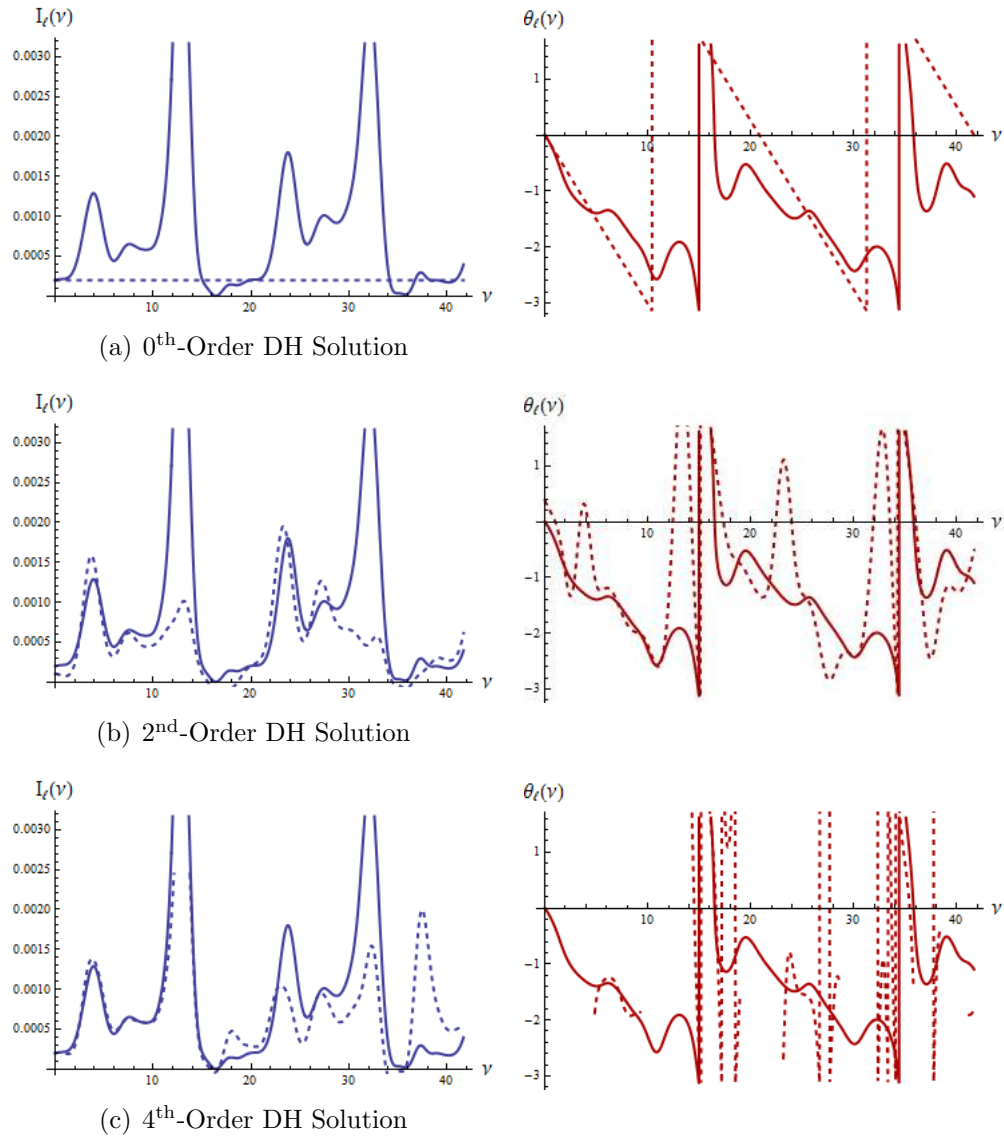
Figure 4.10: DH Convergence for Eccentric Case along $I_\ell$ and $\theta_\ell$

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(a) $0^{\text{th}}$-Order DH Solution



(b) $2^{\text{nd}}$-Order DH Solution



(c) $4^{\text{th}}$-Order DH Solution
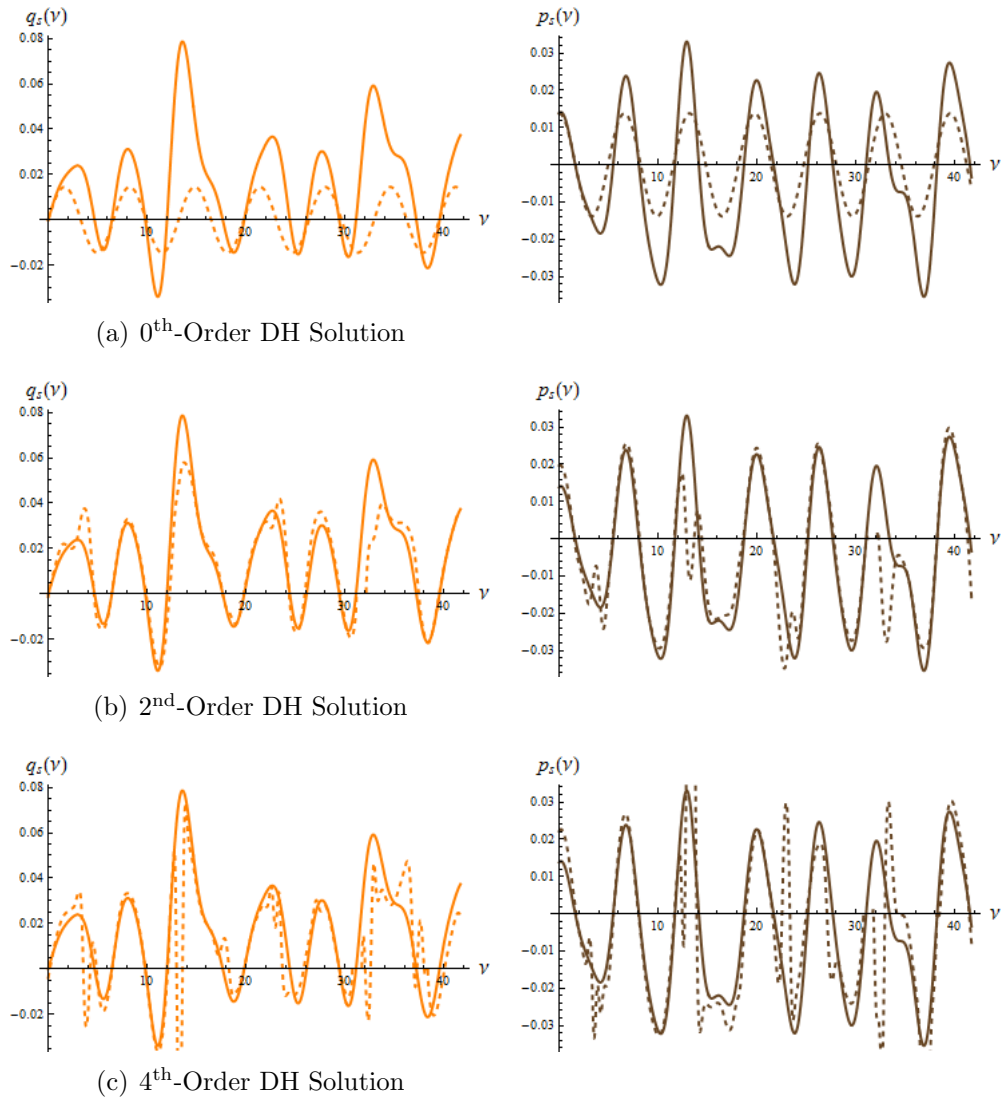
Figure 4.11: DH Convergence for Eccentric Case along $q_s$ and $p_s$

(a) $0^{\text{th}}$-Order DH Solution

(b) $2^{\text{nd}}$-Order DH Solution

(c) $4^{\text{th}}$-Order DH Solution

Figure 4.12: DH Convergence for Eccentric Case along $q_\ell$ and $p_\ell$

## 4.2 Stability Analysis

Since the unperturbed Hamiltonian system $\mathcal{K}_{0,0} = \mathcal{H}_{0,0}^{(0,0)}$ is completely integrable, its phase space is foliated by invariant tori parameterized by the action variables. KAM theory dictates that the perturbed system $\mathcal{K}$ preserves the invariant tori for sufficiently small perturbations.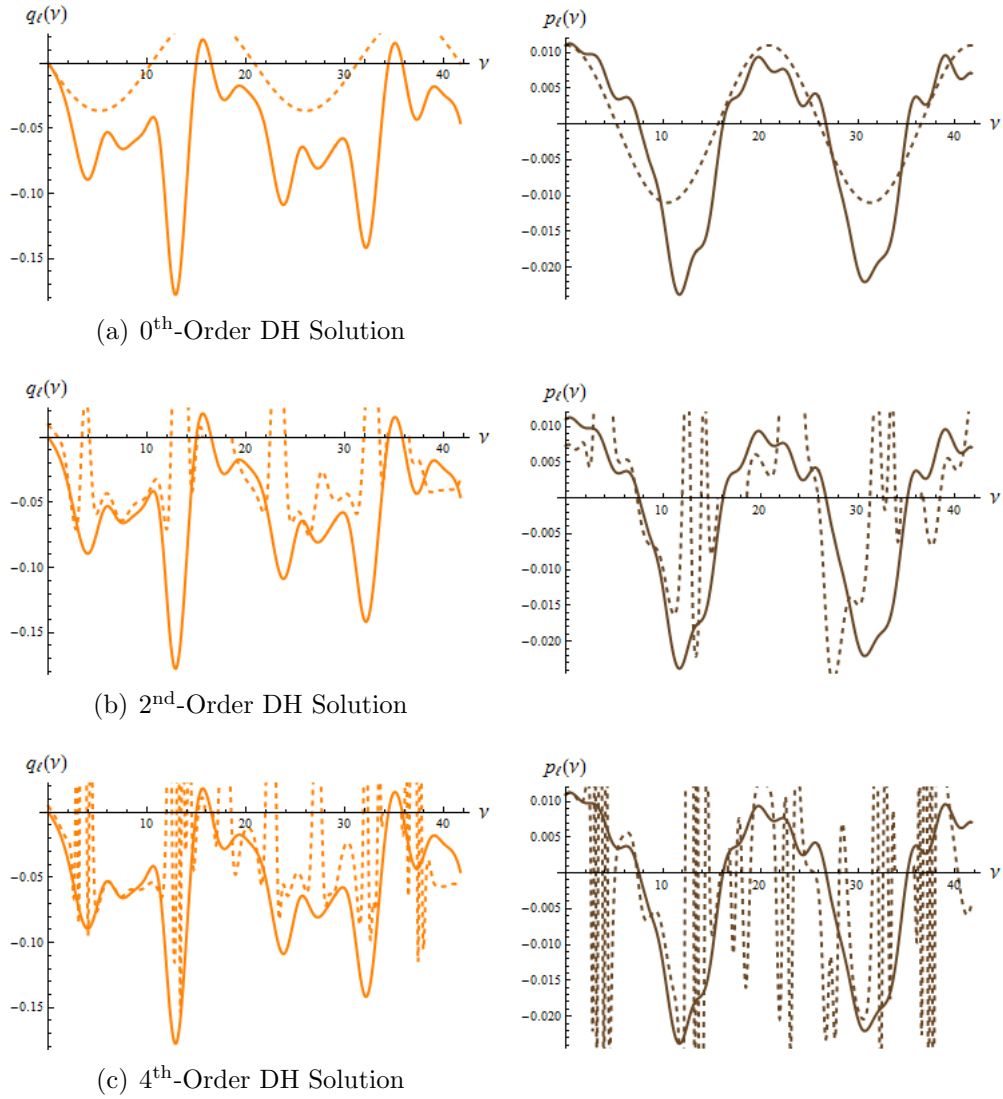 This is further supported by the numerical results shown in Figs. 4.1 through 4.12, which exhibit stable solutions in the vicinity of the Earth-Moon system, even in the exaggerated non-circular case. To rigorously analyze the system stability, one may apply the KAM theorem for Birkhoff normal systems as discussed in Chapter II. However, before treating the nonlinear and non-circular system, one must first extend the linear stability analysis of the CRTBP to the non-circular case.

### 4.2.1 Linearized, Non-circular Stability

The linearized system in the transformed phase space is defined by the reduced Hamiltonian function

$$\mathcal{K}_{0,i} = \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \hat{I}_s - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \hat{I}_\ell + \omega_z \hat{I}_z \qquad (4.41)$$

where $\omega_i$ are the natural frequencies of the unperturbed system and $\tilde{\omega}_{i,j}$ are the non-circular coefficients whose first order terms were given in Eqs. 4.36. The circular system is treated using the classic indirect Lyapunov method (see Section 2.2), which shows linear stability for motion about the triangular Lagrange points for mass ratios in the range $0 < \mu < \mu_c$. The question of whether this linear stability persists to the non-circular system was first addressed by Danby in 1964 who analyzed the system variational equations in order to numerically generate a stability curve in the $\mu - e$ phase space as shown in Fig. 4.13. For values of $\mu$ and $e$ that lie beneath the stability curve, the corresponding triangular Lagrange points are linearly stable.[25]
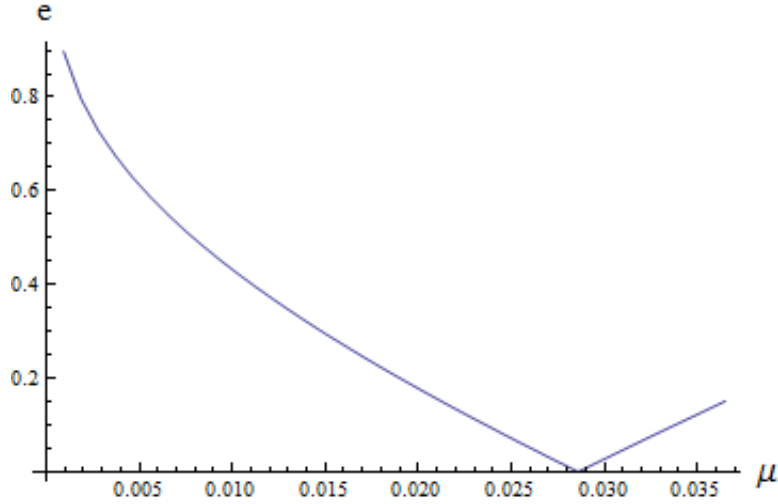
Figure 4.13: Danby's Stability Curve

Danby's stability curve was subsequently verfied analytically by Alfriend and Rand in 1969 and Deprit in 1970.[27;28] In Deprit's approach, the single parameter DH method is applied to the linearized system yielding the same result shown in Eq. 4.41. Being a linear periodic system, one may appeal to Floquet theory in which a linear system is averaged over its characteristic period providing a description of the secular response. The system stability is characterized by the Floquet exponents (akin to the Lyapunov exponents), which are a measure of the secular growth over time. Through the derivation of the linear variational equations, Deprit demonstrated that in linearized Birkhoff normal form, the system Floquet exponents are equivalent to the coefficients of $\hat{I}_s$ and $\hat{I}_\ell$ in Eq. 4.35, that is

$$\sigma = \pm i \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \quad \text{and} \quad \lambda = \mp i \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \tag{4.42}$$

The Floquet multipliers are then $\exp(2\pi\sigma)$ and $\exp(2\pi\lambda)$ whose real parts must be non-positive for linear stability. This provides the necessary check to verify Danby's stability curve in Fig. 4.13.

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### 4.2.2 Nonlinear System Stability

For a rigorous treatment of the nonlinear stability of the elliptic triangular Lagrange points, one may apply KAM theory in the transformed phase space of Eq. 4.35. As mentioned previously, the phenomenon of Arnold diffusion in systems with more than two degrees of freedom effectively eliminates the KAM theorem as a means to prove stability. However, since the out-of-plane dynamics are roughly equivalent to stable harmonic oscillation, it is sufficient for the time being to restrict the system to the planar dynamics and apply KAM theory as a rigorous means of stability analysis. Specifically, since the system is normalized into Birkhoff normal form in Eq. 4.35, stability is achieved under satisfaction of Arnold's condition

$$D_{2k} = \mathcal{K}_{2k}(\hat{I}_s = \omega_\ell, \hat{I}_\ell = \omega_s) \neq 0 \tag{2.94}$$

where $\mathcal{K}_{2k}$ represents terms of order $k$ in the action variables within the transformed Hamiltonian function. Note that this condition also requires non-resonance as does the DH transformation itself. Therefore, potentially unstable solutions are defined as those that satisfy the conditions of resonance (namely $\omega_s/\omega_\ell \in \mathbb{Z} - \{0\}$) or those that do not satisfy Arnold's condition in Eq. 2.94.

Substituting the normalized Hamiltonian function Eq. 4.35 into Arnold's condition for $k = 2$ yields

$$D_4 = \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \omega_\ell^2 + \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \omega_s^2$$
$$+ \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \omega_s \omega_\ell \neq 0 \tag{4.43}$$

For circular systems, this condition further reduces to

$$
\begin{aligned}
D_4 &= \alpha_{ss}\omega_\ell^2 + \alpha_{\ell\ell}\omega_s^2 + \alpha_{s\ell}\omega_s\omega_\ell \\
&= \frac{\omega_\ell^4(81 - 696\omega_s^2 + 124\omega_s^4)}{72(1 - 2\omega_s^2)^2(1 - 5\omega_s^2)} + \frac{\omega_s^4(81 - 696\omega_\ell^2 + 124\omega_\ell^4)}{72(1 - 2\omega_\ell^2)^2(1 - 5\omega_\ell^2)} \\
&\quad - \frac{\omega_s^2\omega_\ell^2(43 + 64\omega_s^2\omega_\ell^2)}{3(1 - 2\omega_s^2)(1 - 2\omega_\ell^2)(1 - 5\omega_s^2)(1 - 5\omega_\ell^2)} \\
&= -\frac{36 - 541\omega_s^2\omega_\ell^2 + 644\omega_s^4\omega_\ell^4}{8(1 - 4\omega_s^2\omega_\ell^2)(4 - 25\omega_s^2\omega_\ell^2)} \neq 0
\end{aligned}
\tag{4.44}
$$

which is in agreement with the solution derived by Deprit and Deprit-Bartholomé.[13] Eq. 4.44 shows that in addition to the resonant cases $\mu_2 = 0.0242939$ and $\mu_3 = 0.013516$, a critical mass ratio occurs at the root of the function $36 - 541\omega_s^2\omega_\ell^2 + 644\omega_s^4\omega_\ell^4$ given by

$$
\mu_4 = \frac{1449 - \sqrt{483(3265 + 2\sqrt{199945})}}{2898} \approx 0.0109137
\tag{4.45}
$$

This does not necessarily prove instability at this mass ratio, but rather the necessity to check the next order of Arnold's condition at $k = 3$ as defined by

$$
\begin{aligned}
D_6 &= \left( \alpha_{sss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!}\tilde{\alpha}_{sss,n} \right) \omega_\ell^3 + \left( \alpha_{\ell\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!}\tilde{\alpha}_{\ell\ell\ell,n} \right) \omega_s^3 \\
&\quad + \left( \alpha_{ss\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!}\tilde{\alpha}_{ss\ell,n} \right) \omega_s\omega_\ell^2 + \left( \alpha_{s\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!}\tilde{\alpha}_{s\ell\ell,n} \right) \omega_s^2\omega_\ell \neq 0
\end{aligned}
\tag{4.46}
$$

which for the circular case reduces to

$$
D_6 = \alpha_{sss}\omega_\ell^3 + \alpha_{\ell\ell\ell}\omega_s^3 + \alpha_{ss\ell}\omega_s\omega_\ell^2 + \alpha_{s\ell\ell}\omega_s^2\omega_\ell \neq 0
\tag{4.47}
$$

wherein the $\alpha_{iii}$ coefficients represent the coefficients of the nonlinear circular system

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at order $I_i^3$ as listed in Eqs. 4.48

$$
\begin{aligned}
\alpha_{sss} &= \omega_\ell^2\Big(-18522432 - 221117724\omega_\ell^2 + 1834402891\omega_\ell^4 - 5330237408\omega_\ell^6 \\
&\quad + 8326473644\omega_\ell^8 - 7970990576\omega_\ell^{10} + 4915656752\omega_\ell^{12} - 1885370432\omega_\ell^{14} \\
&\quad + 349789120\omega_\ell^{16}\Big)\Big/\Big(62208\omega_s(-\omega_\ell^2 + \omega_s^2)^5(-\omega_\ell^2 + 4\omega_s^2)^3(-\omega_\ell^2 + 9\omega_s^2)\Big) \\
\alpha_{\ell\ell\ell} &= -\omega_s^2\Big(83835 - 2577978\omega_\ell^2 + 28794339\omega_\ell^4 - 179112048\omega_\ell^6 + 703645324\omega_\ell^8 \\
&\quad - 1518361584\omega_\ell^{10} + 1512159088\omega_\ell^{12} - 912942528\omega_\ell^{14} + 349789120\omega_\ell^{16}\Big) \\
&\quad \Big/\Big(62208\omega_\ell(\omega_\ell - \omega_s)^5(\omega_\ell + \omega_s)^5(4\omega_\ell^2 - \omega_s^2)^3(9\omega_\ell^2 - \omega_s^2)\Big) \\
\alpha_{ss\ell} &= \Big(13993776\omega_\ell - 291589800\omega_\ell^3 + 2566329143\omega_\ell^5 - 14020325316\omega_\ell^7 \\
&\quad + 51393703020\omega_\ell^9 - 124356412922\omega_\ell^{11} + 196481798617\omega_\ell^{13} - 202885849514\omega_\ell^{15} \\
&\quad + 137831914700\omega_\ell^{17} - 62566226600\omega_\ell^{19} + 19008544000\omega_\ell^{21} - 3176280000\omega_\ell^{23}\Big) \\
&\quad \Big/\Big(1728(\omega_\ell^2 - 9\omega_s^2)(\omega_\ell^2 - \omega_s^2)^5(4\omega_\ell^4 - 17\omega_\ell^2\omega_s^2 + 4\omega_s^4)^3\Big) \\
\alpha_{s\ell\ell} &= -\omega_s\Big(-400896 + 31676688\omega_\ell^2 - 473884444\omega_\ell^4 + 3415134306\omega_\ell^6 \\
&\quad - 15654797925\omega_\ell^8 + 48114061414\omega_\ell^{10} - 95635690781\omega_\ell^{12} + 119761809514\omega_\ell^{14} \\
&\quad - 93965844700\omega_\ell^{16} + 47176186600\omega_\ell^{18} - 15930536000\omega_\ell^{20} + 3176280000\omega_\ell^{22}\Big) \\
&\quad \Big/\Big(1728(\omega_\ell^2 - \omega_s^2)^5(9\omega_\ell^2 - \omega_s^2)(4\omega_\ell^4 - 17\omega_\ell^2\omega_s^2 + 4\omega_s^4)^3\Big)
\end{aligned} \tag{4.48}
$$

Substituting the symbolic representation for these coefficients into Eq. 4.47 yields

$$
\begin{aligned}
D_6 &= \Big(-16096320 + 578209968\omega_\ell^2\omega_s^2 - 5879019660\omega_\ell^4\omega_s^4 + 23361243081\omega_\ell^6\omega_s^6 \\
&\quad - 32843706320\omega_\ell^8\omega_s^8 - 104264873152\omega_\ell^{10}\omega_s^{10} + 481275622400\omega_\ell^{12}\omega_s^{12} \\
&\quad + 94280800000\omega_\ell^{14}\omega_s^{14}\Big)\Big/\Big(20736\omega_\ell\omega_s(\omega_\ell^2 - \omega_s^2)^5(-4 + 25\omega_\ell^2\omega_s^2)^3(-9 + 100\omega_\ell^2\omega_s^2)\Big) \\
&\neq 0
\end{aligned} \tag{4.49}
$$

At the critical mass ratio $\mu = \mu_4$ in Eq. 4.45, $D_6 = 66$ such that condition Eq. 4.49 is

indeed satisfied and the circular triangular Lagrange points are nonlinearly stable at the critical mass ratio. Furthermore, they are nonlinearly stable for all mass ratios in the range $\mu < \mu_c$ save for those exhibiting resonance. All of the preceding results are consistent with those derived in Deprit and Deprit-Bartholomé and Meyer et al.[13;16]

Returning to the non-circular case, the fourth-order Arnold condition is given by Eq. 4.43

$$
\begin{aligned}
D_4 = &\left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \omega_\ell^2 + \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \omega_s^2 \\
&+ \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \omega_s \omega_\ell \neq 0
\end{aligned}
\tag{4.43}
$$

and the sixth-order condition is given by Eq. 4.46

$$
\begin{aligned}
D_6 = &\left( \alpha_{sss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sss,n} \right) \omega_\ell^3 + \left( \alpha_{\ell\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell\ell,n} \right) \omega_s^3 \\
&+ \left( \alpha_{ss\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss\ell,n} \right) \omega_s \omega_\ell^2 + \left( \alpha_{s\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell\ell,n} \right) \omega_s^2 \omega_\ell \neq 0
\end{aligned}
\tag{4.46}
$$

Unfortunately, no symbolic representation of the stability equations is available due to the complexity in simplifying and representing the coefficients in the infinite perturbation series in $e$. Instead, the DH transformation method is numerically applied to the system within the ranges $\mu < \mu_c$ and $0 < e < 1$ and for each pair, Arnold's conditions are checked up to a maximum order of $e^6$ in the expansion as dictated by the limits of the author's processing system. The resultant stability profile is shown in Fig. 4.14 along with Danby's linear stability curve shown in blue. The red curve includes all the values of $\mu$ and $e$ that fail Arnold's fourth-order stability condition while the green curve includes those that fail the sixth-order condition. The black dots represent the three critical mass ratios including the two resonant cases ($\mu_2$ and $\mu_3$) and Danby's critical mass ratio $\mu_4$. Note that these curves were generated from
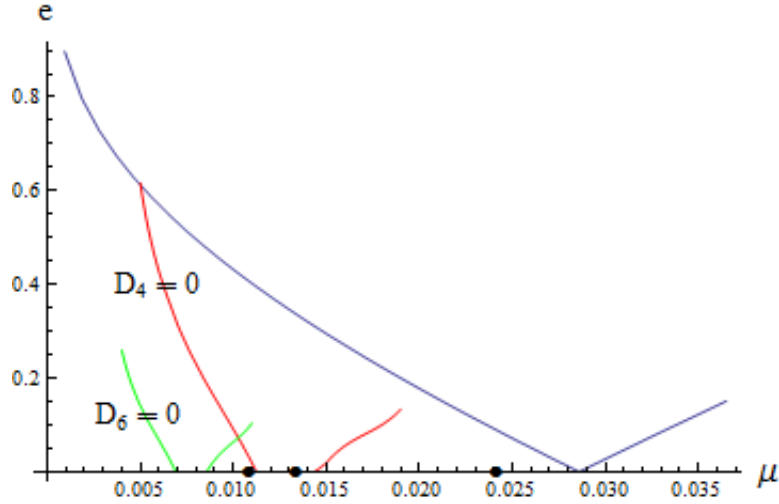
Figure 4.14: Nonlinear and Non-circular Stability Curves

a DH transformation with a maximum order of eccentricity of $e^6$, which was dictated by the limitations of the author's processing system. Nonetheless, the results display a reasonable approximation of the instability curves defined by Arnold's KAM conditions. They are also consistent with the circular results wherein the left-branch of the fourth-order curve intersects the zero eccentricity axis at roughly $\mu = \mu_4$. Furthermore, the general form of the higher-order stability curves resembles the form of Danby's linear stability curve.

Of all the results shown in Fig. 4.14, the only location at which the nonlinear stability remains in question is the point where the green and red curves intersect near $\mu = 0.0102$ and $e = 0.075$. However, for a DH expansion through order $e^6$, the eighth-order Arnold condition is

$$
D_8 = \left( \alpha_{ssss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ssss,n} \right) \omega_\ell^4 + \left( \alpha_{\ell\ell\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell\ell\ell,n} \right) \omega_s^4
$$
$$
+ \left( \alpha_{sss\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sss\ell,n} \right) \omega_s \omega_\ell^3 + \left( \alpha_{ss\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss\ell\ell,n} \right) \omega_s^2 \omega_\ell^2
$$
$$
+ \left( \alpha_{s\ell\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell\ell\ell,n} \right) \omega_s^3 \omega_\ell \neq 0 \tag{4.50}
$$

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which is numerically non-zero at the intersection point between the green and red curves. Thus, a higher-order treatment would provide a more accurate representation of the nonlinear stability curves, but the approximation up to order $e^6$ suggests that the entire region under Danby's stability curve is nonlinearly stable except the aforementioned cases of resonance. This then implies that the elliptic triangular Lagrange points are nonlinearly stable in the sense of Lyapunov, but only when neglecting the coupled effects caused by the out-of-plane dynamics.

## 4.3 Local Integrals of Motion

The transformed Hamiltonian function

$$
\begin{aligned}
\mathcal{K} = & \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) \hat{I}_s \quad - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) \hat{I}_\ell + \omega_z \hat{I}_z \\
& + \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s^2 \quad + \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \hat{I}_\ell^2 \\
& + \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_z^2 \quad + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_s \hat{I}_\ell \\
& + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_s \hat{I}_z + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_\ell \hat{I}_z + \mathrm{O}\left( I_i^3 \right) \quad (4.35)
\end{aligned}
$$

is in integrable Birkhoff normal form up the order of expansion. Thus, by truncating the higher-order, non-integrable terms, the normalized equations of motion are

expressed within the transformed local phase space by

$$
\begin{aligned}
\frac{d\hat{\theta}_s}{d\nu} =\ & \frac{\partial \mathcal{K}}{\partial \hat{I}_s} = \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) + 2 \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_s \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_\ell + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_z + \mathrm{O}\left( I_i^2 \right) \\
\frac{d\hat{\theta}_\ell}{d\nu} =\ & \frac{\partial \mathcal{K}}{\partial \hat{I}_\ell} = - \left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{\ell,n} \right) + 2 \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \hat{I}_\ell \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_s + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_z + \mathrm{O}\left( I_i^2 \right) \\
\frac{d\hat{\theta}_z}{d\nu} =\ & \frac{\partial \mathcal{K}}{\partial \hat{I}_z} = \omega_z + 2 \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_z \\
& + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_s + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_\ell + \mathrm{O}\left( I_i^2 \right)
\end{aligned}
$$

$$(4.38)$$

and

$$
\begin{aligned}
\frac{d\hat{I}_s}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_s} = 0 \\
\frac{d\hat{I}_\ell}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_\ell} = 0 \\
\frac{d\hat{I}_z}{d\nu} &= -\frac{\partial \mathcal{K}}{\partial \hat{\theta}_z} = 0
\end{aligned}
$$

$$(4.39)$$

As such, up to the order of truncation in the normalization, the action-type variables represent local integrals of motion in the transformed phase space corresponding to local invariant tori about the Lagrange point. The angular state variables wind the tori according to the linear trajectories

$$
\hat{\theta}_i = \hat{\theta}_{i,0} \pm \Omega_i(\nu - \nu_0) \quad \mathrm{mod}\ 2\pi
$$

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where $\Omega_i$ are the perturbed natural frequencies, which are constant for a given system and defined by

$$
\begin{aligned}
\Omega_s =\; & \left( \omega_s + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) + 2 \left( \alpha_{ss} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{ss,n} \right) \hat{I}_{s,0} \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_{\ell,0} + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) \hat{I}_{z,0} + \mathrm{O}\left( \hat{I}_{i,0}^2 \right) \\
\Omega_\ell =\; & -\left( \omega_\ell + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\omega}_{s,n} \right) + 2 \left( \alpha_{\ell\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell\ell,n} \right) \hat{I}_{\ell,0} \\
& + \left( \alpha_{s\ell} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{s\ell,n} \right) \hat{I}_{s,0} + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_{z,0} + \mathrm{O}\left( \hat{I}_{i,0}^2 \right) \\
\Omega_z =\; & \omega_z + 2 \left( \alpha_{zz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{zz,n} \right) \hat{I}_{z,0} \\
& + \left( \alpha_{sz} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{sz,n} \right) I_{sl,0} + \left( \alpha_{\ell z} + \sum_{n=1}^{\infty} \frac{e^{2n}}{(2n)!} \tilde{\alpha}_{\ell z,n} \right) \hat{I}_{\ell,0} + \mathrm{O}\left( \hat{I}_{i,0}^2 \right) \quad (4.51)
\end{aligned}
$$

where $\hat{I}_{s,0}$, $\hat{I}_{\ell,0}$, and $\hat{I}_{z,0}$ are the fixed values of the action variables that parameterize the local invariant tori.

For the CRTBP, Hill introduced a method of foliating the phase space by level sets of the Jacobi integral, which was described in detail in Chapter II.[9;11] Since the action-type variables represent local integrals of motion, one may similarly generate local level sets that foliate the phase space in the vicinity of the Lagrange point. To do so, one must first provide an adequate velocity relation in the tradition of Hill's curves of zero velocity. From the transformation equations introduced in Eqs. 4.21, one may derive the mixed variable equation

$$
\hat{p}_s^2 + \hat{p}_\ell^2 + \hat{p}_z^2 = 2 \left( \omega_s \hat{I}_s + \omega_\ell \hat{I}_\ell + \omega_z \hat{I}_z \right) - \left( \omega_s^2 \hat{q}_s^2 + \omega_\ell^2 \hat{q}_\ell^2 + \omega_z^2 \hat{q}_z^2 \right) \qquad (4.52)
$$

where the term on the left-hand side is the magnitude of the generalized momentum vector, which must be nonnegative for real solutions. Therefore, one may define a

constant $C = \omega_s \hat{I}_s + \omega_\ell \hat{I}_\ell + \omega_z \hat{I}_z$ to serve as a local Jacobi-type integral whose level sets foliate the transformed phase space into impassable curves of zero momentum.

For the Earth-Moon system ($\mu = 0.0124$, $e = 0.0549$), the local integral is plotted along a typical state trajectory in Fig. 4.15 for the first two orders of truncation in the expansion. As the order of truncation increases, $C = \omega_s \hat{I}_s + \omega_\ell \hat{I}_\ell + \omega_z \hat{I}_z$ converges



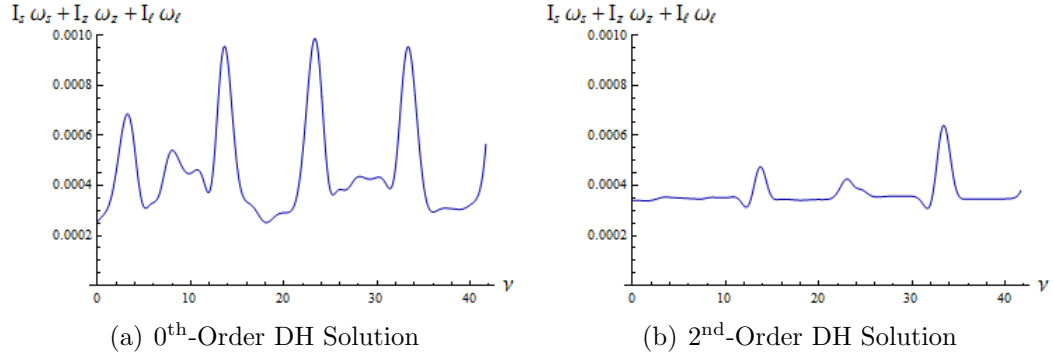(a) $0^{\text{th}}$-Order DH Solution        (b) $2^{\text{nd}}$-Order DH Solution

Figure 4.15: Local Integral of Motion for the Nearly-Circular Earth-Moon System

to a constant value such that it does indeed represent an approximate local integral of motion.

To exaggerate the non-circular effect, the eccentricity of the Earth-Moon system is increased to 0.2. The corresponding local integral of motion is plotted for the first two orders of truncation in Fig. 4.16. The non-circular effects degrade the convergence properties, which would dictate the need for a higher-order expansion for higher eccentricities.



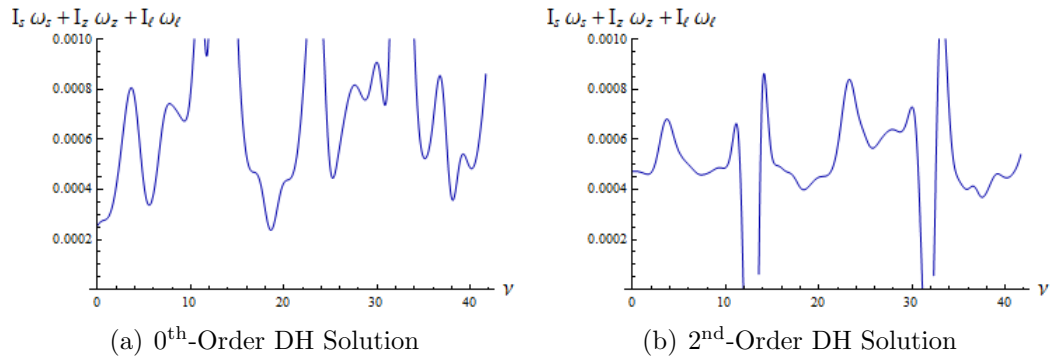(a) $0^{\text{th}}$-Order DH Solution        (b) $2^{\text{nd}}$-Order DH Solution

Figure 4.16: Local Integral of Motion for the Eccentric Earth-Moon System

By mapping the levels sets of the transformed local integral back into the original phase space, the corresponding foliation of the local phase space for the Earth-Moon system is generated in Fig. 4.17 for the real eccentricity of $e = 0.0549$ and in Fig. 4.18 for the eccentric case with $e = 0.2$. In both sets of figures, the foliation of the local phase space is displayed at various values of $\nu$ with the green curves representing level sets of the local integral of motion, $\omega_s I_s + \omega_\ell I_\ell + \omega_z I_z$. In addition, an example state trajectory is plotted in blue up to the given value of $\nu$ and its corresponding level set, as defined by its initial conditions, is shown as a dashed blue curve. The level sets pulsate as $\nu$ varies due to the non-autonomous, $\nu$-dependency in the state transformation equations while the state trajectory remains confined within its corresponding level set (dashed blue curve) within the span of $\nu$ values represented in the figures.

For this study, the order of truncation in Figs. 4.17 and 4.18 was relatively low $(\mathrm{O}\,(e^6))$ due to memory limitations in the author's processing system. Thus, it is difficult to draw any definitive conclusions without first conducting a higher-order analysis for longer durations of $\nu$. Nonetheless, even in the low-order models of Figs. 4.17 and 4.18, the non-circular level suggest a rough measure of the region of motion and stability for a given set of initial conditions in the vicinity of the elliptic triangular Lagrange points. Further study is required to validate this approach and generate higher-accuracy simulations.
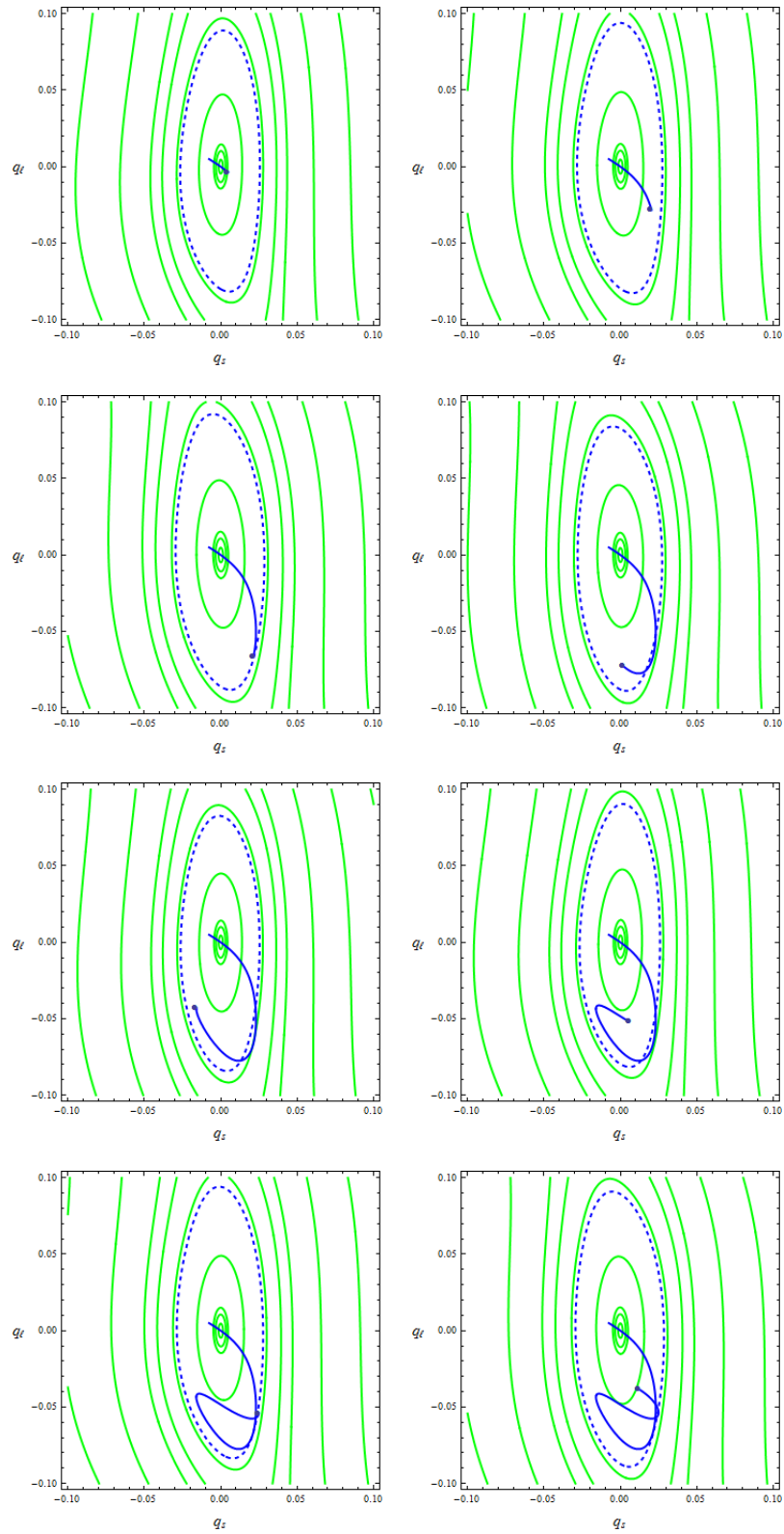
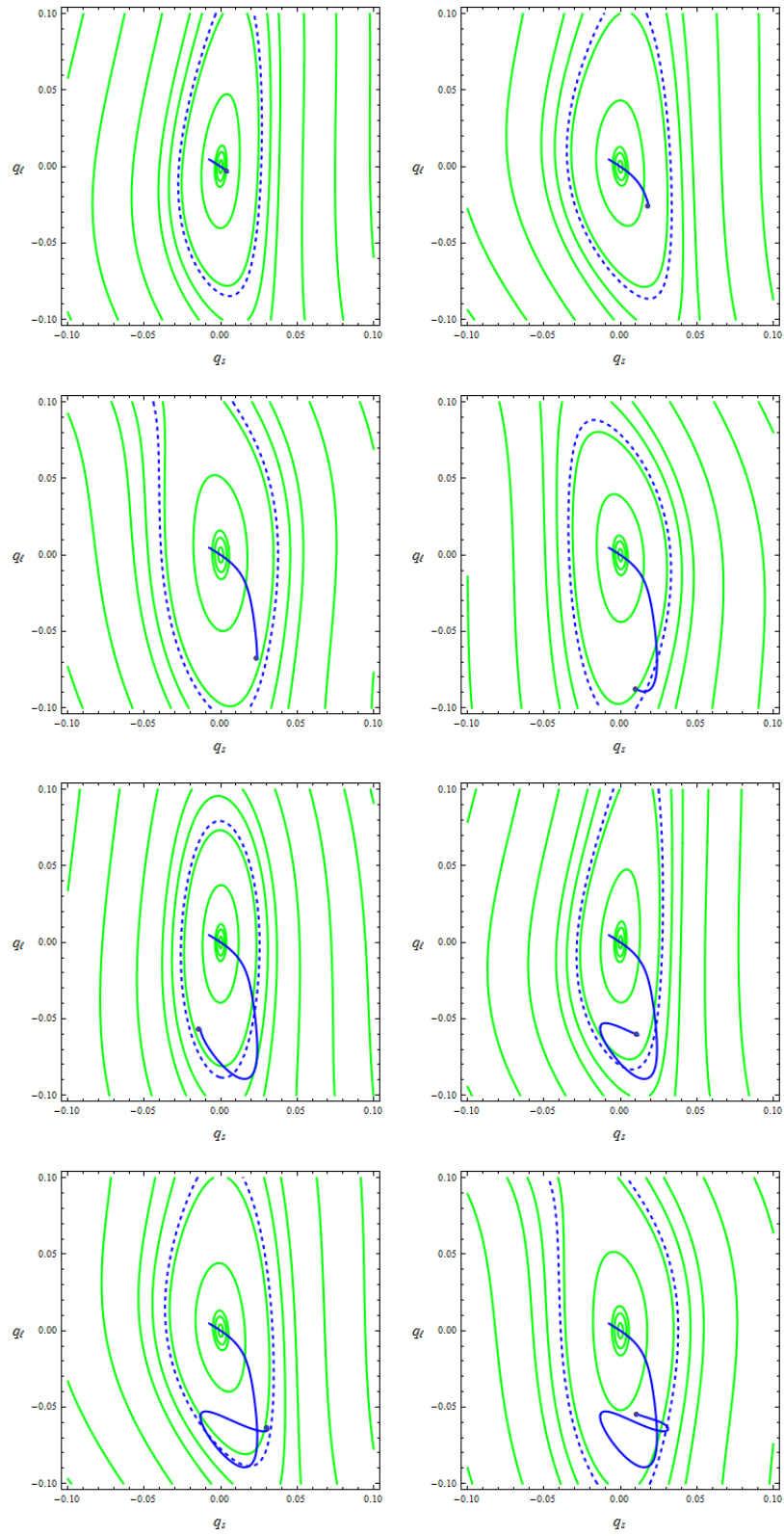Figure 4.17: Pulsating Level Sets of the Nearly-Circular Earth-Moon System

Figure 4.18: Pulsating Level Sets of the Eccentric Earth-Moon System

## 4.4 Control

The preceding sections focused on the transformation and analysis of the natural dynamics of the ERTBP in the local phase space of the triangular Lagrange points. Within Danby's stability curve and away from resonance, the triangular Lagrange points are neutrally stable such that a spacecraft placed near enough to the Lagrange point equilibrium conditions will remain nearby as long as the higher-order perturbations are sufficiently small. Within the region of stability the spacecraft flies around the Lagrange point in an irregular, but bounded relative orbit as demonstrated previously in Figs. 4.1 through 4.12.

Since the Lagrange points are not asymptotically stable, higher-order perturbations may eventually cause the spacecraft to diverge away from the Lagrange point and escape the local phase space. Thus, even under the most ideal conditions, the stability is fragile. To compensate for higher-order perturbations or to stabilize a naturally unstable orbit, feedback control is incorporated into the system and normalized along with the Hamiltonian function through the DH method. While this approach is certainly not the only way to stabilize the system, it is the most relevant to the present study since it acts directly on the transformed Hamiltonian function. Further, no assertions are made as to the optimality of the control laws provided or their performance in comparison to other control methods.

The basic approach of the DH method control strategy was derived and demonstrated for the case of the damped oscillator at the end of Chapter III. A control input is appended to Hamiltonian function to define the controlled Hamiltonian function

$$
\begin{aligned}
\mathcal{H}_c(q, p, \epsilon, \gamma, \nu) &= \mathcal{H}(q, p, \epsilon, \gamma, \nu) + \mathcal{U}(q, p, \epsilon, \gamma, \nu) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \frac{\epsilon^n}{n!} \left( \mathcal{H}_{m,n}^{(0,0)}(q, p, \nu) + \mathcal{U}_{m,n}^{(0,0)}(q, p, \nu) \right)
\end{aligned} \tag{3.107}
$$

which is then normalized through the DH method in the same manner as the un-

controlled case. If no control input is applied to the unperturbed system, that is $\mathcal{U}_{0,0}^{(0,0)} = 0$, and the control law is provided through the controlled homological equation

$$
\begin{aligned}
\mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} = {}& -\frac{\partial \mathcal{W}_{u,i+1,j}}{\partial \nu} + \mathcal{L}_{\mathcal{W}_{u,i+1,j}} \mathcal{H}_{0,0}^{(0,0)} + \mathcal{Q}_{u,i,j} \\
& + \mathcal{U}_{i,j}^{(0,0)} + \mathcal{Q}_{c,i,j}
\end{aligned}
\tag{3.117}
$$

where $\mathcal{K}_{i,j}$ is the uncontrolled, transformed Hamiltonian function, $\mathcal{Q}_{u,i+1,j}$ and $\mathcal{Q}_{c,i+1,j}$ are known *a priori*, and $\mathcal{W}_{u,i+1,j}$ and $\mathcal{U}_{i,j}$ are derived in order to achieve a prescribed form for the controlled, transformed Hamiltonian function $\mathcal{K}_{c,i,j}$.

Consider the control problem of stabilizing motion about an unstable elliptic triangular Lagrange point or a stable point with a small region of stability. This corresponds to a system with relatively high eccentricity for which the regions of stability about the triangular Lagrange points is small compared to the circular case (see Figs. 4.1 and 4.7 for example). The desired solution is defined as the trajectory of the circular solution, the CRTBP, as demonstrated in Figs. 4.19 and 4.20 starting from a small displacement of roughly $||\delta q_i|| \approx 0.01$ in the Cartesian coordinates. Fig. 4.19 includes the planar trajectory and phase portrait in the original Cartesian phase space and the planar trajectory and phase portrait in the normalized short/long-period (SLP) phase space. The phase portraits are represented along each reference direction with red, green and yellow corresponding to the $x$, $y$ and $z$ dynamics in the Cartesian phase space and cyan, brown and yellow corresponding to the $s$, $\ell$ and $z$ dynamics in the normalized SLP phase space. Fig. 4.20 shows the corresponding Cartesian state trajectories as functions of $\nu$.
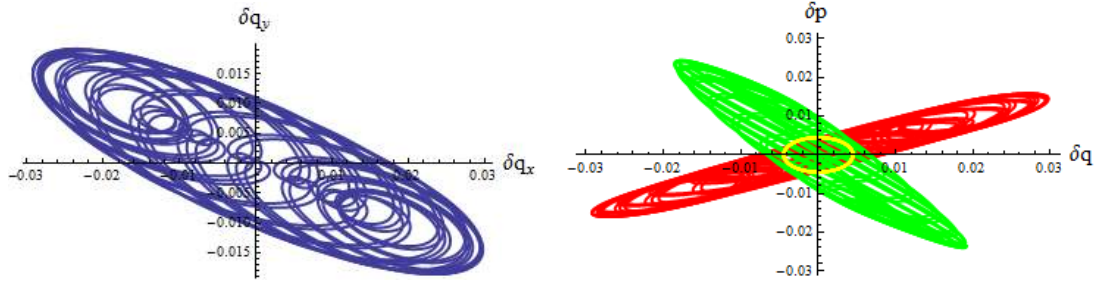
(a) Planar Trajectory in Cartesian Phase Space

(b) Phase Portrait in Cartesian Phase Space
(Red = $x$, Green = $y$, Yellow = $z$)
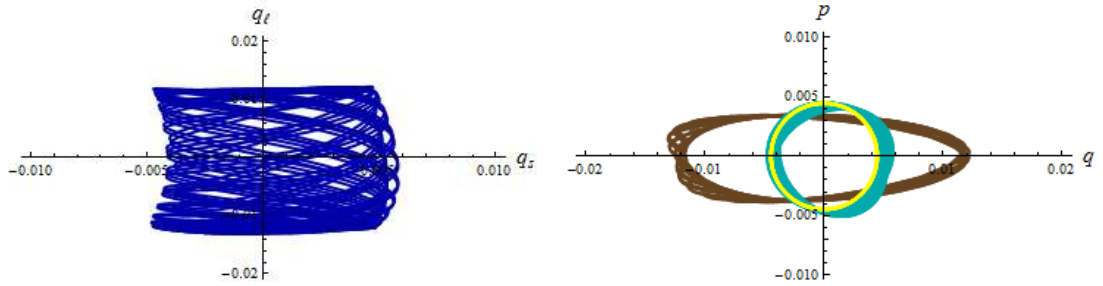
(c) Planar Trajectory in SLP Phase Space

(d) Phase Portrait in SLP Phase Space
(Cyan = $s$, Brown = $\ell$, Yellow = $z$)

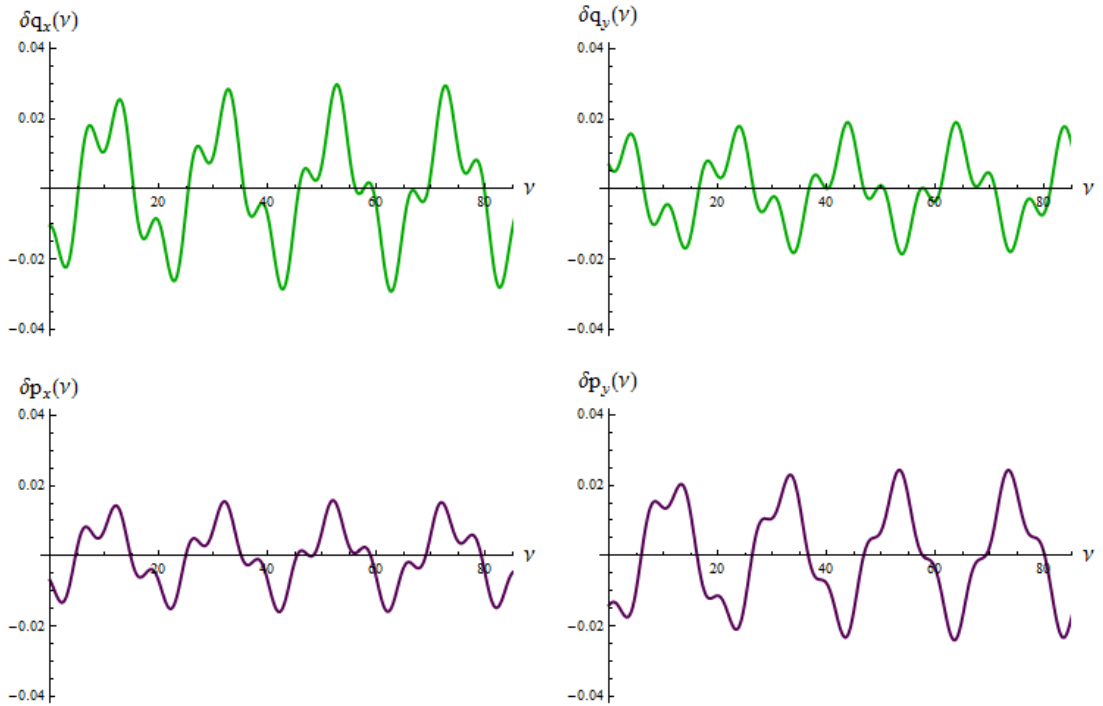Figure 4.19: Motion near $L_5$ for the Earth-Moon CRTBP



Figure 4.20: Motion near $L_5$ for the Earth-Moon CRTBP

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As a first attempt at achieving the desired circular solution within the non-circular system, the control generating functions are defined as $\mathcal{W}_u = \mathcal{V}_u = 0$ such that Eq. 3.117 reduces to

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{K}_{i,j} - \mathcal{Q}_{c,i,j} \tag{3.118}$$

which corresponds to the first control strategy applied to the damped oscillator in Section 3.5. This approach only imposes control on the secular terms within the averaged Hamiltonian function and does not affect the periodic behavior of the system. Since the goal is to compensate for the non-circular effects, the controlled Hamiltonian function is prescribed by the corresponding circular form, that is,

$$\mathcal{K}_{c,i,j} = \begin{cases} 0 & j > 0 \\ \mathcal{K}_{i,j} & j = 0 \end{cases} \tag{4.53}$$

which effectively eliminates all the non-circular perturbation terms from the transformed Hamiltonian function. Note that the circular nonlinear perturbations are retained, that is, the control is forcing the system to the circular system not the unperturbed system (linearized CRTBP) as was done for the damped oscillator. For the Earth-Moon system with $\mu = 0.0124$, the resultant control law is expressed within the transformed phase space as

$$\begin{aligned} \mathcal{U}(\hat{\theta}, \hat{I}, \nu) = e^2 \big( &-0.0155318\hat{I}_s + 0.849718\hat{I}_\ell + 0.512556\hat{I}_s^2 + 69.7551\hat{I}_\ell^2 \\ &+ 0.000941744\hat{I}_z^2 + 11.2013\hat{I}_\ell\hat{I}_s + 0.0706855\hat{I}_s\hat{I}_z - 1.64154\hat{I}_\ell\hat{I}_z + \dots \big) \\ &+ e^4\big(0.0355463\hat{I}_s + 2.43243\hat{I}_\ell + \dots\big) + \dots \end{aligned} \tag{4.54}$$

Unfortunately, since the secular terms only appear at orders of $\mathrm{O}\left(I^2\right) = \mathrm{O}\left(||\delta q, \delta p||^4\right)$, the lower-order perturbations tend to drown out the control function and the control strategy is ineffective at high eccentricities and large initial displacements. However,

for sufficiently small displacements the secular control law does eventually force the response toward the circular solution as demonstrated in Figs. 4.21 and 4.22 for an eccentricity of $e = 0.2$ and the same initial conditions used in Figs. 4.19 and 4.20. Fig. 4.21 shows the phase portrait of the uncontrolled and controlled response. The difference is somewhat subtle, but in reference to Fig. 4.19, the controlled response is evidently forcing the system toward the circular solution. This is further evident in the Cartesian state trajectories shown in Fig.4.22 which explicitly includes the circular solution as represented by dashed lines. The controlled response is again seen to force the system toward the desired circular response. The control function is itself plotted against $\nu$ in Fig. 4.23. The magnitude of the control is relatively small since it only includes higher-order terms of order $O(I^2) = O(||\delta q, \delta p||^4)$. It is also limited to terms of order $e^4$ or less as dictated by the processing power of the author's computer. Higher-order terms would likely produce even better convergence properties, but would not increase the magnitude of the control significantly.



(a) Uncontrolled Phase Portrait          (b) Controlled Phase Portrait
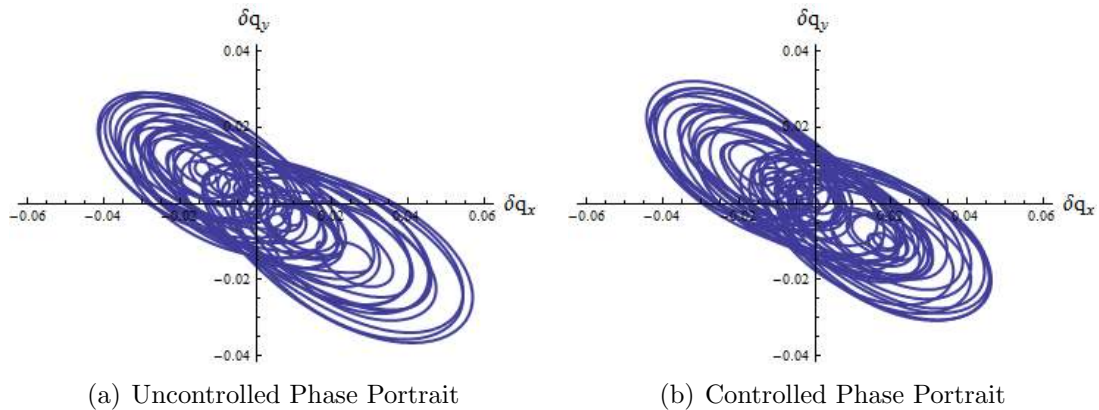
Figure 4.21: Uncontrolled and Controlled Response for Earth-Moon with $e = 0.2$

(a) Uncontrolled $\delta q_x$ Trajectory

(b) Controlled $\delta q_x$ Trajectory

(c) Uncontrolled $\delta p_x$ Trajectory

(d) Controlled $\delta p_x$ Trajectory

(e) Uncontrolled $\delta q_y$ Trajectory

(f) Controlled $\delta q_y$ Trajectory

(g) Uncontrolled $\delta p_y$ Trajectory
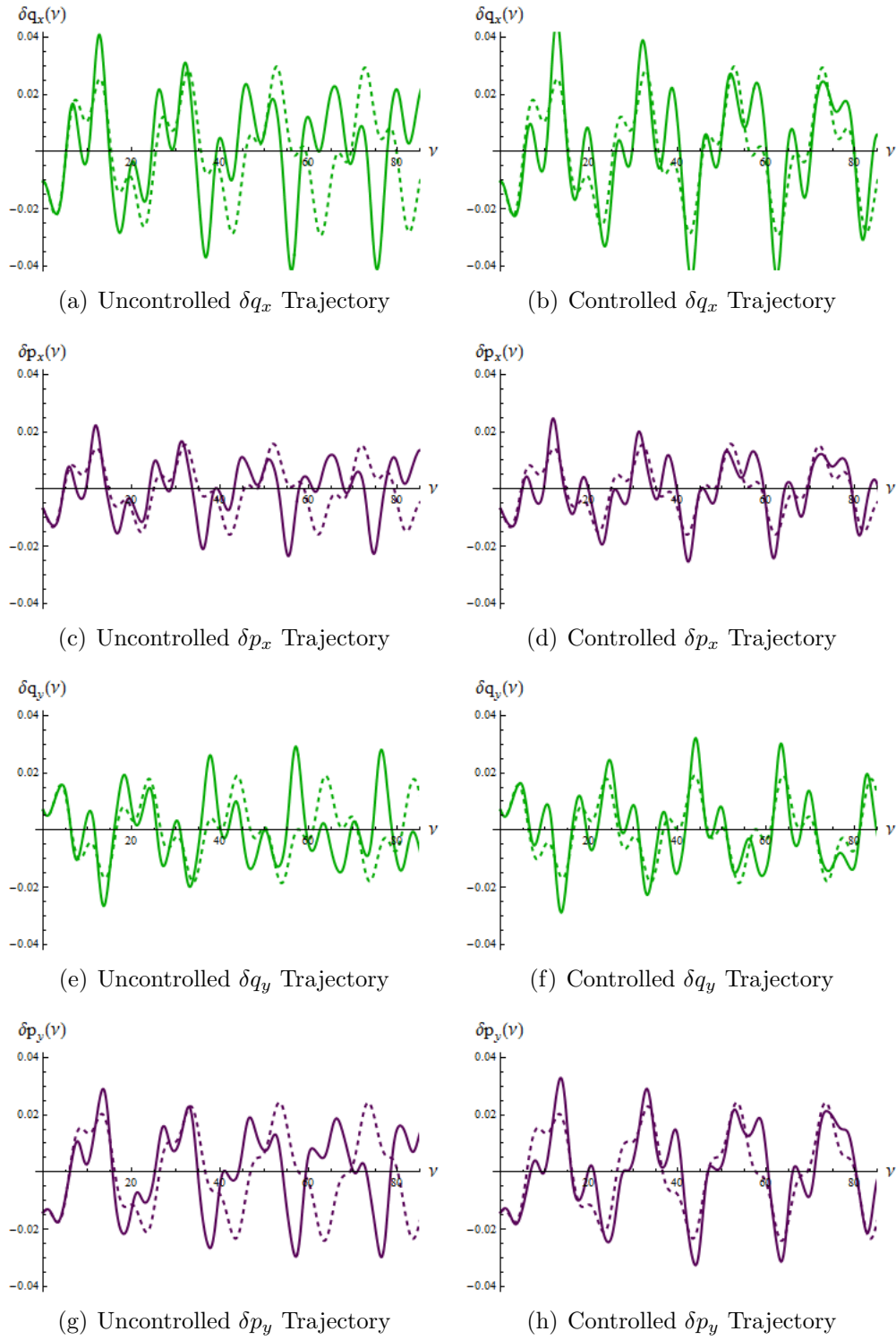
(h) Controlled $\delta p_y$ Trajectory

Figure 4.22: Uncontrolled and Controlled Response for Earth-Moon with $e = 0.2$
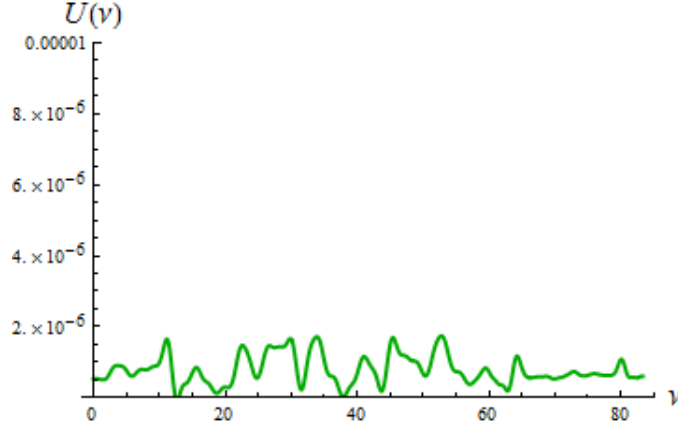
Figure 4.23: Control Function

As an alternative to the previous result, consider the case where the generating functions are defined as $\mathcal{W}_{u,i,j} = -\mathcal{W}_{i,j}$ and $\mathcal{V}_{u,i,j} = -\mathcal{V}_{i,j}$ for the non-circular terms $(j > 0)$ and zero otherwise. The control law in Eq. 3.117 reduces to

$$\mathcal{U}_{i,j}^{(0,0)} = \mathcal{K}_{c,i,j} - \mathcal{H}_{i,j}^{(0,0)} \tag{3.121}$$

where again the controlled transformed Hamiltonian function is prescribed as

$$\mathcal{K}_{c,i,j} = \begin{cases} 0 & j > 0 \\ \mathcal{K}_{i,j} & j = 0 \end{cases} \tag{4.55}$$

in order to eliminate all the non-circular effects. In this case, the control law is somewhat trivial since it is expressed directly in terms of the original Hamiltonian function and therefore does not necessitate the application of the DH method. Nonetheless, it is included for illustration purposes with the uncontrolled and controlled trajectories shown in Fig. 4.24, the corresponding control function plotted in Fig. 4.25, and the uncontrolled and controlled Cartesian state trajectories compared side-by-side in Fig. 4.26. Even when limited to terms of order $e^4$ or less, the control law effectively forces the non-circular solution to the circular case to the point where the two are

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nearly indistinguishable in Fig. 4.26. Since the control law now includes terms of odd-power in the eccentricity, the magnitude of the control is significantly increased and the convergence to the desired circular solution is much better than in the previous results derived solely from the secular control law. However, neither case is necessarily optimal and further study is required to characterize additional control strategies and to incorporate optimal control theory.



(a) Uncontrolled Phase Portrait          (b) Controlled Phase Portrait
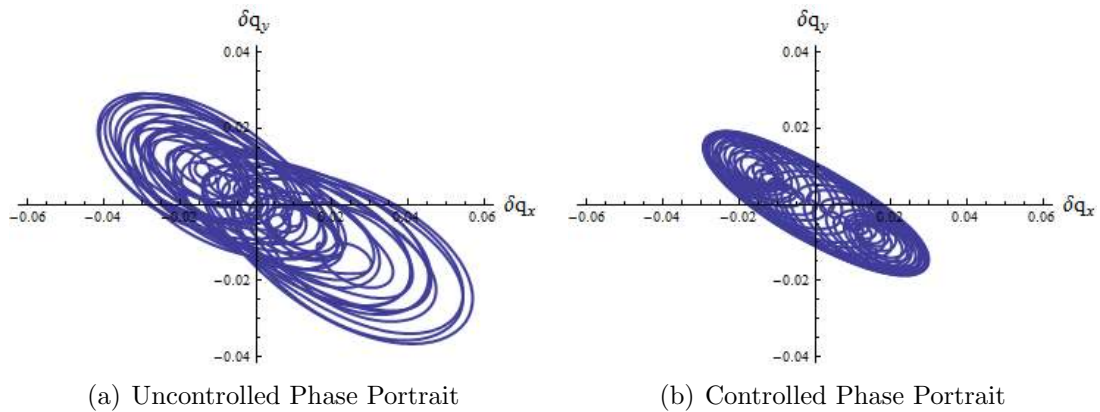
Figure 4.24: Uncontrolled and Controlled Response for Earth-Moon with $e = 0.2$



Figure 4.25: Control Function

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(a) Uncontrolled $\delta q_x$ Trajectory

(b) Controlled $\delta q_x$ Trajectory

(c) Uncontrolled $\delta p_x$ Trajectory

(d) Controlled $\delta p_x$ Trajectory

(e) Uncontrolled $\delta q_y$ Trajectory

(f) Controlled $\delta q_y$ Trajectory

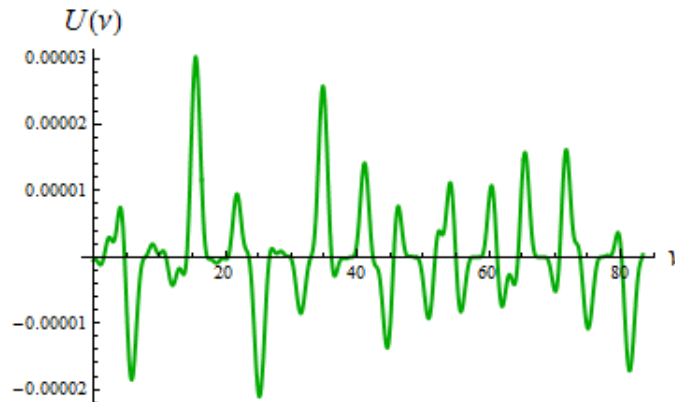(g) Uncontrolled $\delta p_y$ Trajectory
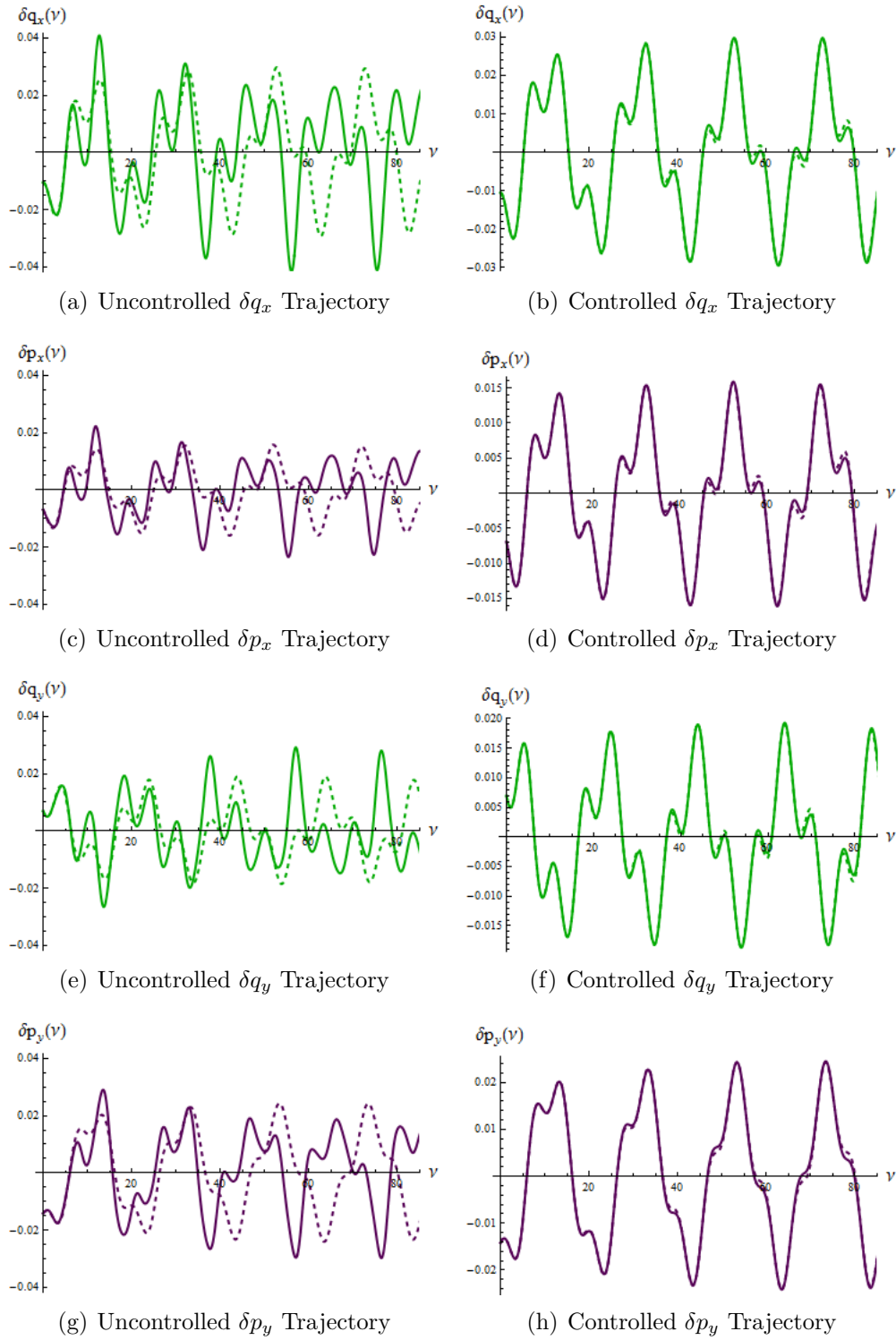
(h) Controlled $\delta p_y$ Trajectory

Figure 4.26: Uncontrolled and Controlled Response for Earth-Moon with $e = 0.2$

# Chapter V

# Conclusions

The elliptic restricted three-body problem (ERTBP) represents a fundamental generalization of the circular restricted three-body problem (CRTBP) that warrants further study within the astrodynamics community. Unfortunately, the mathematical distinction between the ERTBP and CRTBP is non-trivial and has serious implications for the method of solution. The key distinction is that while the CRTBP potential field is fixed relative to the synodic reference frame, the ERTBP potential field pulsates in rhythm with the distance between the primaries. This results in a non-autonomous Hamiltonian function even after normalizing the system such that the distance between the primaries is fixed. Being non-autonomous, the Hamiltonian function is not conserved such that Jacobi's integral no longer exists and the ERTBP is non-integrable.

Despite the mathematical complications associated with the ERTBP, it still represents a close relative to the CRTBP, particularly when the eccentricity is relatively small. To this effect, perturbation theory is applied in this study to represent the former as a close expansion about the latter. In fact, by expanding the ERTBP about zero eccentricity and about a triangular Lagrange point, it may be further be represented as a close expansion of the linearized CRTBP, which is completely integrable in the form of harmonic oscillators. In the absence of resonance, the system

is normalized using the classic Deprit-Hori Lie transform method (DH method) introduced in the 1960s.[36;37] Since their formulation only applies to expansions about a single small parameter, an extension to the method is presented in this study for applications to non-autonomous Hamiltonian functions expanded about two small parameters. While the extension to two parameters is not trivial, it follows the same general approach as used in Deprit and Hori's original method. The theorem, corollary and associated proofs presented in Chapter III of this study are intentionally formulated for an arbitrary, real-analytic Hamiltonian function. It may be compared to similar methods introduced by Varadi and Andrade and applied to other systems of interest.[40;42]

As demonstrated in this study, the extended DH method effectively normalizes the expanded ERTBP about the linearized CRTBP such that the transformed dynamics are in the form of perturbed harmonic oscillators. The corresponding Hamiltonian function is expressed in Birkhoff normal form wherein only the action-type variables explicitly appear in the function (up to the order of truncation). The coefficients of the action variables are in the form of infinite series dependent on the mass ratio of the system and even-powers in the eccentricity. Being in Birkhoff normal form, the system stability may be treated in terms of Kolmogorov-Arnold-Moser (KAM) theory, which dictates that perturbations to an integrable Hamiltonian system preserve the unperturbed integrals of motion for sufficiently small perturbations. As expressed in Birkhoff normal form, the preservation of the integrals of motion is evident directly in terms of the action-type variables, which are constant when the un-normalized, higher-order terms are neglected. The linear stability of the system is defined under Danby's $\mu - e$ stability curve, which extends to the nonlinear system using Arnold's conditions in the transformed phase space. The triangular Lagrange points are shown to be stable for all values of $\mu$ and $e$ under Danby's curve excepting those that exhibit resonance in the natural frequencies. The local foliations in the

phase space are represented in the spirit of Hill's curves of zero velocity by applying the action-type variables as local integrals of motion. The corresponding curves of zero momentum effectively foliate the normalized phase space such that regions of stability are approximated within the pulsating synodic frame.

The incorporation of control terms into the DH transformation results in an additional series of homological equations for the control terms and the corresponding control generating functions. Since the transformed Hamiltonian function encompasses the secular variations in the system with all periodic variations averaged out, a control law may be defined solely in terms of the secular response and used to either eliminate higher-order secular perturbations or confine them within a region of stability. The control method is demonstrated on a damped oscillator system and shown to force the system to the unperturbed, harmonic oscillator form. Further, the control method is applied to the ERTBP to likewise track the system to the circular solution of the CRTBP. While no conditions of optimality have been considered, the results presented in this study provide a means of comparison to alternative control methods such as linear feedback control or control derived from the CRTBP simplification.

## 5.1 Original Contributions

The original contributions presented in this study fall under two categories: the extension of the DH method and the subsequent analysis of the ERTBP. The original DH method applies to non-autonomous Hamiltonian systems, but only for expansions about a single small parameter. In the 1980s, Varadi provided an extension for expansions about two parameters, but in the form of diffeomorphisms and only for autonomous systems.[40] Andrade further extended the DH method to expansions about $N$ parameters, but also limited the method to explicitly autonomous systems. The theorem, corollary and proofs provided in Chapter III generalize the two-parameter method to non-autonomous Hamiltonian systems by explicitly incorporating the non-

autonomous remainder functions. The proof of the methodology relies heavily on the original proof presented in Deprit's paper as well as the concise proof provided in the book by Meyer et al.[16;37] However, the formulation of the theorem and corollary are original and to the author's knowledge, the extension of the DH method to non-autonomous two-parameter systems is a novel development. In addition, the implementation of control within the DH transformation further represents a novel achievement in the formulation of the theory.

The motivation for extending the DH method to non-autonomous two parameter systems was borne out of the stability analysis of the triangular elliptic Lagrange points. The nonlinear stability of the elliptic collinear points as well as the linear stability of the triangular CRTBP Lagrange points may be determined using the Lyapunov theory. However, since the linearized triangular points are not hyperbolic, Lyapunov's indirect method may not be applied in order to infer the stability of the nonlinear system. Instead one must analyze the nonlinear system directly using KAM theory. Previous studies have applied canonical transformation theory and KAM theory to demonstrate the nonlinear stability in the planar circular problem as well as the linearized elliptic case. The analysis included in Chapter IV regarding the nonlinear stability of the planar elliptic Lagrange points thereby extends previous results to the general case of a nonlinear, non-circular system. Further, the results regarding the curves of zero momentum, as derived from the local integrals of motion, compliment previous work conducted in the areas of local integrals in the ERTBP, but are entirely original and provide a novel means of describing the foliations in the local phase space and the approximate regions of stability around the elliptic triangular Lagrange points. The last section in Chapter IV introduces a novel method of feedback stabilization for the ERTBP using the formulation of the DH transformation as a means to derive control laws that effectively eliminate or otherwise modify the secular variations in the system response.

## 5.2 Future Work

A number of outstanding questions remain regarding the ERTBP and its treatment using the DH transformation and KAM theory. Some regard the theory itself while others are focused on the analysis of the ERTBP specifically. In the former, since KAM theory only provides rigorous proof of stability in the planar, 2-degree of freedom case, an advanced analysis is required to determine the onset of Arnold diffusion in the full, three-dimensional problem. Many recent studies have focused on the analysis of Arnold diffusion in general and as the theory develops it may become possible to extend the stability study into the three-dimensional domain. In addition, in extension to the preliminary control studies conducted here, a full union of the presented theory with control theory would be useful as a means of deriving novel control strategies for non-integrable, non-autonomous and otherwise complicated systems. This could then be studied in comparison to, and incorporation with, existing methods of control using linearized feedback, Lyapunov theory, optimal control theory, and controlled Hamiltonian functions.

With regards to the ERTBP, a great deal of study remains in extension to the work presented here and the many ongoing studies being conducted for the CRTBP. For mission design purposes, an expansion of the analysis beyond the local phase space of the Lagrange point is necessary to realize transfer trajectories starting from one of the two bodies or from a point exterior to the three-body system. Unfortunately, the homological equations derived in the DH method take a particularly difficult form for motion away from the Lagrange point. One option is to expand the system dynamics about a different point such as a collinear Lagrange point or upon regularizing the system, one of the primaries. Otherwise, a novel approach is required to normalize the non-autonomous ERTBP across the entire phase space of the three-body system.

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