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ANALYTICAL MODEL FOR TILTING PROPROTOR AIRCRAFT DYNAMICS, INCLUDING BLADE TORSION AND COUPLED BENDING MODES, AND CONVERSION MODE OPERATION

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SUMMARY

An analytical model is developed for proprotor aircraft dynamics. The rotor model includes coupled flap-lag bending modes, and blade torsion degrees of freedom. The rotor aerodynamic model is generally valid for high and low inflow, and for axial and nonaxial flight. For the rotor support, a cantilever wing is considered; incorporation of a more general support with this rotor model will be a straight-forward matter.

INTRODUCTION

This report presents the development of an analytical model for tilting proprotor aircraft dynamics. The emphasis in this model is on the rotor. The rotor support for the present is limited to a cantilever wing, but the incorporation of a more general support model with this rotor model will be a straight-forward matter.

The rotor motion is represented by: coupled flap and lag bending modes; rigid pitch (control system flexibility) and blade elastic torsion deflection; gimbal tilt and rotor speed perturbation degrees of freedom (optional). The six components of shaft linear and angular motion are included, and rotor blade pitch control. The rotor aerodynamic model is

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generally valid for high and low inflow, and for axial and nonaxial flight. The effects of compressibility and static stall may be included, but reverse flow and unsteady wake aerodynamic interference effects are neglected. Three components of aerodynamic gust are included as external excitation. The rotor model includes gimbal undersling, torque offset, precone, droop, sweep, and feathering axis offset (for the case with blade bending flexibility inboard of the pitch bearing). Center of gravity, aerodynamic center, and tension center offsets are included; but the elastic axis is assumed to be a straight line, and only offset from the pitch axis by the droop and sweep rotations. For the equations of motion in the nonrotating frame it is assumed the rotor has three or more blades. The equations of motion are derived for the rotor degrees of freedom, along with the forces and moments acting on the hub.

This rotor model may be coupled with any support model. The present derivation is restricted to a cantilever wing support (fig. 1). The wing motion is represented by three degrees of freedom: wing vertical bending, wing chordwise bending, and wing torsion. Wing aerodynamic forces are included, and a wing trailing edge flap in the controls.

The differential equations of motion for the proprotor and support system are presented in matrix form, for three cases: axial flow, which is a constant coefficient system; nonaxial flow, which is properly a periodic coefficient system; and a constant coefficient approximation for the nonaxial flow equations, using the mean of the coefficients in the nonrotating frame. The axial flow case is applicable to the proprotor aircraft in airplane mode cruise and in helicopter mode hover flight. The nonaxial flow case is applicable to helicopter mode forward flight, and to conversion mode flight of the proprotor aircraft.

Solutions and results for proprotor dynamics from these equations are not presented in this report, but are left to a later work.

The body of this report is composed of the following sections:

- Bending/Torsion of Highly Twisted Beam
- Equations of Motion for a Rotating Blade
- Aerodynamics
- Rotor Trim
- Blade Bending and Torsion Modes
- Support Equations of Motion: Cantilever Wing Equations of Motion

BENDING/TORSION OF HIGHLY TWISTED BEAM

This section presents an engineering beam theory model for the coupled flap/lag bending and torsion of a rotor blade, with large pitch and twist. A high aspect ratio (of the structural elements) is assumed so the beam model is applicable. The object is to relate the bending moments at the section, and the torsion moment, to the blade deflection and elastic torsion at that section. The analysis follows the work of references 1-3.

The basic assumptions are i) an elastic axis exists, and the undeformed elastic axis is a straight line, and ii) the blade has a high aspect ratio (of the structural elements), so engineering beam theory applies. Figure 2 shows the geometry of the undeformed blade. The span vaiable is r, measured from the center of rotation along the straight elastic axis. The section coordinates are x and z, the principle axes of the section, with origin at the elastic axis. So by definition

$$\int_{section} xz \, dA = 0$$

Really the integral is over the tension carring elements, i.e. modulus weighted, $\int xzE \, dA = 0$; so x and z are modulus principle axes. This remark holds for all the section integrals in this section. The tension center (modulus weighted centroid) is x_C aft of the elastic axis, and on the x axis, i.e.

$$\int x \, dA = x_c A$$
$$\int z \, dA = 0$$

Again, these are modulus weighted integrals. If E is uniform over the section, then x_C is the area centroid; and if the section mass distribution is the same as the E distribution, then x_C equals the section center of gravity location.

The angle of the major principle axis (the x axis) with respect to the hub plane is Θ . The existence of the elastic axis means that elastic twist about the EA occurs without bending; we may, and shall, include the elastic torsion deflection in Θ . The blade feathering axis (FA) is at

r_{FA}; the blade pitch is described by root pitch θ^o (rigid pitch about the FA, including that due to elastic distortion of the control system), built in twist θ_{tw}, and elastic torsion about the EA θ_e:

$$\theta = \theta^o + \theta_{tw} + \theta_e$$

$\theta^o(\psi)$ = root pitch; command collective and cyclic and control system flexibility; rigid pitch about the FA; equals θ at r_{FA}

$\theta_{tw}(r)$ = built in twist, $\theta_{tw}(r_{FA}^{+}) = 0$

$\theta_e(r,\psi)$ = elastic torsion, $\theta_e(r_{FA},\psi) = 0$

There is stress in the blade due to θ_e only. It is assumed that θ_e is small, but θ^o and θ_{tw} are allowed to be large.

The unit vectors in the hub plane (HP) axis system (rotating) are $\vec{i}_B, \vec{j}_B, \vec{k}_B$. The unit vectors for the principle axes of the section (x,r,z) are $\vec{i}, \vec{j}, \vec{k}$; these are for no bending, but including the elastic torsion in the pitch angle θ. So the principle unit vectors are rotated by θ from the HP:

$$\vec{i} = \vec{i}_B \cos\theta - \vec{k}_B \sin\theta$$
$$\vec{j} = \vec{j}_B$$
$$\vec{k} = \vec{i}_B \sin\theta + \vec{k}_B \cos\theta$$

Description of the bending

Now the engineering beam theory assumption is introduced: iii) plane sections perpendicular to the EA remain so after the bending of the blade. Figure 3 shows the geometry of the deformed section. The deformation of the blade is described by

i) deflection of the EA: x_o, r_o, z_o

ii) rotation of the section: ϕ_x, ϕ_z

iii) twist about the EA, implicit in $\vec{i}, \vec{k}$

The quantities x_o, r_o, z_o, ϕ_x, ϕ_z, θ_e are assumed to be small.

The unit vectors of the unbent cross section are $\vec{i}, \vec{j}, \vec{k}$. The unit vectors in the deformed cross section are $\vec{i}_{xs}, \vec{j}_{xs}, \vec{k}_{xs}$: $\vec{i}_{xs}$, $\vec{k}_{xs}$ are the priciple axes of the section, and $\vec{j}_{xs}$ is tangent to the deformed EA. It follows then that

$$\vec{i}_{xs} = \vec{i} + \phi_z \vec{j}$$
$$\vec{j}_{xs} = \vec{j} - \phi_z \vec{i} + \phi_x \vec{k}$$
$$\vec{k}_{xs} = \vec{k} - \phi_x \vec{j}$$

Now by definition $\vec{j}_{xs} = \partial \vec{r} / \partial s$ where $\vec{r} = x_o \vec{i} + (r + r_o)\vec{j} + z_o \vec{k}$ and s = arc length along the deformed EA. Hence to first order

$$\vec{j}_{xs} = \vec{j} + (x_o \vec{i} + z_o \vec{k})'$$
$$= \vec{j} + (x_o' + z_o \theta')\vec{i} + (z_o' - x_o \theta')\vec{k}$$

It follows the rotation of the section is

$$-\phi_z = x_o' + z_o \theta'$$
$$\phi_x = z_o' - x_o \theta'$$

or

$$\phi_x \vec{i} + \phi_z \vec{k} = (z_o \vec{i} - x_o \vec{k})'$$

The undeflected position of the blade element is

$$\vec{r} = r\vec{j} + x\vec{i} + z\vec{k}$$

and the deflected position

$$\vec{r} = (r + r_o)\vec{j} + x_o \vec{i} + z_o \vec{k} + x\vec{i}_{xs} + z\vec{k}_{xs}$$
$$= \underbrace{r\vec{j} + x_o\vec{i}}_{\text{radial stretching}} + \underbrace{r_o\vec{j} + z_o\vec{k}}_{\text{deflection of EA}} + \underbrace{(x\phi_z - z\phi_x)\vec{j}}_{\text{rotation of section}} + \underbrace{x\vec{i} + z\vec{k}}_{\text{position in section}}$$

We shall neglect r_o for now. The strain analysis is simplified then since to first order s = r; r_o just gives uniform strain over the section, so it may be simply added back later.

Analysis of strain

The metric of the undeformed blade -- no bending, and no torsion so $\overset{\vee}{\theta} = \overset{\vee}{\theta}_{tw}$ -- is

$$\vec{r} = x\vec{i} + r\vec{j} + z\vec{k}$$

$$\frac{\partial \vec{r}}{\partial r} = -x\overset{\vee}{\theta}_{tw}\vec{i} + \vec{j} + z\overset{\vee}{\theta}_{tw}\vec{k}$$

$$g_{rr} = \frac{\partial \vec{r}}{\partial r}\cdot\frac{\partial \vec{r}}{\partial r} = 1 + \overset{\vee}{\theta}{}_{tw}^{2}(x^2+z^2)$$

The metric of the deformed blade, with bending and torsion, is

$$\vec{R} = (x+x_o)\vec{i} + (r + x\phi_z - z\phi_x)\vec{j} + (z+z_o)\vec{k}$$

$$\frac{\partial \vec{R}}{\partial r} = (\overset{\vee}{x}_o + \overset{\vee}{\theta}(z+z_o))\vec{i} + (1 + x\overset{\vee}{\phi}_z - z\overset{\vee}{\phi}_x)\vec{j} + (\overset{\vee}{z}_o - \overset{\vee}{\theta}(x+x_o))\vec{k}$$

$$G_{rr} = \frac{\partial \vec{R}}{\partial r}\cdot\frac{\partial \vec{R}}{\partial r} = (1 + x\overset{\vee}{\phi}_z - z\overset{\vee}{\phi}_x)^2 + (\overset{\vee}{x}_o + \overset{\vee}{\theta}(z+z_o))^2 + (\overset{\vee}{z}_o - \overset{\vee}{\theta}(x+x_o))^2$$

Then the axial component of the stain tensor:

$$\begin{aligned}\gamma_{rr} &= \frac{1}{2}(G_{rr} - g_{rr}) \\ &= \frac{1}{2}\Big[(1 + x\overset{\vee}{\phi}_z - z\overset{\vee}{\phi}_x)^2 - 1 + (\overset{\vee}{x}_o + \overset{\vee}{\theta}(z+z_o))^2 - \overset{\vee}{\theta}{}_{tw}^{2} z^2 \\ &\qquad + (\overset{\vee}{z}_o - \overset{\vee}{\theta}(x+x_o))^2 - \overset{\vee}{\theta}{}_{tw}^{2} x^2\Big]\end{aligned}$$

The linear strain, for small x_o, z_o, θ_e, ϕ_x, ϕ_z, is

$$\begin{aligned}\gamma_{rr} \cong \epsilon_{rr} = {} & x\overset{\vee}{\phi}_z - z\overset{\vee}{\phi}_x + \overset{\vee}{\theta}{}_{tw}^{2}(xx_o + zz_o) \\ & + \overset{\vee}{\theta}_{tw}(z\overset{\vee}{x}_o - x\overset{\vee}{z}_o + \overset{\vee}{\theta}_e(x^2+z^2))\end{aligned}$$

The strain due to the blade tension, ϵ_T, is a constant such that the tension is given by

$$T = \int_{section} E\epsilon_{rr}\, \delta A = \epsilon_T \int E\delta A$$

Substituting for ϵ_{rr} and using $\int z\, dA = 0$, $\int x\, dA = x_c A$, and

$$\int (x^2+z^2)\delta A = I_p = k_p^2 A$$

where k_p is the (modulus weighted) radius of gyration about the EA, obtain for ϵ_T :

$$\epsilon_T = \frac{T}{EA} = \phi_z' x_c + \theta_{tw}'^2 x_o x_c - \theta_{tw}' z_o' x_c + \theta' \theta_e' k_p^2 + r_o'$$

In this expression, the strain due to the blade extension r_o has been added. It follows the strain may be written, with $\epsilon_T = T/EA$:

$$\epsilon_{rr} = \epsilon_T + (x - x_c)(\phi_z' - \theta_{tw}' \phi_x) - z(\phi_x' + \theta_{tw}' \phi_z) + \theta_{tw}' \theta_e' (x^2 + z^2 - k_p^2)$$

Section moments

To find the moments on the section, the second engineering beam theory assumption is introduced: iv) all stresses except σ_{rr} are negligible. The axial stress is given by $\sigma_{rr} = E\epsilon_{rr}$. The direction of σ_{rr} is

$$\hat{e} = \frac{\partial \vec{R}}{\partial r} \Big/ \left|\frac{\partial \vec{R}}{\partial r}\right|$$

The moment on the deformed cross section (figure 4) is

$$\vec{M} = M_x \vec{i}_{xs} + M_r \vec{j}_{xs} + M_z \vec{k}_{xs}$$

To find $\vec{M}$, integrate the moment about the EA due to the elemental force $\sigma_{rr}\delta A$ on the cross section:

$$\delta\vec{M} = (x\vec{i}_{xs} + z\vec{k}_{xs}) \times \vec{\sigma}_{rr}\, \delta A$$

$$= \left[-z\vec{i}_{xs} + x\vec{k}_{xs} + \theta_{tw}'(x^2+z^2)\vec{j}_{xs}\right]\sigma_{rr}\,\delta A$$

Integrating over the blade section, there follows the total moments due to bending and elastic torsion:

$$M_x = -\int_{\text{section}} z\,\sigma_{rr}\,\delta A$$

$$M_z = \int_{\text{section}} x\,\sigma_{rr}\,\delta A$$

$$M_r = GJ\,\theta_e^{\prime} + \int_{\text{section}} (x^2+z^2)\,\theta_{TW}^{\prime}\,\sigma_{rr}\,\delta A$$

To M_r has been added the torsion moment $GJ\,\theta_e^{\prime}$, due to shear stresses produced by elastic torsion. These moments are about the EA. For bending it is more convenient to work with moments about the tension center x_c:

$$M_x = -\int z\,\sigma_{rr}\,\delta A$$

$$M_z = \int (x-x_c)\,\sigma_{rr}\,\delta A$$

Substituting for σ_{rr} and integrating, the moments are:

$$M_x = EI_{zz}\,(\phi_x^{\prime} + \theta^{\prime}\phi_z) - \theta^{\prime}\theta_e^{\prime}\,EI_{zp}$$

$$M_z = EI_{xx}\,(\phi_z^{\prime} - \theta^{\prime}\phi_x) + \theta^{\prime}\theta_e^{\prime}\,EI_{xp}$$

$$M_r = (GJ + k_p^2 T + \theta^{\prime 2} EI_{pp})\,\theta_e^{\prime} + \theta_{TW}^{\prime}\,k_p^2 T$$
$$+\,\theta^{\prime}\left[EI_{xp}(\phi_z^{\prime} - \theta^{\prime}\phi_x) - EI_{zp}(\phi_x^{\prime} + \theta^{\prime}\phi_z)\right]$$

where

$$I_{zz} = \int z^2\,\delta A$$

$$I_{xx} = \int (x-x_c)^2\,\delta A$$

$$I_p = k_p^2 A = \int (x^2+z^2)\,\delta A$$

$$I_{xp} = \int (x-x_c)(x^2+z^2)\,\delta A$$

$$I_{zp} = \int z\,(x^2+z^2)\,\delta A$$

$$I_{pp} = \int (x^2+z^2-k_p^2)^2\,\delta A$$

The integrals are all over the tension carrying elements of course, i.e. modulus weighted. The tension T acts at the tension center x_C; hence the bending moments about the EA are given from those about x_C by

$$(M_z)_{EA} = M_z + x_c T$$
$$(M_x)_{EA} = M_x$$

The bending/torsion coupling is due to EI_{XP} and EI_{ZP}; for a symmetrical section $EI_{ZP} = 0$.

Vector formulation

Define the bending moment vector

$$\vec{M}_E^{(2)} = M_x \vec{i} + M_z \vec{k}$$

and the flap/lag deflection $\qquad \vec{w} = (z_0 \vec{i} - x_0 \vec{k})$

($\vec{M}_E^{(2)}$ is not quite the moment on the section, because M_x and M_z are really the $\vec{i}_{xs}$ and $\vec{k}_{xs}$ components of the moment). The derivatives of $\vec{w}$ are

$$(z_0 \vec{i} - x_0 \vec{k})' = (z_0' - x_0 \theta')\vec{i} - (x_0' + z_0 \theta')\vec{k}$$
$$= \phi_x \vec{i} + \phi_z \vec{k}$$
$$(z_0 \vec{i} - x_0 \vec{k})'' = (\phi_x' + \theta' \phi_z)\vec{i} + (\phi_z' - \theta' \phi_x)\vec{k}$$

Then the result for the bending and torsion moments may be written:

$$\vec{M}_E^{(2)} = (EI_{zz}\vec{i}\vec{i} + EI_{xx}\vec{k}\vec{k})\cdot(z_0\vec{i} - x_0\vec{k})'' + \theta_{tw}'\theta_e'\,(EI_{xp}\vec{k} - EI_{zp}\vec{i})$$

$$M_T = \left[GJ + k_P^2 T + \theta_{tw}'^2 EI_{PP}\right]\theta_e' + \theta_{tw}' k_P^2 T + \theta_{tw}'(EI_{xp}\vec{k} - EI_{zp}\vec{i})\cdot(z_0\vec{i} - x_0\vec{k})''$$

This is the result sought in this section.

Writing the EI dyadic as $\underline{\underline{EI}} = EI_{zz}\vec{i}\vec{i} + EI_{xx}\vec{k}\vec{k}$ and the coupling as $\overrightarrow{EI_p} = EI_{xp}\vec{k} - EI_{zp}\vec{i}$, this result becomes

$$\vec{M}_E^{(2)} = \underline{\underline{EI}}\,\vec{w}'' + \theta_{tw}' \theta_e' \,\overrightarrow{EI_p}$$

$$M_r = \left[GJ + k_p^2 T + \theta_{tw}'^2 EI_{pp}\right]\theta_e' + k_p^2 T \theta_{tw}' + \theta_{tw}' \,\overrightarrow{EI_p} \cdot \vec{w}''$$

This form is an obvious extension of the engineering beam theory result for uncoupled bending and torsion ($\theta_{tw}' = 0$ case). The vector formulation of the result is a major simplification. The vector form allows an easy transformation from one axis system to another. In fact, the vector form is independent of the axis system used (the base of the vectors), which is the source of the simplification. Working with the vector form simplifies the analysis to follow; the base of the vectors (for example, either the hub plane system, $\vec{i}_B \,\&\, \vec{k}_B$, or the principle axis system, $\vec{i} \,\&\, \vec{k}$) will be considered only when come to evaluate the coefficients of the equations of motion, never in the derivation.

This is a linearized result. So the $\vec{i}, \vec{k}$ appearing in EI and in $\vec{w}$ are based on the trim pitch angle $\theta = \theta^{\circ} + \theta_{tw}$. The perturbation of $\vec{i}, \vec{k}$ due to θ_e gives second order moments, which have already been neglected in the derivation. The net torsion modulus is

$$GJ_{eff} = GJ + k_p^2 T + \theta_{tw}'^2 EI_{pp}$$

where $T = \Omega^2 \int_r^1 \rho m \, d\rho$ = centrifugal tension in the blade. For the elastic torsion stiffness characteristic of rotor blades, the GJ term usually dominates. The $k_p^2 T$ term is only important for very soft (torsionally) blades, near the root. The $\theta_{tw}'^2 EI_{pp}$ term is only important for very soft, high twist blades.

EQUATIONS OF MOTION FOR A ROTATING BLADE

This section derives the equations of motion for a helicopter rotor blade. The blade motion considered includes coupled flap/lag bending (including the rigid modes if the blade is articulated), rigid pitch, elastic torsion, gimbal pitch and roll, and rotation speed perturbation. The analysis includes the effects of precone, droop, and sweep; feathering axis offset; and torque offset and gimbal undersling. The effects of shaft motion, and the hub forces and moments are included, so this analysis may be combined with the equations of motion for a body or support to give the complete aeroelastic model for the system. Numerous approximations are made in the course of the analysis, in order to obtain a tractable set of equations.

Rotor Configuration

Consider an N-bladed rotor, rotating at speed Ω (figure 5). The mth rotor blade is at

$$\psi_m = \psi + m\Delta\psi, \quad m = 1 \cdots N$$

where $\Delta\psi = 2\pi/N$ and $\psi = \Omega t$ is a nondimensional time variable. The S system ($\vec{i}_S, \vec{j}_S, \vec{k}_S$) is a nonrotating, hub plane coordinate system; it is an inertial frame. The B system ($\vec{i}_B, \vec{j}_B, \vec{k}_B$) is a coordinate frame rotating with the mth blade. The acceleration, angular velocity, and angular acceleration of the hub, and the forces and moments exerted by the rotor on the hub are defined in the nonrotating HP frame -- the S system. The rotor blade equations of motion are derived in the rotating frame -- the B system. Figure 6(a) shows the definition of the rotor shaft motion, linear and angular displacement in an inertial frame. Figure 6(b) shows the definition of forces and moments on the hub, in the nonrotating frame.

Blade root geometry

Figure 7 shows the blade root geometry considered (undistorted). The origin of the B system is the location of the gimbal; if there is no gimbal, this is just the point where the shaft motion and hub forces are defined. The gimbal is at the center of the B and S frames. The hub of the rotor is z_{FA} below the gimbal (gimbal undersling). The torque offset x_{FA} is positive in the $-\vec{i}_B$ direction. The azimuth ψ_m is measured to the feathering axis line (its projection in the HP), so the FA is parallel to the $\vec{j}_B$ axis, and offset x_{FA} from the center of rotation. The precone angle δ_{FA_1} gives the orientation of the FA with respect to the hub plane; δ_{FA_1} is positive upward, and is assumed to be a small angle. The FA is offset from the center of rotation by r_{FA}; the FA is located at $r = r_{FA}$ along the blade. The rigid pitch rotation of the blade about the feathering axis occurs at r_{FA}. The droop angle δ_{FA_2} and sweep angle δ_{FA_3} occur at r_{FA}, just outboard of the feather bearing; δ_{FA_2} and δ_{FA_3} give the orientation of the EA of the blade with respect to the FA. Note that these angles are measured in the HP frame; δ_{FA_2} is positive downward, and δ_{FA_3} is positive aft. Both δ_{FA_2} and δ_{FA_3} are assumed to be small angles.

From the gimbal to the blade root is the hub, underslung by z_{FA} and torque offset by x_{FA}. From the root to the FA is a shank of length r_{FA}, which undistorted is a straight line an an angle δ_{FA_1} to the hub plane (small precone). The blade outboard of the FA at r_{FA}, undistorted, is a straight elastic axis, with small droop and sweep (δ_{FA_2} and δ_{FA_3}) with respect to the FA direction.

From the gimbal to the root is a rigid hub. The shank (inboard of the FA at r_{FA}) and the blade (outboard of r_{FA}) are flexible in bending. The shank is assumed to be rigid in torsion, i.e. the effect of torsion of the hub inboard of the feathering axis is neglected. The blade outboard of the FA is flexible in torsion as well as bending. There is rigid pitch rotation of the blade about the FA, which takes place at r_{FA}, about the local direction of the FA at r_{FA}, including the bending of the shank. Inclusion of the bending flexibility of the blade inboard of the feathering axis means the the general rotor configuration is considered: the articulated rotor with the FA inboard or outboard of the hinges, or the cantilever blade with or without flexibility inboard of the FA. The special case of a rigid shank can be considered as well of course.

Geometry of the blade

Figure 8 shows the undeformed geometry of the blade. It is assumed that i) an elastic axis exists, and the undeformed EA is a straight line; and ii) the blade has a high aspect ratio, so engineering beam theory and lifting line theory are applicable. The following notation is used:

FA		feathering axis
EA		elastic axis
CG	x_I	locus of section center of gravity
AC	x_A	locus of section aerodynamic center
TC	x_C	locus of section tension center

The distances x_I, x_A, and x_C are positive aft, measured from the EA; they are in general a function of r. The corresponding z displacements are neglected, i.e. taken as zero.

The $i_o, j_o, \vec{k}_o$ system is the EA/principle axis system of the section. The subscript "o" is for the undeformed frame, i.e. no elastic torsion in θ , or gimbal or rotor speed degrees of freedom. The subscript will be dropped when it is obvious what is meant. The direction of the undeformed EA is j_o ; i_o & $\vec{k}_o$ are the directions of the local principle axes of the section, undeformed (no bending or torsion).

The span variable is r, measured from the center of rotation to the tip. This variable is dimensionless, r = 0 at the root to r = 1 at the tip. The section coordinates x and z are mass principle axes, with origin at the EA. It is assumed that the direction of the mass principle axes isthe same as the modulus principle axes (used in engineering beam theory for the structural moments). The CG is at z = 0, x = x_I. Usually x_I and x_C should be close. By definition then

$$\int_{section} dm = m \qquad \text{section mass}$$

$$\int z\,dm = \int xz\,dm = 0$$

$$\int x\,dm = x_I m \qquad \text{CG location}$$

and

$$\int x^2 + z^2\,dm = I_\theta \qquad \text{section polar moment of inertia, about EA}$$

The blade pitch angle is θ ; here undistorted, denoted by the subscript "m". The angle θ is measured from the HP to the section principle axis. It is then the angle of rotation of $\vec{i}_t \vec{k}_o$ from the HP axes. The undeformed pitch angle is the collective plus the built in twist

$$\theta = \theta_m = \theta_{coll} + \theta_{tw}$$

where θ_{coll} = collective pitch

$\theta_{tw}(r)$ = twist

Define θ_{coll} as the pitch at r_{FA}, so $\theta_{tw}(r_{FA}^+) = 0$. The root pitch is then $\theta^o = \theta_{coll}$. Inboard of r_{FA}, do not have the θ_{coll} rotation of the blade, but there can be pitch of the local prinicple axes with respect to the HP, which is included in θ_{tw} for $r < r_{FA}$. Note $\theta_{tw}(r_{FA}^-)$ is not necessarily zero, hence there is a jump in θ at r_{FA} of magnitude

$$\theta(r_{FA}^+) - \theta(r_{FA}^-) = \theta_{coll} - \theta_{tw}(r_{FA}^-)$$

The trim pitch angle is then

$$\theta = \theta_m = \begin{cases} \theta_{coll} + \theta_{tw}(r) & r > r_{FA} \\ \theta^o = \theta_{coll} & r = r_{FA} \\ \theta_{tw}(r) & r < r_{FA} \end{cases}$$

It is assumed that θ_m is steady, constant in time, so independent of ψ . Cyclic variations in θ , as may be required to trim the rotor, are included in the perturbation to the pitch angle. We shall allow the trim pitch angle to be large, hence θ_{coll} and θ_{tw} may be large angles.

The physical sweep and droop angles are defined with respect to the blade outboard of the FA, i.e. rotated by θ^o about the FA. Let $\delta_{FA_2}^*$ and $\delta_{FA_3}^*$ be defined with respect to the principle axes at the root (at the FA, $r = r_{FA}$); these angles will be equivalent to δ_{FA_2} and δ_{FA_3} when there is zero root pitch. Hence the droop and sweep angles are

$$\delta_{FA_2} = \delta^{*}_{FA_2} \cos\theta^{o} + \delta^{*}_{FA_3} \sin\theta^{o}$$
$$\delta_{FA_3} = -\delta^{*}_{FA_2} \sin\theta^{o} + \delta^{*}_{FA_3} \cos\theta^{o}$$

The angles $\delta^{*}_{FA_2}$ and $\delta^{*}_{FA_3}$ are fixed geometrical constants. It follows then that δ_{FA_2} and δ_{FA_3} vary with the root pitch θ^{o}. This must be accounted for when there are perturbations to θ due to the rigid pitch motion of the blade. In addition, the droop and sweep only affect the blade outboard of the FA, i.e. for $r > r_{FA}$. This may be accounted for by including with δ_{FA_2} and δ_{FA_3} the factor $U(r-r_{FA})$, where

$$U(r) = \begin{cases} 1 & r>0 \\ 0 & r<0 \end{cases}$$

We shall follow the convention of assuming the factor U is present whenever writing δ_{FA_2} or δ_{FA_3}.

From the B system (mth blade, rotating HP axes) to the o system (undistorted EA/XS axes) there is

1) rotation $\delta_{FA_1}-\delta_{FA_2}$ about $\vec{i}_B$ (small precone and droop)
2) rotation δ_{FA_3} about $\vec{k}_B$ (small sweep)
3) then rotation θ_m about $\vec{j}_{EA}$ (large pitch angle)

Hence

$$\vec{i}_o = \cos\theta_m \vec{i}_B - \sin\theta_m \vec{k}_B + \vec{j}_B\left[(\delta_{FA_1}-\delta_{FA_2})\sin\theta_m - \delta_{FA_3}\cos\theta_m\right]$$
$$\vec{k}_o = \sin\theta_m \vec{i}_B + \cos\theta_m \vec{k}_B + \vec{j}_B\left[-(\delta_{FA_1}-\delta_{FA_2})\cos\theta_m - \delta_{FA_3}\sin\theta_m\right]$$
$$\vec{j}_o = \vec{j}_{EA} = \vec{j}_B + \delta_{FA_3}\vec{i}_B + (\delta_{FA_1}-\delta_{FA_2})\vec{k}_B$$

where δ_{FA_2} and δ_{FA_3} are based on $\theta^{o}_m = \theta_{coll}$, and are absent for $r < r_{FA}$. We shall drop the subscripts "o" and "m", denoting the trim and undistorted geometry, when it is obvious what is meant.

Motion

The rotor blade motion (degrees of freedom of the rotor) is described by:

1) gimbal motion (optional); pitch and roll of the rotor disk.
2) rotor speed perturbation.
3) Then elastic torsion about the EA, and rigid pitch about the FA.
4) Followed by bending deflection of the FA, including rigid flap and lag motion if the blade is articulated.

Gimbal motion/rotor speed perturbation

Figure 9(a) shows the gimbal motion and rotor speed perturbation in the nonrotating frame. The gimbal degrees of freedom are β_{Gc} and β_{Gs} : rotation of the rotor disk, in the nonrotating frame (S system), with the same convention as β_{1c} and β_{1s} tip path plane tilt. The rotor rotational speed perturbation is $\dot{\psi}_s$. The degree of freedom ψ_s is a rotation about the shaft axis $\vec{k}_s$; so the azimuth angle of the mth blade is really $\psi_m + \psi_s$.

Figure 9(b) shows the gimbal motion in the rotating frame. The degrees of freedom are β_G and θ_G , given by

$$\beta_G = \beta_{Gc} \cos\psi_m + \beta_{Gs} \sin\psi_m$$
$$\theta_G = -\beta_{Gc} \sin\psi_m + \beta_{Gs} \cos\psi_m$$

The main effects are due to β_G , the flapwise rotation about the $\vec{i}_B$ axis; θ_G , the rotation about $\vec{j}_B$, only introduces a translation of the hub due to z_{FA} and x_{FA}. The blade pitch θ is defined with respect to the hub plane, so only the blade inboard of the FA sees the pitch rotation due to θ_G , and that effect will be neglected.

Blade motion

Figure 3 shows the geometry of the deformed blade. The blade deformation is described by:

1) twist about the EA: θ

2) deflection of the EA: x_o, z_o

3) rotation of the section: ϕ_x, ϕ_z

The pitch angle θ, including perturbations, is implicit in the $\vec{i}, \vec{j}, \vec{k}$ system; $\vec{i}$ & $\vec{k}$ are the principle axes of the blade with no bending, but now with the blade elastic torsion and rigid pitch motion in θ. The XS system ($\vec{i}_{xs}, \vec{j}_{xs}, \vec{k}_{xs}$) are the principle axes and EA of the deformed blade, including torsion and bending. The tangent to the deformed EA is $\vec{j}_{xs}$; the rotation of the cross section from $\vec{i}$ & $\vec{k}$ is given by ϕ_x and ϕ_z:

$$\phi_x \vec{i} + \phi_z \vec{k} = (z_o' - x_o \theta')\vec{i} - (x_o' + z_o \theta')\vec{k}$$
$$= (z_o \vec{i} - x_o \vec{k})'$$

The blade position, relative the root, is then:

$$\vec{r} = (r + r_o)\vec{j} + x_o \vec{i} + z_o \vec{k} + x \vec{i}_{xs} + z \vec{k}_{xs}$$
$$= (\underbrace{r}_{\text{radial station}} + \underbrace{r_o}_{\text{axial extension}} + \underbrace{x\phi_z - z\phi_x}_{\text{rotation of section}})\vec{j} + \underbrace{(x_o \vec{i} + z_o \vec{k})}_{\text{deflection of EA (bending)}} + \underbrace{x\vec{i} + z\vec{k}}_{\text{cross section position}}$$

We will neglect the perturbation of the radial position, $r_o + x\phi_z - z\phi_x \ll r$.

<u>Blade pitch</u>

The angle θ is the angle of the major principle axis of the section (the x axis, chordwise), measured from the hub plane. The blade pitch is composed of:

1) $\theta^o(\psi)$ = root pitch, the pitch of the blade at the FA at $r = r_{FA}^+$; due to commanded collective and control, control system flexibility, and mechanical feedback.

2) $\theta_{tw}(r)$ = built-in twist; $\theta_{tw}(r_{FA}^+) = 0$.

3) $\theta_e(r,\psi)$ = elastic torsion about the EA; zero at the FA, $\theta_e(r_{FA},\psi) = 0$; only θ_e produces torsion shear stress in the blade.

For the shank, $r < r_{FA}$, elastic torsion is neglected, and it does not see the root pitch θ°. Then $\theta_{tw}(r)$ is used for the pitch of the principle axes with respect to the hub plane in the shank. There is no perturbation to θ inboard of the FA, the pitch and torsion degrees of freedom are only for outboard of the FA. Since probably $\theta_{tw}(r_{FA}^{-})$ is not zero, there is a jump in θ at the FA. So the blade pitch is

$$\theta = \begin{cases} \theta^{\circ} + \theta_{tw} + \theta_{e} & r > r_{FA} \\ \theta^{\circ} & r = r_{FA}^{+} \\ \theta_{tw} & r < r_{FA} \end{cases}$$

The commanded root pitch angle is

$$\theta^{c} = \theta_{coll} + \theta_{con}$$

where

θ_{coll} = collective pitch angle; the trim value, which may be large but is assumed to be steady in time.

θ_{con} = control input; time dependent, but assumed to be a small angle; includes cyclic to trim the rotor; and for dynamics analyses this is the control variable.

The blade root pitch commanded by the control system is θ^{c} ; θ° is the actual blade root pitch. The difference $(\theta^{\circ} - \theta^{c})$ is the rigid pitch motion due to control system flexibility or mechanical coupling in the control system (i.e. δ_3 effects). Hence we may write the blade pitch as:

$$\theta = \begin{cases} (\theta_{coll} + \theta_{tw}) + (\theta^{\circ} - \theta^{c}) + \theta_{con} + \theta_{e} & r > r_{FA} \\ \theta^{\circ} = \theta_{coll} + (\theta^{\circ} - \theta^{c}) + \theta_{con} & r = r_{FA}^{+} \\ \theta_{tw} & r < r_{FA} \end{cases}$$

Now the pitch θ may be separated into trim and perturbation contributions:

$$\theta = \begin{cases} \theta_{m} + \tilde{\theta} & r > r_{FA} \\ \theta_{m}^{\circ} + \tilde{\theta}^{\circ} & r = r_{FA}^{+} \\ \theta_{m} & r < r_{FA} \end{cases}$$

where the trim terms are (as above)

$$\theta_m = \begin{cases} \theta_{coll} + \theta_{tw} \\ \theta_{coll} \\ \theta_{tw} \end{cases}$$

and the perturbations

$$\tilde{\theta} = \begin{cases} (\theta^o - \theta^c) + \theta_{con} + \theta_e \\ \tilde{\theta}^o = (\theta^o - \theta^c) + \theta_{con} \\ 0 \end{cases}$$

The trim value of the pitch is θ_m, composed of θ_{coll} and θ_{tw}; it is a large, steady angle. The perturbation of the pitch angle is $\tilde{\theta}$, composed of the blade motion $(\theta^o - \theta^c)$, θ_{con}. and θ_e; all are small angles, so $\tilde{\theta}$ is small. For the rigid pitch degree of freedom we shall use p_o, defined as

$$p_o = \tilde{\theta}^o = (\theta^o - \theta^c) + \theta_{con}$$

and for the elastic pitch θ_e an expansion as a series in the normal modes (described in the sections to follow). Note that p_o is the total rigid pitch perturbation, including the control θ_{con}.

Coordinate Frames and Axes

S System: nonrotating, hub plane frame

rotation $\psi_m - 90$ about $\vec{k}_S$ shaft rotation

B system: rotating, mth blade, hub plane

β_G about $\vec{i}_B$

θ_G about $\vec{j}_B$ gimbal

ψ_s about $\vec{k}_B$ speed perturbation

H system: hub frame

δ_{FA_1} about $\vec{i}_H$ precone

FA system: blade FA (EA for $r < r_{FA}$)

$-\delta_{FA_2}$ about $\vec{i}_{FA}$ droop

$-\delta_{FA_3}$ about $\vec{k}_{FA}$ sweep

EA system: EA outboard of FA

θ about $\vec{j}_{EA}$

$-\theta_G$ about $\vec{j}_{EA}$ pitch/torsion

blade system: principle axes, including torsion

ϕ_x about $\vec{i}$

ϕ_z about $\vec{k}$ bending

XS system: principle axes, torsion and bending

B system

$$\vec{i}_B = \sin\psi_m \, \vec{i}_S - \cos\psi_m \, \vec{j}_S$$

$$\vec{j}_B = \cos\psi_m \, \vec{i}_S + \sin\psi_m \, \vec{j}_S$$

$$\vec{k}_B = \vec{k}_S$$

Blade system

From the B system to the blade system, there is first rotation $\beta_G + \delta_{FA_1} - \delta_{FA_2}$ about $\vec{i}_B$ and rotation $\psi_s - \delta_{FA_3}$ about $\vec{k}_B$; then rotation θ about $\vec{j}_{EA}$. Hence:

$$\vec{i} = \cos\theta\, \vec{i}_B - \sin\theta\, \vec{k}_B + \vec{j}_B\left[(\beta_G + \delta_{FA_1} - \delta_{FA_2})\sin\theta + (\psi_S - \delta_{FA_3})\cos\theta\right]$$

$$\vec{k} = \sin\theta\, \vec{i}_B + \cos\theta\, \vec{k}_B + \vec{j}_B\left[-(\beta_G + \delta_{FA_1} - \delta_{FA_2})\cos\theta + (\psi_S - \delta_{FA_3})\sin\theta\right]$$

$$\vec{j} = \vec{j}_{EA} = \vec{j}_B - (\psi_S - \delta_{FA_3})\vec{i}_B + (\beta_G + \delta_{FA_1} - \delta_{FA_2})\vec{k}_B$$

For $r < r_{FA}$, the δ_{FA_2} and δ_{FA_3} terms drop; in particular:

$$\vec{j} = \vec{j}_{FA} = \vec{j}_B - \psi_S \vec{i}_B + (\beta_G + \delta_{FA_1})\vec{k}_B$$

XS system

$$\vec{i}_{xs} = \vec{i} + \phi_z \vec{j}$$

$$\vec{j}_{xs} = \vec{j} - \phi_z \vec{i} + \phi_x \vec{k} = \vec{j} + (x_o\vec{i} + z_o\vec{k})'$$

$$\vec{k}_{xs} = \vec{k} - \phi_x \vec{j}$$

Undisturbed blade system

The undisturbed blade system is $\vec{i}, \vec{j}, \vec{k}$ without β_G, ψ_S, or the pitch perturbations in θ (and δ_{FA_2} & δ_{FA_3} based on $\theta_m^\circ = \theta_{coll}$); hence:

$$\vec{i}_o = \cos\theta_m\, \vec{i}_B - \sin\theta_m\, \vec{k}_B + \vec{j}_B\left[(\delta_{FA_1} - \delta_{FA_2})\sin\theta_m - \delta_{FA_3}\cos\theta_m\right]$$

$$\vec{k}_o = \sin\theta_m\, \vec{i}_B + \cos\theta_m\, \vec{k}_B + \vec{j}_B\left[-(\delta_{FA_1} - \delta_{FA_2})\cos\theta_m - \delta_{FA_3}\sin\theta_m\right]$$

$$\vec{j}_o = \vec{j}_B + \delta_{FA_3}\vec{i}_B + (\delta_{FA_1} - \delta_{FA_2})\vec{k}_B$$

Now since the blade motion $\tilde{\theta}$, β_G, and ψ_S is small, it is possible to expand the blade system in terms of the undisturbed frame:

$$\vec{i} = \vec{i}_o - \tilde{\theta}\vec{k}_o + \vec{j}_B\left[(\beta_G - \tilde{\theta}^\circ\delta_{FA_3})\sin\theta + (\psi_S + \tilde{\theta}^\circ\delta_{FA_2})\cos\theta\right]$$

$$\vec{k} = \vec{k}_o + \tilde{\theta}\vec{i}_o + \vec{j}_B\left[-(\beta_G - \tilde{\theta}^\circ\delta_{FA_3})\cos\theta + (\psi_S + \tilde{\theta}^\circ\delta_{FA_2})\sin\theta\right]$$

There follows then:

$$(x_o\vec{i} + z_o\vec{k}) = (x_o\vec{i}_o + z_o\vec{k}_o) + \tilde{\theta}(z_o\vec{i}_o - x_o\vec{k}_o)$$
$$+ \vec{j}_B\left[(\psi_s + \tilde{\theta}^{\circ}\delta_{FA_2})\vec{i}_B - (\beta_G - \tilde{\theta}^{\circ}\delta_{FA_3})\vec{k}_B\right]\cdot(x_o\vec{i}_o + z_o\vec{k}_o)$$

which is an expansion of the bending/torsion deflection of the blade in terms of the undisturbed axis system.

Blade position, velocity, and acceleration

Position

The distance from the gimbal to a point on the blade section is

$$\vec{r} = \underbrace{-z_{FA}\vec{k}_H - x_{FA}\vec{i}_H}_{\text{pt of gimbal / blade}} + \underbrace{r_{FA}(\vec{j}_{FA} - \vec{j}_{BA})}_{\text{[illegible]}} + \underbrace{r\vec{j}}_{\text{radial position}} + \underbrace{x_o\vec{i} + z_o\vec{k}}_{\text{bending deflection}} + \underbrace{x\vec{i}_{xs} + z\vec{k}_{xs}}_{\text{cross section point}}$$

which may be written

$$\vec{r} = \vec{i}_B(-x_{FA} - z_{FA}\theta_G - r_{FA}\delta_{FA_3})$$
$$+ \vec{j}_B(z_{FA}\beta_G - x_{FA}\psi_s)$$
$$+ \vec{k}_B(-z_{FA} + x_{FA}\theta_G + r_{FA}\delta_{FA_2})$$
$$+ r\vec{j} + (x_o\vec{i} + z_o\vec{k}) + (x\vec{i} + z\vec{k})$$

$$= \vec{i}_B(-x_{FA} - z_{FA}\theta_G - r\psi_s + (r - r_{FA})\delta_{FA_3})$$
$$+ \vec{j}_B(r + z_{FA}\beta_G - x_{FA}\psi_s)$$
$$+ \vec{k}_B(-z_{FA} + x_{FA}\theta_G + r(\beta_G + \delta_{FA_1}) - (r - r_{FA})\delta_{FA_2})$$
$$+ (x_o\vec{i} + z_o\vec{k}) + (x\vec{i} + z\vec{k})$$

Velocity

The velocity of a point on the blade, relative to the rotating frame (the B system) is:

$$
\begin{aligned}
\vec{v}_r = \left(\frac{\partial}{\partial t}\vec{r}\right)_B = \ & \vec{i}_B\,(-z_{FA}\,\dot{\theta}_G - r\,\dot{\psi}_s) \\
& + \vec{j}_B\,(z_{FA}\,\dot{\beta}_G - x_{FA}\,\dot{\psi}_s) \\
& + \vec{k}_B\,(x_{FA}\,\dot{\theta}_G + r\,\dot{\beta}_G) \\
& + (r - r_{FA})\,\dot{\theta}^\circ\,(-\vec{i}_B\,\delta_{FA_2} - \vec{k}_B\,\delta_{FA_3}) \\
& + ((x_0+x)\vec{i} + (z_0+z)\vec{k})^{\cdot}
\end{aligned}
$$

where

$$
\begin{aligned}
((x_0+x)\vec{i} + (z_0+z)\vec{k})^{\cdot} \cong \ & (x_0\vec{i}_0 + z_0\vec{k}_0)^{\cdot} \\
& + \dot{\theta}\,((z_0+z)\vec{i}_0 - (x_0+x)\vec{k}_0)
\end{aligned}
$$

Acceleration

The acceleration of a point on the blade, relative to the rotating frame, and neglecting the squares of velocities, is:

$$
\begin{aligned}
\vec{a}_r = \left(\frac{\partial}{\partial t}\vec{v}_r\right)_B = \ & \vec{i}_B\,(-z_{FA}\,\ddot{\theta}_G - r\,\ddot{\psi}_s) \\
& + \vec{j}_B\,(z_{FA}\,\ddot{\beta}_G - x_{FA}\,\ddot{\psi}_s) \\
& + \vec{k}_B\,(x_{FA}\,\ddot{\theta}_G + r\,\ddot{\beta}_G) \\
& + (r - r_{FA})\,\ddot{\theta}^\circ\,(-\vec{i}_B\,\delta_{FA_2} - \vec{k}_B\,\delta_{FA_3}) \\
& + ((x_0+x)\vec{i} + (z_0+z)\vec{k})^{\cdot\cdot}
\end{aligned}
$$

where

$$
\begin{aligned}
((x_0+x)\vec{i} + (z_0+z)\vec{k})^{\cdot\cdot} \cong \ & (x_0\vec{i}_0 + z_0\vec{k}_0)^{\cdot\cdot} \\
& + \ddot{\theta}\,((z_0+z)\vec{i} - (x_0+x)\vec{k})
\end{aligned}
$$

Acceleration of the blade

The acceleration of the blade is required with respect to an inertia frame, i.e. in the S system. The B system rotates at a constant angular velocity $\vec{\Omega} = \Omega\vec{k}_B$ with respect to the S frame. The shaft motion is composed of linear and angular velocity and acceleration of the origin of the S frame (the gimbal point at the hub center of rotation). The acceleration, angular velocity, and angular acceleration of the S system, with respect to the nonrotating, inertial frame, are:

$$\vec{a}_0 = \ddot{x}_h \vec{i}_s + \ddot{y}_h \vec{j}_s + \ddot{z}_h \vec{k}_s$$
$$\vec{\omega}_0 = \dot{\alpha}_x \vec{i}_s + \dot{\alpha}_y \vec{j}_s + \dot{\alpha}_z \vec{k}_s$$
$$\dot{\vec{\omega}}_0 = \ddot{\alpha}_x \vec{i}_s + \ddot{\alpha}_y \vec{j}_s + \ddot{\alpha}_z \vec{k}_s$$

It is assumed that $\vec{a}_0$, $\vec{\omega}_0$, and $\dot{\vec{\omega}}_0$ are all small quantities.

Given above is the motion of the blade in the B frame, the acceleration and velocity of the blade $\vec{a}_r$ and $\vec{v}_r$. Now we shall derive the acceleration of a blade point in inertial space ($\vec{a}$), in terms of the motion of the shaft, the rotation of the rotor, and the blade motion in the B frame. From the result for the acceleration in a rotating coordinate frame, there follows:

$$\vec{a} = \vec{a}_0 + \vec{a}_{r,s} + 2\vec{\omega}_0 \times \vec{v}_{r,s} + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}$$

where $\vec{a}_{r,s}$ and $\vec{v}_{r,s}$ are the acceleration and velocity of a point in the S frame. The B system rotates at angular velocity $\vec{\Omega} = \Omega \vec{k}_s$ with respect to the S frame. Hence with Ω constant and no angular acceleration or acceleration of B with respect to S, there follows:

$$\vec{a}_{r,s} = \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$
$$\vec{v}_{r,s} = \vec{v}_r + \vec{\Omega} \times \vec{r}$$

where $\vec{a}_r$ and $\vec{v}_r$ are the acceleration and velocity in the B frame. Thus:

$$\vec{a} = \vec{a}_0 + \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\omega}_0 \times \vec{v}_r + 2\vec{\omega}_0 \times (\vec{\Omega} \times \vec{r}) + \vec{\omega}_0 \times (\vec{\omega}_0 \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}$$

To first order in the velocity and angular velocity, this becomes:

$$\vec{a} \cong \vec{a}_0 + \vec{a}_r + 2\vec{\Omega} \times \vec{v}_r + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + 2\vec{\omega}_0 \times (\vec{\Omega} \times \vec{r}) + \dot{\vec{\omega}}_0 \times \vec{r}$$

The six terms are respectively: the acceleration of the origin; the relative acceleration in the rotating frame; the relative coriolis acceleration; the centrifugal acceleration; the coriolis acceleration due to the angular velocity of the origin; and the angular acceleration of the origin. In dyadic operator form, and with $\vec{\Omega} = \Omega \vec{k}_B$, this result is

$$\vec{a} = \vec{a}_o + \vec{a}_r + 2\Omega(\vec{k}_B \times)\vec{v}_r - \Omega^2(\vec{i}_B\vec{i}_B + \vec{j}_B\vec{j}_B)\vec{r}$$
$$+ 2\Omega(\vec{k}_B\vec{r} - \vec{r}\vec{k}_B)\vec{\omega}_o - (\vec{r}\times)\dot{\vec{\omega}}_o$$

To obtain the forces and moments and equations of motion, the acceleration is multiplied by the density of the blade point (dmdr) and integrated over the volume of the blade, to produce the total acceleration of the blade.

Equations of Motion and Forces

The equations of motion for elastic bending, torsion, and rigid pitch of the blade are obtained from equilibrium of inertial, aerodynamic, and elastic moments on the portion of the blade outboard of r:

$$-\vec{M}_E + \vec{M}_A = \vec{M}_I$$

where

M_E = structural moment on deformed cross section, on the inboard face; so $-M_E$ is the external force on the outboard face.

M_A = total aerodynamic force on blade surface outboard of r.

M_I = total acceleration of the blade outboard of r.

M_E is a general elastic constraint: from engineering beam theory for bending and torsion; from control system flexibility for rigid pitch; hub spring for gimbal motion; or it is the force or moment on the hub due to the rotor (so $-M_E$ is the force on the rotor). M_I is the angular acceleration of the blade outboard of r, about the point $\vec{r}_o(r)$:

$$\vec{M}_I = \int_r^1 \int_{\text{section}} (\vec{r}(\rho) - \vec{r}_o(r)) \times \vec{a}\, dm\, d\rho$$

For bending, engineering beam theory gives

$$\vec{M}_E^{(2)} = M_x\vec{i} + M_z\vec{k} = (\vec{i}\,\vec{i}_{xs} + \vec{k}\,\vec{k}_{xs})\vec{M}_E$$

So this operator is applied to $\vec{M}_I$ and $\vec{M}_A$ also. For bending the moments about the tension center x_C are required. Then the desired PDE for bending is obtained from $\frac{\partial^2}{\partial r^2}\vec{M}^{(2)}$.

For elastic torsion, engineering beam theory gives $M_{r_E} = \vec{j}_{xs}\cdot\vec{M}_E$. So this same operator is applied to $\vec{M}_I$ and $\vec{M}_A$. For torsion require moments about the section EA (x = 0) at r; also, elastic torsion involves only $r > r_{FA}$. The desired PDE for torsion is then obtained from $\frac{\partial}{\partial r} M_r$.

The equation of motion for rigid pitch degree of freedom $p_o = \tilde{\theta}_o$ is obtained from equilibrium of moments about the FA:

$$M_{FA} = \vec{j}_{xs}(r_{FA}^-)\cdot\vec{M}(r_{FA}^-)$$

where $\vec{M}$ is the moment about the FA (x = 0) at $r = r_{FA}$. The elastic restraint from the control system flexibility gives the restoring moment about the FA, completing the desired equation of motion.

The equations of motion for the gimbal degrees of freedom β_{Gc} and β_{Gs} are obtained from equilibrium of moments about the gimbal:

$$M_x = \vec{i}_s\cdot\vec{M}$$
$$M_y = \vec{j}_s\cdot\vec{M}$$

where $\vec{M}$ is the total moments (from all N blades) about the gimbal point, in the nonrotating frame.

The equation of motion for the speed perturbation degree of freedom ψ_s is obtained from equilibrium of torque moments $Q = -M_z = \vec{k}_s\cdot\vec{M}$ where again $\vec{M}$ is the total moment about the gimbal point.

The total rotor force and moment on the hub (at the gimbal point) are obtained from a sum over the N blades of $\vec{F}^{(m)}$ and $\vec{M}^{(m)}$, the force and moment due to the mth blade:

$$\vec{F} = \sum_{m=1}^{N} \vec{F}^{(m)}$$

$$\vec{M} = \sum_{m=1}^{N} \vec{M}^{(m)}$$

Since $-\vec{F}^{(m)}$ and $-\vec{M}^{(m)}$ are the forces on the blade, there follows from force and moment equilibrium of the entire blade:

$$-\vec{F}^{(m)} + \vec{F}_A = \vec{F}_I$$

$$-\vec{M}^{(m)} + \vec{M}_A = \vec{M}_I$$

The hub force and moment are required in the nonrotating hub plane frame (the S system); the components are defined as:

$$\vec{F} = H\vec{i}_s + Y\vec{j}_s + T\vec{k}_s$$

$$\vec{M} = M_x\vec{i}_s + M_y\vec{j}_s - Q\vec{k}_s$$

Note $\vec{M}$ produces the gimbal and rotor speed perturbation motion, if those degrees of freedom are used, but it is also transmitted throught the gimbal to the helicopter body or support.

Aerodynamics

The aerodynamic forces and moments on the blade are obtained from the integral over the span of the aerodynamic forces and pitch moments on the blade section. The aerodynamic forces and moment on the section are:

F_x	in hub plane, positive in drag direction, ($\vec{i}_B$ direction), at the EA
F_z	normal to the hub plane, positive up ($\vec{k}_B$ direction), at the EA
F_r	radial, positive outward ($\vec{j}_B$ direction), at the EA
M_a	moment about the EA, positive nose up

The forces on the section are F_x, F_z, and F_r; these are the component of the aerodynamic lift and drag forces in the hub plane axis system (the B frame). F_r is here just the radial drag force; the radial components of F_x and F_z due to tilt of the blade when it is bent are included explicitly in the results below.

The aerodynamic force on the section, at the deformed EA, including the effect of the rotation of the section due to bending, is thus:

$$\vec{F}_{aero} = F_x \vec{i}_B + F_z \vec{k}_B - \vec{j}_B \vec{j}_{xs} \cdot (F_x \vec{i}_B + F_z \vec{k}_B) + F_r \vec{j}_{xs}$$

$$\cong F_x \vec{i}_B + F_z \vec{k}_B + \vec{j}_B \left(F_r - F_z \left(\beta_G + \delta_{FA_1} - \delta_{FA_2} + \vec{k}_B \cdot (x_o \vec{i} + z_o \vec{k})' \right) \right)$$

and the aerodynamic moment:

$$\vec{M}_{aero} = M_a \vec{j}_{xs}$$

Equations of Motion and Hub Forces/Moments

Bending

The equation of motion comes from

$$\frac{\partial^2}{\partial r^2}\vec{M}_E^{(2)} + \frac{\partial^2}{\partial r^2}\vec{M}_I^{(2)} = \frac{\partial^2}{\partial r^2}\vec{M}_A^{(2)}$$

where $\vec{M}$ is the moment about the tension center ($x = x_0$) at r, and

$$\vec{M}^{(2)} = (\vec{i}\vec{i}_{xs} + \vec{k}\vec{k}_{xs})\vec{M} = (\vec{i}\vec{i} + \vec{k}\vec{k} - (x_0\vec{i} + z_0\vec{k})'\vec{j}\,)\vec{M}$$

Inertia: Considering first $r > r_{FA}$, the moment is

$$\vec{M}_I = \int_r^1 \int_{\text{section}} \left(\vec{r}\big|_{\rho x z} - \vec{r}\big|_{r x_c 0} \right) \times \vec{a}\, dm\, d\rho$$

$$= \int_r^1 \int \left[\begin{matrix} (\rho - r)\vec{j} + (x_0 + x)\vec{i} + (z_0 + z)\vec{k} \\ -((x_0 + x_c)\vec{i} + z_0\vec{k})\big|_r \end{matrix} \right] \times \vec{a}\, dm\, d\rho$$

So

$$\frac{\partial \vec{M}}{\partial r} = -(\vec{j} + (x_0\vec{i} + z_0\vec{k} + x_c\vec{i})')\times \int_r^1 \int \vec{a}\, dm\, d\rho$$
$$- \int (x\vec{i} + z\vec{k} - x_c\vec{i}) \times \vec{a}\, dm$$

$$\frac{\partial^2 \vec{M}}{\partial r^2} = \vec{j} \times \int \vec{a}\, dm$$
$$- \left[(x_0\vec{i} + z_0\vec{k} + x_c\vec{i})' \times \int_r^1 \int \vec{a}\, dm\, d\rho \right]'$$
$$- \left[\int (x\vec{i} + z\vec{k} - x_c\vec{i}) \times \vec{a}\, dm \right]'$$

$$\vec{j}\cdot\vec{M} = \int_r^1 \int \left[\begin{matrix} (z_0 + z)\vec{i} - (x_0 + x)\vec{k} \\ -(z_0\vec{i} - x_0\vec{k} - x_c\vec{k})\big|_r \end{matrix} \right] \cdot \vec{a}\, dm\, d\rho$$

So

$$\frac{\partial^2}{\partial r^2}\vec{M}_I^{(2)} = (\vec{i}\vec{i}+\vec{k}\vec{k})\frac{\partial^2 \vec{M}_I}{\partial r^2} - \left[(x_0\vec{i}+z_0\vec{k})^{v}\,\vec{j}\cdot\vec{M}_I\right]^{vv}$$

$$= \vec{j}\times\int \vec{a}\,dm$$

$$+ \left[\int (z\vec{i} - x\vec{k} + x_c\vec{k})\,\vec{j}\cdot\vec{a}\,dm\right]^{v}$$

$$+ \left[(z_0\vec{i} - x_0\vec{k} - x_c\vec{k})^{v}\int_r^1\int \vec{j}\cdot\vec{a}\,dm\,d\xi\right]^{v}$$

$$- \left[(x_0\vec{i}+z_0\vec{k})^{v}\int_r^1\left[\begin{matrix}(z_0+\zeta)\vec{i} - (x_0+x)\vec{k} \\ -(z_0\vec{i} - x_0\vec{k} - x_c\vec{k})|_r\end{matrix}\right]\cdot\vec{a}\,dm\,d\rho\right]^{vv}$$

We shall neglect the last term in this result, $[(x_0\vec{i}+z_0\vec{k})^{v}\,\vec{j}\cdot\vec{M}_I]^{vv}$, as order $(c/R)^2$ smaller than the first term. Including the case $r<r_{FA}$, which only introduces an effect of droop and sweep, the result is:

$$\frac{\partial^2}{\partial r^2}\vec{M}_I^{(2)} = \vec{j}\times\int\vec{a}\,dm$$

$$+ \left[\int (z\vec{i} - x\vec{k} + x_c\vec{k})\,\vec{j}\cdot\vec{a}\,dm\right]^{v}$$

$$+ \left[(z_0\vec{i} - x_0\vec{k} - x_c\vec{k})^{v}\int_r^1\int \vec{j}\cdot\vec{a}\,dm\,d\rho\right]^{v}$$

$$- \delta(r-r_{FA})\,(\delta_{FA_2}\vec{i}_B + \delta_{FA_3}\vec{k}_B)\int_{r_{FA}}^1\int \vec{j}\cdot\vec{a}\,dm\,d\rho$$

where $\delta(r)$ is the delta function, i.e. an impulse at $r=0$.

a) shaft motion: with $\vec{r}\cong r\vec{j}_B$ have

$$\vec{a} = \vec{a}_0 + 2\Omega(\vec{k}_B\vec{r} - \vec{r}\vec{k}_B)\vec{\omega}_0 - \vec{r}\times\dot{\vec{\omega}}_0$$

$$\cong \vec{a}_0 + 2\Omega r(\vec{k}_B\vec{j}_B - \vec{j}_B\vec{k}_B)\vec{\omega}_0 - r\,\vec{j}_B\times\dot{\vec{\omega}}_0$$

So

$$\frac{\partial^2\vec{M}^{(2)}}{\partial r^2} \cong m\,\vec{j}_B\times\vec{a} = m(\vec{i}_B\vec{k}_B - \vec{k}_B\vec{i}_B)\vec{a}_0$$

$$+ 2\Omega r m(\vec{i}_B\vec{j}_B)\vec{\omega}_0$$

$$+ mr(\vec{i}_B\vec{i}_B + \vec{k}_B\vec{k}_B)\dot{\vec{\omega}}_0$$

b) relative acceleration:

$$\frac{\partial^2 \vec{M}^{(2)}}{\partial r^2} \cong \vec{j} \times \int \vec{a}_r \, dm$$

$$= m \begin{bmatrix} -\vec{k}_B \left(-z_{FA} \ddot{\theta}_G - r \ddot{\psi}_S \right) \\ +\vec{i}_B \left(x_{FA} \ddot{\theta}_G + r \ddot{\beta}_G \right) \\ + \left(z_0 \vec{i} - x_0 \vec{k} \right)^{\cdot\cdot} \\ - \ddot{\theta} \left((x_0 + x_I) \vec{i} + z_0 \vec{k} \right) \\ - \ddot{\theta}^{\circ} \left(r - r_{FA} \right) \left(\delta_3 \vec{i}_B - \delta_2 \vec{k}_B \right) \end{bmatrix}$$

c) centrifugal acceleration:

$$\vec{a} = -\Omega^2 \left(\vec{i}_B \vec{i}_B + \vec{j}_B \vec{j}_B \right) \vec{r}$$

so

$$\vec{j} \times \vec{a} = \Omega^2 \vec{k}_B \left[-x_{FA} - z_{FA} \theta_G - r_{FA} \delta_3 + \vec{i}_B \cdot \left((x_0 + x) \vec{i} + (z_0 + z) \vec{k} \right) \right]$$
$$+ \Omega^2 \vec{i}_B \, r \left(\beta_G + \delta_1 - \delta_2 \right)$$

$$\vec{j} \cdot \vec{a} \cong -\Omega^2 r$$

so

$$\frac{\partial^2 \vec{M}^{(2)}}{\partial r^2} = -\Omega^2 \left[\begin{array}{l} \left[(z_0 \vec{i} - x_0 \vec{k})' \int_r^1 \rho m \, d\rho \right]' + m \vec{k}_B \vec{k}_B \cdot (z_0 \vec{i} - x_0 \vec{k}) \\ - \left[\left(\tilde{\theta} (x_0 \vec{i} + z_0 \vec{k} + x_c \vec{i}) \right)' \int_r^1 \rho m \, d\rho \right]' \\ + \left[(x_c - x_I) \tilde{\theta} \vec{i} r m \right]' - m \vec{k}_B \tilde{\theta} \vec{k}_B \cdot (x_0 \vec{i} + z_0 \vec{k} + x_I \vec{i}) \\ - \delta(r - r_{FA}) \tilde{\theta}^{\circ} (\delta_3 \vec{i}_B - \delta_2 \vec{k}_B) \int_{r_{FA}}^1 \rho m \, d\rho \\ + \tilde{\theta}^{\circ} m \left[r \delta_3 \vec{i}_B - r_{FA} \delta_2 \vec{k}_B \right] \\ + \vec{k}_B m z_{FA} \theta_G - \vec{i}_B m r \beta_G \end{array} \right.$$

$$-\Omega^2 \left[\begin{array}{l} \left[(x_c - x_I) \vec{k} r m \right]' - \left[(x_c \vec{k})' \int_r^1 \rho m \, d\rho \right]' \\ - \delta(r - r_{FA}) \left(\delta_2 \vec{i}_B + \delta_3 \vec{k}_B \right) \int_{r_{FA}}^1 \rho m \, d\rho \\ + \vec{k}_B m \left[x_{FA} + r_{FA} \delta_3 - x_I \cos\theta \right] \\ - \vec{i}_B m r \left(\delta_1 - \delta_2 \right) \end{array} \right.$$

d) coriolis acceleration: $\vec{a} = 2\Omega \vec{k}_B \times \vec{v}_r$

So $\quad \vec{J}\cdot\vec{a} = 2\Omega \left\{ -r\dot{\psi}_s - \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k})^{\cdot} \right\}$

$$\vec{J}\times\vec{a} = 2\Omega \left\{ \vec{k}_B\, \vec{J}_B\cdot\vec{v}_r + \left[-(\delta_1-\delta_2)\vec{i}_B + \delta_3\vec{k}_B\right]\left[-r\dot{\psi}_s - \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k})^{\cdot}\right]\right\}$$

For $\vec{J}\cdot\vec{v}_r$, it is here necessary to include the effect of the change in the radial position of the blade due to bending:

$$\Delta\vec{r} = -\vec{J}_B \frac{1}{2}\int_0^r \begin{bmatrix} (x_0\vec{i} + z_0\vec{k} + x_I\vec{i})' - (\psi_s - \delta_3)\vec{i}_B \\ +(\beta_G + \delta_1 - \delta_2)\vec{k}_B \end{bmatrix}^2 d\rho$$

So

$$\begin{aligned} \vec{J}\cdot\vec{v}_r = & -\int_0^r (z_0\vec{i} - x_0\vec{k})'^{\cdot}\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k})'\, d\rho \\ & -(z_0\vec{i} - x_0\vec{k})^{\cdot}\cdot((\delta_1-\delta_2)\vec{i}_B - \delta_3\vec{k}_B) \\ & -\dot{\beta}_G\left[-z_{FA} + \vec{i}_B\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) + r\delta_1 - (r-r_{FA})\delta_2\right] \\ & -\dot{\psi}_S\left[x_{FA} + \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) - (r-r_{FA})\delta_3\right] \end{aligned}$$

then

$$\frac{\partial^2\vec{M}^{(2)}}{\partial r^2} = 2\Omega \left[\begin{aligned} & -\vec{k}_B m \int_0^r (z_0\vec{i} - x_0\vec{k})'^{\cdot}\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k})'\, d\rho \\ & -\vec{k}_B m \dot{\beta}_G\left[-z_{FA} + \vec{i}_B\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) + r\delta_1 - (r-r_{FA})\delta_2\right] \\ & -\vec{k}_B m \dot{\psi}_S\left[x_{FA} + \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) + r_{FA}\delta_3\right] \\ & -\left[(x_c - x_I)\vec{k} m\, (r\dot{\psi}_S + \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k})^{\cdot})\right]' \\ & -\left[(z_0\vec{i} - x_0\vec{k} - x_c\vec{k})'\int_r^1 (\rho\dot{\psi}_s + \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k})^{\cdot})\, m\, d\rho\right]' \\ & +\delta(r-r_{FA})(\delta_2\vec{i}_B + \delta_3\vec{k}_B)\int_{r_{FA}}^1 (\rho\dot{\psi}_s + \vec{k}_B\cdot(z_0\vec{i} - x_0\vec{k})^{\cdot})\, m\, d\rho \\ & +m(\delta_1-\delta_2)\vec{J}\times\left[(z_0\vec{i} - x_0\vec{k})^{\cdot} + r\dot{\psi}_s\vec{k}_B\right] \end{aligned}\right.$$

elastic:

$$\frac{\partial^2 \vec{M}_E^{(2)}}{\partial r^2} = \left[(EI_{zz}\vec{i}\vec{i} + EI_{xx}\vec{k}\vec{k})(z_0\vec{i} - x_0\vec{k})'' \right]'' + \left[(EI_{xp}\vec{k} - EI_{zp}\vec{i})\theta_{tw}' \theta_e' \right]''$$

aerodynamic: The moment about the tension center ($x = x_C$) at r due to the blade loading at the EA at ρ :

$$\vec{M}_A = \int_r^1 (\vec{r}|_{\rho 00} - \vec{r}|_{rx_C0}) \times \vec{F}_{aero}\, d\rho$$

$$\cong \int_r^1 (\rho - r)(F_z\vec{i}_B - F_x\vec{k}_B)\, d\rho$$

So

$$\frac{\partial^2 \vec{M}_A^{(2)}}{\partial r^2} = \vec{j} \times \vec{F}_{aero} = F_z\vec{i}_B - F_x\vec{k}_B$$

<u>Elastic torsion</u>

The equation of motion is obtained from

$$-\frac{\partial}{\partial r} M_{r_E} - \frac{\partial}{\partial r} M_{r_I} = -\frac{\partial}{\partial r} M_{r_A}$$

where $\vec{M}$ is the moment about the EA ar r, and

$$\frac{\partial}{\partial r} M_r = \frac{\partial}{\partial r} \vec{j}_{xs} \cdot \vec{M} = \vec{j} \cdot \frac{\partial \vec{M}}{\partial r} + \left[(x_0\vec{i} + z_0\vec{k})' \cdot \vec{M} \right]'$$

inertia:

$$\vec{M}_I = \int_r^1 \int_{section} (\vec{r}|_{\rho xz} - \vec{r}|_{r00}) \times \vec{a}\, dm\, d\rho$$

$$= \int_r^1 \int \left[\begin{matrix} (\rho - r)\vec{j} + (x_0 + x)\vec{i} + (z_0 + z)\vec{k} \\ -(x_0\vec{i} + z_0\vec{k})|_r \end{matrix} \right] \times \vec{a}\, dm\, d\rho$$

So $$\frac{\partial \vec{M}}{\partial r} = -(\vec{j} + (x_0\vec{i} + z_0\vec{k})') \times \int_r^1 \int \vec{a}\, dm\, d\rho - \int (x\vec{i} + z\vec{k}) \times \vec{a}\, dm$$

So
$$\frac{\partial M_{rx}}{\partial r} = \int (x\vec{k} - z\vec{\imath})\cdot\vec{a}\,dm - (z_0\vec{\imath} - x_0\vec{k})^{''}\cdot\int_r^1 (\rho - r)\int \vec{a}\,dm\,d\rho$$
$$- (x_0\vec{\imath} + z_0\vec{k})^{'}\cdot \int (x\vec{k} - z\vec{\imath})\vec{J}\cdot\vec{a}\,dm$$
$$- (x_0\vec{\imath} + z_0\vec{k})^{''}\cdot\int_r^1\int \begin{bmatrix} (z_0+z)\vec{\imath} - (x_0+x)\vec{k} \\ -(z_0\vec{\imath} - x_0\vec{k})|_r \end{bmatrix}\vec{J}\cdot\vec{a}\,dm\,d\rho$$

The ODE for the kth torsion node of the mth blade is obtained by operating with $\int_{r_{FA}}^1 \xi_k(\cdots)\,dr$; where ξ_k is the elastic torsion mode shape. It is most convenient to apply this operator at this point:

$$\int_{r_{FA}}^1 \xi_k \frac{\partial M_{rx}}{\partial r}\,dr =$$
$$\int_{r_{FA}}^1 \int \Big\{ \xi_k (x\vec{k} - z\vec{\imath}) - \int_{r_{FA}}^r \xi_k (z_0\vec{\imath} - x_0\vec{k})^{''}(r-\rho)\,d\rho \Big\}\cdot\vec{a}\,dm\,dr$$

$$- \int_{r_{FA}}^1 \xi_k \left\{ \begin{aligned} &(x_0\vec{\imath} + z_0\vec{k})^{'}\cdot\int (x\vec{k} - z\vec{\imath})\vec{J}\cdot\vec{a}\,dm \\ &+ (x_0\vec{\imath} + z_0\vec{k})^{''}\cdot\int_r^1\int \begin{bmatrix} (z_0+z)\vec{\imath} - (x_0+x)\vec{k} \\ -(z_0\vec{\imath} - x_0\vec{k})|_r \end{bmatrix}\vec{J}\cdot\vec{a}\,dm\,d\rho \end{aligned} \right\} dr$$

and we shall use the notation:

$$\vec{X}_k = \xi_k x_I \vec{k} - \int_{r_{FA}}^r \xi_k (z_0\vec{\imath} - x_0\vec{k})^{''}(r-\rho)\,d\rho$$

a) shaft motion:

$$\int_{r_{FA}}^1 \xi_k \frac{\partial M_r}{\partial r}\,dr = \left(\int_{r_{FA}}^1 \vec{X}_k\, m\,dr\right)(\vec{\imath}_B\vec{\imath}_B + \vec{k}_B\vec{k}_B)\vec{a}_0$$
$$+ 2\Omega\left(\int_{r_{FA}}^1 \vec{X}_k\, r m\,dr\right)\cdot\vec{k}_B\,\vec{J}_B\cdot\vec{\omega}_0$$
$$+ \left(\int_{r_{FA}}^1 \vec{X}_k\, r m\,dr\right)(\vec{k}_B\vec{\imath}_B - \vec{\imath}_B\vec{k}_B)\,\dot{\vec{\omega}}_0$$

b) relative acceleration:

$$
\begin{aligned}
\int_{r_{FA}}^{1} \xi_k \frac{\partial M_r}{\partial r}\,dr = {} & \left(\int_{r_{FA}}^{1} \vec{X}_k\, m\,dr\right)\cdot\left(-z_{FA}\,\ddot{\theta}_G\,\vec{\imath}_B + x_{FA}\,\ddot{\theta}_G\,\vec{k}_B\right) \\
& + \left(\int_{r_{FA}}^{1} \vec{X}_k\, r\, m\,dr\right)\cdot\left(-\ddot{\psi}_s\,\vec{\imath}_B + \ddot{\beta}_G\,\vec{k}_B\right) \\
& + \int_{r_{FA}}^{1} \vec{X}_k\cdot\left(x_0\vec{\imath} + z_0\vec{k}\right)^{\cdot\cdot}\, m\,dr \\
& + \int_{r_{FA}}^{1} \left[\vec{X}_k\cdot\left(z_0\vec{\imath} - x_0\vec{k} - x_I\vec{k}\right) + \xi_k\, x_I^2\right]\ddot{\theta}\, m\,dr \\
& - \int_{r_{FA}}^{1} \vec{X}_k\cdot\left(\delta_2\vec{\imath}_B + \delta_3\vec{k}_B\right)\left(r - r_{FA}\right)\ddot{\theta}^{\circ}\, m\,dr \\
& - \int_{r_{FA}}^{1} \xi_k\,\ddot{\theta}\, I_\theta\,dr
\end{aligned}
$$

where $I_\theta = \int x^2+z^2\,dm$ = section pitch moment of inertia, about EA.

c) centrifugal acceleration: neglecting a number of terms due to blade torsion and pitch (of the same order as the propeller moment), compared to the structural stiffening; there follows:

$$
\begin{aligned}
\int_{r_{FA}}^{1} \xi_k \frac{\partial M_r}{\partial r}\,dr = -\Omega^2 \Bigg[& -\left(\int_{r_{FA}}^{1} \vec{X}_k\, m\,dr\right)\cdot\vec{\imath}_B\; z_{FA}\,\theta_G \\
& -\left(\int_{r_{FA}}^{1} \vec{X}_k\, r\, m\,dr\right)\cdot\vec{k}_B\;\beta_G \\
& + \int_{r_{FA}}^{1} \xi_k\,\tilde{\theta}\, I_\theta\left(\cos^2\theta - \sin^2\theta\right)dr \\
& - \int_{r_{FA}}^{1} \vec{X}_k\cdot\vec{k}_B\;\vec{k}_B\cdot\left(x_0\vec{\imath} + z_0\vec{k} + x_I\vec{\imath}\right) m\,dr \\
& - \int_{r_{FA}}^{1} \xi_k \left\{ \begin{array}{l} \left(x_0\vec{\imath} + z_0\vec{k}\right)^{\prime}\cdot\vec{k}\; x_I\, m\, r \\ -\left(x_0\vec{\imath} + z_0\vec{k}\right)^{\prime\prime}\cdot\left(z_0\vec{\imath} - x_0\vec{k}\right)\int_r^1 \rho\, m\,d\rho \end{array} \right\} dr \\
& + \int_{r_{FA}}^{1} \xi_k \left\{ \begin{array}{l} \left(x_0\vec{\imath} + z_0\vec{k}\right)\cdot\vec{k}\; x_I\, m \\ - r\left(x_0\vec{\imath} + z_0\vec{k}\right)^{\prime\prime}\cdot\int_r^1 \left(z_0\vec{\imath} - x_0\vec{k} - x_I\vec{k}\right) m\,d\rho \end{array} \right\} dr \\
& + \int_{r_{FA}}^{1} \vec{X}_k\cdot\left[\vec{\imath}_B\left(-x_{FA} - r_{FA}\,\delta_3\right) - \vec{k}_B\, r\left(\delta_1 - \delta_2\right)\right] m\,dr \Bigg]
\end{aligned}
$$

elastic: using $T = \Omega^2 \int_r^1 \rho m \, d\rho$,

$$\frac{\partial M_{rE}}{\partial r} = \left[\left(GJ + k_P^2 \Omega^2 \int_r^1 \rho m \, d\rho + \theta_{tw}'^2 EI_{PP}\right)\theta_e' \right]' + \left(\theta_{tw}' k_P^2 \Omega^2 \int_r^1 \rho m \, d\rho \right)' + \left[\theta_{tw}' \left(EI_{xP}\vec{k} - EI_{zP}\vec{i}\right)\cdot\left(z_0\vec{i} - x_0\vec{k}\right)'' \right]'$$

aerodynamic: the moment about the EA at r is

$$\vec{M}_A = \int_r^1 M_a \vec{i}_{xs} \, d\rho + \int_r^1 \left[\begin{matrix} (\rho - r)\vec{i} + (x_0\vec{i} + z_0\vec{k}) \\ - (x_0\vec{i} + z_0\vec{k})|_r \end{matrix} \right] \times \vec{F}_{aero} \, d\rho$$

so

$$\frac{\partial \vec{M}_A}{\partial r} = - M_a \vec{i}_{xs} - \vec{i}_{xs} \times \int_r^1 \vec{F}_{aero} \, d\rho$$

$$\frac{\partial M_{rA}}{\partial r} = \frac{\partial}{\partial r} \vec{i}_{xs} \cdot \vec{M}_A = \vec{i}_{xs} \cdot \frac{\partial \vec{M}_A}{\partial r} + (x_0\vec{i} + z_0\vec{k})'' \cdot \vec{M}_A$$

$$= - M_a - (z_0\vec{i} - x_0\vec{k})'' \cdot \int_r^1 (\rho - r)(F_x\vec{i}_B + F_z\vec{k}_B) \, d\rho$$

thus

$$\int_{r_{FA}}^1 \eta_k \frac{\partial M_{rA}}{\partial r} \, dr = - \int_{r_{FA}}^1 \eta_k M_a \, dr + \int_{r_{FA}}^1 (F_x\vec{i}_B + F_z\vec{k}_B)\cdot\vec{X}_{Ak} \, dr$$

where $\vec{X}_{Ak} = \vec{X}_k - \eta_k x_I \vec{k}$

Rigid pitch

The equation of motion comes from $M_{FA_E} + M_{FA_I} = M_{FA_A}$, where

$$M_{FA} = \vec{J}_{xs}(r_{FA}^-)\cdot\vec{M} = (\vec{J}_{FA} + (x_0\vec{i}+z_0\vec{k})|_{r_{FA}})\cdot\vec{M}$$

and $\vec{M}$ is the moment about the FA, at $r = r_{FA}$.

inertia:

$$\vec{M}_I = \int_{r_{FA}}^1 \int (\vec{r}|_{rxz} - \vec{r}|_{r_{FA}^-\,00})\times\vec{a}\,dm\,dr$$

$$= \int_{r_{FA}}^1 \int \left[\begin{matrix}(r-r_{FA})\vec{J} + (x_0+x)\vec{i} + (z_0+z)\vec{k} \\ -(x_0\vec{i}+z_0\vec{k})|_{r_{FA}}\end{matrix}\right]\times\vec{a}\,dm\,dr$$

So

$$M_{FA_I} = \int_{r_{FA}}^1 \int \left[\begin{matrix}(z_0+z)\vec{i} - (x_0+x)\vec{k} - (\delta_2\vec{i}_B + \delta_3\vec{k}_B)(r-r_{FA}) \\ -(z_0\vec{i}-x_0\vec{k})|_{r_{FA}} - (z_0\vec{i}-x_0\vec{k})'|_{r_{FA}}(r-r_{FA})\end{matrix}\right]\cdot\vec{a}\,dm\,dr$$

$$+\int_{r_{FA}}^1 \int \left[\begin{matrix}-(x_0\vec{i}+z_0\vec{k})'|_{r_{FA}}\cdot\left\{\begin{matrix}(z_0+z)\vec{i}-(x_0+x)\vec{k} \\ -(z_0\vec{i}-x_0\vec{k})|_{r_{FA}}\end{matrix}\right\} \\ \\ +(\delta_3\vec{i}_B - \delta_2\vec{k}_B)\cdot\left\{\begin{matrix}(z_0+z)\vec{i}-(x_0+x)\vec{k} \\ +(z_0\vec{i}-x_0\vec{k})')_{r_{FA}}(r-r_{FA})\end{matrix}\right\}\end{matrix}\right]\vec{J}\cdot\vec{a}\,dm\,dr$$

and we shall use the notation:

$$\vec{X}_0 = -(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) + (\delta_2\vec{i}_B + \delta_3\vec{k}_B)(r-r_{FA})$$
$$+ (z_0\vec{i}-x_0\vec{k})|_{r_{FA}} + (z_0\vec{i}-x_0\vec{k})')_{r_{FA}}(r-r_{FA})$$

a) shaft motion:

$$M_{FA} = -\left(\int_{r_{FA}}^1 \vec{X}_0\,m\,dr\right)(\vec{i}_B\vec{i}_B + \vec{k}_B\vec{k}_B)\ddot{\vec{a}}_0$$
$$-2\Omega\left(\int_{r_{FA}}^1 \vec{X}_0\,r\,m\,dr\right)\cdot\vec{k}_B\,\vec{J}_B\cdot\dot{\vec{\omega}}_0$$
$$-\left(\int_{r_{FA}}^1 \vec{X}_0\,r\,m\,dr\right)(\vec{k}_B\vec{i}_B - \vec{i}_B\vec{k}_B)\dot{\vec{\omega}}_0$$

b) relative acceleration:

$$
\begin{aligned}
M_{FA} = & -\left(\int_{r_{FA}}^{1} \vec{X}_o m\,dr\right)\cdot\left(-z_{FA}\ddot{\theta}_G \vec{i}_B + x_{FA}\ddot{\theta}_G \vec{k}_B\right) \\
& -\left(\int_{r_{FA}}^{1} \vec{X}_o r m\,dr\right)\cdot\left(-\ddot{\psi}_s \vec{i}_B + \ddot{\beta}_s \vec{k}_B\right) \\
& -\int_{r_{FA}}^{1} \vec{X}_o \cdot (x_o\vec{i} + z_o\vec{k})'' m\,dr \\
& -\int_{r_{FA}}^{1} \left[\vec{X}_o\cdot(z_o\vec{i} - x_o\vec{k} - x_I\vec{k}) + x_I^2\right]\ddot{\theta}\, m\,dr \\
& +\int_{r_{FA}}^{1} \vec{X}_o\cdot(\delta_2\vec{i}_B + \delta_3\vec{k}_B)(r-r_{FA})\ddot{\theta}^{o}\, m\,dr \\
& +\int_{r_{FA}}^{1} \ddot{\theta} I_\theta\,dr
\end{aligned}
$$

c) centrifugal acceleration:

$$
\begin{aligned}
M_{FA} = -\Omega^2 \Bigg[& \left(\int_{r_{FA}}^{1} \vec{X}_o m\,dr\right)\cdot\vec{i}_B\, z_{FA}\,\theta_G \\
& +\left(\int_{r_{FA}}^{1} \vec{X}_o r m\,dr\right)\cdot\vec{k}_B\,\beta_G \\
& -\int_{r_{FA}}^{1} \tilde{\theta} I_\theta (\cos^2\theta - \sin^2\theta)\,dr \\
& -\int_{r_{FA}}^{1} \vec{X}_o\cdot\left[\vec{i}_B(-x_{FA} - r_{FA}\delta_3) - \vec{k}_B r(\delta_1-\delta_2)\right] m\,dr \\
& +\int_{r_{FA}}^{1} \vec{X}_o\cdot\vec{k}_B\,\vec{k}_B\cdot(x_o\vec{i} + z_o\vec{k} + x_I\vec{i})\, m\,dr \\
& +(x_o\vec{i} + z_o\vec{k})'\big|_{r_{FA}}\cdot(z_o\vec{i} - x_o\vec{k})\big|_{r_{FA}} \int_{r_{FA}}^{1} r m\,dr \\
& +(x_o\vec{i} + z_o\vec{k})'\big|_{r_{FA}}\cdot(\delta_2\vec{i}_B + \delta_3\vec{k}_B)\int_{r_{FA}}^{1} r(r-r_{FA}) m\,dr \\
& -\int_{r_{FA}}^{1} (x_o\vec{i} + z_o\vec{k} + x_I\vec{i})\cdot\begin{bmatrix} (z_o\vec{i} - x_o\vec{k})\big|_{r_{FA}} \\ -r_{FA}(z_o\vec{i} - x_o\vec{k})'\big|_{r_{FA}} \\ -r_{FA}(\delta_2\vec{i}_B + \delta_3\vec{k}_B) \end{bmatrix} m\,dr \Bigg]
\end{aligned}
$$

aerodynamics: moment about the FA at r_{FA} is

$$\vec{M}_A = \int_{r_{FA}}^{1} M_a \vec{\jmath}_{xs}\, dr + \int_{r_{FA}}^{1} \begin{bmatrix} (r - r_{FA})\vec{\jmath} + (x_o\vec{\imath} + z_o\vec{k}) \\ -(x_o\vec{\imath} + z_o\vec{k})|_{r_{FA}} \end{bmatrix} \times \vec{F}_{aero}\, dr$$

So

$$M_{FA_A} = \int_{r_{FA}}^{1} M_a\, dr - \int_{r_{FA}}^{1} (F_x \vec{\imath}_B + F_z \vec{k}_B) \cdot \vec{X}_{A_o}\, dr$$

where $$\vec{X}_{A_o} = \vec{X}_o - x_I \vec{k}$$

elastic:

The aerodynamic and intertial moments about the FA are reacted by moments due to the deformation of the control system, due to commanded pitch angle, and due to feedback (mechanical or kinematic) from the blade bending or gimbal motion. The restoring moment about the feathering axis on the blade is $-M_{con}$; it is given by the control system flexibility, i.e. the elastic deformation in the control system θ_{ec} times the control system stiffness K_{con}. Hence:

$$M_{con} = K_{con}\theta_{ec} = K_{con}\left(\delta^{\circ} - \theta_{con} + \sum_i K_{P_i} q_i + K_{P_G}\beta_G\right)$$

The q_i are the bending degrees of freedom, so K_{P_i} are the pitch/flap and pitch/lag coupling, mechanical or kinematic feedback due to the control system and blade root geometry. Similarly, K_{P_G} is the pitch/flap coupling for the gimbal motion. For the rigid flap motion of the blade, this coupling is given by the δ_3 angle, such that $K_P = \tan\delta_3$. For a rigid control system, $K_{con} \longrightarrow \infty$, the rigid pitch equation of motion reduces to

$$p_o = \delta^{\circ} \longrightarrow \theta_{con} - \sum_i K_{P_i} q_i - K_{P_G}\beta_G$$

So p_o becomes just the control input, and pitch/bending coupling.

Now we write the control system stiffness K_{con} in terms of the nonrotating natural frequency of the rigid pitch motion of the blade, ω_o :

$$K_{con} = \left(\int_{r_{FA}}^{1} I_\theta \, dr \right) \omega_o^2$$

Then:

$$M_{FA_E} = M_{con} = \left(\int_{r_{FA}}^{1} I_\theta \, dr \right) \omega_o^2 \Big[p_i - \theta_{con} + \sum K_{P_i} q_i + K_{P_G} \beta_G \Big]$$

Force

The net force of the mth blade on the hub is

$$\vec{F}^{(m)} = \vec{F}_A - \vec{F}_I$$

where $\vec{F}$ is the force due to the blade, at the hub.

inertia: $$\vec{F}_I = \int_0^1 \int_{section} \vec{a} \, dm \, dr$$

a) shaft motion:

$$\vec{F} = \left(\int_0^1 m \, dr \right) \vec{a}_o + 2\Omega \left(\int_0^1 r m \, dr \right) (\vec{k}_B \vec{j}_B - \vec{j}_B \vec{k}_B) \vec{\omega}_o + \left(\int_0^1 r m \, dr \right) (\vec{k}_B \vec{i}_B - \vec{i}_B \vec{k}_B) \dot{\vec{\omega}}_o$$

b) relative acceleration:

$$\vec{F} = \left(\int_0^1 r m \, dr \right) (-\vec{i}_B \ddot{\psi}_s + \vec{k}_B \ddot{\beta}_G) + \int_0^1 (x_o \vec{i} + z_o \vec{k})'' m \, dr$$

c) coriolis acceleration:

$$\vec{F} = 2\Omega \int_0^1 \int \vec{k}_B \times \vec{v}_r \, dm \, dr$$

$$= 2\Omega \vec{j}_B \Big[- \left(\int_0^1 r m \, dr \right) \dot{\psi}_s + \int_0^1 \vec{i}_B \cdot (x_o \vec{i} + z_o \vec{k})^{\cdot} m \, dr \Big]$$

d) centrifugal acceleration:

$$\vec{F} = -\Omega^2 \int_0^1 \int (\vec{i}_B\vec{i}_B + \vec{j}_B\vec{j}_B)\,\vec{r}\,dm\,dr$$

$$= -\Omega^2 \Big[\vec{i}_B \Big[\int_0^1 (-x_{FA} + (r - r_{FA})\delta_3)\,m\,dr \Big] + \vec{j}_B \Big(\int_0^1 rm\,dr \Big) - \vec{i}_B \Big(\int_0^1 rm\,dr \Big) \psi_s + \vec{i}_B \int_0^1 \vec{i}_B \cdot (x_0\vec{i} + z_0\vec{k} + x_I\vec{i})\,m\,dr \Big]$$

aerodynamics:

$$\vec{F}_A = \int_0^1 \vec{F}_{aero}\,dr$$

$$= \int_0^1 \Big[F_x\vec{i}_B + F_z\vec{k}_B + \vec{j}_B \big(F_r - F_z(\beta_G + \delta_1 - \delta_2 + \vec{k}_B \cdot (x_0\vec{i} + z_0\vec{k})') \big) \Big] dr$$

<u>Moment</u>

The net moment of the mth blade on the hub, about the gimbal point, is:

$$\vec{M}^{(m)} = \vec{M}_A - \vec{M}_I$$

inertia: $$\vec{M}_I = \int_0^1 \int \vec{r} \times \vec{a}\,dm\,dr$$

a) shaft motion:

$$\vec{M} = \Big(\int_0^1 rm\,dr \Big) (\vec{i}_B\vec{k}_B - \vec{k}_B\vec{i}_B)\,\vec{a}_0 + 2\Omega \Big(\int_0^1 r^2 m\,dr \Big) \vec{i}_B\,\vec{j}_B \cdot \vec{\omega}_0 + \Big(\int_0^1 r^2 m\,dr \Big) (\vec{i}_B\vec{i}_B + \vec{k}_B\vec{k}_B)\,\dot{\vec{\omega}}_0$$

b) relative acceleration:

$$\vec{M} = \left[\begin{array}{l} \left(\int_0^1 rm\,dr\right)\left(z_{FA}\,\ddot{\theta}_G\,\vec{k}_B + y_{FA}\,\ddot{\theta}_G\,\vec{\imath}_B\right) \\ + \left(\int_0^1 r^2 m\,dr\right)\left(k_B\,\ddot{\Psi}_s + \vec{\imath}_B\,\ddot{\beta}_G\right) \\ + \int_0^1 \left(z_0\vec{\imath} - x_0\vec{k}\right)^{\cdot\cdot} rm\,dr \\ - \int_{r_{FA}}^1 \ddot{\theta}\left(\left(x_0 + x_I\right)\vec{\imath} + z_0\vec{k}\right) rm\,dr \\ - \ddot{\theta}^0\left(\delta_3\vec{\imath}_B - \delta_2\vec{k}_B\right)\int_{r_{FA}}^1 \left(r - r_{FA}\right) rm\,dr \end{array} \right.$$

c) centrifugal acceleration:

$$\vec{r}\times\left(\vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right) = \vec{r}\times\vec{\Omega}\;\vec{\Omega}\cdot\vec{r} = -\Omega^2\,\vec{k}_B\times\vec{r}\;\vec{k}_B\cdot\vec{r}$$

$$\cong -\Omega^2 r\left(-\vec{\imath}_B\,\vec{k}_B\cdot\vec{r}\right)$$

So

$$\vec{M} = \Omega^2\vec{\imath}_B \left[\begin{array}{l} \int_0^1 \left(-z_{FA} + r\,\delta_1 - \left(r - r_{FA}\right)\delta_2\right) rm\,dr \\ \quad + \left(\int_0^1 r^2 m\,dr\right)\beta_G \\ + \int_0^1 \vec{k}_B\cdot\left(x_0\vec{\imath} + z_0\vec{k} + x_I\vec{\imath}\right) rm\,dr \\ + \int_{r_{FA}}^1 \tilde{\theta}\,\vec{k}_B\cdot\left(z_0\vec{\imath} - x_0\vec{k} - x_I\vec{k}\right) rm\,dr \\ - \tilde{\theta}^0\,\delta_{FA_3}\left(\int_{r_{FA}}^1 \left(r - r_{FA}\right) rm\,dr\right) \end{array} \right.$$

d) coriolis acceleration:

$$\vec{r} \times \vec{a} = 2\Omega \left[\vec{j}_B \times \vec{r} \left(r\dot{\Psi}_s + \vec{k}_B \cdot (z_0\vec{i} - x_0\vec{k})^{\cdot} \right) + r\vec{k}_B \; \vec{j}_B \cdot \vec{v}_r \right]$$

and

$$\vec{j}_B \times \vec{r} = -\vec{k}_B \left(-x_{FA} + (r - r_{FA})\delta_3 \right) + \vec{i}_B \left(-z_{FA} + r\delta_1 - (r - r_{FA})\delta_2 \right. \\ \left. + (z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) \right)$$

So

$$\vec{M} = 2\Omega \Bigg[\vec{i}_B \int_0^1 \left(r\dot{\Psi}_s + \vec{k}_B \cdot (x_0\vec{i} + z_0\vec{k})^{\cdot} \right) \begin{bmatrix} -z_{FA} + r\delta_1 - (r - r_{FA})\delta_2 \\ + \vec{i}_B \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) \end{bmatrix} m\,dr$$

$$+ \vec{k}_B \Bigg[-\int_0^1 \left(\int_0^r (z_0\vec{i} - x_0\vec{k})^{\cdot\cdot} \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k})^{\cdot}\, d\rho \right) r m\,dr$$

$$+ \int_0^1 \vec{i}_B \cdot (z_0\vec{i} - x_0\vec{k})^{\cdot} \left(-r\delta_1 + (r - r_{FA})\delta_2 \right) m\,dr$$

$$+ \int_0^1 \vec{k}_B \cdot (z_0\vec{i} - x_0\vec{k})^{\cdot} \left(x_{FA} + r_{FA}\delta_3 \right) m\,dr$$

$$+ \int_0^1 \vec{k}_B \cdot (z_0\vec{i} - x_0\vec{k})^{\cdot}\; \vec{k}_B \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k})\, m\,dr$$

$$- \dot{\beta}_G \int_0^1 \left[-z_{FA} + \vec{i}_B \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k}) + r\delta_1 - (r - r_{FA})\delta_2 \right] r m\,dr \Bigg] \Bigg]$$

aerodynamics:

$$\vec{M}_A = \int_0^1 \vec{r} \times \vec{F}_{aero}\, dr$$

$$\cong \int_0^1 \left(F_z \vec{i}_B - F_x \vec{k}_B \right) dr$$

Gimbal

The equations of motion for the gimbal degrees of freedom are obtained from the $\vec{i}_s$ & $\vec{j}_s$ components of $\vec{M} = \Sigma \vec{M}^{(m)}$; thus:

$$\vec{M}_{HS} + \vec{M}_I = \vec{M}_A$$

where $\vec{M}_{HS}$ is the spring and damper moment at the gimbal, reacting the rotor applied moment. The gimbal spring and damper are assumed to be in the nonrotating frame. Hence:

$$\vec{M}_{HS} = \vec{i}_s \left(K_G \beta_{GS} + C_G \dot{\beta}_{GS} \right) - \vec{j}_s \left(K_G \beta_{GC} + C_G \dot{\beta}_{GC} \right)$$

Taking the $\vec{i}_s$ and $\vec{j}_s$ components of $\vec{M}$, the gimbal equations of motion are:

$$M_y + C_G \dot{\beta}_{GC} + K_G \beta_{GC} = 0$$

$$-M_x + C_G \dot{\beta}_{GS} + K_G \beta_{GS} = 0$$

We shall write the gimbal hub spring and damper as:

$$K_G = \frac{N}{2} I_o \Omega^2 (\nu_G^2 - 1)$$

$$C_G = \frac{N}{2} I_o \Omega C_G^*$$

where $I_o = \int_0^R r^2 m\, dr$ and ν_G is the natural frequency of the gimbal flap motion.

Modal Equations

Bending

Consider the equilibrium of the elastic, inertial, and centrifugal bending moments. From the above analysis, these terms give the homogeneous equation for bending of the blade:

$$\begin{aligned}&\left[(EI_{zz}\vec{\tau}\vec{\tau} + EI_{xx}\vec{k}\vec{k})(z_0\vec{\tau} - x_0\vec{k})''\right]'' \\ &\quad - \Omega^2\left[\int_r^1 \rho m\, d\rho\,(z_0\vec{\tau} - x_0\vec{k})'\right]' \\ &\quad - \vec{\Omega}\, m\vec{\Omega}\cdot(z_0\vec{\tau} - x_0\vec{k}) \\ &\quad + m\,(z_0\vec{\tau} - x_0\vec{k})^{\cdot\cdot} = 0\end{aligned}$$

This equation may be solved by the method of separation of variables. Writing

$$(z_0\vec{\tau} - x_0\vec{k}) = \vec{\eta}(r)\, e^{i\nu t}$$

it becomes

$$(EI\vec{\eta}'')'' - \Omega^2\left[\int_r^1 \rho m\, d\rho\, \vec{\eta}'\right]' - \vec{\Omega}\, m\vec{\Omega}\cdot\vec{\eta} - m\nu^2\vec{\eta} = 0$$

This is the modal equation for coupled flap/lag bending of the rotating blade. It is an ordinary differential equation for the mode shape $\vec{\eta}(r)$: this mode may be interpreted as the free vibration of the rotating beam at natural frequency ν.

This modal equation, with the appropriate boundary conditions for a cantilever or hinged blade, is a proper Sturm-Liouville eigenvalue problem. It follows that there exists a series of eigensolutions $\vec{\eta}_k(r)$ of this equation, with corresponding eigenvalues ν_k^2. The eigensolutions -- modes -- are orthogonal with weighting function m; so if $i \neq k$,

$$\int_0^1 \vec{\eta}_i\cdot\vec{\eta}_k\, m\, dr = 0$$

These modes form a complete series, so it is possible to expand the rotor blade bending as a series in the modes:

$$z_0\vec{\tau} - x_0\vec{k} = \sum_{i=1}^{\infty} q_i(t)\,\vec{\eta}_i(r)$$

We shall normalize the bending modes to unit amplitude (nondimensional) at the tip: $|\vec{\eta}_k(1)| = 1$.

Torsion

Consider the homogeneous equation for the elastic torsion motion of the nonrotating blade; i.e. the balance of structural and inertial torsion moments, which from the above analysis is:

$$-(GJ\,\theta_e')' + I_\theta \ddot{\theta}_e = 0$$

We could consider the equation for the torsion motion of the rotating blade, i.e. including centrifugal forces and some additional structural torsion moments. For the usual torsion stiffness of rotor blades these terms have little effect however, and the nonrotating torsion modes are an accurate representation of the blade motion. Solving this equation by separation of variables, write $\theta_e = \xi(r)e^{i\omega t}$, so:

$$-(GJ\,\xi')' - I_\theta \omega^2 \xi = 0$$

This equation is a proper Sturm-Liouville eigenvalue problem, so it follows that there exists a series of eigensolutions $\xi_k(r)$, and corresponding eigenvalues ω_k^2 $(k = 1 \ldots \infty)$. The modes are orthogonal with weighting function I_θ, so if $i \neq k$

$$\int_{r_{FA}}^{1} \xi_i \xi_k I_\theta \, dr = 0$$

The modes form a complete set, so the elastic torsion of the blade may be expanded as a series in the modes:

$$\theta_e = \sum_{i=1}^{\infty} p_i(t)\,\xi_i(r)$$

These modes are the free vibration shape of the nonrotating blade in torsion, at natural frequency ω_k. We shall normalize the torsion modes to unity at the tip, $\xi_k(1) = 1$.

Expansion in modes

The bending and torsion motion of the blade is now expanded as series in the natural modes. By this means the partial differential equations for the motion (in r and t) are converted to ordinary differential equations (in t) for the degrees of freedom.

For the blade bending we write

$$(z_o\vec{\imath} - x_o\vec{k}) = (z_o\vec{\imath} - x_o\vec{k})_{trim} + \sum_{i=1}^{\infty} q_i(t)\,\vec{\eta}_i(r)$$

where $\vec{\eta}_i$ are the rotating, coupled bending modes defined above. These modes are orthogonal, and satisfy the modal equation given above. The q_i are the degrees of freedom for the bending motion of the blade. It is assumed (for the inertial terms) that the trim bending deflection is steady, independent of time; and when the substitution for the modal expansion is made, the subscript "trim" will be dropped, as that is all that will be meant by $(z_o\vec{\imath} - x_o\vec{k})$ then.

For the blade elastic torsion we write

$$\theta_e = \sum_{i=1}^{\infty} p_i(t)\,\xi_i(r)$$

where ξ_i are the nonrotating elastic torsion modes. These modes are orthogonal, and satisfy the modal equation given above. The p_i $(i \geq 1)$ are the degrees of freedom for the elastic torsion motion of the blade.

We also have the rigid pitch degree of freedom

$$p_o = \tilde{\theta}^{\circ} = (\theta^{\circ} - \theta^{c}) + \theta_{con}$$

which is the total rigid pitch motion of the blade. Since it is rigid pitch, rotation about the FA, it has mode shape $\xi_o \equiv 1$. Thus the total pitch perturbation of the blade is expanded as a series:

$$\tilde{\theta} = \sum_{i=0}^{\infty} p_i(t)\,\xi_i(r)$$

For the blade pitch θ then, the mean plus the perturbation is

$$\theta = \theta_m + \tilde{\theta} = (\theta_{coll} + \theta_{tw}) + \sum_0 p^i \xi_i$$

The subscript "m" on the trim pitch angle will be dropped when the substitution for the modal expansion is made, since that is all that will be meant by θ then.

Nonrotating Frame

The equations of motion and the hub forces and moments are in the rotating frame yet. To get to the nonrotating frame, we introduce a coordinate transformation of the Fourier type; i.e., introduce the new degrees of freedom:

$$\beta_0 = \frac{1}{N} \sum_{m=1}^{N} q^{(m)}$$

$$\beta_{nc} = \frac{2}{N} \sum_{m=1}^{N} q^{(m)} \cos n\psi_m$$

$$\beta_{ns} = \frac{2}{N} \sum_{m=1}^{N} q^{(m)} \sin n\psi_m$$

$$\beta_{\frac{N}{2}} = \frac{1}{N} \sum_{m=1}^{N} q^{(m)} (-1)^m$$

where β_0 is the coning mode; β_{1c} & β_{1s} the tip path plane tilt coordinates; and $\beta_{\frac{N}{2}}$ is the reactionless flap mode -- for the out of plane bending of the blade. Then:

$$q^{(m)} = \beta_0 + \sum_n (\beta_{nc} \cos n\psi_m + \beta_{ns} \sin n\psi_m) + \beta_{\frac{N}{2}} (-1)^m$$

where the summation over n goes from 1 to (N-1)/2 for N odd; and from 1 to (N-2)/2 for N even; the $\beta_{\frac{N}{2}}$ degree of freedom appears only if N is even.

The quantities β_0, β_{nc}, β_{ns}, and $\beta_{\frac{N}{2}}$ are degrees of freedom, i.e. functions of time, just as the quantities $q^{(m)}$ are. These degrees of freedom describe the rotor motion as seen in the nonrotating frame, while the $q^{(m)}$ describe the motion in the rotating frame.

This coordinate transform must be accompanied by a conversion of the equations of motion for $q^{(m)}$ from the rotating to the nonrotating frame. This is accomplished by operating on the equations of motion with the following summation operators:

$$\frac{1}{N}\sum_m(\ldots),\quad \frac{2}{N}\sum_m(\ldots)\cos n\psi_m,\quad \frac{2}{N}\sum_m(\ldots)\sin n\psi_m,\quad \frac{1}{N}\sum_m(\ldots)(-1)^m$$

Reference 4 gives more details of this transformation.

Similarly, the degrees of freedom for the blade pitch and gimbal motion are transformed to the nonrotating frame. The corresponding degrees of freedom for the rotating and nonrotating frames are:

rotating	nonrotating
$q_i^{(m)}$	$\beta_{0,1c,1s}^{(i)}$
$p_i^{(m)}$	$\theta_{0,1c,1s}^{(i)}$
$\beta_G, \theta_G, \psi_s$	$\beta_{Gc}, \beta_{Gs}, \psi_s$

When the transformation of the equations and degrees of freedom is accomplished, there is a decoupling of the inertial and structural terms as follows (for $N \geq 3$):

a) 0, 1C, 1S degrees of freedom; β_{Gc}, β_{Gs}, and ψ_s ; and the rotor shaft motion.

b) 2C, 2S, ... , nc, ns, N/2 degrees of freedom (as present).

The first set couples with the fixed system motion. The latter set is just internal rotor motion. For $N = 3$, the first set is the complete description of the motion of course. Nonaxial flow aerodynamics couples all the rotor degrees of freedom and shaft motion; i.e. the two sets above are coupled for helicopter forward flight or conversion mode operation. For axial flow -- hover or proprotor airplane mode cruise operation -- the aerodynamic terms decouple also.

We shall assume here that the rotor has three or more blades, $N \geq 3$. For $N = 2$, there are periodic coefficients even in the inertia terms, so that is a special case. For the case of periodic coefficients in the aerodynamics, i.e. helicopter forward flight or conversion mode flight, it is necessary to specify N; we shall take N = 3 for that case. (The periodic coefficients depend on N.) For the case of axial flow, or for the constant coefficient approximation for the nonaxial flow case, the equations obtained will be valid for all N greater than or equal to 3.

Reference 4 discusses these points further.

Equations of Motion/ Hub Forces and Moments

The elements are available now to obtain the equations of motion for the blade bending and torsion modes, in the rotating frame; and the forces and moments acting on the hub due to the mth blade. The steps required are:

a) Substitute for the expansion of the bending and torsion motion as a series in the modes.

b) Use the appropriate modal equation to introduce the mode natural frequency into the bending or torsion equation, replacing the structural stiffness terms (and for bending also some of the centrifugal stiffness terms).

c) For the bending equation, operate with $\int_0^1 \vec{\eta}_k \cdot (\cdots) dr$ to obtain the ordinary differential equation for the kth mode of the mth blade (the q_k equation).

d) For the torsion equation, operate with $\int_{r_{FA}}^1 \xi_k (\cdots) dr$ to obtain the ordinary differential equation fo the kth mode of the mth blade (the p_k equation).

The result is the equations of motion and hub forces in the rotating frame. The transformation to the nonrotating frame involves the following steps:

a) Operate on the hub force and moment with $\sum_m (\cdots)$; i.e. sum over all N blades to obtain the total force and moment on the hub.

b) Find the $\vec{i}_S, \vec{j}_S, \vec{k}_S$ components of the force and moment in the nonrotating frame (the S system).

c) Write the shaft motion $\vec{a}_0$, $\vec{\omega}_0$, and $\dot{\vec{\omega}}_0$ in terms of the $\vec{i}_S, \vec{j}_S, \vec{k}_S$ components in the nonrotating frame (the S system).

d) Apply the Fourier coordinate transform to the equations of motion and rotor degrees of freedom; operate on the equations for bending and torsion with

$$\frac{1}{N}\sum_m (\cdots), \quad \frac{2}{N}\sum_m (\cdots)\cos\psi_m, \quad \frac{2}{N}\sum_m (\cdots)\sin\psi_m, \cdots$$

to obtain the nonrotating equations of motion (0, 1C, 1S, etc.). $N \geq 3$ is assumed for this transformation.

The transformation of the equations to the nonrotating frame will be delayed however, so the rotating modal equations may be presented first.

We add at this point structural damping terms, modelled as equivalent viscous damping; the structural damping parameter is g_s (which may be different for each degree of freedom), equal to twice the equivalent damping ratio.

Names are given to all the inertial constants now. The equations of motion, hub forces and moments, and inertia constants are also normalized at this point. The inertia constants are divided by the characteristic inertia $I_b = \int_0^R r^2 m \, dr$, and we introduce the blade Lock number $\gamma = \rho acR^4/I_b$. This normalization of the inertia constants is denoted by a superscript *. The rotating equations of motion are divided by I_b; the hub forces and moments are divided by $(N/2)I_b$ for M_x, M_y, H, and Y, and by NI_b for Q and T. The result is that the forces and moments are obtained in coefficient form. More details of this normalization procedure are given in reference 4.

Equations

The resulting hub forces, hub moments, gimbal equations, and equations of motion for coupled flap/lag bending and for elastic torsion/rigid pitch of the rotating blade are as follows.

Forces:

$$\gamma \frac{2C_H}{\sigma a} = \gamma \left(\frac{2C_H}{\sigma a}\right)_{aero} - 2M_b^* \ddot{x}_h + \Sigma S_{q_i}^* \cdot \vec{k}_B \; \ddot{\beta}_{1s}^{(i)}$$

$$\gamma \frac{2C_Y}{\sigma a} = \gamma \left(\frac{2C_Y}{\sigma a}\right)_{aero} - 2M_b^* \ddot{y}_h - \Sigma S_{q_i}^* \cdot \vec{k}_B \; \ddot{\beta}_{1c}^{(i)}$$

$$\gamma \frac{C_T}{\sigma a} = \gamma \left(\frac{C_T}{\sigma a}\right)_{aero} - M_b^* \ddot{z}_h - \Sigma S_{q_i}^* \cdot \vec{i}_B \; \ddot{\beta}_0^{(i)}$$

Moments:

$$\gamma \frac{2C_{Mx}}{\sigma a} = \gamma \left(\frac{2C_{Mx}}{\sigma a}\right)_{aero} - I_0^* (\ddot{\alpha}_x + 2\dot{\alpha}_y) - I_0^* (\ddot{\beta}_{Gs} - 2\dot{\beta}_{Gc})$$
$$- \Sigma I_{q_i\alpha}^* \cdot \vec{i}_B \; (\ddot{\beta}_{1s}^{(i)} - 2\dot{\beta}_{1c}^{(i)})$$
$$+ \Sigma S_{p_i\alpha}^* \cdot \vec{i}_B \; (\ddot{\theta}_{1s}^{(i)} - 2\dot{\theta}_{1c}^{(i)})$$
$$- 2 \Sigma I_{q_i\alpha}^* \; (\dot{\beta}_{1s}^{(i)} - \beta_{1c}^{(i)})$$

$$\gamma \frac{2C_{My}}{\sigma a} = \gamma \left(\frac{2C_{My}}{\sigma a}\right)_{aero} - I_0^* (\ddot{\alpha}_y - 2\dot{\alpha}_x) + I_0^* (\ddot{\beta}_{Gc} + 2\dot{\beta}_{Gs})$$
$$+ \Sigma I_{q_i\alpha}^* \cdot \vec{i}_B \; (\ddot{\beta}_{1c}^{(i)} + 2\dot{\beta}_{1s}^{(i)})$$
$$- \Sigma S_{p_i\alpha}^* \cdot \vec{i}_B \; (\ddot{\theta}_{1c}^{(i)} + 2\dot{\theta}_{1s}^{(i)})$$
$$+ 2 \Sigma I_{q_i\alpha}^* \; (\dot{\beta}_{1c}^{(i)} + \beta_{1s}^{(i)})$$

$$\gamma \frac{C_Q}{\sigma a} = \gamma \left(\frac{C_Q}{\sigma a}\right)_{aero} + I_0^* \ddot{\alpha}_z + I_0^* \ddot{\psi}_s$$
$$+ \Sigma I_{q_i\alpha}^* \cdot \vec{k}_B \; \ddot{\beta}_0^{(i)} - \Sigma S_{p_i\alpha}^* \cdot \vec{k}_B \; \ddot{\theta}_0^{(i)}$$
$$- 2 \Sigma I_{q_i\psi}^* \; \dot{\beta}_0^{(i)}$$

Gimbal:

$$\gamma \frac{2C_{My}}{\sigma a} + I_0^* C_G^* \dot{\beta}_{Gc} + I_0^* (\nu_G^2 - 1) \beta_{Gc} = 0$$

$$-\gamma \frac{2C_{Mx}}{\sigma a} + I_0^* C_G^* \dot{\beta}_{Gs} + I_0^* (\nu_G^2 - 1) \beta_{Gs} = 0$$

Bending:

$$
\begin{aligned}
& I^{*}_{q_k}\left(\ddot{q}_k + g_s \nu_k \dot{q}_k + \nu_k^2 q_k\right) + 2\sum I^{*}_{q_k \dot{q}_i}\,\dot{q}_i \\
& - \sum S^{*}_{q_k \ddot{p}_i}\,\ddot{p}_i - \sum S^{*}_{q_k p_i}\,p_i \\
& + I^{*}_{q_k\alpha}\cdot\vec{k}_B\,\ddot{\psi}_s + I^{*}_{q_k\alpha}\cdot\vec{i}_B\left(\ddot{\beta}_G + \beta_G\right) \\
& + 2I^{*}_{q_k\dot{\psi}}\,\dot{\psi}_s - I^{*}_{\dot{q}_k\alpha}\left(\ddot{\theta}_G - \theta_G + 2\dot{\beta}_G\right) \\
& + S^{*}_{q_k}\cdot\vec{i}_B\,\ddot{z}_h - S^{*}_{q_k}\cdot\vec{k}_B\left(\ddot{x}_h \sin\psi_m - \ddot{y}_h \cos\psi_m\right) \\
& + I^{*}_{q_k\alpha}\cdot\vec{k}_B\,\ddot{\alpha}_z + I^{*}_{q_k\alpha}\cdot\vec{i}_B\Big(\left(\ddot{\alpha}_x + 2\dot{\alpha}_y\right)\sin\psi_m \\
& \qquad\qquad - \left(\ddot{\alpha}_y - 2\dot{\alpha}_x\right)\cos\psi_m\Big) \\
& = \gamma\,\frac{M_{q_k\,aero}}{ac} + I^{*}_{q_k o}
\end{aligned}
$$

Torsion/pitch:

$$
\begin{aligned}
& I^{*}_{p_k}\left(\ddot{p}_k + g_s \omega_k \dot{p}_k + \omega_k^2 p_k\right) \\
& + \sum I^{*}_{p_k \ddot{p}_i}\,\ddot{p}_i + \sum I^{*}_{p_k p_i}\,p_i \\
& - \sum S^{*}_{p_k \ddot{q}_i}\,\ddot{q}_i - \sum S^{*}_{p_k q_i}\,q_i \\
& + I^{*}_{p_k\alpha}\cdot\vec{i}_B\,\ddot{\psi}_s - I^{*}_{p_k\alpha}\cdot\vec{k}_B\left(\ddot{\beta}_G + \beta_G\right) \\
& - S^{*}_{p_k\ddot{\alpha}}\left(\ddot{\theta}_G + 2\dot{\beta}_G - \theta_G\right) + S^{*}_{p_k\alpha}\,\beta_G \\
& - S^{*}_{p_k}\cdot\vec{k}_B\,\ddot{z}_h - S^{*}_{p_k}\cdot\vec{i}_B\left(\ddot{x}_h \sin\psi_m - \ddot{y}_h \cos\psi_m\right) \\
& + I^{*}_{p_k\alpha}\cdot\vec{i}_B\,\ddot{\alpha}_z - I^{*}_{p_k\alpha}\cdot\vec{k}_B\Big(\left(\ddot{\alpha}_x + 2\dot{\alpha}_y\right)\sin\psi_m \\
& \qquad\qquad - \left(\ddot{\alpha}_y - 2\dot{\alpha}_x\right)\cos\psi_m\Big) \\
& = \gamma\,\frac{M_{p_k\,aero}}{ac} + \left(I^{*}_{p_o}\,\omega_o^2\,\xi_k(r_{pA})\right)\theta_{can}
\end{aligned}
$$

Aerodynamic forces

$$\frac{2C_H}{\sigma a} = \frac{2}{N}\sum_m \Big[\sin\psi_m \int_0^1 \frac{F_x}{ac}\,dr + \cos\psi_m \int_0^1 \frac{F_r}{ac} - \frac{F_z}{ac}\big(\beta_G + \delta_1 - \delta_2$$
$$+ \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\big)\,dr\Big]$$

$$-\frac{2C_Y}{\sigma a} = \frac{2}{N}\sum_m \Big[\cos\psi_m \int_0^1 \frac{F_x}{ac}\,dr - \sin\psi_m \int_0^1 \frac{F_r}{ac} - \frac{F_z}{ac}\big(\beta_G + \delta_1 - \delta_2$$
$$+ \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\big)\,dr\Big]$$

$$\frac{C_T}{\sigma a} = \frac{1}{N}\sum_m \int_0^1 \frac{F_z}{ac}\,dr$$

$$\frac{2C_{Mx}}{\sigma a} = \frac{2}{N}\sum_m \sin\psi_m \int_0^1 \frac{F_z}{ac}\,r\,dr$$

$$-\frac{2C_{My}}{\sigma a} = \frac{2}{N}\sum_m \cos\psi_m \int_0^1 \frac{F_z}{ac}\,r\,dr$$

$$\frac{C_Q}{\sigma a} = \frac{1}{N}\sum_m \int_0^1 \frac{F_x}{ac}\,r\,dr$$

$$\frac{M_{q_k\,aero}}{ac} = \int_0^1 \vec{\eta}_k\cdot\left(\frac{F_z}{ac}\vec{i}_B - \frac{F_x}{ac}\vec{k}_B\right)dr$$

$$\frac{M_{p_k\,aero}}{ac} = \int_{r_{FA}}^1 \xi_k M_a\,dr - \int_{r_{FA}}^1 \left(\frac{F_x}{ac}\vec{i}_B + \frac{F_z}{ac}\vec{k}_B\right)\cdot\vec{X}_{Ak}\,dr$$

where $$\vec{X}_{Ak} = \vec{X}_k - \xi_k x_I \vec{k}$$

and then for the nonrotating equations:

$$M_{\beta_0\,aero} = \frac{1}{N}\sum_m M_{q_k\,aero}$$
$$M_{\beta_{1c}\,aero} = \frac{2}{N}\sum_m M_{q_k\,aero}\cos\psi_m$$
$$M_{\beta_{1s}\,aero} = \frac{2}{N}\sum_m M_{q_k\,aero}\sin\psi_m$$

$$M_{\theta_0\,aero} = \frac{1}{N}\sum_m M_{p_k\,aero}$$
$$M_{\theta_{1c}\,aero} = \frac{2}{N}\sum_m M_{p_k\,aero}\cos\psi_m$$
$$M_{\theta_{1s}\,aero} = \frac{2}{N}\sum_m M_{p_k\,aero}\sin\psi_m$$

Inertia constants

$$M_b^* = \int_0^1 m\,dr / I_b$$

$$S_{q_i}^* = \int_0^1 \vec{\eta}_i\, m\,dr / I_b$$

$$I_0^* = \int_0^1 r^2 m\,dr / I_b$$

$$I_{q_i\alpha}^* = \int_0^1 \vec{\eta}_i\, r m\,dr / I_b$$

$$S_{p_i\alpha}^* = \frac{1}{I_b} \int_{r_{FA}}^1 \Big(\xi_i (x_0\vec{i} + z_0\vec{k} + x_I\vec{i}) + \xi_i(r_{FA})(\delta_3 \vec{i}_B - \delta_2 \vec{k}_B)(r - r_{FA})\Big) r m\,dr$$

$$I_{\dot{q}_i\alpha}^* = \frac{1}{I_b} \int_0^1 \vec{k}_B \cdot \vec{\eta}_i \Big(-z_{FA} + r\delta_1 - (r - r_{FA})\delta_2 + \vec{i}_B \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k})\Big) m\,dr$$

$$I_{\dot{q}_i\psi}^* = \frac{1}{I_b} \left[\int_0^1 r m \int_0^r \vec{\eta}_i{}' \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k})' \,d\rho\,dr - \int_0^1 \vec{i}_B \cdot \vec{\eta}_i \big(-r\delta_1 + (r - r_{FA})\delta_2\big) m\,dr - \int_0^1 \vec{k}_B \cdot \vec{\eta}_i \big(x_{FA} + r_{FA}\delta_3 + \vec{k}_B \cdot (z_0\vec{i} - x_0\vec{k} - x_I\vec{k})\big) m\,dr \right.$$

$$I_0^*(\nu_G^2 - 1) = K_G / \left(\tfrac{N}{2} I_b \Omega^2\right)$$

$$I_0^*(C_G^*) = C_G / \left(\tfrac{N}{2} I_b \Omega\right)$$

$$I_{q_k}^* = \int_0^1 \eta_k^2\, m\,dr / I_b$$

$$S_{q_k\ddot{p}_i}^* = \frac{1}{I_b} \int_{r_{FA}}^1 \vec{\eta}_k \cdot \Big(\xi_i (x_0\vec{i} + z_0\vec{k} + x_I\vec{i}) - \xi_i(r_{FA})(\delta_2 \vec{k}_B - \delta_3 \vec{i}_B)(r - r_{FA})\Big) m\,dr$$

$$S^{*}_{q_k p_i} = \frac{1}{I_b}\left[\begin{array}{l} \int_{r_{FA}}^{1} \vec{\eta}_k \cdot \left\{\begin{array}{l} -\left[\left(\xi_i\,(x_o\vec{\imath}+z_o\vec{k}+x_I\vec{\imath})\right)' \int_r^1 \rho\, m\, d\rho\right]' \\ +\left[(x_c-x_I)\,\xi_i\,\vec{\imath}\, r\, m\right]' \\ -m\,\vec{k}_B\,\xi_i\,\vec{k}_B\cdot(x_o\vec{\imath}+z_o\vec{k}+x_I\vec{\imath}) \\ +\xi_i(r_{FA})\,m\,(r\delta_3\vec{\imath}_B - r_{FA}\delta_2\vec{k}_B) \end{array}\right\} dr \\ -\vec{\eta}_k(r_{FA})\cdot(\delta_3\vec{\imath}_B-\delta_2\vec{k}_B)\,\xi_i(r_{FA})\int_{r_{FA}}^{1} r\,m\,dr \\ -\int_{r_{FA}}^{1}\vec{\eta}_k\cdot\left[\left(\frac{EI_{xP}}{\Omega^2}\vec{k}-\frac{EI_{zP}}{\Omega^2}\vec{\imath}\right)\partial_m'\,\xi_i''\right]''dr \end{array}\right]$$

$$I^{*}_{q_k \dot{q}_i} = \frac{1}{I_b}\left[\begin{array}{l} -\int_0^1 \vec{\eta}_k\cdot\vec{k}_B\, m \int_0^r \vec{\eta}_i'\cdot(z_o\vec{\imath}-x_o\vec{k}-x_I\vec{k})'\,d\rho\, dr \\ +\int_0^1 \vec{\eta}_i\cdot\vec{k}_B\, m\int_0^r \vec{\eta}_k'\cdot(z_o\vec{\imath}-x_o\vec{k}-x_c\vec{k})'\,d\rho\,dr \\ -\int_0^1\vec{\eta}_k\cdot\left[\left((x_c-x_I)\vec{k}\,m\,\vec{k}_B\cdot\vec{\eta}_i\right)' \right. \\ \qquad\qquad \left. -m\,(\delta_1-\delta_2)\,\vec{\jmath}\times\vec{\eta}_i\right]dr \\ +\vec{\eta}_k(r_{FA})\cdot(\delta_2\vec{\imath}_B+\delta_3\vec{k}_B)\int_{r_{FA}}^{1}\vec{k}_B\cdot\vec{\eta}_i\, m\, dr \end{array}\right]$$

$$I^{*}_{q_k\dot{\psi}} = \frac{1}{I_b}\left[\begin{array}{l} \int_0^1 r\,m\int_0^r\vec{\eta}_k'\cdot(z_o\vec{\imath}-x_o\vec{k}-x_c\vec{k})'\,d\rho\,dr \\ -\int_0^1\vec{\eta}_k\cdot\vec{k}_B\left[x_{FA}+r_{FA}\delta_3+\vec{k}_B\cdot(z_o\vec{\imath}-x_o\vec{k}-x_I\vec{k})\right]m\,dr \\ -\int_0^1\vec{\eta}_k\cdot\left[\left((x_c-x_I)\vec{k}\,r\,m\right)'-m\,(\delta_1-\delta_2)\,\vec{\imath}_B\, r\right]dr \\ +\vec{\eta}_k(r_{FA})\cdot(\delta_2\vec{\imath}_B+\delta_3\vec{k}_B)\int_{r_{FA}}^{1} r\,m\,dr \end{array}\right]$$

$$I^{*}_{q_k\theta} = \frac{1}{I_b}\left[\begin{array}{l} -\int_0^1\vec{\eta}_k\cdot\left\{\begin{array}{l} \left((x_c\vec{k})'\int_r^1\rho\,m\,d\rho\right)'+\left((x_I-x_c)\vec{k}\,m\,r\right)' \\ +m\,\vec{\imath}_B\, r\,(\delta_1-\delta_2)-m\,\vec{k}_B\,(x_{FA}+r_{FA}\delta_3 \\ \qquad\qquad -x_I\cos\theta) \end{array}\right\}dr \\ -\vec{\eta}_k(r_{FA})\cdot(\delta_2\vec{\imath}_B+\delta_3\vec{k}_B)\int_{r_{FA}}^{1} r\,m\,dr \end{array}\right]$$

and with
$$\vec{X}_k = \xi_k x_I \vec{k} - \int_{r_{FA}}^{r} \xi_k \left(z_0\vec{i} - x_0\vec{k}\right)'' (r-\rho)\, d\rho$$

$$I^*_{pk} = \int_{r_{FA}}^{1} \xi_k^2\, I_\theta\, dr \,/\, I_b$$

$$S^*_{pk} = \int_{r_{FA}}^{1} \vec{X}_k\, m\, dr \,/\, I_b$$

$$I^*_{pk\alpha} = \int_{r_{FA}}^{1} \vec{X}_k\, r\, m\, dr \,/\, I_b$$

$$S^*_{pk\ddot{\alpha}} = \frac{1}{I_b} \int_{r_{FA}}^{1} \vec{X}_k \cdot \vec{i}_B \Big[-z_{FA} + r\delta_1 - (r - r_{FA})\delta_2 + \vec{i}_B \cdot \left(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}\right)\Big] m\, dr$$

$$S^*_{pk\ddot{q}_i} = \frac{1}{I_b} \int_{r_{FA}}^{1} \vec{X}_k \cdot \left(-\vec{j} \times \vec{\eta}_i\right) m\, dr$$

$$I^*_{pk\ddot{p}_i} = \frac{1}{I_b} \left[\begin{aligned} & -\int_{r_{FA}}^{1} \xi_i \left[\vec{X}_k \cdot \left(z_0\vec{i} - x_0\vec{k} - x_I\vec{k}\right) + \xi_k x_I^2 \right] m\, dr \\ & + \int_{r_{FA}}^{1} \vec{X}_k \cdot \left(\delta_2 \vec{i}_B + \delta_3 \vec{k}_B\right)(r - r_{FA})\, \xi_i(r_{FA})\, m\, dr \\ & + \int_{r_{FA}}^{1} \Big[\left(\xi_k - \xi_k(r_{FA})\right) \xi_i(r_{FA}) \\ & \qquad + \left(\xi_i - \xi_i(r_{FA})\right) \xi_k(r_{FA}) \Big] I_\theta\, dr \end{aligned} \right.$$

$$I^*_{pkp_i} = \frac{1}{I_b} \left[\begin{aligned} & \int_{r_{FA}}^{1} \xi_k \xi_i\, I_\theta \left(\cos^2\theta - \sin^2\theta\right) dr \\ & - \int_{r_{FA}}^{1} \left(\xi_k - \xi_k(r_{FA})\right) \Big[\Big(k_P^2 \int_r^1 \rho\, m\, d\rho \\ & \qquad + \theta_{Tw}'^2 \frac{EI_{pp}}{\Omega^2} \Big) \xi_i' \Big]' dr \end{aligned} \right.$$

$$S^*_{pk\alpha} = k_{P_G}\, I^*_{p_0}\, \omega_0^2\, \xi_k(r_{FA})$$

$$
\begin{aligned}
S^{*}_{p_k q_i} = \frac{1}{I_b}\Bigg[& \int_{r_{FA}}^{1} \vec{X}_k \cdot \vec{k}_B \, \vec{i}_B \cdot \vec{\eta}_i \, m \, dr \\
& + \int_{r_{FA}}^{1} \xi_k \, x_I \, m \, \vec{i} \cdot \left(\vec{\eta}_i^{\,\prime} r - \vec{\eta}_i \right) dr \\
& + \int_{r_{FA}}^{1} \xi_k \left(x_0 \vec{i} + z_0 \vec{k} \right)'' \cdot \int_{r}^{1} \left(r \vec{\eta}_i - \rho \, \vec{\eta}_i(r) \right) m \, d\rho \, dr \\
& - \int_{r_{FA}}^{1} \xi_k \, \vec{\eta}_i^{\,\prime\prime} \cdot \int_{r}^{1} \left\{ \begin{array}{l} r \, \vec{i}_B \, \vec{i}_B \cdot \left(x_0 \vec{i} + z_0 \vec{k} + x_I \vec{i} \right) \\ + \rho \, \vec{k}_B \, \vec{k}_B \cdot \left(x_0 \vec{i} + z_0 \vec{k} + x_I \vec{i} \right) \\ - \rho \left(x_0 \vec{i} + z_0 \vec{k} \right) \big|_r \\ + (\rho - r) \, \vec{i}_B \left(x_{FA} + r_{FA} \delta_3 \right) \\ + (\rho - r) \, \vec{k}_B \, \rho \left(\delta_1 - \delta_2 \right) \end{array} \right\} m \, d\rho \, dr \\
& + \int_{r_{FA}}^{1} \xi_k \left[\theta_{tw}' \left(\frac{EI_{xr}}{\Omega^2} \vec{k} - \frac{EI_{zr}}{\Omega^2} \vec{i} \right) \cdot \vec{\eta}_i^{\,\prime\prime} \right]' dr \Bigg]
\end{aligned}
$$

except for rigid pitch (k = 0), where

$$
\begin{aligned}
\vec{X}_0 = & -\left(z_0 \vec{i} - x_0 \vec{k} - x_I \vec{k} \right) + \left(\delta_2 \vec{i}_B + \delta_3 \vec{k}_B \right) \left(r - r_{FA} \right) \\
& + \left(z_0 \vec{i} - x_0 \vec{k} \right) \big|_{r_{FA}} \\
& + \left(z_0 \vec{i} - x_0 \vec{k} \right)' \big|_{r_{FA}} \left(r - r_{FA} \right)
\end{aligned}
$$

and:

$$S^{*}_{po q_i} = \frac{1}{I_b}\left[\begin{array}{l}
\int_{r_{FA}}^{1} \vec{X}_0 \cdot \vec{k}_B \, \vec{i}_B \cdot \vec{\eta}_i \, m\,dr \\[2ex]
+ \int_{r_{FA}}^{1} (\vec{\eta}_i - r\vec{\eta}_i{}^{\blacktriangledown}(r_{FA})) \cdot \left\{\begin{array}{l}
-\vec{k}_B \vec{k}_B \cdot (x_0\vec{i} + z_0\vec{k} + x_I\vec{i}) \\
\quad + (x_0\vec{i} + z_0\vec{k})|_{r_{FA}} \\
\quad - r_{FA}(x_0\vec{i} + z_0\vec{k})^{\blacktriangledown}|_{r_{FA}} \\
-\vec{i}_B (x_{FA} + r_{FA}\delta_3) \\
-\vec{k}_B \, r(\delta_1 - \delta_2) \\
+ r_{FA}(\delta_3 \vec{i}_B - \delta_2 \vec{k}_B)
\end{array}\right\} m\,dr \\[2ex]
+ \int_{r_{FA}}^{1} (\vec{\eta}_i - r\vec{\eta}_i{}^{\blacktriangledown})|_{r_{FA}} \cdot \left\{\begin{array}{l}
-\vec{i}_B \vec{i}_B \cdot (x_0\vec{i} + z_0\vec{k} + x_I\vec{i}) \\
\quad + r(x_0\vec{i} + z_0\vec{k})^{\blacktriangledown}|_{r_{FA}} \\
+ \vec{i}_B (x_{FA} + r_{FA}\delta_3) \\
+ \vec{k}_B \, r(\delta_1 - \delta_2)
\end{array}\right\} m\,dr \\[2ex]
+ \vec{\eta}_i{}^{\blacktriangledown}(r_{FA}) \cdot (\delta_3 \vec{i}_B - \delta_2 \vec{k}_B) \int_{r_{FA}}^{1} r^2 m\,dr \\[2ex]
- K_{P_i} \, \omega_0^2 \, I_{P_0}
\end{array}\right.$$

Nonrotating frame equations

The equations of motion for the rotor in the nonrotating frame, i.e. after application of the Fourier coordinate transformation, are

$$A_2 \ddot{x}_R + A_1 \dot{x}_R + A_0 x_R + \tilde{A}_2 \ddot{\alpha} + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha = B v_R + M_{aero}$$

and the hub forces and moments

$$F = C_2 \ddot{x}_R + C_1 \dot{x}_R + C_0 x_R + \tilde{C}_2 \ddot{\alpha} + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + F_{aero}$$

where the rotor degrees of freedom ($\vec{x}_R$), shaft motion ($\vec{\alpha}$), rotor blade pitch input ($\vec{v}_R$), and the hub forces and moments ($\vec{F}$) are:

$$\vec{x}_R = \begin{bmatrix} \beta_0^{(k)} \\ \beta_{1c}^{(k)} \\ \beta_{1s}^{(k)} \\ \theta_0^{(k)} \\ \theta_{1c}^{(k)} \\ \theta_{1s}^{(k)} \\ \beta_{Gc} \\ \beta_{Gs} \\ \psi_s \end{bmatrix} \qquad \vec{\alpha} = \begin{bmatrix} x_h \\ y_h \\ z_h \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$\vec{v}_R = \begin{bmatrix} \theta_0^{con} \\ \theta_{1c}^{con} \\ \theta_{1s}^{con} \end{bmatrix} \qquad \vec{F} = \gamma \begin{bmatrix} \frac{C_T}{\sigma a} \\ \frac{2C_H}{\sigma a} \\ -\frac{2C_Y}{\sigma a} \\ \frac{C_Q}{\sigma a} \\ \frac{2C_{My}}{\sigma a} \\ -\frac{2C_{Mx}}{\sigma a} \end{bmatrix}$$

The matrices of the coefficients, and the aerodynamic forcing vectors, follow.

$$A_2 = \begin{bmatrix}
I^*_{q_k} & & & -S^*_{q_k \ddot{p}_i} & & & & & I^*_{q_k\alpha}\cdot\vec{k}_B \\
& I^*_{q_k} & & & -S^*_{q_k \ddot{p}_i} & & I^*_{q_k\alpha}\cdot\vec{i}_B & -I^*_{\dot{q}_k\alpha} & \\
& & I^*_{q_k} & & & -S^*_{q_k \ddot{p}_i} & I^*_{\dot{q}_k\alpha} & I^*_{q_k\alpha}\cdot\vec{i}_B & \\
-S^*_{p_k \ddot{q}_i} & & & \substack{I^*_{p_k} \\ + I^*_{p_k \ddot{p}_i}} & & & & & I^*_{p_k\alpha}\cdot\vec{i}_B \\
& -S^*_{p_k \ddot{q}_i} & & & \substack{I^*_{p_k} \\ + I^*_{p_k \ddot{p}_i}} & & -I^*_{p_k\alpha}\cdot\vec{k}_B & -S^*_{p_k \ddot{\alpha}} & \\
& & -S^*_{p_k \ddot{q}_i} & & & \substack{I^*_{p_k} \\ + I^*_{p_k \ddot{p}_i}} & S^*_{p_k \ddot{\alpha}} & -I^*_{p_k\alpha}\cdot\vec{k}_B & \\
& I^*_{q_i\alpha}\cdot\vec{i}_B & & & -S^*_{p_i\alpha}\cdot\vec{i}_B & & I^*_0 & & \\
& & I^*_{q_i\alpha}\cdot\vec{i}_B & & & -S^*_{p_i\alpha}\cdot\vec{i}_B & & I^*_0 &
\end{bmatrix}$$

$A_1 =$

$$
\begin{bmatrix}
I^*_{q_k} g_s \nu_k + 2 I^*_{q_k} \dot{q}_i & & & & & & & & 2 I^*_{q_k} \dot{\psi} \\
& I^*_{q_k} g_s \nu_k + 2 I^*_{q_k} \dot{q}_i & 2 I^*_{q_k} & & & -2 S^*_{q_k} \ddot{p}_i & & 2 I^*_{q_k \alpha} \cdot \vec{\iota}_B & \\
& -2 I^*_{q_k} & I^*_{q_k} g_s \nu_k + 2 I^*_{q_k} \dot{q}_i & & 2 S^*_{q_k} \ddot{p}_i & & -2 I^*_{q_k \alpha} \cdot \vec{\iota}_B & & \\
& & & I^*_{p_k} g_s \omega_k & & & & & \\
& & -2 S^*_{p_k} \ddot{q}_i & & I^*_{p_k} g_s \omega_k & 2 I^*_{p_k} + 2 I^*_{p_k} \ddot{p}_i & & -2 I^*_{p_k \alpha} \cdot \vec{\kappa}_B & \\
& 2 S^*_{p_k} \ddot{q}_i & & & -2 I^*_{p_k} - 2 I^*_{p_k} \ddot{p}_i & I^*_{p_k} g_s \omega_k & 2 I^*_{p_k \alpha} \cdot \vec{\kappa}_B & & \\
& 2 I^*_{q_i \alpha} & 2 I^*_{q_i \alpha} \cdot \vec{\iota}_B & & & -2 S^*_{p_i \alpha} \cdot \vec{\iota}_B & I^*_0 C^*_G & 2 I^*_0 & \\
& -2 I^*_{q_i \alpha} \cdot \vec{\iota}_B & 2 I^*_{q_i \alpha} & & 2 S^*_{p_i \alpha} \cdot \vec{\iota}_B & & -2 I^*_0 & I^*_0 C^*_G &
\end{bmatrix}
$$

$$A_0 = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\overset{*}{I}_{q_K}\nu_K^2 & & & -\overset{*}{S}_{q_K p_i} & & & & & \\
\hline
 & \overset{*}{I}_{q_K}(\nu_K^2-1) & \begin{array}{c}\overset{*}{I}_{q_K q_s}\nu_K \\ +2\overset{*}{I}_{q_K \dot{q}_i}\end{array} & & \begin{array}{c}\overset{*}{S}_{q_K \ddot{p}_i} \\ -\overset{*}{S}_{q_K p_i}\end{array} & & & & \\
\hline
 & \begin{array}{c}-\overset{*}{I}_{q_K q_s}\nu_K \\ -2\overset{*}{I}_{q_K \dot{q}_i}\end{array} & \overset{*}{I}_{q_K}(\nu_K^2-1) & & & \begin{array}{c}\overset{*}{S}_{q_K \ddot{p}_i} \\ -\overset{*}{S}_{q_K p_i}\end{array} & & & \\
\hline
-\overset{*}{S}_{p_K q_i} & & & \begin{array}{c}\overset{*}{I}_{p_K}\omega_K^2 \\ +\overset{*}{I}_{p_K p_i}\end{array} & & & & & \\
\hline
 & \begin{array}{c}\overset{*}{S}_{p_K \ddot{q}_i} \\ -\overset{*}{S}_{p_K q_i}\end{array} & & & \begin{array}{c}\overset{*}{I}_{p_K}(\omega_K^2-1) \\ +\overset{*}{I}_{p_K p_i} \\ -\overset{*}{I}_{p_K \ddot{p}_i}\end{array} & \overset{*}{I}_{p_K q_s}\omega_K & \overset{*}{S}_{p_K \alpha} & & \\
\hline
 & & \begin{array}{c}\overset{*}{S}_{p_K \ddot{q}_i} \\ -\overset{*}{S}_{p_K q_i}\end{array} & & -\overset{*}{I}_{p_K q_s}\omega_K & \begin{array}{c}\overset{*}{I}_{p_K}(\omega_K^2-1) \\ +\overset{*}{I}_{p_K p_i} \\ -\overset{*}{I}_{p_K \ddot{p}_i}\end{array} & & \overset{*}{S}_{p_K \alpha} & \\
\hline
 & & 2\overset{*}{I}_{\dot{q}_i \alpha} & & & & \overset{*}{I}_0(\nu_G^2-1) & & \\
\hline
 & -2\overset{*}{I}_{\dot{q}_i \alpha} & & & & & & \overset{*}{I}_0(\nu_G^2-1) &
\end{array}\right]$$

$$\tilde{A}_2 = \begin{bmatrix}
 & & S^*_{q\kappa}\cdot\vec{l}_B & & & I^*_{q\kappa\alpha}\cdot\vec{k}_B \\
 & S^*_{q\kappa}\cdot\vec{k}_B & & & -I^*_{q\kappa\alpha}\cdot\vec{l}_B & \\
-S^*_{q\kappa}\cdot\vec{k}_B & & & I^*_{q\kappa\alpha}\cdot\vec{l}_B & & \\
 & & -S^*_{p\kappa}\cdot\vec{k}_B & & & I^*_{p\kappa\alpha}\cdot\vec{l}_B \\
 & S^*_{p\kappa}\cdot\vec{l}_B & & & I^*_{p\kappa\alpha}\cdot k_B & \\
-S^*_{p\kappa}\cdot\vec{l}_B & & & -I^*_{p\kappa\alpha}\cdot\vec{k}_B & & \\
 & & & & -I^*_0 & \\
 & & & I^*_0 & &
\end{bmatrix}$$

$$\tilde{A}_1 = \left[\begin{array}{c|c|c|c|c|c}
 & & & & & \\
\hline
 & & & 2I^*_{qk\alpha}\cdot\vec{i}_B & & \\
\hline
 & & & & 2I^*_{qk\alpha}\cdot\vec{i}_B & \\
\hline
 & & & & & \\
\hline
 & & & -2I^*_{pk\alpha}\cdot\vec{k}_B & & \\
\hline
 & & & & -2I^*_{pk\alpha}\cdot\vec{k}_B & \\
\hline
 & & & 2I^*_0 & & \\
\hline
 & & & & 2I^*_0 &
\end{array}\right]$$

$B =$

$$
\begin{bmatrix}
 & & \\
 & & \\
 & & \\
I^{*}_{p_o}\,\omega_o^2\,\xi_k(r_{FA}) & & \\
 & I^{*}_{p_o}\,\omega_o^2\,\xi_k(r_{FA}) & \\
 & & I^{*}_{p_o}\,\omega_o^2\,\xi_k(r_{FA}) \\
 & & \\
 & &
\end{bmatrix}
$$

$C_2 =$

$$
\begin{bmatrix}
-S^{*}_{q_i}\cdot\vec{l}_{B} & & & & & & & & \\
 & & S^{*}_{q_i}\cdot\vec{k}_{B} & & & & & & \\
 & S^{*}_{q_i}\cdot\vec{k}_{B} & & & & & & & \\
I^{*}_{q_i\alpha}\cdot\vec{k}_{B} & & & -S^{*}_{p_i\alpha}\cdot\vec{k}_{B} & & & & & I^{*}_{0} \\
 & I^{*}_{q_i\alpha}\cdot\vec{l}_{B} & & & -S^{*}_{p_i\alpha}\cdot\vec{l}_{B} & & I^{*}_{0} & & \\
 & & I^{*}_{q_i\alpha}\cdot\vec{l}_{B} & & & -S^{*}_{p_i\alpha}\cdot\vec{l}_{B} & & I^{*}_{0} &
\end{bmatrix}
$$

$C_1 =$

$$
\begin{bmatrix}
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
-2I^*_{q,\psi} & & & & & & & & \\
 & 2I^*_{q,\alpha} & 2I^*_{q,\alpha}\cdot\vec{i}_B & & & -2S^*_{p,\alpha}\cdot\vec{i}_B & & 2I^*_o & \\
 & -2I^*_{q,\alpha}\cdot\vec{i}_B & 2I^*_{q,\alpha} & & 2S^*_{p,\alpha}\cdot\vec{i}_B & & -2I^*_o & &
\end{bmatrix}
$$

$C_0 =$

$$\begin{bmatrix}
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & 2\overset{*}{I}_{q:\alpha} & & & & & & \\
 & -2\overset{*}{I}_{q:\alpha} & & & & & & &
\end{bmatrix}$$

$$\tilde{C}_2 = \begin{bmatrix} & & -M_b^* & & & \\ -2M_b^* & & & & & \\ & 2M_b^* & & & & \\ & & & & & I_0^* \\ & & & & -I_0^* & \\ & & & I_0^* & & \end{bmatrix}$$

$$\tilde{C}_1 = \begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & 2I_0^* & & \\ & & & & 2I_0^* & \end{bmatrix}$$

$$\tilde{A}_0 = 0$$

$$\tilde{C}_0 = 0$$

and:

$$M_{aero} = \gamma \begin{bmatrix} \frac{M_{\beta 0}}{ac} \\ \frac{M_{\beta 1c}}{ac} \\ \frac{M_{\beta 1s}}{ac} \\ \frac{M_{\partial 0}}{ac} \\ \frac{M_{\partial 1c}}{ac} \\ \frac{M_{\partial 1s}}{ac} \\ -\frac{2C_{My}}{\sigma a} \\ \frac{2C_{Mx}}{\sigma a} \end{bmatrix} \qquad F_{aero} = \gamma \begin{bmatrix} \frac{C_T}{\sigma a} \\ \frac{2C_H}{\sigma a} \\ -\frac{2C_Y}{\sigma a} \\ \frac{C_Q}{\sigma a} \\ \frac{2C_{My}}{\sigma a} \\ -\frac{2C_{Mx}}{\sigma a} \end{bmatrix}$$

AERODYNAMICS

In this section, the aerodynamic forces and moments on the rotor blade are derived. We shall consider the general case of high or low inflow, and axial or nonaxial flow. The aerodynamic terms in the rotor equations of motion and the hub forces and moments are obtained for three cases: axial flow (hover or high inflow cruise); nonaxial flow with periodic coefficients (helicopter forward flight, or conversion mode flight), and a constant coefficient approximation for nonaxial flow.

The principle assumptions in the aerodynamic analysis are: reverse flow is neglected (good to an advance ratio of about 0.4 or 0.5, which is sufficient for the tilting proprotor aircraft); the wing wake (near field and far field) effect on the rotor, and other wing/rotor interferences are neglected; the unsteady rotor wake effects are neglected; the virtual mass aerodynamic forces and moments are neglected; the order c (rotor chord) terms in the aerodynamic lift expression are neglected; the order c^3 terms in the aerodynamic moment expression are neglected; and only first order velocity terms are retained. The derivation and notation are an extension of that in reference 4.

Section Aerodynamic Forces

A hub plane reference frame is used for the aerodynamic forces. All forces and velocities are resolved in the hub plane then, i.e. in the B system. The hub plane reference frame is fixed with respect to the shaft, hence it is tilted and displaced by the shaft motion. Figure 10 illustrates the forces and velocities of the blade section aerodynamics. The velocity of the air seen by the blade, the pitch angle, and the angle of attack are:

θ = blade pitch, measured from the reference plane

u_T, u_R, u_P = air velocity seen by the blade, resolved with respect to the reference plane; u_T is in the hub plane, positive in the blade drag direction; u_R

is in the hub plane, positive radially outward along the blade; and u_P is normal to the hub plane, positive down through the rotor disk.

U = resultant air velocity in the plane of the section.

ϕ = induced angle

α = section angle of attack

where

$$U^2 = u_T^2 + u_P^2$$

$$\phi = \tan^{-1} u_P/u_T$$

$$\alpha = \theta - \phi$$

The aerodynamic forces and moment on the section, at the EA, are:

L, D = aerodynamic lift and drag forces on the section, normal and parallel to the resultant velocity U

F_z, F_x = section L and D (total aerodynamic force on the section) resolved with respect to the hub plane, normal to and in the plane of the rotor

F_r = radial drag force on the blade, in the plane of the disk, positive outward (the same direction as positive u_R); the radial forces due to the tilt of F_z and F_x have been considered separately.

M_a = section aerodynamic moment about the EA, positive nose up.

Aerodynamic forces -- wind axes

The section lift and drag are

$$L = \tfrac{1}{2}\rho U^2 c c_\ell$$

$$D = \tfrac{1}{2}\rho U^2 c c_d$$

where U = resultant velocity at the section

ρ = air density

c = chord of blade

The air density is dropped at this point, in the process of making the quantities dimensionless with ρ, Ω, and R. The section lift and drag coefficients, $c_l = c_l(\alpha, M)$ and $c_d = c_d(\alpha, M)$ are functions of the section angle of attack and Mach number:

$$\alpha = \theta - \phi = \theta - \tan^{-1} u_P/u_T$$
$$M = M_{TIP}\, U$$

where M_{TIP} is the tip Mach number, the rotor tip speed ΩR divided by the speed of sound. The dependence of c_l and c_d on other quantities, such as the local yaw angle or unsteady angle of attack changes, is neglected. The radial force, due to the radial drag, is

$$F_r = \frac{u_R}{U} D = \frac{1}{2} U u_R c c_d$$

The radial drag force is derived assuming that the viscous drag force on the section has the same sweep angle as the local section velocity. The moment about the EA is

$$M_a = -x_A L + M_{AC} + M_{US}$$
$$= -x_A \frac{1}{2} U^2 c c_l + \frac{1}{2} U^2 c^2 c_{m_{ac}} + M_{US}$$

where x_A = distance aerodynamic center (AC) behind EA

$c_{m_{ac}}$ = section moment about the AC, positive nose up.

M_{US} = unsteady aerodynamic moment.

For the section aerodynamic moment it is necessary to include the unsteady aerodynamic terms, which from thin airfoil theory are

$$\frac{M_{US}}{ac} = -\frac{c^2}{32}\Big[(VB)\left(1 + 8\frac{x_A}{c} + 16\left(\frac{x_A}{c}\right)^2\right) + (\dot{w} + u_R w')\left(1 + 4\frac{x_A}{c}\right)\Big]$$

where w = mean upwash along the blade chord, i.e. normal blade section

$= u_T \sin\theta - u_P \cos\theta$

$B = \partial w/\partial x$, basically the pitch rate $\dot{\theta}$

$V = u_T \cos\theta + u_P \sin\theta$

Hence in the aerodynamic model we have neglected the following effects: reverse flow; shed wake aerodynamic interference (e.g. lift deficiency function set to unity); terms in L order c and above; terms in M order c^3 and above; virtual mass terms in the unsteady aerodynamic moment.

Aerodynamic forces -- hub plane axes

With respect to the hub plane then

$$F_z = L\cos\phi - D\sin\phi = \frac{L u_T - D u_P}{u}$$

$$F_x = L\sin\phi + D\cos\phi = \frac{L u_P + D u_T}{u}$$

Substituting for L and D, and dividing by ac (where a is the two-dimensional section lift curve slope, and c the section chord; which enter the Lock number γ also), we obtain:

$$\frac{F_z}{ac} = u\left(u_T \frac{c_l}{2a} - u_P \frac{c_d}{2a}\right)$$

$$\frac{F_x}{ac} = u\left(u_P \frac{c_l}{2a} + u_T \frac{c_d}{2a}\right)$$

$$\frac{F_r}{ac} = u\, u_R \frac{c_d}{2a}$$

$$\frac{M_a}{ac} = -x_A u^2 \frac{c_l}{2a} + u^2 \frac{c\, c_m}{2a} + \frac{M_{us}}{ac}$$

The net rotor aerodynamic forces are obtained by integration of the section forces over the span of the blade, an then summation over all N blades.

Perturbation forces

Each component of the velocity seen by the blade has a trim term, due to operation of the rotor in its trim equilibrium state; and a perturbation term due to the perturbed motion of the system. The latter is due to the system degrees of freedom, and is assumed to be small in obtaining the linear differential equations describing the dynamics. We shall write the blade pitch and section velocities as trim plus perturbation terms:

$$\theta \Rightarrow \theta + \delta\theta$$

$$u_T \Rightarrow u_T + \delta u_T$$

$$u_P \Rightarrow u_P + \delta u_P$$

$$u_R \Rightarrow u_R + \delta u_R$$

then there follows the perturbations of α, U, and M:

$$\delta\alpha = \delta\theta - \frac{u_T\,\delta u_P - u_P\,\delta u_T}{U^2}$$

$$\delta U = \frac{u_T\,\delta u_T + u_P\,\delta u_P}{U}$$

$$\delta M = M_{tip}\,\delta U$$

and of the aerodynamic coeffients

$$\delta c_l = \frac{\partial c_l}{\partial\alpha}\delta\alpha + \frac{\partial c_l}{\partial M}\delta M = c_{l_\alpha}\delta\alpha + c_{l_M}\delta M$$

(and similarly for c_m and c_d). The perturbations of the section aerodynamic forces may then be obtained by carrying out the differential operation on the expressions above for F_z, F_x, F_r, and M_a, using the above results to express the perturbations in terms of $\delta\theta$, δu_T, δu_P, and δu_R. The coefficients of the perturbation quantities are then evaluated at the trim state. The results are:

$$
\begin{aligned}
\delta\frac{F_z}{ac} = {} & \left(U u_T \frac{c_{l_\alpha}}{2a} - U u_P \frac{c_{d_\alpha}}{2a}\right)\delta\theta \\
& + \Big[-\frac{u_T}{U}\left(u_T\frac{c_{l_\alpha}}{2a} - u_P\frac{c_{d_\alpha}}{2a}\right) + \left(\frac{c_l}{2a} + M\frac{c_{l_M}}{2a}\right)\frac{u_T u_P}{U} \\
& \qquad - \left(\frac{c_d}{2a} + M\frac{c_{d_M}}{2a}\right)\frac{u_P^2}{U} - \frac{c_d}{2a}U\Big]\delta u_P \\
& + \Big[\frac{u_P}{U}\left(u_T\frac{c_{l_\alpha}}{2a} - u_P\frac{c_{d_\alpha}}{2a}\right) + \left(\frac{c_l}{2a} + M\frac{c_{l_M}}{2a}\right)\frac{u_T^2}{U} \\
& \qquad + \frac{c_l}{2a}U - \left(\frac{c_d}{2a} + M\frac{c_{d_M}}{2a}\right)\frac{u_T u_P}{U}\Big]\delta u_T \\
= {} & F_{z_\theta}\delta\theta + F_{z_P}\delta u_P + F_{z_T}\delta u_T
\end{aligned}
$$

$$\delta \frac{F_x}{ac} = \left(u u_P \frac{C_{l\alpha}}{2a} + u u_T \frac{C_{d\alpha}}{2a} \right) \delta\theta$$

$$+ \left[-\frac{u_T}{u}\left(u_P \frac{C_{l\alpha}}{2a} + u_T \frac{C_{d\alpha}}{2a} \right) + \left(\frac{C_l}{2a} + M \frac{C_{lM}}{2a} \right) \frac{u_P^2}{u} \right.$$

$$\left. + \frac{C_l}{2a} u + \left(\frac{C_d}{2a} + M \frac{C_{dM}}{2a} \right) \frac{u_T u_P}{u} \right] \delta u_P$$

$$+ \left[\frac{u_P}{u}\left(u_P \frac{C_{l\alpha}}{2a} + u_T \frac{C_{d\alpha}}{2a} \right) + \left(\frac{C_l}{2a} + M \frac{C_{lM}}{2a} \right) \frac{u_P u_T}{u} \right.$$

$$\left. + \left(\frac{C_d}{2a} + M \frac{C_{dM}}{2a} \right) \frac{u_T^2}{u} + \frac{C_d}{2a} u \right] \delta u_T$$

$$= F_{x_\theta} \delta\theta + F_{x_P} \delta u_P + F_{x_T} \delta u_T$$

$$\delta \frac{F_r}{ac} = \left(u u_R \frac{C_{d\alpha}}{2a} \right) \delta\theta$$

$$+ \left[-\frac{u_T u_R}{u} \frac{C_{d\alpha}}{2a} + \left(\frac{C_d}{2a} + M \frac{C_{dM}}{2a} \right) \frac{u_R u_P}{u} \right] \delta u_P$$

$$+ \left[\frac{u_P u_R}{u} \frac{C_{d\alpha}}{2a} + \left(\frac{C_d}{2a} + M \frac{C_{dM}}{2a} \right) \frac{u_R u_T}{u} \right] \delta u_T$$

$$+ \left[u \frac{C_d}{2a} \right] \delta u_R$$

$$= F_{r_\theta} \delta\theta + F_{r_P} \delta u_P + F_{r_T} \delta u_T + F_{r_R} \delta u_R$$

and:

$$\delta \frac{M_a}{ac} = \left[u^2 \left(-x_A \frac{C_{\ell\alpha}}{2a} + c \frac{C_{m\alpha}}{2a} \right) \right] \delta\theta$$

$$+ \left[u_P \left(-x_A \frac{C_{\ell\alpha}}{2a} + c \frac{C_{m\alpha}}{2a} \right) - x_A u_T \left(2 \frac{C_\ell}{2a} + M \frac{C_{\ell M}}{2a} \right) \right.$$

$$\left. + c u_T \left(2 \frac{C_m}{ac} + M \frac{C_{mM}}{2a} \right) \right] \delta u_T$$

$$+ \left[-u_T \left(-x_A \frac{C_{\ell\alpha}}{2a} + c \frac{C_{m\alpha}}{2a} \right) - x_A u_P \left(2 \frac{C_\ell}{2a} + M \frac{C_{\ell M}}{2a} \right) \right.$$

$$\left. + c \cdot u_P \left(2 \frac{C_m}{ac} + M \frac{C_{mM}}{ac} \right) \right] \delta u_P$$

$$+ \frac{M_{us}}{ac}$$

$$= M_{a\theta}\, \delta\theta + M_{aP}\, \delta u_P + M_{aT}\, \delta u_T + \frac{M_{us}}{ac}$$

Velocity of the Blade

Now we obtain the velocity of the air seen by the blade section. There is the trim velocity, composed of the forward speed, rotor rotation, and rotor induced velocity; and the perturbation velocities, due to the rotor degrees of freedom and the shaft motion, and due to the aerodynamic gust velocity.

The rotor is rotating at constant speed Ω. The steady velocity of the rotor with respect to the air, is described by (figure 11):

V = trim velocity of the rotor in inertial axes, in the rotor x-z plane.

α_{HP} = angle of attack (undisturbed) of the rotor hub plane with respect to V, positive for disk tilt forward (for V down through the disk); this is the shaft angle.

There are then the following cases: $\alpha_{HP} = 90^{\circ}$ for cruise (high inflow axial flight); α_{HP} small for helicopter forward flight; α_{HP} large but less than 90° for conversion mode; and $V = 0$ is the hover case. The rotor induced velocity is v, due to the thrust T (figure 11); v is assumed to be normal to the hub plane, and uniform over the disk. Now the rotor advance ratio μ and inflow ratio λ are defined:

$$\mu = \frac{V \cos \alpha_{HP}}{\Omega R}$$

$$\lambda = \frac{V \sin \alpha_{HP} + v}{\Omega R}$$

The cases are then: for hover $\mu = 0$ and λ small: for helicopter forward flight $\mu \neq 0$ and λ small; for conversion mode flight $\mu \neq 0$ and λ order 1; and for cruise flight $\mu = 0$ and λ order 1.

For the rotor induced velocity we use the Glauert result:

$$\lambda = \mu \tan \alpha_{HP} + \frac{C_T}{2\sqrt{\mu^2 + \lambda^2}}$$

For high speed ($V^2 \gg \frac{1}{2} C_T (\Omega R)^2$ or about $V/\Omega R > 0.15$) in inflow ratio is approximately

$$\lambda = \frac{V}{\Omega R} \sin \alpha_{HP} + \frac{C_T}{2 V/\Omega R}$$

The induced velocity is thus quite small, $v/V \ll 1$, for typical proprotor cruise and conversion mode operation. The induced velocity is not generally an important factor in proprotor aerodynamics at high inflow; hence the assumption of uniform induced inflow is acceptable for an investigation of the proprotor aeroelastic behavior. (See reference 4.) The mutual aerodynamic interference of the rotors is neglected.

The trim velocity V is steady, at an angle α_{HP} to the rotor hub plane. The uniform induced velocity v is normal to the hub plane. The advance ratio and inflow ratio, μ and λ, are the nondimensional components parallel and normal to the hub plane. In body axes, V would be fixed in the reference frame, and would tilt with it. Here an inertial frame (the S system) is used however, so it follows that tilt of the rotor by the shaft motion gives a small change in the direction of V as seen in the reference frame.

The shaft motion consists of small linear and angular velocity, with components defined in the nonrotating frame:

$$\Delta\vec{V}_0 = \dot{x}_h \vec{i}_S + \dot{y}_h \vec{j}_S + \dot{z}_h \vec{k}_S$$
$$\vec{\omega}_0 = \dot{\alpha}_x \vec{i}_S + \dot{\alpha}_y \vec{j}_S + \dot{\alpha}_z \vec{k}_S$$

The aerodynamic gust velocity has components u_G, v_G, and w_G (longitudinal, lateral, and vertical) defined with respect to the body or earth axes (figure 11); these components are the velocity seen by the aircraft, and are assumed to be small compared to ΩR. The gust components are defined with respect to V, i.e. α_{HP} from the disk plane, so that with V usually horizontal (level flight) w_G and u_G are always the vertical and longitudinal components with respect to the flight path. The gust components are normalized by dividing by ΩR, not by V as is often the convention for airplane analyses. The aerodynamic gust is assumed to be uniform throughout space.

<u>Trim terms</u>

The result for the trim velocity terms is:

$$u_T = r + \mu\sin\psi - \mu\cos\psi\left(\delta\alpha_3 - \Sigma q_i \vec{k}_B\cdot\vec{\eta}_i^{\,\prime}\right) + \Sigma \dot{q}_i \vec{k}_B\cdot\vec{\eta}_i$$

$$u_P = \lambda + \Sigma \dot{q}_i \vec{i}_B\cdot\vec{\eta}_i + r\dot{\beta}_G + \mu\cos\psi\left(\delta\alpha_1 - \delta\alpha_2 + \Sigma q_i \vec{i}_B\cdot\vec{\eta}_i^{\,\prime} + \beta_G\right)$$

$$u_R = \mu\cos\psi + (x_{FA} + r_{FA}\,\delta_{FA_3}) + \Sigma q_i\,\vec{k}_B\cdot(\vec{\eta}_i - r\vec{\eta}_i{}')$$
$$-\lambda(\delta_{FA_1} - \delta_{FA_2} + \Sigma q_i\,\vec{i}_B\cdot\vec{\eta}_i{}' + \beta_G)$$
$$+\mu\sin\psi\,(\delta_{FA_3} - \Sigma q_i\,\vec{k}_B\cdot\vec{\eta}_i{}')$$

and

$$\theta = \theta_{coll} + \theta_{TW} + \theta_{cyc} - K_{P_G}\beta_G - \Sigma K_{P_i} q_i$$

where θ_{cyc} is the input cyclic pitch required to trim the rotor. For the trim velocity, the blade bending and gimbal motion is periodic. For axial flight, $\mu = 0$, the trim velocities are constant; for nonaxial flow, $\mu > 0$, these velocities are periodic in ψ_m, due to the rotation of the blade with respect to the rotor forward velocity.

Perturbation terms

The result for the perturbations of the velocity components and the blade pitch, due to the rotor and shaft motion and the aerodynamic gust, is then:

$$\delta u_T = (\lambda\alpha_x + \dot{y}_h + v_G)\cos\psi_m$$
$$+(\lambda\alpha_y - \dot{x}_h + u_G\cos\alpha_{HP} + w_G\sin\alpha_{HP})\sin\psi_m$$
$$+\mu\cos\psi_m\,(\alpha_z + \psi_s)$$
$$+r\,(\dot{\alpha}_z + \dot{\psi}_s)$$
$$+\Sigma\dot{q}_i\,(\vec{k}_B\cdot\vec{\eta}_i)$$
$$+\mu\cos\psi_m\,\Sigma q_i\,(\vec{k}_B\cdot\vec{\eta}_i{}')$$

$$\delta u_P = (\dot{z}_h - \mu\alpha_y + u_G \sin\alpha_{HP} - w_G \cos\alpha_{HP})$$
$$+ \mu\cos\psi_m \, (\beta_G)$$
$$+ r\,(\dot{\beta}_G + \dot{\alpha}_x \sin\psi_m - \dot{\alpha}_y \cos\psi_m)$$
$$+ \Sigma\, \dot{q}_i \, (\tau_B \cdot \vec{\eta}_i)$$
$$+ \mu\cos\psi_m \, \Sigma\, q_i \, (\tau_B \cdot \vec{\eta}_i^{\,*})$$

$$\delta u_R = -(\lambda\alpha_x + \dot{y}_h + v_G)\sin\psi_m$$
$$+ (\lambda\alpha_y - \dot{x}_h + u_G \cos\alpha_{HP} + w_G \sin\alpha_{HP})\cos\psi_m$$
$$- \lambda\beta_G - \mu\sin\psi_m \, (\alpha_z + \psi_s)$$
$$+ \Sigma\, q_i \left[\vec{k}_B \cdot (\vec{\eta}_i - r\vec{\eta}_i^{\,*} - \mu\sin\psi_m \vec{\eta}_i^{\,*}) - \lambda\tau_B \cdot \vec{\eta}_i^{\,*} \right]$$

$$\delta\theta = \tilde{\theta} = \Sigma\, p_i \xi_i$$

and for M_{HS}

$$V_B = (u_T\cos\theta + u_P\sin\theta)(\dot{\theta} + \beta_G + \Sigma q_i \tau_B \cdot \vec{\eta}_i^{\,*})$$

$$\dot{w} + u_R w^* = \Sigma\, \dot{p}_i \xi_i (u_T\cos\theta + u_P\sin\theta)$$
$$+ u_R \, \Sigma\, p_i \left(\xi_i^* (u_T\cos\theta + u_P\sin\theta) + 2\xi_i\cos\theta \right)$$
$$- \dot{\beta}_G \, (2u_R\cos\theta)$$
$$+ \beta_G \, (\mu\sin\psi_m \cos\theta)$$
$$+ (\dot{\alpha}_z + \dot{\psi}_s)(2u_R\sin\theta)$$
$$- (\alpha_z + \psi_s)(\mu\sin\psi_m \sin\theta)$$
$$- 2u_R \, \Sigma\, \dot{q}_i \, \tau \cdot \vec{\eta}_i^{\,*}$$
$$+ \Sigma\, q_i \, (\mu\sin\psi_m \, \tau \cdot \vec{\eta}_i^{\,*} - u_R^2 \, \tau \cdot \vec{\eta}_i^{\,**})$$

Rotor Aerodynamic Forces -- Rotating Blade

With now the expansions for the section forces and moment in terms of the velocity perturbations, and the velocity in terms of the motion of the rotor and shaft, we may obtain the perturbations of the aerodynamic forces on the blade. These are the blade forces expanded as linear combinations of the degrees of freedom. Giving names to the aerodynamic coefficients at this point, the results for the required aerodynamic forces on the rotating blade are as follows.

Bending:

$$\int_0^1 \vec{\eta}_k \cdot \left(\frac{F_z}{ac}\vec{i}_B - \frac{F_x}{ac}\vec{k}_B \right) dr =$$

$$
\begin{aligned}
& M_{q_k 0} + M_{q_k \mu}\Big[(\lambda \alpha_x + \dot{y}_h + v_G)\cos\psi_m \\
& \qquad + (\lambda \alpha_y - \dot{x}_h + u_G \cos\alpha_{HP} + w_G \sin\alpha_{HP})\sin\psi_m \Big] \\
& + M_{q_k \dot{\zeta}} \ (\dot{\alpha}_z + \dot{\psi}_s) \\
& + M_{q_k \zeta} \ (\alpha_z + \psi_s) \\
& + M_{q_k \lambda} \ (\dot{z}_h - \mu \alpha_y + u_G \sin\alpha_{HP} - w_G \cos\alpha_{HP}) \\
& + M_{q_k \dot{\beta}} \ (\dot{\beta}_G + \dot{\alpha}_x \sin\psi_m - \dot{\alpha}_y \cos\psi_m) \\
& + M_{q_k \beta} \ \beta_G \\
& + \sum M_{q_k \dot{q}_i} \ \dot{q}_i \\
& + \sum M_{q_k q_i} \ q_i \\
& + \sum M_{q_k p_i} \ p_i
\end{aligned}
$$

radial force:

$$\int_0^1 \frac{F_r}{ac} - \frac{F_z}{ac}\left(\beta_G+\delta_1-\delta_2+\vec{k}_B\cdot(x_0\vec{i}+z_0\vec{k})'\right)dr$$

$$= R_\mu\Big[-(\lambda\alpha_x+\dot{y}_n+v_G)\sin\Psi_m + (\lambda\alpha_y-\dot{x}_n+u_G\cos\alpha_{HP}+w_G\sin\alpha_{HP})\cos\Psi_m\Big]$$

$$+ R_r\Big[(\lambda\alpha_x+\dot{y}_n+v_G)\cos\Psi_m + (\lambda\alpha_y-\dot{x}_n+u_G\cos\alpha_{HP}+w_G\sin\alpha_{HP})\sin\Psi_m\Big]$$

$$+ R_{\dot{\zeta}}(\dot{\alpha}_z+\dot{\Psi}_s) + R_\zeta(\alpha_z+\Psi_s)$$

$$+ R_\lambda(\dot{z}_n-\mu\alpha_y+u_G\sin\alpha_{HP}-w_G\cos\alpha_{HP})$$

$$+ R_{\dot{\beta}}(\dot{\beta}_G+\dot{\alpha}_x\sin\Psi_m-\dot{\alpha}_y\cos\Psi_m) + R_\beta\beta_G$$

$$+ \Sigma R_{\dot{q}_i}\dot{q}_i + \Sigma R_{q_i} q_i$$

$$+ \Sigma R_{p_i} p_i$$

Torsion/pitch:

$$\int_{r_{FA}}^1 \xi_K M_a\,dr - \int_{r_{FA}}^1\left(\frac{F_x}{ac}\vec{i}_B+\frac{F_z}{ac}\vec{k}_B\right)\cdot\vec{X}_{AK}\,dr$$

$$= M_{pK\mu}\Big[(\lambda\alpha_x+\dot{y}_n+v_G)\cos\Psi_m + (\lambda\alpha_y-\dot{x}_n+u_G\cos\alpha_{HP}+w_G\sin\alpha_{HP})\sin\Psi_m\Big]$$

$$+M_{pK\dot{\zeta}}(\dot{\alpha}_z+\dot{\Psi}_s) + M_{pK\zeta}(\alpha_z+\Psi_s)$$

$$+ M_{pK\lambda}\Big[\dot{z}_n-\mu\alpha_y+u_G\sin\alpha_{HP}-w_G\cos\alpha_{HP}\Big]$$

$$+ M_{pK\dot{\beta}}(\dot{\beta}_G+\dot{\alpha}_x\sin\Psi_m-\dot{\alpha}_y\cos\Psi_m) + M_{pK\beta}\beta_G$$

$$+ \Sigma M_{pK\dot{q}_i}\dot{q}_i + \Sigma M_{pKq_i} q_i$$

$$+ \Sigma M_{pK\dot{p}_i}\dot{p}_i + \Sigma M_{pKp_i} p_i$$

Hub forces and moments: similar to the bending case, but with notation

	integrand	notation
flap moment	rF_z	M
torque	rF_x	Q
blade drag force	F_x	H
thrust	F_z	T

Aerodynamic coefficients

Applying the results for the expansion of the aerodynamic forces, and the expansion of the velocities, the aerodynamic coefficients may be evaluated. These coefficients of the degrees of freedom in the aerodynamic forces are constant for axial flow, the $\mu = 0$ case. For the general nonaxial flow case, $\mu > 0$, the coefficients are however periodic functions of ψ_m. The results follow.

Bending:

$$M_{q_k 0} = \int_0^1 \vec{\eta}_k \cdot \left(\frac{F_z}{ac} \vec{i}_B - \frac{F_x}{ac} \vec{k}_B \right) dr$$

$$M_{q_k \mu} = \int_0^1 \vec{\eta}_k \cdot (F_{zT} \vec{i}_B - F_{xT} \vec{k}_B) dr$$

$$M_{q_k \dot{\zeta}} = \int_0^1 \vec{\eta}_k \cdot (F_{zT} \vec{i}_B - F_{xT} \vec{k}_B) r\, dr$$

$$M_{q_k \zeta} = \mu \cos\psi_m \, M_{q_k \mu}$$

$$M_{q_k \lambda} = \int_0^1 \vec{\eta}_k \cdot (F_{zP} \vec{i}_B - F_{xP} \vec{k}_B) dr$$

$$M_{q_k \dot{\beta}} = \int_0^1 \vec{\eta}_k \cdot (F_{zP} \vec{i}_B - F_{xP} \vec{k}_B) r\, dr$$

$$M_{q_k \beta} = \mu \cos\psi_m \, M_{q_k \lambda}$$

$$M_{q_k \dot{q}_i} = \int_0^1 \vec{\eta}_k \cdot (F_{zT} \vec{i}_B - F_{xT} \vec{k}_B) \vec{k}_B \cdot \vec{\eta}_i \, dr + \int_0^1 \vec{\eta}_k \cdot (F_{zP} \vec{i}_B - F_{xP} \vec{k}_B) \vec{i}_B \cdot \vec{\eta}_i \, dr$$

$$M_{q_k q_i} = \mu \cos\psi_m \left[\int_0^1 \vec{\eta}_k \cdot (F_{zT} \vec{i}_B - F_{xT} \vec{k}_B) \vec{k}_B \cdot \vec{\eta}_i' \, dr + \int_0^1 \vec{\eta}_k \cdot (F_{zP} \vec{i}_B - F_{xP} \vec{k}_B) \vec{i}_B \cdot \vec{\eta}_i' \, dr \right]$$

$$M_{q_k p_i} = \int_0^1 \vec{\eta}_k \cdot (F_{z\theta} \vec{i}_B - F_{x\theta} \vec{k}_B) \xi_i \, dr$$

Flap moment:

$$M_\mu = \int_0^1 F_{zT}\, r\, dr$$

$$M_{\dot\zeta} = \int_0^1 F_{zT}\, r^2\, dr$$

$$M_\zeta = \mu \cos\psi_m\, M_\mu$$

$$M_\lambda = \int_0^1 F_{zP}\, r\, dr$$

$$M_{\dot\beta} = \int_0^1 F_{zP}\, r^2\, dr$$

$$M_\beta = \mu \cos\psi_m\, M_\lambda$$

$$M_{\dot q_i} = \int_0^1 \left(F_{zT}\, \vec{k}_B \cdot \vec{\eta}_i + F_{zP}\, \vec{i}_B \cdot \vec{\eta}_i \right) r\, dr$$

$$M_{q_i} = \mu \cos\psi_m \int_0^1 \left(F_{zT}\, \vec{k}_B \cdot \vec{\eta}_i{}' + F_{zP}\, \vec{i}_B \cdot \vec{\eta}_i{}' \right) r\, dr$$

$$M_{p_i} = \int_0^1 F_{z\theta}\, \xi_i\, r\, dr$$

Other hub forces and moments: similar to flap moment, with

	coefficient	integrand
flap moment	M	rF_z
torque	Q	rF_x
blade drag force	H	F_x
thrust	T	F_z

Radial force:

$$R_\mu = \int_0^1 F_{rR}\, dr$$

$$R_r = \int_0^1 \left[F_{rT} - F_{zT} \left(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{i} + z_0 \vec{k})' \right) \right] dr$$

$$R_{\dot\zeta} = \int_0^1 \left[F_{rT} - F_{zT} \left(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{i} + z_0 \vec{k})' \right) \right] r\, dr$$

$$R_\zeta = \mu \cos\psi_m\, R_r - \mu \sin\psi_m\, R_\mu$$

$$R_\lambda = \int_0^1 \left[F_{rP} - F_{zP} \left(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{i} + z_0 \vec{k})' \right) \right] dr$$

$$R_{\dot\beta} = \int_0^1 \left[F_{rP} - F_{zP} \left(\delta_1 - \delta_2 + \vec{k}_B \cdot (x_0 \vec{i} + z_0 \vec{k})' \right) \right] r\, dr$$

$$R_{\beta} = \mu\cos\psi_m \, R_{\lambda} - \lambda R_{\mu} - \int_0^1 \frac{F_z}{ac}\, dr$$

$$\begin{aligned} R_{\dot q_i} = {} & \int_0^1 \left[F_{r_T} - F_{z_T}\left(\delta_1 - \delta_2 + \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\right)\right] \vec{k}_B\cdot\vec{\eta}_i \, dr \\ & + \int_0^1 \left[F_{r_P} - F_{z_P}\left(\delta_1 - \delta_2 + \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\right)\right] \vec{i}_B\cdot\vec{\eta}_i \, dr \end{aligned}$$

$$\begin{aligned} R_{q_i} = {} & \mu\cos\psi_m \Big\{ \int_0^1 \left[F_{r_T} - F_{z_T}\left(\delta_1 - \delta_2 + \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\right)\right] \vec{k}_B\cdot\vec{\eta}_i{}' \, dr \\ & + \int_0^1 \left[F_{r_P} - F_{z_P}\left(\delta_1 - \delta_2 + \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\right)\right] \vec{i}_B\cdot\vec{\eta}_i{}' \, dr \Big\} \\ & + \int_0^1 F_{r_R}\left[\vec{k}_B\cdot\left(\vec{\eta}_i - r\vec{\eta}_i{}' - \mu\sin\psi_m\,\vec{\eta}_i{}'\right) - \lambda\,\vec{i}_B\cdot\vec{\eta}_i{}'\right] dr \\ & - \int_0^1 \frac{F_z}{ac}\,\vec{i}_B\cdot\vec{\eta}_i{}' \, dr \end{aligned}$$

$$R_{p_i} = \int_0^1 \left[F_{r_\theta} - F_{z_\theta}\left(\delta_1 - \delta_2 + \vec{k}_B\cdot(x_0\vec{i} + z_0\vec{k})'\right)\right] \xi_i \, dr$$

Torsion/pitch:

$$M_{p_k\mu} = \int_{r_{FA}}^1 \left[\xi_k M_{a_T} - (F_{x_T}\vec{i}_B + F_{z_T}\vec{k}_B)\cdot\vec{X}_{A_k}\right] dr$$

$$\begin{aligned} M_{p_k\dot\zeta} = {} & \int_{r_{FA}}^1 \left[\xi_k M_{a_T} - (F_{x_T}\vec{i}_B + F_{z_T}\vec{k}_B)\cdot\vec{X}_{A_k}\right] r\, dr \\ & - \int_{r_{FA}}^1 \xi_k \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right) 2u_R\sin\theta \, dr \end{aligned}$$

$$M_{p_k\zeta} = \mu\cos\psi_m \, M_{p_k\mu} + \int_{r_{FA}}^1 \xi_k \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right)\mu\sin\psi_m \sin\theta \, dr$$

$$M_{p_k\lambda} = \int_{r_{FA}}^1 \left[\xi_k M_{a_P} - (F_{x_P}\vec{i}_B + F_{z_P}\vec{k}_B)\cdot\vec{X}_{A_k}\right] dr$$

$$\begin{aligned} M_{p_k\dot\beta} = {} & \int_{r_{FA}}^1 \left[\xi_k M_{a_P} - (F_{x_P}\vec{i}_B + F_{z_P}\vec{k}_B)\cdot\vec{X}_{A_k}\right] r\, dr \\ & + \int_{r_{FA}}^1 \xi_k \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right) 2u_R\cos\theta \, dr \end{aligned}$$

$$\begin{aligned} M_{p_k\beta} = {} & \mu\cos\psi_m \, M_{p_k\lambda} \\ & - \int_{r_{FA}}^1 \xi_k \frac{c^2}{32}\left(u_T\cos\theta + u_P\sin\theta\right)\left(1 + 8\frac{x_A}{c} + 16\left(\frac{x_A}{c}\right)^2\right) dr \\ & - \int_{r_{FA}}^1 \xi_k \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right)\mu\sin\psi_m\cos\theta \, dr \end{aligned}$$

$$M_{p\kappa \dot{q}_i} = \int_{r_{PA}}^{1} \left[\xi_\kappa M_{aT} - (F_{xT}\vec{i}_B + F_{zT}\vec{k}_B)\cdot\vec{X}_{A\kappa}\right]\vec{k}_B\cdot\vec{\eta}_i\, dr + \int_{r_{PA}}^{1} \left[\xi_\kappa M_{aP} - (F_{xP}\vec{i}_B + F_{zP}\vec{k}_B)\cdot\vec{X}_{A\kappa}\right]\vec{i}_B\cdot\vec{\eta}_i\, dr + \int_{r_{PA}}^{1} \xi_\kappa \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right) 2u_R\, \vec{i}\cdot\vec{\eta}_i{}'\, dr$$

$$M_{p\kappa q_i} = \mu\cos\psi_m \left\{ \int_{r_{PA}}^{1} \left[\xi_\kappa M_{aT} - (F_{xT}\vec{i}_B + F_{zT}\vec{k}_B)\cdot\vec{X}_{A\kappa}\right]\vec{k}_B\cdot\vec{\eta}_i{}''\, dr + \int_{r_{PA}}^{1} \left[\xi_\kappa M_{aP} - (F_{xP}\vec{i}_B + F_{zP}\vec{k}_B)\cdot\vec{X}_{A\kappa}\right]\vec{i}_B\cdot\vec{\eta}_i{}''\, dr \right\} - \int_{r_{PA}}^{1} \left(\frac{F_x}{\partial c}\vec{i}_B + \frac{F_z}{\partial c}\vec{k}_B\right)\cdot\vec{X}_{A\kappa q_i}\, dr - \int_{r_{PA}}^{1} \xi_\kappa \frac{c^2}{32}(u_T\cos\theta + u_P\sin\theta)\left(1 + 8\frac{x_A}{c} + 16\left(\frac{x_A}{c}\right)^2\right)\vec{i}_B\cdot\vec{\eta}_i{}'\, dr - \int_{r_{PA}}^{1} \xi_\kappa \frac{c^2}{32}\left(1 + 4\frac{x_A}{c}\right)\left(\mu\sin\psi\, \vec{i}\cdot\vec{\eta}_i{}' - (\mu\cos\psi)^2\, \vec{i}\cdot\vec{\eta}_i{}'''\right) dr$$

$$M_{p\kappa p_i} = \int_{r_{PA}}^{1} \left[\xi_\kappa M_{a\theta} - (F_{x\theta}\vec{i}_B + F_{z\theta}\vec{k}_B)\cdot\vec{X}_{A\kappa}\right]\xi_i\, dr - \int_{r_{PA}}^{1} \xi_\kappa \frac{c^2}{32} u_R\left[\xi_i'(u_T\cos\theta + u_P\sin\theta) + 2\xi_i\cos\theta\right]\left(1 + 4\frac{x_A}{c}\right) dr$$

$$M_{p\kappa \dot{p}_i} = -\int_{r_{PA}}^{1} \xi_\kappa \xi_i \frac{c^2}{16}(u_T\cos\theta + u_P\sin\theta)\left(1 + 6\frac{x_A}{c} + 8\left(\frac{x_A}{c}\right)^2\right) dr$$

where

$$\vec{X}_{A\kappa} = -\int_{r_{PA}}^{r} \xi_\kappa (z_0\vec{i} - x_0\vec{k})''(r-\rho)\, d\rho$$

$$\vec{X}_{A0} = -(z_0\vec{i} - x_0\vec{k}) + (\delta_2\vec{i}_B + \delta_3\vec{k}_B)(r - r_{PA}) + (z_0\vec{i} - x_0\vec{k})\big|_{r_{PA}} + (z_0\vec{i} - x_0\vec{k})'\big|_{r_{PA}}(r - r_{PA})$$

and

$$\vec{X}_{A\kappa q_i} = -\int_{r_{PA}}^{r} \xi_\kappa \vec{\eta}_i{}''(r-\rho)\, d\rho$$

$$\vec{X}_{A0 q_i} = -\vec{\eta}_i + \vec{\eta}_i(r_{PA}) + \vec{\eta}_i{}'(r_{PA})(r - r_{PA})$$

Rotor Aerodynamic Forces -- Nonrotating Frame

The aerodynamic forcing functions for the rotor equations of motion in the nonrotating frame, and the hub forces and moments are now required. These are obtained by summing the blade rotating forces (given above) over all N blades. The Fourier coordinate transform of the rotor degrees of freedom is introduced as required.

Axial Flow

First consider the case of axial flow, $\mu = 0$; for either high inflow ratio λ (order 1, i.e. proprotor cruise flight), or low inflow (small λ, i.e. hover in helicopter mode). In this case the aerodynamic coefficients in the blade forces are constant, independent of ψ_m. The coefficients are also independent of m (the blade index) then, so the summation over N blades operates only on the blade degrees of freedom and shaft motion variables. The result for the required aerodynamic forces, in matrix form, is

$$-M_{aero} = A_1 \dot{x}_R + A_0 x_R + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha - B_G g$$
$$F_{aero} = C_1 \dot{x}_R + C_0 x_R + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + D_G g$$

where the rotor degrees of freedom ($\vec{x}_R$), shaft motion ($\vec{\alpha}$), and aerodynamic gust input ($\vec{g}$) are:

$$\vec{x}_R = \begin{bmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{1s} \\ \theta_0 \\ \theta_{1c} \\ \theta_{1s} \\ \beta_{Gc} \\ \beta_{Gs} \\ \psi_s \end{bmatrix} \qquad \vec{\alpha} = \begin{bmatrix} x_h \\ y_h \\ z_h \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} u_G \\ v_G \\ w_G \end{bmatrix}$$

These coefficients simply add to the inertial coefficients already derived, to complete the equations of motion. The matrices of the aerodynamic coefficients follow.

$$A_1 = \begin{bmatrix}
-\gamma M_{q_k \dot{q}_i} & & & & & & & & -\gamma M_{q_k \dot{\xi}} \\
& -\gamma M_{q_k \dot{q}_i} & & & & & -\gamma M_{q_k \dot{\beta}} & & \\
& & -\gamma M_{q_k \dot{q}_i} & & & & & -\gamma M_{q_k \dot{\beta}} & \\
-\gamma M_{p_k \dot{q}_i} & & & -\gamma M_{p_k \dot{p}_i} & & & & & -\gamma M_{p_k \dot{\xi}} \\
& -\gamma M_{p_k \dot{q}_i} & & & -\gamma M_{p_k \dot{p}_i} & & -\gamma M_{p_k \dot{\beta}} & & \\
& & -\gamma M_{p_k \dot{q}_i} & & & -\gamma M_{p_k \dot{p}_i} & & -\gamma M_{p_k \dot{\beta}} & \\
& -\gamma M_{\dot{q}_i} & & & & & -\gamma M_{\dot{\beta}} & & \\
& & -\gamma M_{\dot{q}_i} & & & & & -\gamma M_{\dot{\beta}} &
\end{bmatrix}$$

$$
A_0 =
\begin{bmatrix}
-\gamma M_{q\kappa q}: & & & -\gamma M_{q\kappa p}: & & & & & -\gamma M_{q\kappa \zeta} \\
 & -\gamma M_{q\kappa q}: & -\gamma M_{q\kappa \dot q}: & & -\gamma M_{q\kappa p}: & & -\gamma M_{q\kappa \beta} & -\gamma M_{q\kappa \dot\beta} & \\
 & \gamma M_{q\kappa \dot q}: & -\gamma M_{q\kappa q}: & & & -\gamma M_{q\kappa p}: & \gamma M_{q\kappa \dot\beta} & -\gamma M_{q\kappa \beta} & \\
-\gamma M_{p\kappa q}: & & & -\gamma M_{p\kappa p}: & & & & & -\gamma M_{p\kappa \zeta} \\
 & -\gamma M_{p\kappa q}: & -\gamma M_{p\kappa \dot q}: & & -\gamma M_{p\kappa p}: & -\gamma M_{p\kappa \dot p}: & -\gamma M_{p\kappa \beta} & -\gamma M_{p\kappa \dot\beta} & \\
 & \gamma M_{p\kappa \dot q}: & -\gamma M_{p\kappa q}: & & \gamma M_{p\kappa \dot p}: & -\gamma M_{p\kappa p}: & \gamma M_{p\kappa \dot\beta} & -\gamma M_{p\kappa \beta} & \\
 & -\gamma M_{q}: & -\gamma M_{\dot q}: & & -\gamma M_{p}: & & -\gamma M_{\beta} & -\gamma M_{\dot\beta} & \\
 & \gamma M_{\dot q}: & -\gamma M_{q}: & & & -\gamma M_{p}: & \gamma M_{\dot\beta} & -\gamma M_{\beta} &
\end{bmatrix}
$$

$$
\tilde{A}_1 =
\begin{bmatrix}
 & & -\gamma M_{q\kappa\lambda} & & & -\gamma M_{q\kappa\dot{\xi}} \\
 & -\gamma M_{q\kappa\mu} & & & \gamma M_{q\kappa\dot{\beta}} & \\
\gamma M_{q\kappa\mu} & & & -\gamma M_{q\kappa\dot{\beta}} & & \\
 & & -\gamma M_{p\kappa\lambda} & & & -\gamma M_{p\kappa\dot{\xi}} \\
 & -\gamma M_{p\kappa\mu} & & & \gamma M_{p\kappa\dot{\beta}} & \\
\gamma M_{p\kappa\mu} & & & -\gamma M_{p\kappa\dot{\beta}} & & \\
 & -\gamma M_{\mu} & & & \gamma M_{\dot{\beta}} & \\
\gamma M_{\mu} & & & -\gamma M_{\dot{\beta}} & &
\end{bmatrix}
$$

$$\tilde{A}_0 =$$

$$\begin{bmatrix}
 & & & & \gamma_\mu M_{q\times\lambda} & -\gamma M_{q\times\zeta} \\
 & & & -\gamma\lambda M_{q\times\mu} & & \\
 & & & & -\gamma\lambda M_{q\times\mu} & \\
 & & & & \gamma_\mu M_{p\times\lambda} & -\gamma M_{p\times\zeta} \\
 & & & -\gamma\lambda M_{p\times\mu} & & \\
 & & & & -\gamma\lambda M_{p\times\mu} & \\
 & & & -\gamma\lambda M_\mu & & \\
 & & & & -\gamma\lambda M_\mu &
\end{bmatrix}$$

$$
B_G = \left[\begin{array}{c|c|c}
\gamma \sin\alpha \, M_{q\kappa\lambda} & & -\gamma \cos\alpha \, M_{q\kappa\lambda} \\
\hline
 & \gamma M_{q\kappa\mu} & \\
\hline
\gamma \cos\alpha \, M_{q\kappa\mu} & & \gamma \sin\alpha \, M_{q\kappa\mu} \\
\hline
\gamma \sin\alpha \, M_{p\kappa\lambda} & & -\gamma \cos\alpha \, M_{p\kappa\lambda} \\
\hline
 & \gamma M_{p\kappa\mu} & \\
\hline
\gamma \cos\alpha \, M_{p\kappa\mu} & & \gamma \sin\alpha \, M_{p\kappa\mu} \\
\hline
 & \gamma M_{\mu} & \\
\hline
\gamma \cos\alpha \, M_{\mu} & & \gamma \sin\alpha \, M_{\mu}
\end{array}\right]
$$

$$
C_1 = \begin{bmatrix}
\gamma T_{\dot q_i} & & & & & & & & \gamma T_{\dot \zeta} \\
 & \gamma R_{\dot q_i} & \gamma H_{\dot q_i} & & & & \gamma R_{\dot \beta} & \gamma H_{\dot \beta} & \\
 & \gamma H_{\dot q_i} & -\gamma R_{\dot q_i} & & & & \gamma H_{\dot \beta} & -\gamma R_{\dot \beta} & \\
\gamma Q_{\dot q_i} & & & & & & & & \gamma Q_{\dot \zeta} \\
 & -\gamma M_{\dot q_i} & & & & & -\gamma M_{\dot \beta} & & \\
 & & -\gamma M_{\dot q_i} & & & & & -\gamma M_{\dot \beta} &
\end{bmatrix}
$$

$$C_0 = \begin{bmatrix} \gamma T_{q_i} & & & \gamma T_{p_i} & & & & & \gamma T_5 \\ & \begin{matrix}\gamma R_{q_i} \\ -\gamma H_{\dot q_i}\end{matrix} & \begin{matrix}\gamma H_{q_i} \\ +\gamma R_{\dot q_i}\end{matrix} & & \gamma R_{p_i} & \gamma H_{p_i} & \begin{matrix}\gamma R_\beta \\ -\gamma H_{\dot\beta}\end{matrix} & \begin{matrix}\gamma H_\beta \\ +\gamma R_{\dot\beta}\end{matrix} & \\ & \begin{matrix}\gamma H_{q_i} \\ +\gamma R_{\dot q_i}\end{matrix} & \begin{matrix}-\gamma R_{q_i} \\ +\gamma H_{\dot q_i}\end{matrix} & & \gamma H_{p_i} & -\gamma R_{p_i} & \begin{matrix}\gamma H_\beta \\ +\gamma R_{\dot\beta}\end{matrix} & \begin{matrix}-\gamma R_\beta \\ +\gamma H_{\dot\beta}\end{matrix} & \\ \gamma Q_{q_i} & & & \gamma Q_{p_i} & & & & & \gamma Q_5 \\ & -\gamma M_{q_i} & -\gamma M_{\dot q_i} & & -\gamma M_{p_i} & & -\gamma M_\beta & -\gamma M_{\dot\beta} & \\ & \gamma M_{\dot q_i} & -\gamma M_{q_i} & & & -\gamma M_{p_i} & \gamma M_{\dot\beta} & -\gamma M_\beta & \end{bmatrix}$$

$$\tilde{C}_1 = \begin{bmatrix} & & \gamma T_\lambda & & & \gamma T_{\dot\xi} \\ -\gamma(H_\mu + R_\mu) & \gamma R_r & & \gamma H_{\dot\beta} & -\gamma R_{\dot\beta} & \\ \gamma R_r & \gamma(H_\mu + R_\mu) & & -\gamma R_{\dot\beta} & -\gamma H_{\dot\beta} & \\ & & \gamma Q_\lambda & & & \gamma Q_{\dot\xi} \\ & -\gamma M_\mu & & & \gamma M_{\dot\beta} & \\ \gamma M_\mu & & & -\gamma M_{\dot\beta} & & \end{bmatrix}$$

$$\tilde{C}_o =$$

$$\left[\begin{array}{c|c|c|c|c|c}
 & & & & -\gamma_\mu T_\lambda & \gamma T_5 \\ \hline
 & & & \gamma\lambda R_r & \gamma\lambda(H_\mu+R_\mu) & \\ \hline
 & & & \gamma\lambda(H_\mu+R_\mu) & -\gamma\lambda R_r & \\ \hline
 & & & & -\gamma_\mu Q_\lambda & \gamma Q_5 \\ \hline
 & & & -\gamma\lambda M_\mu & & \\ \hline
 & & & & -\gamma\lambda M_\mu &
\end{array}\right]$$

$$D_\Theta = \begin{bmatrix} \gamma \sin\alpha \, T_\lambda & & -\gamma \cos\alpha \, T_\lambda \\ \gamma \cos\alpha \, (H_\mu + R_\mu) & \gamma R_r & \gamma \sin\alpha \, (H_\mu + R_\mu) \\ -\gamma \cos\alpha \, R_r & \gamma (H_\mu + R_\mu) & -\gamma \sin\alpha \, R_r \\ \gamma \sin\alpha \, Q_\lambda & & -\gamma \cos\alpha \, Q_\lambda \\ & -\gamma M_\mu & \\ -\gamma \cos\alpha \, M_\mu & & -\gamma \sin\alpha \, M_\mu \end{bmatrix}$$

Nonaxial flow

Consider now the case of nonaxial flow, $\mu > 0$. This case includes helicopter mode forward flight, and conversion mode flight for the tilting proprotor aircraft. The aerodynamic coefficients are then periodic functions of ψ_m. Hence the equations of motion for the system have periodic coefficients, due to the periodically varying aerodynamics of the edgewise moving rotor.

One can express the aerodynamic coefficients as Fourier series, and then obtain the coefficients of the nonrotating equations of motion in terms of these harmonics. For the general rotor considered here, it would be necessary to evaluate the harmonics of the aerodynamic coefficients numerically, however. It is simplest therefore to just sum (numerically) the coefficients over m = 1...N as is required in finding the nonrotating equations of motion and the net hub forces and moments. The nonrotating coordinates for the rotor motion (Fourier coordinate transformation) are also introduced.

For the periodic coefficient case, it is necessary to specify the number of blades N, since the periodic coefficients depend on N; also, the periodic coefficients couple all the rotor nonrotating degrees of freedom, so more than the 0, 1C, and 1S variables are involved with the shaft motion (if $N > 3$). We shall consider only the case $N = 3$; then the 0, 1C, and 1S degrees of freedom are the complete set, even for the periodic coefficient case. The period of the equations in the nonrotating frame is $\Delta\psi = 2\pi/N$.

Again we write the aerodynamic forces in matrix form, as

$$-M_{aero} = A_1 \dot{x}_R + A_0 x_R + \tilde{A}_1 \dot{\alpha} + \tilde{A}_0 \alpha - B_G g$$

$$F_{aero} = C_1 \dot{x}_R + C_0 x_R + \tilde{C}_1 \dot{\alpha} + \tilde{C}_0 \alpha + D_G g$$

where now the coefficients A, B, C, and D are periodic functions of ψ (period $2\pi/N$). The matrices of the aerodynamic coefficients follow. The notation

$$C = \cos \psi_m$$

$$S = \sin \psi_m$$

is used ($\psi_m = \psi + m \Delta\psi$). Note that each matrix is a summation over all N blades ($N = 3$ in this case).

$$A_1 = \gamma \frac{1}{N} \sum_m$$

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c}
-M_{q\kappa\dot q_i} & & & & & & -M_{q\kappa\dot\beta}C & & \\
& -M_{q\kappa\dot q_i}C & & & & & & -M_{q\kappa\dot\beta}S & \\
& & -M_{q\kappa\dot q_i}S & & & & & & -M_{q\kappa\dot\zeta} \\
\hline
-M_{q\kappa\dot q_i}2C & & & & & & -M_{q\kappa\dot\beta}2C^2 & & \\
& -M_{q\kappa\dot q_i}2C^2 & & & & & & -M_{q\kappa\dot\beta}2CS & \\
& & -M_{q\kappa\dot q_i}2CS & & & & & & -M_{q\kappa\dot\zeta}2C \\
\hline
-M_{q\kappa\dot q_i}2S & & & & & & -M_{q\kappa\dot\beta}2CS & & \\
& -M_{q\kappa\dot q_i}2CS & & & & & & -M_{q\kappa\dot\beta}2S^2 & \\
& & -M_{q\kappa\dot q_i}2S^2 & & & & & & -M_{q\kappa\dot\zeta}2S \\
\hline
-M_{p\kappa\dot q_i} & & & -M_{p\kappa\dot p_i} & & & -M_{p\kappa\dot\beta}C & & \\
& -M_{p\kappa\dot q_i}C & & & -M_{p\kappa\dot p_i}C & & & -M_{p\kappa\dot\beta}S & \\
& & -M_{p\kappa\dot q_i}S & & & -M_{p\kappa\dot p_i}S & & & -M_{p\kappa\dot\zeta} \\
\hline
-M_{p\kappa\dot q_i}2C & & & -M_{p\kappa\dot p_i}2C & & & -M_{p\kappa\dot\beta}2C^2 & & \\
& -M_{p\kappa\dot q_i}2C^2 & & & -M_{p\kappa\dot p_i}2C^2 & & & -M_{p\kappa\dot\beta}2CS & \\
& & -M_{p\kappa\dot q_i}2CS & & & -M_{p\kappa\dot p_i}2CS & & & -M_{p\kappa\dot\zeta}2C \\
\hline
-M_{p\kappa\dot q_i}2S & & & -M_{p\kappa\dot p_i}2S & & & -M_{p\kappa\dot\beta}2CS & & \\
& -M_{p\kappa\dot q_i}2CS & & & -M_{p\kappa\dot p_i}2CS & & & -M_{p\kappa\dot\beta}2S^2 & \\
& & -M_{p\kappa\dot q_i}2S^2 & & & -M_{p\kappa\dot p_i}2S^2 & & & -M_{p\kappa\dot\zeta}2S \\
\hline
-M_{\dot q_i}2C & & & & & & -M_{\dot\beta}2C^2 & & \\
& -M_{\dot q_i}2C^2 & & & & & & -M_{\dot\beta}2CS & \\
& & -M_{\dot q_i}2CS & & & & & & -M_{\dot\zeta}2C \\
\hline
-M_{\dot q_i}2S & & & & & & -M_{\dot\beta}2CS & & \\
& -M_{\dot q_i}2CS & & & & & & -M_{\dot\beta}2S^2 & \\
& & -M_{\dot q_i}2S^2 & & & & & & -M_{\dot\zeta}2S
\end{array}\right]$$

$$A_0 = \gamma \frac{1}{N} \sum_m$$

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c}
-M_{q_k q_i} & \begin{matrix} M_{q_k \dot q_i} S \\ -M_{q_k q_i} C \end{matrix} & \begin{matrix} -M_{q_k \dot q_i} C \\ -M_{q_k q_i} S \end{matrix} & -M_{q_k p_i} & -M_{q_k p_i} C & -M_{q_k p_i} S & \begin{matrix} M_{q_k \dot p} S \\ -M_{q_k p} C \end{matrix} & \begin{matrix} -M_{q_k \dot p} C \\ -M_{q_k p} S \end{matrix} & -M_{q_k \zeta} \\
\hline
-M_{q_k q_i} 2C & \begin{matrix} M_{q_k \dot q_i} 2CS \\ -M_{q_k q_i} 2C^2 \end{matrix} & \begin{matrix} -M_{q_k \dot q_i} 2C^2 \\ -M_{q_k q_i} 2CS \end{matrix} & -M_{q_k p_i} 2C & -M_{q_k p_i} 2C^2 & -M_{q_k p_i} 2CS & \begin{matrix} M_{q_k \dot p} 2CS \\ -M_{q_k p} 2C^2 \end{matrix} & \begin{matrix} -M_{q_k \dot p} 2C^2 \\ -M_{q_k p} 2CS \end{matrix} & -M_{q_k \zeta} 2C \\
\hline
-M_{q_k q_i} 2S & \begin{matrix} M_{q_k \dot q_i} 2S^2 \\ -M_{q_k q_i} 2CS \end{matrix} & \begin{matrix} -M_{q_k \dot q_i} 2CS \\ -M_{q_k q_i} 2S^2 \end{matrix} & -M_{q_k p_i} 2S & -M_{q_k p_i} 2CS & -M_{q_k p_i} 2S^2 & \begin{matrix} M_{q_k \dot p} 2S^2 \\ -M_{q_k p} 2CS \end{matrix} & \begin{matrix} -M_{q_k \dot p} 2CS \\ -M_{q_k p} 2S^2 \end{matrix} & -M_{q_k \zeta} 2S \\
\hline
-M_{p_k q_i} & \begin{matrix} +M_{p_k \dot q_i} S \\ -M_{p_k q_i} C \end{matrix} & \begin{matrix} -M_{p_k \dot q_i} C \\ -M_{p_k q_i} S \end{matrix} & -M_{p_k p_i} & \begin{matrix} -M_{p_k p_i} C \\ M_{p_k \dot p_i} S \end{matrix} & \begin{matrix} -M_{p_k p_i} S \\ -M_{p_k \dot p_i} C \end{matrix} & \begin{matrix} +M_{p_k \dot p} S \\ -M_{p_k p} C \end{matrix} & \begin{matrix} -M_{p_k \dot p} C \\ -M_{p_k p} S \end{matrix} & -M_{p_k \zeta} \\
\hline
-M_{p_k q_i} 2C & \begin{matrix} +M_{p_k \dot q_i} 2CS \\ -M_{p_k q_i} 2C^2 \end{matrix} & \begin{matrix} -M_{p_k \dot q_i} 2C^2 \\ -M_{p_k q_i} 2CS \end{matrix} & -M_{p_k p} 2C & \begin{matrix} -M_{p_k p_i} 2C^2 \\ M_{p_k \dot p_i} 2CS \end{matrix} & \begin{matrix} -M_{p_k p_i} 2CS \\ -M_{p_k \dot p_i} 2C^2 \end{matrix} & \begin{matrix} +M_{p_k \dot p} 2CS \\ -M_{p_k p} 2C^2 \end{matrix} & \begin{matrix} -M_{p_k \dot p} 2C^2 \\ -M_{p_k p} 2CS \end{matrix} & -M_{p_k \zeta} 2C \\
\hline
-M_{p_k q_i} 2S & \begin{matrix} +M_{p_k \dot q_i} 2S^2 \\ -M_{p_k q_i} 2CS \end{matrix} & \begin{matrix} -M_{p_k \dot q_i} 2CS \\ -M_{p_k q_i} 2S^2 \end{matrix} & -M_{p_k p_i} 2S & \begin{matrix} -M_{p_k p_i} 2CS \\ M_{p_k \dot p} 2S^2 \end{matrix} & \begin{matrix} -M_{p_k p_i} 2S^2 \\ -M_{p_k \dot p_i} 2CS \end{matrix} & \begin{matrix} +M_{p_k \dot p} 2S^2 \\ -M_{p_k p} 2CS \end{matrix} & \begin{matrix} -M_{p_k \dot p} 2CS \\ -M_{p_k p} 2S^2 \end{matrix} & -M_{p_k \zeta} 2S \\
\hline
-M_{q_i} 2C & \begin{matrix} +M_{\dot q_i} 2CS \\ -M_{q_i} 2C^2 \end{matrix} & \begin{matrix} -M_{\dot q_i} 2C^2 \\ -M_{q_i} 2CS \end{matrix} & -M_{p_i} 2C & -M_{p_i} 2C^2 & -M_{p_i} 2CS & \begin{matrix} +M_{\dot p} 2CS \\ -M_{p} 2C^2 \end{matrix} & \begin{matrix} -M_{\dot p} 2C^2 \\ -M_{p} 2CS \end{matrix} & -M_{\zeta} 2C \\
\hline
-M_{q_i} 2S & \begin{matrix} +M_{\dot q_i} 2S^2 \\ -M_{q_i} 2CS \end{matrix} & \begin{matrix} -M_{\dot q_i} 2CS \\ -M_{q_i} 2S^2 \end{matrix} & -M_{p_i} 2S & -M_{p_i} 2CS & -M_{p_i} 2S^2 & \begin{matrix} +M_{\dot p} 2S^2 \\ -M_{p} 2CS \end{matrix} & \begin{matrix} -M_{\dot p} 2CS \\ -M_{p} 2S^2 \end{matrix} & -M_{\zeta} 2S
\end{array}\right]$$

$$\tilde{A}_1 = \gamma \frac{1}{N} \sum_{m}$$

$$\begin{bmatrix}
M_{q\kappa\mu} S & -M_{q\kappa\mu} C & -M_{q\kappa\lambda} & -M_{q\kappa\dot{\beta}} S & M_{q\kappa\dot{\beta}} C & -M_{q\kappa\dot{\xi}} \\
M_{q\kappa\mu} 2CS & -M_{q\kappa\mu} 2C^2 & -M_{q\kappa\lambda} 2C & -M_{q\kappa\dot{\beta}} 2CS & M_{q\kappa\dot{\beta}} 2C^2 & -M_{q\kappa\dot{\xi}} 2C \\
M_{q\kappa\mu} 2S^2 & -M_{q\kappa\mu} 2CS & -M_{q\kappa\lambda} 2S & -M_{q\kappa\dot{\beta}} 2S^2 & M_{q\kappa\dot{\beta}} 2CS & -M_{q\kappa\dot{\xi}} 2S \\
M_{p\kappa\mu} S & -M_{p\kappa\mu} C & -M_{p\kappa\lambda} & -M_{p\kappa\dot{\beta}} S & M_{p\kappa\dot{\beta}} C & -M_{p\kappa\dot{\xi}} \\
M_{p\kappa\mu} 2CS & -M_{p\kappa\mu} 2C^2 & -M_{p\kappa\lambda} 2C & -M_{p\kappa\dot{\beta}} 2CS & M_{p\kappa\dot{\beta}} 2C^2 & -M_{p\kappa\dot{\xi}} 2C \\
M_{p\kappa\mu} 2S^2 & -M_{p\kappa\mu} 2CS & -M_{p\kappa\lambda} 2S & -M_{p\kappa\dot{\beta}} 2S^2 & M_{p\kappa\dot{\beta}} 2CS & -M_{p\kappa\dot{\xi}} 2S \\
M_{\mu} 2CS & -M_{\mu} 2C^2 & -M_{\lambda} 2C & -M_{\dot{\beta}} 2CS & M_{\dot{\beta}} 2C^2 & -M_{\dot{\xi}} 2C \\
M_{\mu} 2S^2 & -M_{\mu} 2CS & -M_{\lambda} 2S & -M_{\dot{\beta}} 2S^2 & M_{\dot{\beta}} 2CS & -M_{\dot{\xi}} 2S
\end{bmatrix}$$

$$\tilde{A}_o = \gamma \frac{1}{2} \sum_m$$

$$\left[\begin{array}{c|c|c|c|c|c}
 & & & -\lambda M_{q\kappa\mu} C & \begin{matrix} \mu M_{q\kappa\lambda} \\ -\lambda M_{q\kappa\mu} S \end{matrix} & -M_{q\kappa\xi} \\ \hline
 & & & -\lambda M_{q\kappa\mu} 2C^2 & \begin{matrix} \mu M_{q\kappa\lambda} 2C \\ -\lambda M_{q\kappa\mu} 2CS \end{matrix} & -M_{q\kappa\xi} 2C \\ \hline
 & & & -\lambda M_{q\kappa\mu} 2CS & \begin{matrix} \mu M_{q\kappa\lambda} 2S \\ -\lambda M_{q\kappa\mu} 2S^2 \end{matrix} & -M_{q\kappa\xi} 2S \\ \hline
 & & & -\lambda M_{p\kappa\mu} C & \begin{matrix} \mu M_{p\kappa\lambda} \\ -\lambda M_{p\kappa\mu} S \end{matrix} & -M_{p\kappa\xi} \\ \hline
 & & & -\lambda M_{p\kappa\mu} 2C^2 & \begin{matrix} \mu M_{p\kappa\lambda} 2C \\ -\lambda M_{p\kappa\mu} 2CS \end{matrix} & -M_{p\kappa\xi} 2C \\ \hline
 & & & -\lambda M_{p\kappa\mu} 2CS & \begin{matrix} \mu M_{p\kappa\lambda} 2S \\ -\lambda M_{p\kappa\mu} 2S^2 \end{matrix} & -M_{p\kappa\xi} 2S \\ \hline
 & & & -\lambda M_{\mu} 2C^2 & \begin{matrix} \mu M_{\lambda} 2C \\ -\lambda M_{\mu} 2CS \end{matrix} & -M_{\xi} 2C \\ \hline
 & & & -\lambda M_{\mu} 2CS & \begin{matrix} \mu M_{\lambda} 2S \\ -\lambda M_{\mu} 2S^2 \end{matrix} & -M_{\xi} 2S
\end{array}\right]$$

$$B_G = \gamma \frac{1}{N} \sum_m$$

$$\left[\begin{array}{c|c|c}
\begin{matrix}\cos\alpha\, M_{q\kappa\mu} S \\ \sin\alpha\, M_{q\kappa\lambda}\end{matrix} & M_{q\kappa\mu} C & \begin{matrix}\sin\alpha\, M_{q\kappa\mu} S \\ -\cos\alpha\, M_{q\kappa\lambda}\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{q\kappa\mu} 2CS \\ \sin\alpha\, M_{q\kappa\lambda} 2C\end{matrix} & M_{q\kappa\mu} 2C^2 & \begin{matrix}\sin\alpha\, M_{q\kappa\mu} 2CS \\ -\cos\alpha\, M_{q\kappa\lambda} 2C\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{q\kappa\mu} 2S^2 \\ \sin\alpha\, M_{q\kappa\lambda} 2S\end{matrix} & M_{q\kappa\mu} 2CS & \begin{matrix}\sin\alpha\, M_{q\kappa\mu} 2S^2 \\ -\cos\alpha\, M_{q\kappa\lambda} 2S\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{p\kappa\mu} S \\ \sin\alpha\, M_{p\kappa\lambda}\end{matrix} & M_{p\kappa\mu} C & \begin{matrix}\sin\alpha\, M_{p\kappa\mu} S \\ -\cos\alpha\, M_{p\kappa\lambda}\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{p\kappa\mu} 2CS \\ \sin\alpha\, M_{p\kappa\lambda} 2C\end{matrix} & M_{p\kappa\mu} 2C^2 & \begin{matrix}\sin\alpha\, M_{p\kappa\mu} 2CS \\ -\cos\alpha\, M_{p\kappa\lambda} 2C\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{p\kappa\mu} 2S^2 \\ \sin\alpha\, M_{p\kappa\lambda} 2S\end{matrix} & M_{p\kappa\mu} 2CS & \begin{matrix}\sin\alpha\, M_{p\kappa\mu} 2S^2 \\ -\cos\alpha\, M_{p\kappa\lambda} 2S\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{\mu} 2CS \\ \sin\alpha\, M_{\lambda} 2C\end{matrix} & M_{\mu} 2C^2 & \begin{matrix}\sin\alpha\, M_{\mu} 2CS \\ -\cos\alpha\, M_{\lambda} 2C\end{matrix} \\ \hline
\begin{matrix}\cos\alpha\, M_{\mu} 2S^2 \\ \sin\alpha\, M_{\lambda} 2S\end{matrix} & M_{\mu} 2CS & \begin{matrix}\sin\alpha\, M_{\mu} 2S^2 \\ -\cos\alpha\, M_{\lambda} 2S\end{matrix}
\end{array}\right]$$

$$
C_1 =
\left[\begin{array}{c|c|c|c|c|c|c|c|c}
T_{\dot q_i} & T_{\dot q_i} C & T_{\dot q_i} S & & & & T_{\dot\phi} C & T_{\dot\phi} S & T_{\dot\xi} \\
\hline
\begin{matrix} H_{\dot q_i} 2S \\ R_{\dot q_i} 2C \end{matrix} & \begin{matrix} H_{\dot q_i} 2CS \\ R_{\dot q_i} 2C^2 \end{matrix} & \begin{matrix} H_{\dot q_i} 2S^2 \\ R_{\dot q_i} 2CS \end{matrix} & & & & \begin{matrix} H_{\dot\phi} 2CS \\ R_{\dot\phi} 2C^2 \end{matrix} & \begin{matrix} H_{\dot\phi} 2S^2 \\ R_{\dot\phi} 2CS \end{matrix} & \begin{matrix} H_{\dot\xi} 2S \\ R_{\dot\xi} 2C \end{matrix} \\
\hline
\begin{matrix} H_{\dot q_i} 2C \\ -R_{\dot q_i} 2S \end{matrix} & \begin{matrix} H_{\dot q_i} 2C^2 \\ -R_{\dot q_i} 2CS \end{matrix} & \begin{matrix} H_{\dot q_i} 2CS \\ -R_{\dot q_i} 2S^2 \end{matrix} & & & & \begin{matrix} H_{\dot\phi} 2C^2 \\ -R_{\dot\phi} 2CS \end{matrix} & \begin{matrix} H_{\dot\phi} 2CS \\ -R_{\dot\phi} 2S^2 \end{matrix} & \begin{matrix} H_{\dot\xi} 2C \\ -R_{\dot\xi} 2S \end{matrix} \\
\hline
Q_{\dot q_i} & Q_{\dot q_i} C & Q_{\dot q_i} S & & & & Q_{\dot\phi} C & Q_{\dot\phi} S & Q_{\dot\xi} \\
\hline
-M_{\dot q_i} 2C & -M_{\dot q_i} 2C^2 & -M_{\dot q_i} 2CS & & & & -M_{\dot\phi} 2C^2 & -M_{\dot\phi} 2CS & -M_{\dot\xi} 2C \\
\hline
-M_{\dot q_i} 2S & -M_{\dot q_i} 2CS & -M_{\dot q_i} 2S^2 & & & & -M_{\dot\phi} 2CS & -M_{\dot\phi} 2S^2 & -M_{\dot\xi} 2S
\end{array}\right]
$$

$$C_0 = \gamma \frac{1}{N} \sum_{m}$$

$$\begin{bmatrix}
T_{q_i} & \begin{matrix} -T_{\dot q_i} S \\ T_{q_i} C \end{matrix} & \begin{matrix} T_{\dot q_i} C \\ T_{q_i} S \end{matrix} & T_{p_i} & T_{p_i} C & T_{p_i} S & \begin{matrix} -T_{\dot p} S \\ T_{p} C \end{matrix} & \begin{matrix} T_{\dot p} C \\ T_{p} S \end{matrix} & T_{\zeta} \\
\begin{matrix} H_{q_i} 2S \\ R_{q_i} 2C \end{matrix} & \begin{matrix} -H_{\dot q_i} 2S^2 \\ H_{q_i} 2CS \\ -R_{\dot q_i} 2CS \\ R_{q_i} 2C^2 \end{matrix} & \begin{matrix} H_{\dot q_i} 2CS \\ H_{q_i} 2S^2 \\ R_{\dot q_i} 2C^2 \\ R_{q_i} 2CS \end{matrix} & \begin{matrix} H_{p_i} 2S \\ R_{p_i} 2C \end{matrix} & \begin{matrix} H_{p_i} 2CS \\ R_{p_i} 2C^2 \end{matrix} & \begin{matrix} H_{p_i} 2S^2 \\ R_{p_i} 2CS \end{matrix} & \begin{matrix} -H_{\dot p} 2S^2 \\ H_{p} 2CS \\ -R_{\dot p} 2CS \\ R_{p} 2C^2 \end{matrix} & \begin{matrix} H_{\dot p} 2CS \\ H_{p} 2S^2 \\ R_{\dot p} 2C^2 \\ R_{p} 2CS \end{matrix} & \begin{matrix} H_{\zeta} 2S \\ R_{\zeta} 2C \end{matrix} \\
\begin{matrix} H_{q_i} 2C \\ -R_{q_i} 2S \end{matrix} & \begin{matrix} -H_{\dot q_i} 2CS \\ H_{q_i} 2C^2 \\ +R_{\dot q_i} 2S^2 \\ -R_{q_i} 2CS \end{matrix} & \begin{matrix} H_{\dot q_i} 2C^2 \\ H_{q_i} 2CS \\ -R_{\dot q_i} 2CS \\ -R_{q_i} 2S^2 \end{matrix} & \begin{matrix} H_{p_i} 2C \\ -R_{p_i} 2S \end{matrix} & \begin{matrix} H_{p_i} 2C^2 \\ -R_{p_i} 2CS \end{matrix} & \begin{matrix} H_{p_i} 2CS \\ -R_{p_i} 2S^2 \end{matrix} & \begin{matrix} -H_{\dot p} 2CS \\ H_{p} 2C^2 \\ +R_{\dot p} 2S^2 \\ -R_{p} 2CS \end{matrix} & \begin{matrix} H_{\dot p} 2C^2 \\ H_{p} 2CS \\ -R_{\dot p} 2CS \\ -R_{p} 2S^2 \end{matrix} & \begin{matrix} H_{\zeta} 2C \\ -R_{\zeta} 2S \end{matrix} \\
Q_{q_i} & \begin{matrix} -Q_{\dot q_i} S \\ Q_{q_i} C \end{matrix} & \begin{matrix} Q_{\dot q_i} C \\ Q_{q_i} S \end{matrix} & Q_{p_i} & Q_{p_i} C & Q_{p_i} S & \begin{matrix} -Q_{\dot p} S \\ Q_{p} C \end{matrix} & \begin{matrix} Q_{\dot p} C \\ Q_{p} S \end{matrix} & Q_{\zeta} \\
-M_{q_i} 2C & \begin{matrix} M_{\dot q_i} 2CS \\ -M_{q_i} 2C^2 \end{matrix} & \begin{matrix} -M_{\dot q_i} 2C^2 \\ -M_{q_i} 2CS \end{matrix} & -M_{p_i} 2C & -M_{p_i} 2C^2 & -M_{p_i} 2CS & \begin{matrix} M_{\dot p} 2CS \\ -M_{p} 2C^2 \end{matrix} & \begin{matrix} -M_{\dot p} 2C^2 \\ -M_{p} 2CS \end{matrix} & -M_{\zeta} 2C \\
-M_{q_i} 2S & \begin{matrix} M_{\dot q_i} 2S^2 \\ -M_{q_i} 2CS \end{matrix} & \begin{matrix} -M_{\dot q_i} 2CS \\ -M_{q_i} 2S^2 \end{matrix} & -M_{p_i} 2S & -M_{p_i} 2CS & -M_{p_i} 2S^2 & \begin{matrix} M_{\dot p} 2S^2 \\ -M_{p} 2CS \end{matrix} & \begin{matrix} -M_{\dot p} 2CS \\ -M_{p} 2S^2 \end{matrix} & -M_{\zeta} 2S
\end{bmatrix}$$

$$\tilde{C}_1 = \gamma \frac{1}{2} \underset{\sim}{\Sigma}$$

$$\left[\begin{array}{c|c|c|c|c|c}
-T_\mu S & T_\mu C & T_\lambda & T_{\dot\beta} S & -T_{\dot\beta} C & T_{\dot\xi} \\
\hline
\begin{matrix}-H_\mu 2S^2 \\ -R_\mu 2C^2 \\ -R_r 2CS\end{matrix} & \begin{matrix}H_\mu 2CS \\ -R_\mu 2CS \\ R_r 2C^2\end{matrix} & \begin{matrix}H_\lambda 2S \\ R_\lambda 2C\end{matrix} & \begin{matrix}H_{\dot\beta} 2S^2 \\ R_{\dot\beta} 2CS\end{matrix} & \begin{matrix}-H_{\dot\beta} 2CS \\ -R_{\dot\beta} 2C^2\end{matrix} & \begin{matrix}H_{\dot\xi} 2S \\ R_{\dot\xi} 2C\end{matrix} \\
\hline
\begin{matrix}-H_\mu 2CS \\ R_\mu 2CS \\ R_r 2S^2\end{matrix} & \begin{matrix}H_\mu 2C^2 \\ R_\mu 2S^2 \\ -R_r 2CS\end{matrix} & \begin{matrix}H_\lambda 2C \\ -R_\lambda 2S\end{matrix} & \begin{matrix}H_{\dot\beta} 2CS \\ -R_{\dot\beta} 2S^2\end{matrix} & \begin{matrix}-H_{\dot\beta} 2C^2 \\ R_{\dot\beta} 2CS\end{matrix} & \begin{matrix}H_{\dot\xi} 2C \\ -R_{\dot\xi} 2S\end{matrix} \\
\hline
-Q_\mu S & Q_\mu C & Q_\lambda & Q_{\dot\beta} S & -Q_{\dot\beta} C & Q_{\dot\xi} \\
\hline
M_\mu 2CS & -M_\mu 2C^2 & -M_\lambda 2C & -M_{\dot\beta} 2CS & +M_{\dot\beta} 2C^2 & -M_{\dot\xi} 2C \\
\hline
M_\mu 2S^2 & -M_\mu 2CS & -M_\lambda 2S & -M_{\dot\beta} 2S^2 & M_{\dot\beta} 2CS & -M_{\dot\xi} 2S
\end{array}\right]$$

$$\tilde{C}_0 = \gamma \frac{1}{2} \sum_m$$

$$\left[\begin{array}{c|c|c|c|c|c}
 & & & \lambda T_\mu C & \begin{matrix} -\mu T_\lambda \\ \lambda T_\mu S \end{matrix} & T_\xi \\
\hline
 & & & \begin{matrix} \lambda H_\mu 2CS \\ -\lambda R_\mu 2CS \\ \lambda R_r 2C^2 \end{matrix} & \begin{matrix} \lambda R_\mu 2C^2 \\ -\mu H_\lambda 2S \\ \lambda H_\mu 2S^2 \\ -\mu R_\lambda 2C \\ \lambda R_r 2CS \end{matrix} & \begin{matrix} H_\xi 2S \\ R_\xi 2C \end{matrix} \\
\hline
 & & & \begin{matrix} \lambda H_\mu 2C^2 \\ \lambda R_\mu 2S^2 \\ -\lambda R_r 2CS \end{matrix} & \begin{matrix} -\lambda R_\mu 2CS \\ -\mu H_\lambda 2C \\ \lambda H_\mu 2CS \\ \mu R_\lambda 2S \\ -\lambda R_r 2S^2 \end{matrix} & \begin{matrix} H_\xi 2C \\ -R_\xi 2S \end{matrix} \\
\hline
 & & & \lambda Q_\mu C & \begin{matrix} -\mu Q_\lambda \\ \lambda Q_\mu S \end{matrix} & Q_\xi \\
\hline
 & & & -\lambda M_\mu 2C^2 & \begin{matrix} \mu M_\lambda 2C \\ -\lambda M_\mu 2CS \end{matrix} & -M_\xi 2C \\
\hline
 & & & -\lambda M_\mu 2CS & \begin{matrix} \mu M_\lambda 2S \\ -\lambda M_\mu 2S^2 \end{matrix} & -M_\xi 2S
\end{array}\right]$$

$$D_\sigma = \gamma \frac{1}{N} \sum_{m}$$

$$\left[\begin{array}{c|c|c}
\begin{matrix} \cos\alpha \, T_\mu S \\ \\ \sin\alpha \, T_\lambda \end{matrix} & T_\mu C & \begin{matrix} \sin\alpha \, T_\mu S \\ \\ -\cos\alpha \, T_\lambda \end{matrix} \\
\hline
\begin{matrix} \cos\alpha \, H_\mu 2S^2 \\ \cos\alpha \, R_\mu 2C^2 \\ \sin\alpha \, H_\lambda 2S \\ \cos\alpha \, R_r 2CS \\ \sin\alpha \, R_\lambda 2C \end{matrix} & \begin{matrix} H_\mu 2CS \\ R_r 2C^2 \\ -R_\mu 2CS \end{matrix} & \begin{matrix} \sin\alpha \, H_\mu 2S^2 \\ \sin\alpha \, R_\mu 2C^2 \\ -\cos\alpha \, H_\lambda 2S \\ \sin\alpha \, R_r 2CS \\ -\cos\alpha \, R_\lambda 2C \end{matrix} \\
\hline
\begin{matrix} \cos\alpha \, H_\mu 2CS \\ -\cos\alpha \, R_\mu 2CS \\ \sin\alpha \, H_\lambda 2C \\ -\cos\alpha \, R_r 2S^2 \\ -\sin\alpha \, R_\lambda 2S \end{matrix} & \begin{matrix} H_\mu 2C^2 \\ R_\mu 2S^2 \\ -R_r 2CS \end{matrix} & \begin{matrix} \sin\alpha \, H_\mu 2CS \\ -\sin\alpha \, R_\mu 2CS \\ -\cos\alpha \, H_\lambda 2C \\ -\sin\alpha \, R_r 2S^2 \\ \cos\alpha \, R_\lambda 2S \end{matrix} \\
\hline
\begin{matrix} \cos\alpha \, Q_\mu S \\ \\ \sin\alpha \, Q_\lambda \end{matrix} & Q_\mu C & \begin{matrix} \sin\alpha \, Q_\mu S \\ \\ -\cos\alpha \, Q_\lambda \end{matrix} \\
\hline
\begin{matrix} -\cos\alpha \, M_\mu 2CS \\ \\ -\sin\alpha \, M_\lambda 2C \end{matrix} & -M_\mu 2C^2 & \begin{matrix} -\sin\alpha \, M_\mu 2CS \\ \\ \cos\alpha \, M_\lambda 2C \end{matrix} \\
\hline
\begin{matrix} -\cos\alpha \, M_\mu 2S^2 \\ \\ -\sin\alpha \, M_\lambda 2S \end{matrix} & -M_\mu 2CS & \begin{matrix} -\sin\alpha \, M_\mu 2S^2 \\ \\ \cos\alpha \, M_\lambda 2C \end{matrix}
\end{array}\right]$$

Constant Coefficient Approximation

Finally, we consider a constant coefficient approximation for the nonaxial flow case. This approximation uses the mean values of the periodic coefficients of the differential equations. A constant coefficient approximation is desirable (if it is demonstrated to be accurate enough) because the calculation required for the analysis is considerably reduced compared to the periodic coefficient equations, and because the powerful techniques for analyzing time-invariant (constant coefficient) linear differential equations are applicable. It is only an approximation to the correct dynamics however; the accuracy of the approximation must be determined by comparison with the correct periodic coefficient solutions.

To find the mean value of the coefficients, we apply the operator

$$\frac{1}{2\pi}\int_0^{2\pi}(\ldots)d\psi$$

to the periodic coefficients given above. The result is terms of the form

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{N}\sum_m \begin{matrix}1\\ \cos\psi_m\\ \sin\psi_m\\ 2\cos^2\psi_m\\ 2\sin^2\psi_m\\ 2\sin\psi_m\cos\psi_m\end{matrix}\, M(\psi_m)\right)d\psi$$

$$=\frac{1}{N}\sum_m \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}1\\ c\\ s\\ 2c^2\\ 2s^2\\ 2cs\end{pmatrix} M\,d\psi_m$$

$$=\begin{pmatrix}M^0\\ \frac{1}{2}M^{1c}\\ \frac{1}{2}M^{1s}\\ M^0+\frac{1}{2}M^{2c}\\ M^0-\frac{1}{2}M^{2c}\\ \frac{1}{2}M^{2s}\end{pmatrix}$$

where $M^{nc,ns}$ are the harmonics of a Fourier series representation of the rotating blade aerodynamic coefficient M:

$$M(\psi_m) = M^0 + \sum_{n=1}^{\infty} M^{nc}\cos n\psi_m + M^{ns}\sin n\psi_m$$

In the present case, these harmonics must be evaluated numerically. We evaluate M at J points, equally spaced around the azimuth:

$$M^0 = \frac{1}{J}\sum M_j$$

$$M^{nc,ns} = \frac{2}{J}\sum M_j \begin{matrix}\cos n\psi_j \\ \sin n\psi_j\end{matrix}$$

where

$$M_j = M(\psi_j)$$

$$\psi_j = j\Delta\psi, \quad j = 1 \ldots J$$

$$\Delta\psi = \frac{2\pi}{J}$$

The harmonics up to the second (n = 2) are required here. This Fourier interpolation formula requires then for good accuracy about J > 12 (a 30° azimuth increment). Using these expressions, the required harmonics are:

$$\begin{pmatrix} M^0 \\ \frac{1}{2}M^{1c} \\ \frac{1}{2}M^{1s} \\ M^0 + \frac{1}{2}M^{2c} \\ M^0 - \frac{1}{2}M^{2c} \\ \frac{1}{2}M^{2s} \end{pmatrix} = \frac{1}{J}\sum_j \begin{pmatrix} 1 \\ \cos\psi \\ \sin\psi \\ 2\cos^2\psi \\ 2\sin^2\psi \\ 2\sin\psi\cos\psi \end{pmatrix} M(\psi_j)$$

It follows then that the constant coefficient approximation is obtained from the periodic coefficient expressions by the simple transformation:

$$\frac{1}{N}\sum_{m=1}^{N}\begin{pmatrix}1\\ c\\ s\\ 2c^2\\ 2s^2\\ 2cs\end{pmatrix}M(\psi_m) \Rightarrow \frac{1}{J}\sum_{j=1}^{J}\begin{pmatrix}1\\ c\\ s\\ 2c^2\\ 2s^2\\ 2cs\end{pmatrix}M(\psi_j)$$

The summation over N blades ($m = 1\ldots N$, $\Delta\psi = 2\pi/N$) for the periodic coefficient case is replaced by a summation over the rotor azimuth ($j = 1\ldots J$, $\Delta\psi = 2\pi/J$) for the constant coefficient approximation. This is quite convenient, since the same procedure may be used to evaluate the coefficients for the two cases, with simply a change in the azimuth increment. The periodic coefficients must be evaluated throughout the period of $\psi = 0$ to $2\pi/N$ of course; while the constant coefficient approximation (the mean values only) is evaluated only once.

With the substitution $\frac{1}{N}\sum_m \Rightarrow \frac{1}{J}\sum_j$, the results given above for the periodic coefficient matrices are directly applicable to the constant coefficient approximation as well.

ROTOR TRIM

The are two requirements in the dynamics analysis for the trim, equilibrium solution for the rotor blade motion and rotor performance: first, the trim bending deflection $(x_0\hat{i} + z_0\hat{k})$ is required for the coefficients, particularly when the blade torsion dynamics are involved; secondly, the evaluation of the aerodynamic coefficients requires the lift and drag loading of the rotor blade. The trim bending deflection is assumed to be independent of ψ in the analysis, so the mean value must be used when $\mu > 0$; for the aerodynamic coefficients, the periodic variation of the trim blade aerodynamics when $\mu > 0$ will be included however. The dynamics analysis (the evaluation of the coefficients of the equations of motion) must be preceded therefore by a preliminary calculation of the rotor equilibrium motion. The trim solution for the blade motion is periodic in the rotating frame for the general case of nonaxial flow; for $\mu = 0$, axial flow, the blade motion is steady in the rotating frame. For the trim blade motion solution we shall consider only the bending and gimbal degrees of freedom. It is assumed that there is no shaft motion, gusts, rotational speed perturbation, or torsion/pitch motion (except cyclic control and any bending/torsion coupling) in the trim solution.

The trim solution involves the numerical integration of the differential equations of motion for a single blade in the rotating frame, until the blade motion converges to the desired periodic solution. The equations for the blade motion are obtained from the above analysis, and are for the bending and gimbal degrees of freedom:

$$I^*_{q_k}\left(\ddot{q}_k + g_s \nu_k \dot{q}_k + \nu_k^2 q_k\right) + 2\sum I^*_{q_k q_i}\dot{q}_i = I^*_{q_k o} + \gamma\int_0^1 \vec{\eta}_k\cdot\left(\frac{F_z}{ac}\hat{i}_B - \frac{F_x}{ac}\hat{k}_B\right)dr$$

$$I_o^*(\nu_G^2 - 1)\begin{pmatrix}\beta_{GC}\\ \beta_{GS}\end{pmatrix} = \frac{2}{N}\sum_j \begin{pmatrix}\cos\psi_j\\ \sin\psi_j\end{pmatrix}\gamma\int_0^1 \frac{F_z}{ac}\, r\, dr$$

where the inertia constants are defined above, and the aerodynamic forces are evaluated using the trim velocity components (for which expressions are given above).

After the integration of the blade motion converges to a periodic solution, the rotor performance may be evaluated, i.e. the mean aerodynamic forces and moments the rotor produces at the hub, particularly the rotor thrust and torque coefficients. The Fourier harmonics of the blade bending motion are also evaluated. From the zeroth harmonics of the bending motion, the mean bending deflection of the blade may be evaluated.

For axial flow, $\mu = 0$, integration of the blade motion is not required; for the gimbal motion is zero (assuming no cyclic pitch input) and the equation for the blade bending modal deflection reduces to

$$I^*_{q_k} \nu_k^2 q_k = I^*_{q_k o} + \gamma \int_0^1 \vec{\eta}_k \cdot \left(\frac{F_z}{ac} \vec{i}_B - \frac{F_x}{ac} \vec{k}_B \right) dr$$

BLADE BENDING AND TORSION MODES

Coupled bending modes of a rotating blade

Equilibrium of the elastic, inertial, and centrifugal bending moments on the blade gives the differential equation for the coupled flap/lag bending of the rotating blade. For free vibration -- the homogeneous equation (no forcing) with harmonic motion at the natural frequency ν -- we obtain the modal equation for bending of the blade:

$$(EI\,\vec{\eta}'')'' - \Omega^2\left(\int_r^R \rho m\, d\rho\, \vec{\eta}'\right)' - m\vec{\Omega}\,\vec{\Omega}\cdot\vec{\eta} - m\nu^2\vec{\eta} = 0$$

where

$\vec{\eta}(r) = z_0\vec{i} - x_0\vec{k}$	= bending deflection (mode shape)
$EI = EI_{zz}\vec{i}\vec{i} + EI_{xx}\vec{k}\vec{k}$	= bending stiffness dyadic
$\vec{\Omega} = \Omega\vec{k}_B$	= rotor rotational speed
ν	= natural frequency of mode

This is an eigenvalue problem, a differential equation in r for the mode shapes $\vec{\eta}$ and the natural frequencies ν. The equation with the appropriate boundary conditions constitutes a proper Sturm-Liouville eigenvalue problem. It follows that the solution exists: a series of modes $\vec{\eta}_i(r)$ and corresponding natural frequencies ν_i; where the modes are orthogonal with weight m, i.e. if $i \neq k$ then

$$\int_0^R \vec{\eta}_i\cdot\vec{\eta}_k\, m\, dr = 0$$

and the frequencies satisfy the relation (an energy balance):

$$\nu^2 = \frac{\int_0^R\left[\vec{\eta}''\,EI\,\vec{\eta}'' + \Omega^2\int_r^R \rho m\, d\rho\, \eta'^2 - m(\hat{\Omega}\cdot\vec{\eta})^2\right]dr}{\int_0^R \eta^2 m\, dr}$$

The modal equation will be solved by a Galerkin method. The mode shape is expanded as a finite series in the functions $\vec{f}_i(r)$:

$$\vec{\eta} = \sum c_i\vec{f}_i(r)$$

We require that each of the $\vec{f}_i$ satisfy the boundary conditions on $\vec{\eta}$; then the sum automatically does. Since a finite series is required for computation, this is an approximate calculation; the functions $\vec{f}_i$ should then be chosen so that at least the lower frequency modes can be well represented, for best numerical accuracy. Substituting this series in the differential equation and operating with

$$\int_e^1 \vec{f}_k \cdot (\cdots) \, dr$$

reduces the problem (after integration by parts and an application of the boundary conditions) to a set of algebraic equations for $\vec{c} = [c_i]$

$$\left(A - \nu^2 B \right) \vec{c} = 0$$

where the coefficient matrices are

$$A_{ki} = \int_e^1 \left[\vec{f}_k'' \frac{EI}{\Omega^2 R^4} \vec{f}_i'' + \int_r^1 \rho m \, d\rho \, \vec{f}_k' \cdot \vec{f}_i' - m \, \vec{f}_k \cdot \vec{k}_\theta \, \vec{f}_i \cdot \vec{k}_\theta \right] dr$$

$$B_{ki} = \int_e^1 m \, \vec{f}_k \cdot \vec{f}_i \, dr$$

Eigenvalues of the matrix $B^{-1}A$ are the natural frequencies ν^2 of the coupled bending vibration of the blade; and the corresponding eigenvectors $\vec{c}$ give the mode shape $\vec{\eta}$. As a final step, the modes are normalized to unity at the tip: $|\vec{\eta}(1)| = 1$.

A convenient set of functions for f_i are the polynomials (ref 5):

$$f_n = \frac{(n+2)(n+3)}{6} r^{n+1} - \frac{n(n+3)}{3} r^{n+2} + \frac{n(n+1)}{6} r^{n+3}$$

(for a hinged blade $f_1 = r$ is used). These polynomials satisfy the required boundary conditions, but are not orthogonal functions.

Torsion modes of a nonrotating blade

Equilibrium of the elastic and inertial torsion moments gives the modal equation

$$(GJ\,\xi')' + I_\theta \omega^2 \xi = 0$$

The modes are orthogonal with weight I_θ ; i.e. if $i \neq k$ then

$$\int_0^R \xi_i \xi_k \, I_\theta \, dr = 0$$

and the frequencies satisfy the relation

$$\omega^2 = \frac{\int_{r_{FA}}^1 GJ\,\xi'^2 \, dr}{\int_{r_{FA}}^1 I_\theta \, \xi^2 \, dr}$$

These are nonrotating modes, so the solution is independent of Ω or Θ . The equation is solved by a Galerkin method. Writing

$$\xi = \sum c_i f_i(r)$$

where the functions f_i satisfy the boundary conditions on ξ , and operating with $\int_{r_{FA}}^1 \xi_k (\cdots) dr$ on the differential equation, produces a set of algebraic equations for $\vec{c} = [c_i]$:

$$(A - \nu^2 B)\vec{c} = 0$$

where

$$A_{ki} = \int_{r_{FA}}^1 \frac{GJ}{\Omega^2 R^2} f_k' f_i' \, dr$$

$$B_{ki} = \int_{r_{FA}}^1 I_\theta f_k f_i \, dr$$

The eigenvalues of the matrix $B^{-1}A$ give the natural frequencies of the torsion vibration, and the corresponding eigenvectors for $\vec{c}$ give the modes. Finally, the torsion modes are normalized to unity at the tip, $\xi(1) = 1$.

A convenient set of functions to use for f_i is the solution for the torsion modes of a uniform beam:

$$f_n = \sin\left[\left(n - \frac{1}{2}\right)\pi \frac{r - r_{BA}}{1 - r_{BA}} \right]$$

These functions satisfy the boundary conditions, and will usually be close to the actual mode shapes.

SUPPORT EQUATIONS OF MOTION: CANTILEVER WING

For the rotor support we consider a cantilever wing, with the rotor on a mast or pylon attached to the wing tip. Reference 4 discusses the cantilever wing as a representation of the tilting proprotor aircraft dynamics, and develops the equations of motion describing this support. The equations of motion for the wing, and the rotor motion produced by the wing are developed in reference 4; these results are adopted here with only two extensions: to arbitrary angle of attack of the rotor shaft with respect to the forward velocity; and the inclusion of a wing trailing-edge flap among the controls.

Cantilever wing

The cantilever wing and pylon geometry is shown in figure 12. We consider a high aspect ratio, flexible wing, with the rotor on the tip. The wing is attached to an immovable support with cantilever root restraint. A pylon with large mass and moment of inertia is rigidly attached to the wing tip. The rotor is mounted on the pylon with the hub forward of the wing EA, with mast height h. A general pylon angle δ_p is considered, from vertical in helicopter mode to horizontal in airplane mode. The wing motion consists of elastic bending, vertical and chordwise, and elastic torsion. There is no motion of the pylon relative to the wing tip, so the wing tip motion is transmitted directly to the hub, and hub forces and moments transmitted directly to the wing tip, through the mast of height h. The rotor and wing operate in a steady free stream of velocity V. The pylon (or mast, or rotor shaft) angle of attack δ_p may be large, so it covers the entire range of tilting proprotor operation. The cases include: δ_p near 90° for helicopter mode; δ_p between 0 and 90° for conversion mode; $\delta_p = 0$ for cruise mode; and $V = 0$ is the case of hover flight.

The wing angle of attack is δ_{w_2}, defined positive nose up; it is assumed that δ_{w_2} is a small angle. The angle between the wing and the rotor shaft is then $\delta_p - \delta_{w_2}$; it is this angle which determines the transmission of motion and forces between the rotor and the wing. Recall that α_{HP} is the angle of the rotor disk to the forward speed V; here we use δ_P for the shaft angle of attack, hence $\delta_P = 90^\circ - \alpha_{HP}$. We also consider small sweep angle δ_{w_3} (positive aft) and small dihedral angle δ_{w_1} (positive up) of the wing. A major effect of δ_{w_3} and δ_{w_1} is on the position of the effective elastic axis of the wing, hence on the effective mast height for the transmission of motion and forces between the rotor and the wing. The angles δ_{w_1}, δ_{w_2}, and δ_{w_3} are removed from the orientation of the pylon and shaft at the wing tip. So the rotor shaft is in a vertical plane with no sweep or dihedral, parallel to V when $\delta_P = 0$; and then δ_P is the angle of attack of the shaft with respect to V, not with respect to the wing.

The wing is assumed to have a straight spar line, which is the locus of the local EA. The wing root is supported with cantilever restraint, and the rotor shaft is attached rigidly to the wing tip. The wing has no twist, constant chord c_W, length y_T from root to tip (semispan), with the distance y_W measured from the root, along the wing spar. The shaft length (mast height) is h, the distance the rotor hub is forward of the wing tip EA. The wing spar is roughly perpendicular to V, with small wing sweep, dihedral, and angle of attack considered. The wing root is attached to a plane defined by the forward velocity V and the vertical; then the three rotation angles δ_{w_1}, δ_{w_2}, and δ_{w_3} define the orientation of the spar with respect to the free stream velocity. Next the pylon is rotated by $-\delta_{w_1}$, $-\delta_{w_2}$, and $-\delta_{w_3}$ with respect to the wing tip, to keep the shaft parallel to V; finally the pylon is rotated by δ_P with respect to V, defining the orientation of the rotor.

Swept wings are usually built with a center box structure in the fuslage, where the spars are unswept, and only the wing structure outside the fuslage has swept spars. The wing is restrained at several points where the wing box is tied to the fuselage structure, or in this case to the cantilever wing fixed support. There exists an effective elastic axis for vertical bending of the wing tip: some point on the shaft or its extension where the application of a vertical force results in purely vertical displacement of the shaft, with no rotation in pitch. Without sweep this point would be just at the wing tip EA; but with sweep a force there will produce a pitch motion of the shaft also, hence the effective EA is some distance from the wing tip EA. The effective elastic axis for the tip lies between the actual wing tip EA and the extension of the unswept spar line, the actual position depending on the degree of root restraint and sweep, and other structural details. Figure 13 illustrates the geometry involved. Reference 4 develops an elementary model for the wing bending and torsion including the shift of the effective EA due to sweep (and a similar effect due to dihedral), which is adopted here. The effective EA position is described by (figure 13):

h = mast height, distance hub forward wing tip EA.

h_{EA} = effective mast height, distance hub forward effective EA.

z_{EA} = distance hub below effective EA due to dihedral.

Further discussion of this effect, including the estimation of the parameters involved, is given in reference 4.

The aircraft has two contrarotating rotors, one on each wing tip. The direction of rotation of the rotor on the right wing (as in figure 12) may be either clockwise or counterclockwise. The influence of the rotor rotational direction is a few signs in the equations of motion, reflecting how the rotor hub forces and moments excite the wing motion, and how the wing produces motion of the rotor shaft. As in reference 4, the notation Ω is used to carry this influence of the rotor rotation direction, where Ω takes only the values ± 1:

$$\Omega = \begin{cases} +1, & \text{rotor rotation clockwise on right wing, counterclockwise on left.} \\ -1, & \text{rotor rotation counterclockwise on right wing, clockwise on left.} \end{cases}$$

Wing Motion

The wing motion is described by elastic bending and torsion of the spar; the pylon, and with it the rotor shaft, is rigidly attached to the wing tip. Elastic bending results in deflection of the wing spar with components both perpendicular to the wing surface (vertical or beam bending), and parallel to the wing surface (chordwise bending). Vertical and chordwise bending are defined with respect to the direction of the local principle axes of the section. There is no wing twist, so these principle axes are the same all along the span, but they are not vertical and horizontal axes because of the wing sweep, dihedral, and angle of attack. We define (figure 12) the wing bending and torsion deflection as follows:

$z_w(y_w)$ = elastic bending vertical displacement of the spar, normal to the wing surface, positive up.

$x_w(y_w)$ = elastic bending chordwise displacement of the span, in the plane of the wing, positive rearward.

$\theta_w(y_w)$ = pitch change of local wing section, due to elastic torsion about the local EA, positive nose up.

A modal description of the wing elastic deformation is used, and only the lowest frequency modes retained. We consider just three degrees of freedom for the wing: first mode vertical bending, chordwise bending, and torsion. The degrees of freedom representing the wing motion are:

q_{w_1} = wing vertical or beamwise bending, positive upward; $q_{w_1} = z_w/y_{T_w}$ at the wing tip.

q_{w_2} = wing chordwise bending, positive rearward; $q_{w_2} = x_w/y_{T_w}$ at the wing tip.

p_w = wing elastic torsion, positive nose up; $p_w = \theta_w$ at the wing tip.

Associated with these degrees of freedom are mode shapes, $\xi_w(y_w)$ for torsion, and $\eta_w(y_w)$ for bending. These modes are normalized to 1 and to y_{T_w} respectively at the wing tip ($y_w = y_{T_w}$).

From the results of reference 4, generalized to arbitrary pylon angle of attack , δ_P , the rotor hub motion due to the wing degrees of freedom is:

$$
\begin{bmatrix} x_h \\ y_h \\ z_h \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}
=
\begin{bmatrix}
yC - \ell_h \eta \delta_3 & yS + \ell_h \eta \delta_1 & \ell_h + (\ell_{EA} - \ell_h)C - z_{EA} S \\
-\ell_h \eta S - y\delta_1 & \ell_h \eta C - y\delta_3 & -\ell_h \delta_1 C - \ell_h \delta_3 S \\
yS & -yC & (\ell_{EA} - \ell_h)S + z_{EA} C \\
\eta S & -\eta C & \delta_1 C + \delta_3 S \\
-\eta \delta_3 & \eta \delta_1 & 1 \\
-\eta C & -\eta S & \delta_1 S - \delta_3 C
\end{bmatrix}
\begin{bmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{bmatrix}
$$

where we have written

C for $\cos(\delta_P - \delta_{w_2})$
S for $\sin(\delta_P - \delta_{w_2})$
η for $\Omega \gamma_w^*(y_{Tw})$
y for y_{T_W}
δ_1 for $\Omega \delta_{w_1}$
δ_3 for $\Omega \delta_{w_3}$

Wing Equations of Motion

From reference 4, the equations of motion for the q_{w_1}, q_{w_2}, and p_w degrees of freedom of the cantilever wing, excited by the forces and moments at the rotor hub and by the wing aerodynamic forces, are:

$$
\begin{bmatrix} I_{q_w}^{*} + m_p^{*} & 0 & S_w^{*} \\ 0 & I_{q_w}^{*} + I_{p_x}^{*}\eta_w^{2} + m_p^{*} & -S_w^{*}\delta_{w_2} \\ S_w^{*} & -S_w^{*}\delta_{w_2} & I_{p_w}^{*} + I_{p_y}^{*} \end{bmatrix} \begin{pmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{pmatrix}^{\cdot\cdot}
$$

$$
+ \begin{bmatrix} C_{q_1}^{*} & 0 & 0 \\ 0 & C_{q_2}^{*} & 0 \\ 0 & 0 & C_p^{*} \end{bmatrix} \begin{pmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{pmatrix}^{\cdot} + \begin{bmatrix} K_{q_1}^{*} & 0 & 0 \\ 0 & K_{q_2}^{*} & 0 \\ C_{pq}^{*}\eta\gamma\frac{2C_T}{\sigma a}C & C_{pq}^{*}\eta\gamma\frac{2C_T}{\sigma a}S & K_p^{*} \end{bmatrix} \begin{pmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{pmatrix}
$$

$$
= \gamma \begin{pmatrix} M_{q_1 aero} \\ M_{q_2 aero} \\ M_{p_w aero} \end{pmatrix}
$$

$$
+ \left[\begin{array}{c|c|c|c|c|c} 2\eta S & \eta C - h\eta\delta_3 & h\eta S + \eta\delta_1 & 2\eta C & -\eta\delta_3 & -\eta S \\ \hline -2\eta C & \eta S + h\eta\delta_1 & -h\eta C + \eta\delta_3 & 2\eta S & \eta\delta_1 & \eta C \\ \hline 2(h_{EA}-h)S + 2z_{EA}C & h + (h_{EA}-h)C - z_{EA}S & h\delta_3 S + h\delta_1 C & 2\delta_3 C - 2\delta_1 S & 1 & -\delta_3 S - \delta_1 C \end{array}\right] \gamma \begin{bmatrix} \frac{C_T}{\sigma a} \\ \frac{2C_H}{\sigma a} \\ -\frac{2C_Y}{\sigma a} \\ \frac{C_Q}{\sigma a} \\ \frac{2C_{M_y}}{\sigma a} \\ -\frac{2C_{M_x}}{\sigma a} \end{bmatrix}
$$

The wing equations are nomalized by dividing by $(N/2)I_b$, so the rotor exciting forces are in helicopter coefficient form. The inertias are:

$$I^*_{qw} = \frac{1}{\frac{N}{2}I_b}\int_0^{y_{Tw}} m_w \eta_w^2 \, dy_w$$

$$I^*_{pw} = \frac{1}{\frac{N}{2}I_b}\int_0^{y_{Tw}} I_{\theta w} \zeta_w^2 \, dy_w$$

$$m^*_p = \frac{m_p y_{Tw}^2}{\frac{N}{2}I_b}$$

$$I^*_{px} = \frac{I_{px}}{\frac{N}{2}I_b}$$

$$I^*_{py} = \frac{I_{py}}{\frac{N}{2}I_b}$$

$$S^*_w = m^*_p \frac{z_{pEA}}{y_{Tw}}$$

where m_w is the wing mass per unit length; I_{θ_w} is the wing section moment of inertia in pitch; m_p is the pylon mass (without the rotor); I_{p_x} and I_{p_y} are the pylon yaw and pitch moments of inertia, without the rotor, about the wing tip effective EA; and $z_{p_{EA}}$ is the distance the pylon CG (without the rotor) is ahead of the wing tip effective EA. For the proprotor configuration, the pylon mass is so large that it dominates the wing inertias. Hence the inertia is primarily that of the pylon and rotor, with the wing contributing elastic restraint of the motion. The wing structural spring constants are $K^*_{q_1}$, $K^*_{q_2}$, and K^*_p ; these were evaluated by matching the predicted frequencies of the wing modes to the values obtained experimentally. $C^*_{q_1}$, $C^*_{q_2}$, and C^*_p are the structural damping constants for the wing modes. Vertical bending elevates the rotor trim thrust above the inboard sections, and so gives a nose down pitch moment with effectiveness given by C^*_{pq}:

$$C^*_{pq} = \int_0^{y_{Tw}} \zeta_w \eta'_w \, dy_w / y_{Tw} \cong \frac{2}{3}$$

Dimensionally, the spring and damping constants are

$$K^* = K / \tfrac{N}{2} I_b \Omega^2$$
$$C^* = C / \tfrac{N}{2} I_b |\Omega|$$

Hence the relative spring and damping rates vary with the rotor rotational speed; i.e. the wing frequency is really a fixed dimensional value (Hz), so the per-rev values vary with Ω.

Additional discussion and details of the wing equations of motion are given in reference 4.

Wing Aerodynamics

The wing aerodynamic forces exciting bending and torsion motion of the wing are:

$$M_{q_{w_1}aero} = \frac{1}{\gamma \frac{N}{2} I_b} \int_0^{y_{T_w}} F_{z_w} \eta_w \, dy_w$$
$$M_{q_{w_2}aero} = \frac{1}{\gamma \frac{N}{2} I_b} \int_0^{y_{T_w}} F_{x_w} \eta_w \, dy_w$$
$$M_{p_w aero} = \frac{1}{\gamma \frac{N}{2} I_b} \int_0^{y_{T_w}} M_w \xi_w \, dy_w$$

where F_{z_w} and F_{x_w} are the vertical and chordwise aerodynamic forces on the wing section (lift and profile plus induced drag); M_w is the aerodynamic moment about the local EA. The velocity seen by the section has perturbations due to the wing degrees of freedom, and due to aerodynamic gusts. Aerodynamic interference between the rotor and the wing is neglected. From the velocity perturbations, the perturbations of the section forces may be found, and hence the wing aerodynamic coefficients. The derivation of the wing aerodynamic coefficients follows the standard techniques of strip theory in aeroelasticity; more details of the derivation are given in reference 4. We also include here the aerodynamic force due to the deflection of a control surface (flap or aileron) on the wing trailing-edge. The geometry is shown in figure 14. A constant chord (c_F) trailing-edge flap, extending from $y_w = y_{F_I}$ to $y_w = y_{F_O}$ is considered. The flap deflection angle is δ_f, positive downward. So

δ_ξ is a control variable, in addition to the rotor cyclic and collective pitch controls. The result for the wing aerodynamic forces is:

$$\begin{pmatrix} M_{q_{w_1}\,aero} \\ M_{q_{w_2}\,aero} \\ M_{p_w\,aero} \end{pmatrix} = \begin{bmatrix} C_{q_1\dot{q}_1} & C_{q_1\dot{q}_2} & C_{q_1\dot{p}} \\ C_{q_2\dot{q}_1} & C_{q_2\dot{q}_2} & C_{q_2\dot{p}} \\ C_{p\dot{q}_1} & C_{p\dot{q}_2} & C_{p\dot{p}} \end{bmatrix} \begin{pmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{pmatrix}^{\bullet}$$

$$+ \begin{bmatrix} C_{q_1q_1} & C_{q_1q_2} & C_{q_1p} \\ C_{q_2q_1} & C_{q_2q_2} & C_{q_2p} \\ C_{pq_1} & C_{pq_2} & C_{pp} \end{bmatrix} \begin{pmatrix} q_{w_1} \\ q_{w_2} \\ p_w \end{pmatrix}$$

$$+ \begin{bmatrix} C_{q_1u} & C_{q_1v} & C_{q_1w} \\ C_{q_2u} & C_{q_2v} & C_{q_2w} \\ C_{pu} & C_{pv} & C_{pw} \end{bmatrix} \begin{pmatrix} u_G \\ v_G \\ w_G \end{pmatrix}$$

$$+ \begin{bmatrix} C_{q_1\delta} \\ C_{q_2\delta} \\ C_{p\delta} \end{bmatrix} \delta_\xi$$

The aerodynamic coefficients are:

$$C_{q_1u} = \delta_{12} V\, 2 C_{L_0}\, e_1$$

$$C_{q_1w} = \delta_{12} V C_{L_\alpha}\, e_1$$

$$C_{q_1v} = \delta_{w_1} C_{q_1w} + \delta_{w_3} C_{q_1u}$$

$$C_{q_1\dot{q}_1} = -\delta_{13} V C_{L_\alpha}\, e_2$$

$$C_{q_1\dot{q}_2} = -\delta_{13} V C_{L_0} e_2$$

$$C_{q_1 q_1} = -\delta_{12} V^2 \delta_{w_3} C_{L_\alpha} e_3$$

$$C_{q_1 q_2} = -\delta_{12} V^2 \delta_{w_3} C_{L_0} e_3$$

$$C_{q_1\dot{p}} = \delta_{22} \tfrac{1}{2} V \left(\tfrac{3}{4} + \tfrac{x_{AW}}{c_w} \right) C_{L_\alpha} e_4$$

$$C_{q_1 p} = \delta_{12} V^2 C_{L_\alpha} e_4$$

$$C_{q_1\delta} = \delta_{12} V^2 C_{L_\alpha} C^{*}_{\ell_\delta} e_5$$

$$C_{q_2 u} = \delta_{12} V 2 \left(C_{D_0} - \delta_{w_2} C_{L_0} \right) e_1$$

$$C_{q_2 w} = \delta_{12} V \left(C_{D_\alpha} - 2C_{L_0} \right) e_1$$

$$C_{q_2 v} = \delta_{w_1} C_{q_2 w} + \delta_{w_3} C_{q_2 u}$$

$$C_{q_2\dot{q}_1} = -\delta_{13} V \left(C_{D_\alpha} - 2C_{L_0} \right) e_2$$

$$C_{q_2\dot{q}_2} = -\delta_{13} V \left(2C_{D_0} - \delta_{w_2} C_{D_\alpha} \right) e_2$$

$$C_{q_2 q_1} = -\delta_{12} V^2 \delta_{w_3} \left(C_{D_\alpha} - 2C_{L_0} \right) e_3$$

$$C_{q_2 q_2} = -\delta_{12} V^2 \delta_{w_3} \left(2C_{D_0} - \delta_{w_2} C_{D_\alpha} \right) e_3$$

$$C_{q_2\dot{p}} = \delta_{22} \tfrac{1}{2} V \left[\left(\tfrac{1}{2} + \tfrac{x_{AW}}{c_w} \right) \left(C_{D_\alpha} - 2C_{L_0} \right) - \tfrac{1}{4} C_{L_0} \right] e_4$$

$$C_{q_2 p} = \delta_{12} V^2 \left(C_{D_\alpha} - C_{L_0} \right) e_4$$

$$C_{q_2\delta} = \delta_{12} V^2 \left(C_{d_\delta} + \left(C_{D_\alpha} - C_{L_0} \right) C^{*}_{\ell_\delta} \right) e_5$$

$$C_{pu} = \delta_{21} V 2 C_{m_{ac}} f_1$$

$$C_{pw} = -\delta_{21} V \frac{x_{AW}}{c_w} C_{L_\alpha} f_1$$

$$C_{pv} = \delta_{w_1} C_{pw} + \delta_{w_3} C_{pu}$$

$$C_{p\dot{q}_1} = \delta_{22} V \frac{x_{AW}}{c_w} C_{L_\alpha} e_4$$

$$C_{P\dot{q}_2} = -\delta_{22} V 2 C_{m_{ac}} e_4$$

$$C_{Pq_1} = \delta_{12} V^2 C_{m_{ac}} f_2$$

$$C_{Pq_2} = -\delta_{12} V^2 C_{L_0} f_2$$

$$C_{P\dot{p}} = -\delta_{31} \frac{1}{2} V \left(\frac{1}{4} + \frac{1}{2} \frac{x_{A_w}}{c_w} \right) C_{L_\alpha} f_3$$

$$C_{Pp} = -\delta_{21} V^2 \frac{x_{A_w}}{c_w} C_{L_\alpha} f_3$$

$$C_{P\delta} = -\delta_{21} V^2 \left(\frac{x_{A_w}}{c_w} C_{\ell_\delta}^{*} - C_{m_\delta}^{*} \right) C_{L_\alpha} f_4$$

C_{L_0} and C_{D_0} are the aircraft trim lift and drag (profile plus induced) coefficients; and C_{L_α} and C_{D_α} their derivatives with respect to α_w. The section moment characteristics are given by x_{A_w}, the distance the wing AC is behind the EA, and $c_{m_{ac}}$, the nose up moment coefficient about the AC. The constant

$$\delta_{nm} = \frac{c_w^n \, y_{T_w}^m}{\pi \sigma a}$$

accounts for the difference in the normalization of the wing and rotor coefficients. The constants e_n and f_n are integrals of the wing mode shapes, accounting for the way the motion produces forces on the wing:

$$e_1 = \int_0^{y_{T_w}} \eta_w \, dy_w / y_{T_w}^2 \cong \frac{1}{3}$$

$$e_2 = \int_0^{y_{T_w}} \eta_w^2 \, dy_w / y_{T_w}^3 \cong \frac{1}{5}$$

$$e_3 = \int_0^{y_{T_w}} \eta_w \check{\eta}_w \, dy_w / y_{T_w}^2 \cong \frac{1}{2}$$

$$e_4 = \int_0^{y_{T_w}} \eta_w \zeta_w \, dy_w / y_{T_w} \cong \frac{1}{4}$$

$$e_5 = \int_{y_{F_I}}^{y_{F_O}} \eta_w \, dy_w / y_{T_w} \cong \frac{1}{3} \left[(y_{F_O}/y_{T_w})^3 - (y_{F_I}/y_{T_w})^3 \right]$$

$$f_1 = \int_0^{y_{T_w}} \zeta_w \, dy_w / y_{T_w} \cong \frac{1}{2}$$

$$f_2 = \int_0^{y_{T_w}} \zeta_w \tilde{\eta}_w \frac{1}{2} (y_{T_w} - y_w)^2 \, dy_w / y_{T_w} \cong \frac{1}{12}$$

$$f_3 = \int_0^{y_{T_w}} \zeta_w^2 \, dy_w / y_{T_w} \cong \frac{1}{3}$$

$$f_4 = \int_{y_{F_I}}^{y_{F_O}} \zeta_w \, dy_w / y_{T_w} \cong \frac{1}{2} \left[(y_{F_O}/y_{T_w})^2 - (y_{F_I}/y_{T_w})^2 \right]$$

For the flaperon coefficients, we use:

$$c_{\ell_\delta}^{*} = \frac{1}{2\pi}\frac{\partial c_\ell}{\partial \delta_f} = \frac{1}{\pi}\left(\sqrt{1-c^2} + \cos^{-1} c\right)\left(.95 + .05\frac{c_f}{c_w}\right)\left(1 + \frac{1 - c_f/c_w}{2R_w}\right)$$

$$c_{m_\delta}^{*} = \frac{1}{2\pi}\frac{\partial c_m}{\partial \delta_f} = -\frac{1}{4\pi}\left((1+c)\sqrt{1-c^2}\right)\left(.95 + .05\frac{c_f}{c_w}\right)\left(1 + \frac{1 - c_f/c_w}{2R_w}\right)$$

$$c_{d_\delta} = \frac{\partial c_{d_w}}{\partial \delta_f} \cong .02 \text{ to } .06$$

where $\quad c = 1 - 2\frac{c_f}{c_w}$

The first factor in these expressions is the two-dimensional thin airfoil theory result for the lift and moment due to control surface deflection; and the last two factors are corrections for the wing aspect ratio, thickness, and real flow effects on the flap effectiveness (based on ref. 6).

ψ_s Equation of Motion

The rotational speed degree of freedom ($\dot{\psi}_s$) is an important factor in the dynamics, especially with a windmilling rotor. Usually the ψ_s equation of motion will involve the engine, drive train, interconnect shaft, and governor dynamics; here we shall consider only two limiting cases.

The first case is windmilling or autorotation operation of the rotor. The rotor is free to turn on the shaft, so no torque moments are transmitted from the rotor to the shaft, and no pylon roll motion transmitted to the rotor. Both effects are accomplished by using $C_Q = 0$ as the equation of motion for ψ_s . There is no spring term on ψ_s , so the degree of freedom is really $\dot{\psi}_s$, the rotor speed perturbation. It should be noted that ψ_s is defined with respect to the pylon, which has a roll angle α_z ; so the rotor speed perturbation with respect to space is the sum $\dot{\psi}_s + \dot{\alpha}_z$.

The second case considered here is powered operation of the rotor. It is assumed that the rotor hub rotational speed is fixed, at Ω, with no perturbations. This case may be viewed as the limit of operation with a perfect governor on engine or rotor speed. The powered case is treated by dropping the ψ_s degree of freedom and equation; i.e. the solution is just $\psi_s = 0$.

Hence we add to the support equations of motion the equation for $\dot{\psi}_s$:

$$\frac{C_Q}{\sigma a} = 0$$

For the powered case this equation and the ψ_s degree of freedom are dropped from the system (a row and column eliminated from the matrices). For the windmilling case they are retained; note that the ψ_s equation is first order, since there is no spring term.

Reference 4 gives a further discussion of these two cases, windmilling and powered operation, and their effects on the proprotor dynamics.

Support Equations of Motion

We have obtained now the shaft motion and support equations of motion, which in matrix form are:

$$\alpha = C x_w$$

$$a_2 \ddot{x}_w + a_1 \dot{x}_w + a_0 x_w = b\, v_w + b_G\, g + \tilde{a}\, F$$

where the wing degrees of freedom ($\vec{x}_w$) and the wing flap control ($\vec{v}_w$) are

$$\vec{x}_w = \begin{bmatrix} q_{w1} \\ q_{w2} \\ p_w \end{bmatrix}$$

$$\vec{v}_w = \begin{bmatrix} \delta_f \end{bmatrix}$$

and as defined above, the rotor hub forces and moments ($\vec{F}$), shaft motion ($\vec{\alpha}$), and aerodynamic gust ($\vec{g}$) are:

$$\vec{F} = \begin{bmatrix} \gamma \frac{C_T}{\sigma a} \\ \gamma \frac{2C_H}{\sigma a} \\ -\gamma \frac{2C_Y}{\sigma a} \\ \gamma \frac{C_Q}{\sigma a} \\ \gamma \frac{2C_{M_y}}{\sigma a} \\ -\gamma \frac{2C_{M_x}}{\sigma a} \end{bmatrix}$$

$$\vec{\alpha} = \begin{bmatrix} x_h \\ y_h \\ z_h \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} u_G \\ v_G \\ w_G \end{bmatrix}$$

The matrices of the coefficients of the equations of motion follow. The matrix c_s relating the rotor shaft motion to the wing motion, has been given above.

$$
a_2 = \begin{bmatrix} & & \\ I^*_{qw} + m^*_p & & S^*_w \\ & I^*_{qw} + I^*_{px}\eta^2 + m^*_p & -S^*_w \delta_{w2} \\ S^*_w & -S^*_w \delta_{w2} & I^*_{pw} + I^*_{py} \end{bmatrix}
$$

$$
\begin{bmatrix} & & \\ C^*_{q_1} - \gamma C_{q_1 \dot{q}_1} & -\gamma C_{q_1 \dot{q}_2} & -\gamma C_{q_1 \dot{p}} \\ -\gamma C_{q_2 \dot{q}_1} & C^*_{q_2} - \gamma C_{q_2 \dot{q}_2} & -\gamma C_{q_2 \dot{p}} \\ -\gamma C_{p \dot{q}_1} & -\gamma C_{p \dot{q}_2} & C^*_{p} - \gamma C_{p \dot{p}} \end{bmatrix}
$$

$$
a_0 = \left[\begin{array}{c|c|c}
 & & \\
\hline
\begin{array}{c} K^*_{q_1} \\ \\ -\gamma C_{q_1 q_1} \end{array} & -\gamma C_{q_1 q_2} & -\gamma C_{q_1 p} \\
\hline
-\gamma C_{q_2 q_1} & \begin{array}{c} K^*_{q_2} \\ \\ -\gamma C_{q_2 q_2} \end{array} & -\gamma C_{q_2 p} \\
\hline
\begin{array}{c} C^*_{pq}\, y\, \gamma \frac{2c_T}{m} C \\ \\ -\gamma C_{p q_1} \end{array} & \begin{array}{c} C^*_{pq}\, y\, \delta \frac{2c}{m} S \\ \\ -\gamma C_{p q_2} \end{array} & \begin{array}{c} K^*_{p} \\ \\ -\gamma C_{p p} \end{array}
\end{array}\right]
$$

$$
b = \left[\begin{array}{c}
 \\
\hline
\gamma C_{q_1 \delta} \\
\hline
\gamma C_{q_2 \delta} \\
\hline
\gamma C_{p \delta}
\end{array}\right]
$$

$$b_G = \begin{bmatrix} & & \\ \gamma c_{q_1 u} & \gamma c_{q_1 v} & \gamma c_{q_1 w} \\ \gamma c_{q_2 u} & \gamma c_{q_2 v} & \gamma c_{q_2 w} \\ \gamma c_{p u} & \gamma c_{p v} & \gamma c_{p w} \end{bmatrix}$$

$$\tilde{\alpha} = \begin{bmatrix} & & & -1 & & \\ 2yS & \begin{matrix} yC \\ -h\eta\delta_3 \end{matrix} & \begin{matrix} h\eta S \\ +y\delta_1 \end{matrix} & 2\eta C & -\eta\delta_3 & -\eta S \\ -2yC & \begin{matrix} yS \\ +h\eta\delta_1 \end{matrix} & \begin{matrix} -h\eta C \\ +y\delta_3 \end{matrix} & 2\eta S & \eta\delta_1 & \eta C \\ \begin{matrix} 2(h_{EA}-h)S \\ +2z_{EA}C \end{matrix} & \begin{matrix} h \\ +(h_{EA}-h)C \\ -z_{EA}S \end{matrix} & \begin{matrix} h\delta_3 S \\ +h\delta_1 C \end{matrix} & \begin{matrix} 2\delta_3 C \\ -2\delta_1 S \end{matrix} & 1 & \begin{matrix} -\delta_3 S \\ -\delta_1 C \end{matrix} \end{bmatrix}$$

EQUATIONS MOTION

The complete set of equations of motion describing the proprotor and cantilever wing system may now be obtained, by substituting for the shaft motion into the rotor forces and moments, and then for the rotor forces into the wing equations. The result is a set of linear differential equations, of the form:

$$A_2\ddot{x} + A_1\dot{x} + A_0 x = Bv + B_G g$$

where the degrees of freedom (state) vector ($\vec{x}$) and the input vector ($\vec{v}$) are:

$$x = \begin{bmatrix} x_R \\ x_w \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_{1c} \\ \beta_{1s} \\ \theta_0 \\ \theta_{1c} \\ \theta_{1s} \\ \beta_{Gc} \\ \beta_{Gs} \\ \psi_s \\ q_{w_1} \\ q_{w_2} \\ p_w \end{bmatrix}$$

$$v = \begin{bmatrix} v_R \\ v_w \end{bmatrix} = \begin{bmatrix} \theta_0^{con} \\ \theta_{1c}^{con} \\ \theta_{1s}^{con} \\ \delta_{\xi} \end{bmatrix}$$

Recalling the equations for the rotor equations of motion, the rotor hub forces and moments, the shaft motion, and the wing equations of motion:

$$A_2\ddot{x}_R + A_1\dot{x}_R + A_0 x_R + \tilde{A}_2\ddot{\alpha} + \tilde{A}_1\dot{\alpha} + \tilde{A}_0\alpha = B v_R + B_G g$$
$$F = C_2\ddot{x}_R + C_1\dot{x}_R + C_0 x_R + \tilde{C}_2\ddot{\alpha} + \tilde{C}_1\dot{\alpha} + \tilde{C}_0\alpha + D_G g$$
$$\alpha = c x_w$$
$$a_2\ddot{x}_w + a_1\dot{x}_w + a_0 x_w = b v_w + b_G g + \tilde{a} F$$

the coefficient matrices of the complete equations of motion may be identified, as:

$$\mathbf{A}_2 = \left[\begin{array}{c|c} A_2 & \tilde{A}_2 c \\ \hline -\tilde{a} C_2 & a_2 - \tilde{a}\tilde{C}_2 c \end{array}\right]$$

$$\mathbf{A}_1 = \left[\begin{array}{c|c} A_1 & \tilde{A}_1 c \\ \hline -\tilde{a} C_1 & a_1 - \tilde{a}\tilde{C}_1 c \end{array}\right]$$

$$\mathbf{A}_0 = \left[\begin{array}{c|c} A_0 & \tilde{A}_0 c \\ \hline -\tilde{a} C_0 & a_0 - \tilde{a}\tilde{C}_0 c \end{array}\right]$$

$$\mathbf{B} = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & b \end{array}\right] \qquad \mathbf{B}_G = \left[\begin{array}{c} B_G \\ \hline b_G + \tilde{a} D_G \end{array}\right]$$

Treatment of Rotor Pitch/Torsion

The equations of motion have been set up including the rotor pitch and torsion degrees of freedom, $\theta^{(1)}$, and with θ^{con} (the commanded pitch angle) as the rotor control variable. One may not wish to include these degrees of freedom in the system dynamics, but it is not possible to simply drop them at this stage. The pitch control and bending/gimbal feedback enters the system through the rigid pitch degree of freedom (p_o), so it is necessary to first operate on the columns of the equation matrices to account for these effects. Then the degrees of freedom and equations (columns and rows of the matrices) may be dropped as appropriate. We shall consider three options for the treatment of the rotor pitch/torsion motion.

The first option is to include the pitch and torsion degrees of freedom in the system; then the equations are used as derived.

The second option is the case of a rigid control system. It is the limit of infinite control system and blade torsion stiffness. Thus the rotor blade elastic torsion motion is zero, and the response of the rigid pitch motion reduces to

$$p_o = \theta^{con} - \Sigma K_{P_i} q_i - K_{P_G} \beta_G$$

or

$$\begin{pmatrix} \theta_o \\ \theta_{1c} \\ \theta_{1s} \end{pmatrix}_o = \begin{pmatrix} \theta_o \\ \theta_{1c} \\ \theta_{1s} \end{pmatrix}_{con} - \Sigma K_{P_i} \begin{pmatrix} \beta_o \\ \beta_{1c} \\ \beta_{1s} \end{pmatrix}_i - K_{P_G} \begin{pmatrix} 0 \\ \beta_{Gc} \\ \beta_{Gs} \end{pmatrix}$$

Thus we operate on the columns of the A_o matrix as follows:

subtract K_{P_i} times the $\theta_o^{(o)}$ column from the $\beta_o^{(i)}$ column

subtract K_{P_i} times the $\theta_{1c}^{(o)}$ column from the $\beta_{1c}^{(i)}$ column

subtract K_{P_i} times the $\theta_{1s}^{(o)}$ column from the $\beta_{1s}^{(i)}$ column

subtract K_{P_G} times the $\theta_{1c}^{(o)}$ column from the β_{Gc} column

subtract K_{P_G} times the $\theta_{1s}^{(o)}$ column from the β_{Gs} column and reconstruct the control matrix B as follows:

replace the θ_0^{con} column of B with minus the $\theta_0^{(0)}$ column of A_0

replace the θ_{1c}^{con} column of B with minus the $\theta_{1c}^{(0)}$ column of A_0

replace the θ_{1s}^{con} column of B with minus the $\theta_{1s}^{(0)}$ column of A_0

Then the rigid pitch degrees of freedom and equations of motion are dropped from the system. Note that the above transformation is only the result of infinite control system stiffness; it would be possible to retain the elastic torsion degrees of freedom, dropping only the rigid pitch p_0.

The third option is a quasistatic approximation for the effect of the blade torsion and pitch motion. We shall neglect the acceleration and velocity terms in the torsion/pitch equations. The torsion/pitch equations then become just a static substitution relation for θ in the other equations of motion. This treatment retains all the static coupling effects in the A_0 matrix. The required transformation of the equations is accomplished as follows. First the A_0, B, and B_G matrices are partitioned, to separate the θ variables and equations from the rest. Assuming the θ block is in the middle of $\vec{x}$, the state equations take the form:

$$A_2\ddot{x} + A_1\dot{x} + \begin{bmatrix} A_0^{11} & A_0^{12} & A_0^{13} \\ A_0^{21} & A_0^{22} & A_0^{23} \\ A_0^{31} & A_0^{32} & A_0^{33} \end{bmatrix} x = \begin{bmatrix} B^1 \\ B^2 \\ B^3 \end{bmatrix} v + \begin{bmatrix} B_G^1 \\ B_G^2 \\ B_G^3 \end{bmatrix} g$$

Now the acceleration and velocity terms are dropped from the pitch equations; and we write $\vec{x}$ still for the state variable vector, but now with the pitch degrees of freedom dropped. Hence

$$\theta = (A_0^{22})^{-1}\left[-\left[A_0^{21} \quad A_0^{23}\right] x + B^2 v + B_G^2 g\right]$$

which may be substituted into the remaining equations, eliminating θ from A_0 (the pitch acceleration and velocity terms in the remaining equations are dropped). Thus the quasistatic torsion approximation gives the following equations of motion, in terms of the reduced state variable $\vec{x}$ (without the torsion/pitch degrees of freedom):

$$\begin{bmatrix} A_2^{11} & A_2^{13} \\ A_2^{31} & A_2^{33} \end{bmatrix} \ddot{x} + \begin{bmatrix} A_1^{11} & A_1^{13} \\ A_1^{31} & A_1^{33} \end{bmatrix} \dot{x}$$

$$+ \begin{bmatrix} A_0^{11} - A_0^{12}(A_0^{22})^{-1}A_0^{21} & A_0^{13} - A_0^{12}(A_0^{22})^{-1}A_0^{23} \\ A_0^{31} - A_0^{32}(A_0^{22})^{-1}A_0^{21} & A_0^{33} - A_0^{32}(A_0^{22})^{-1}A_0^{23} \end{bmatrix} x$$

$$= \begin{bmatrix} B^1 - A_0^{12}(A_0^{22})^{-1}B^2 \\ B^3 - A_0^{32}(A_0^{22})^{-1}B^2 \end{bmatrix} v$$

$$+ \begin{bmatrix} B_G^1 - A_0^{12}(A_0^{22})^{-1}B_G^2 \\ B_G^3 - A_0^{32}(A_0^{22})^{-1}B_G^2 \end{bmatrix} \delta$$

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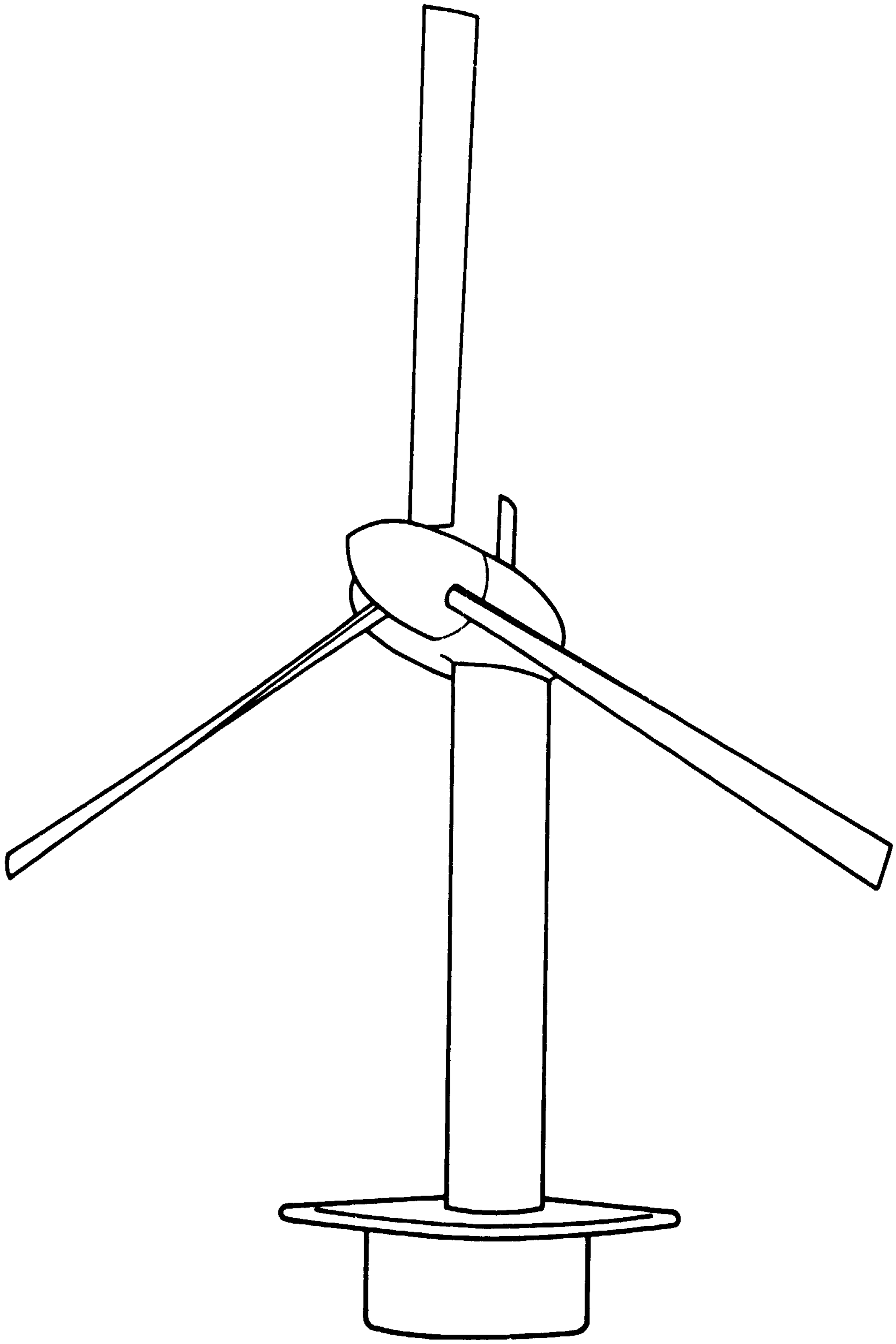


Figure 1. Proprotor and cantilever wing configuration.

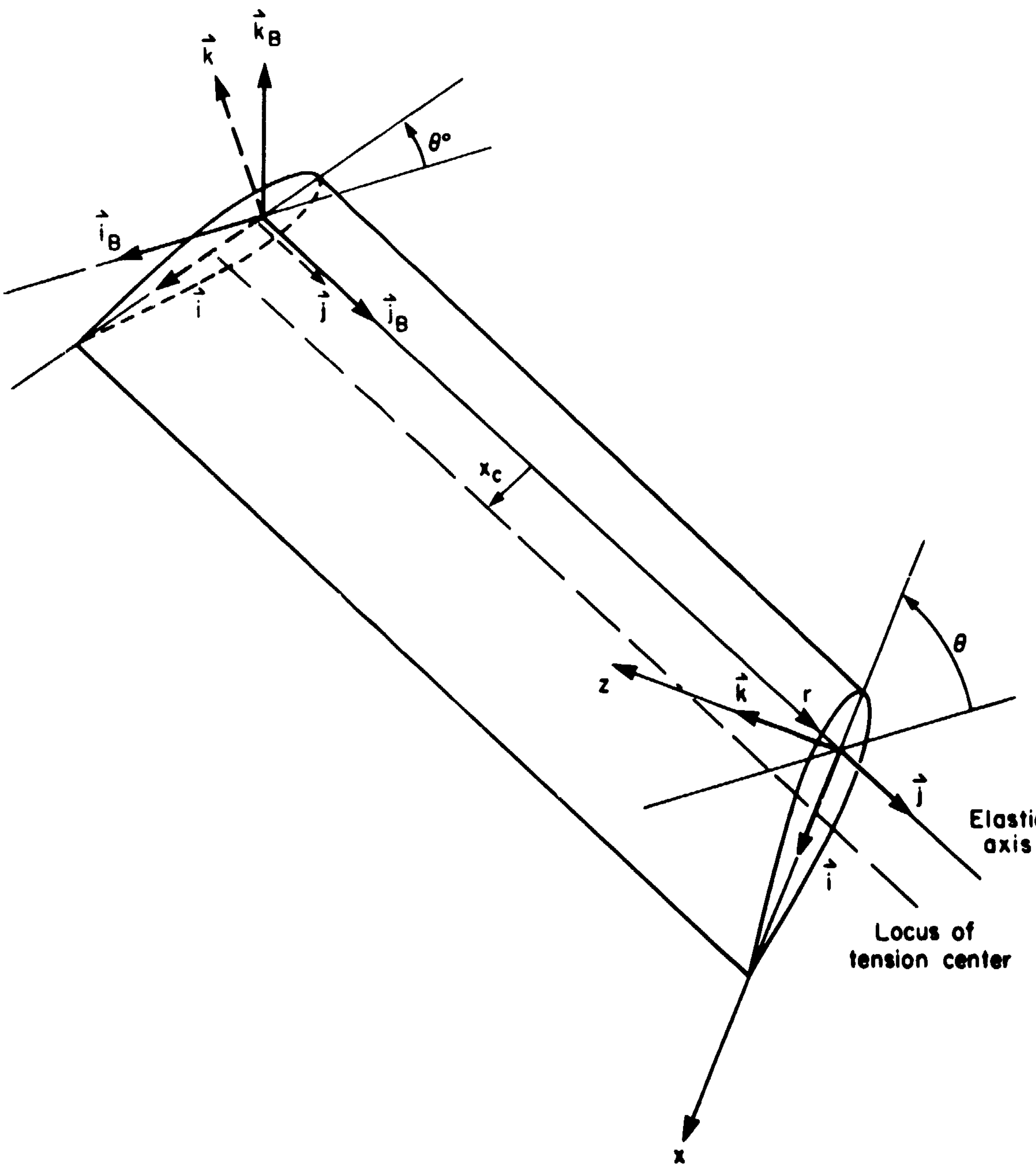


Figure 2. Geometry of undeformed blade.

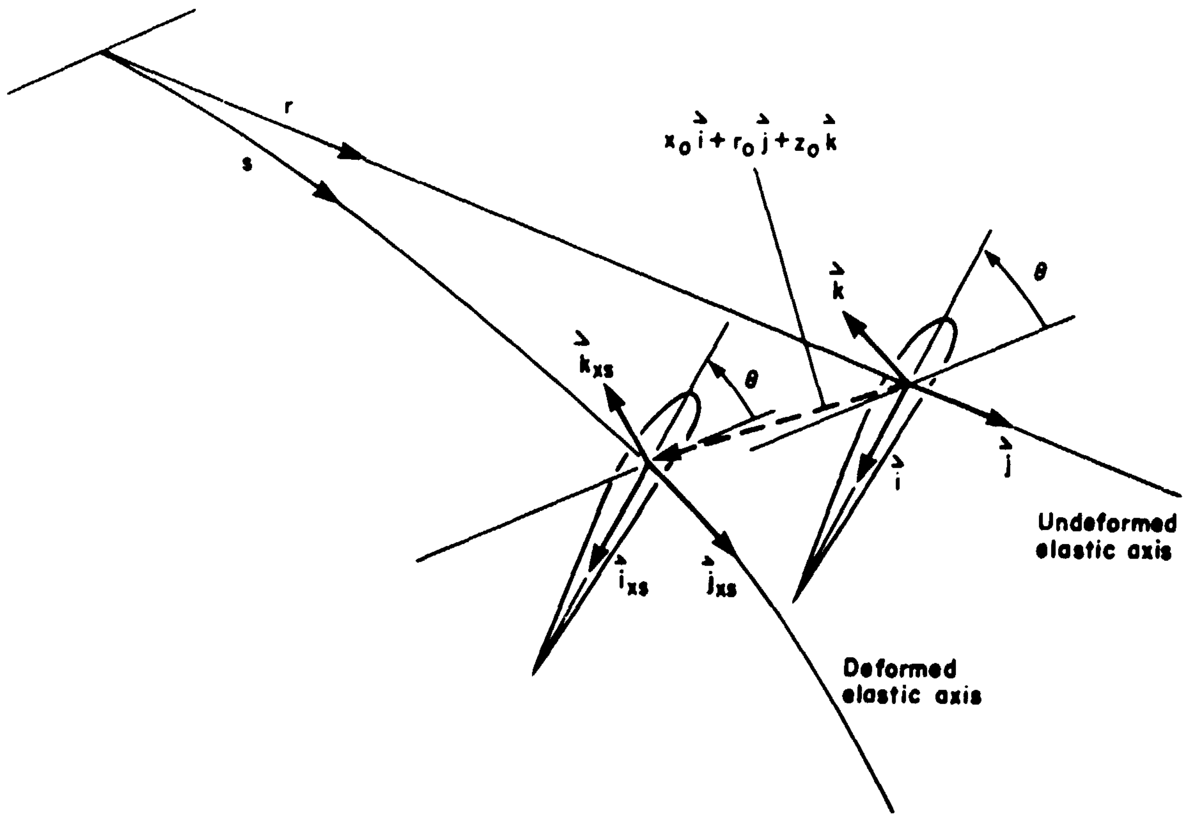


Figure 3. Geometry of deformed blade.

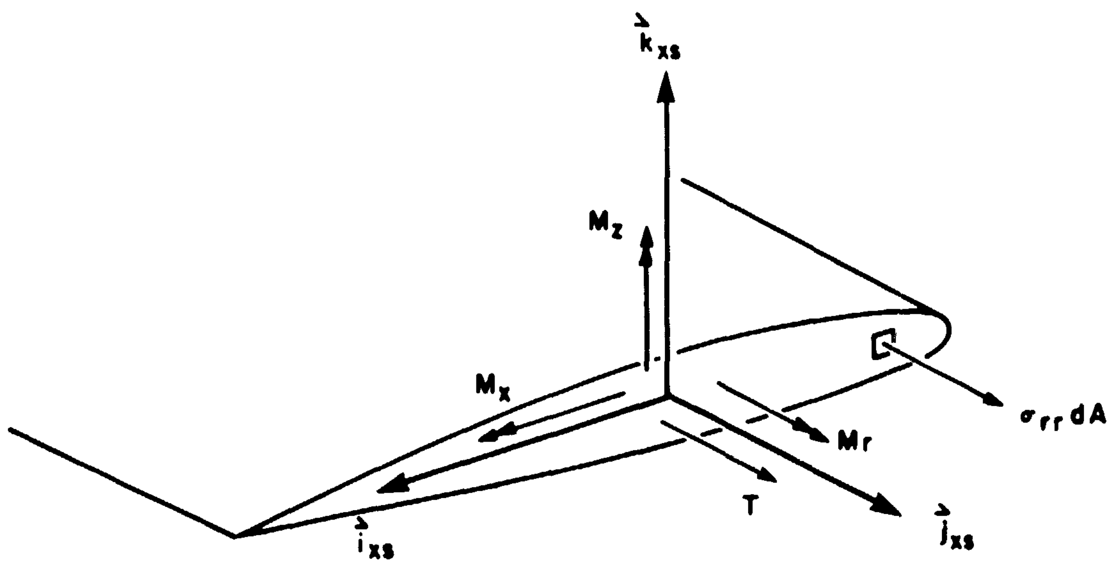


Figure 4. Moments on blade section.

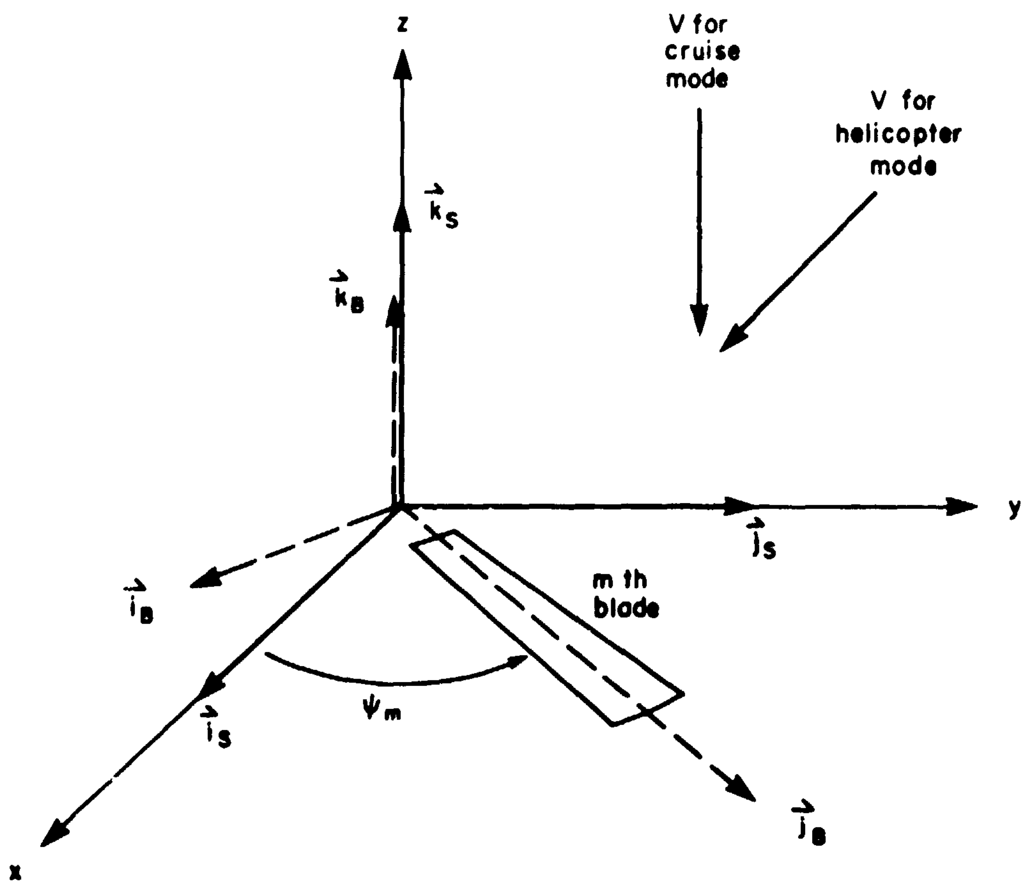


Figure 5. Hub frame coordinate systems (rotating and nonrotating).

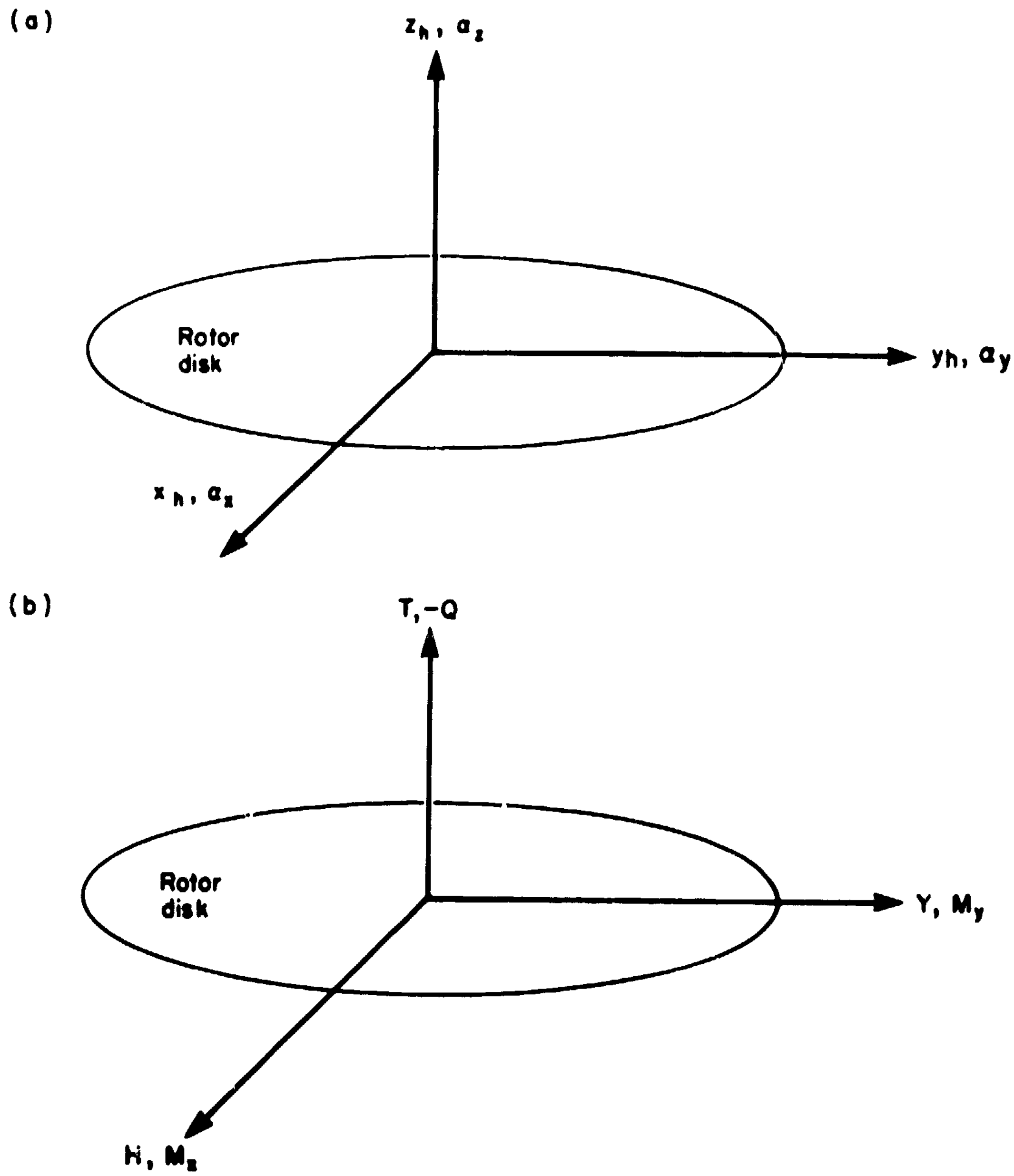
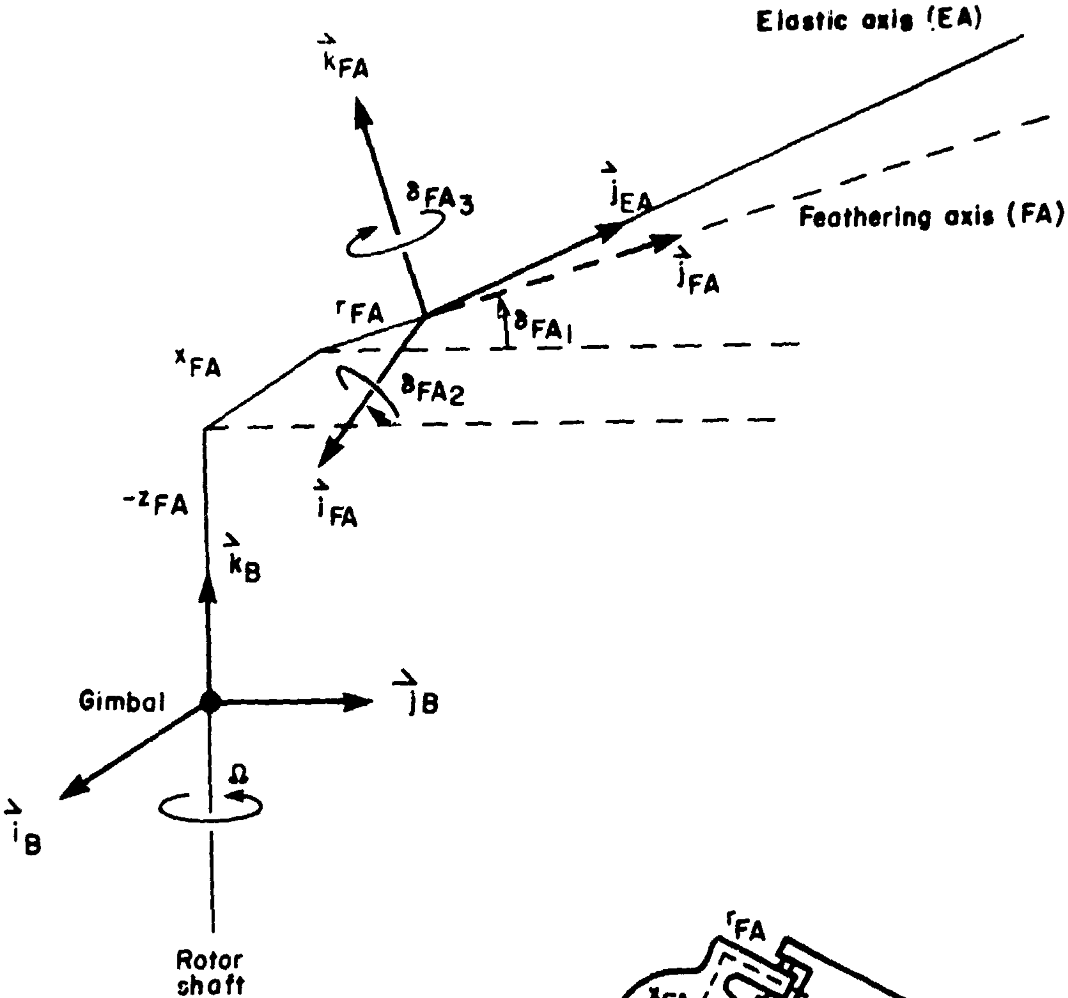


Figure 6. Notation and sign conventions for (a) shaft motion, angular and linear displacement in an inertial frame; and (b) hub forces and moments, on the hub in a nonrotating frame.



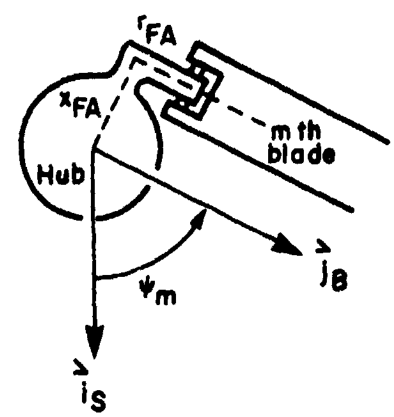


Figure 7. Rotor hub and root geometry (undistorted), showing gimbal undersling (z_{FA}), torque offset (x_{FA}), feathering axis offset (r_{FA}), precone (δ_{FA_1}), droop (δ_{FA_2}), and sweep (δ_{FA_3}).

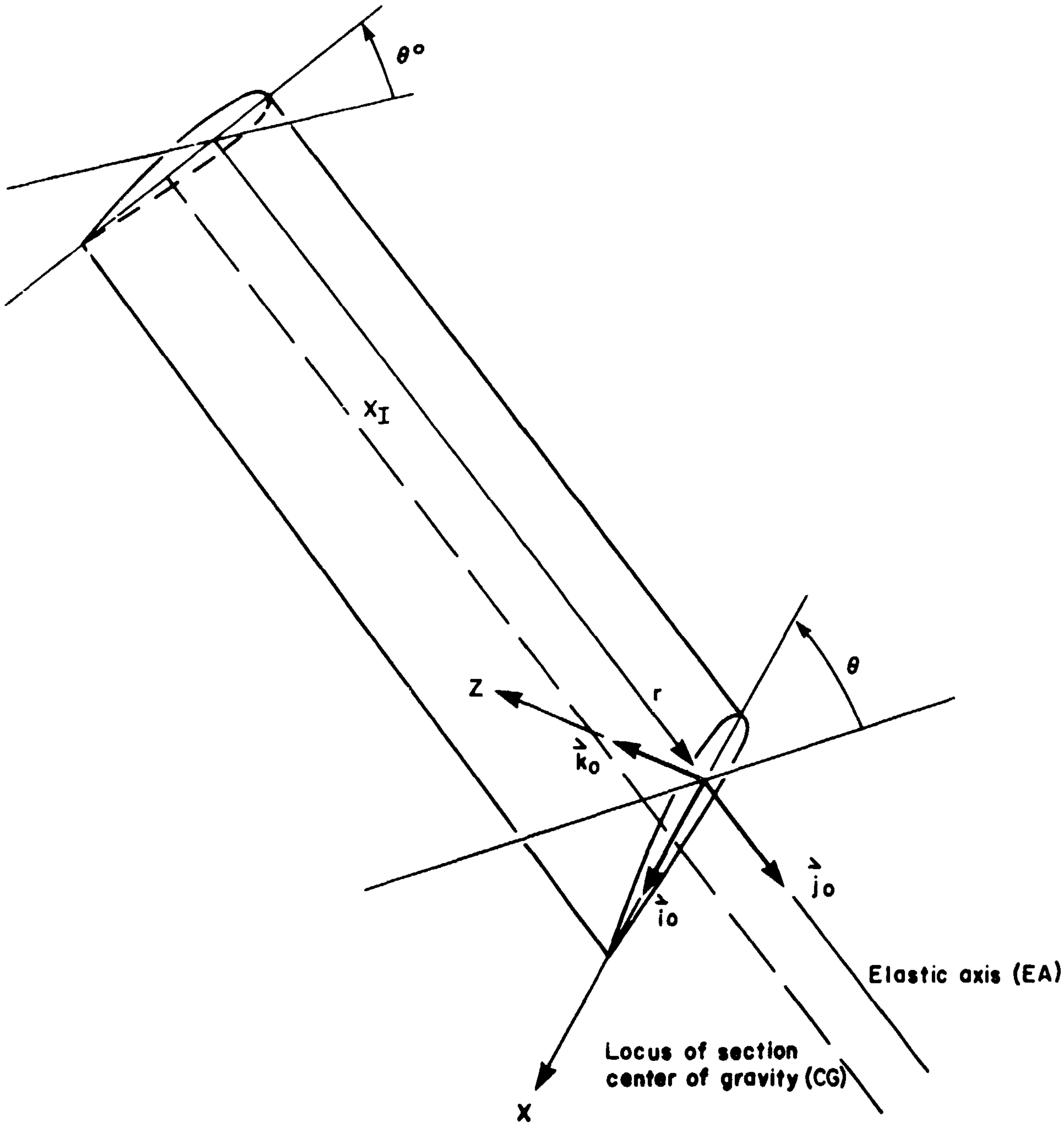


Figure 8. Geometry of undeformed blade.

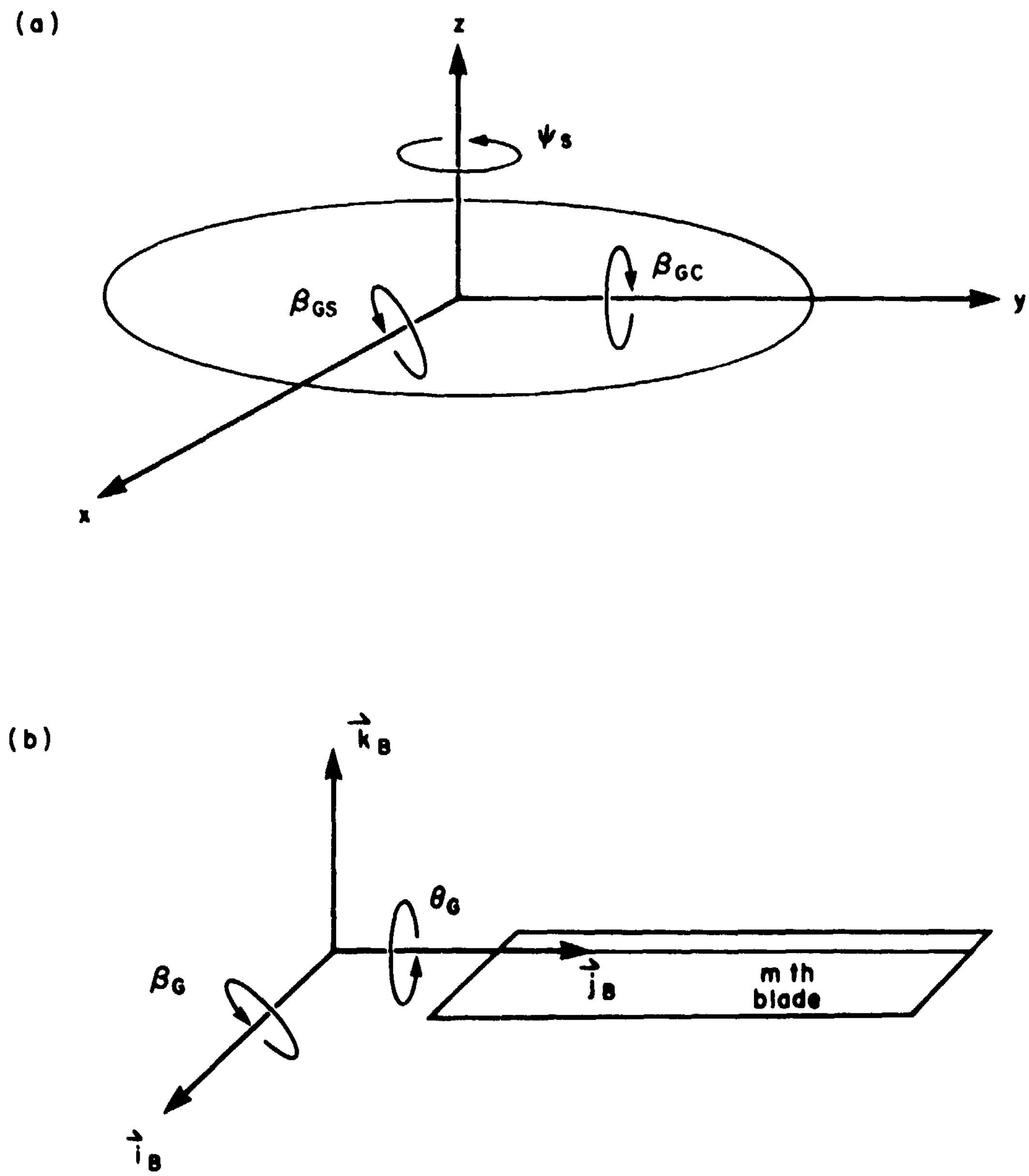


Figure 9. Notation and sign convention for gimbal motion, (a) in the nonrotating frame, and (b) in the rotating frame.

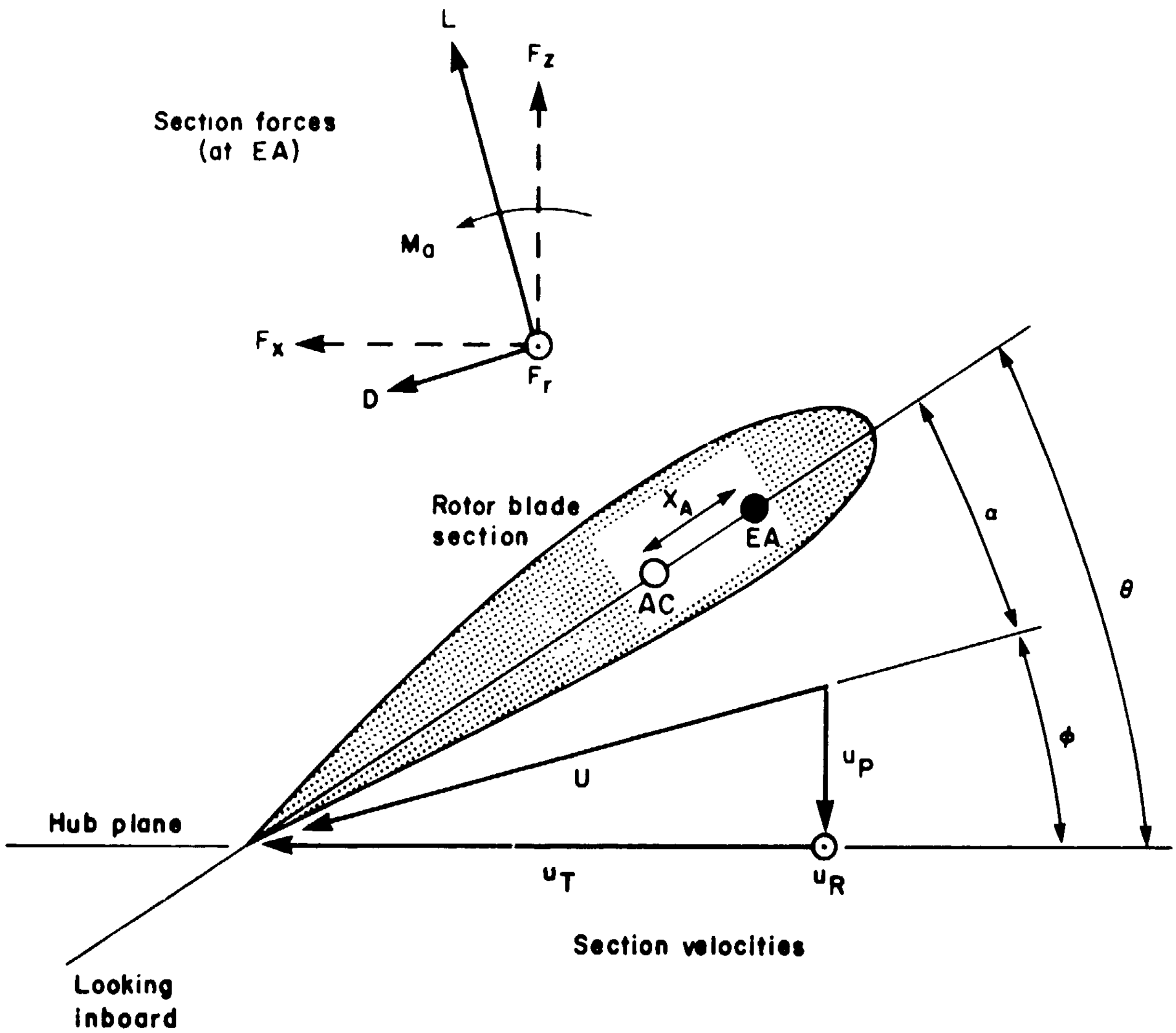


Figure 10. Rotor blade section aerodynamics; notation and sign conventions for section forces and velocities.

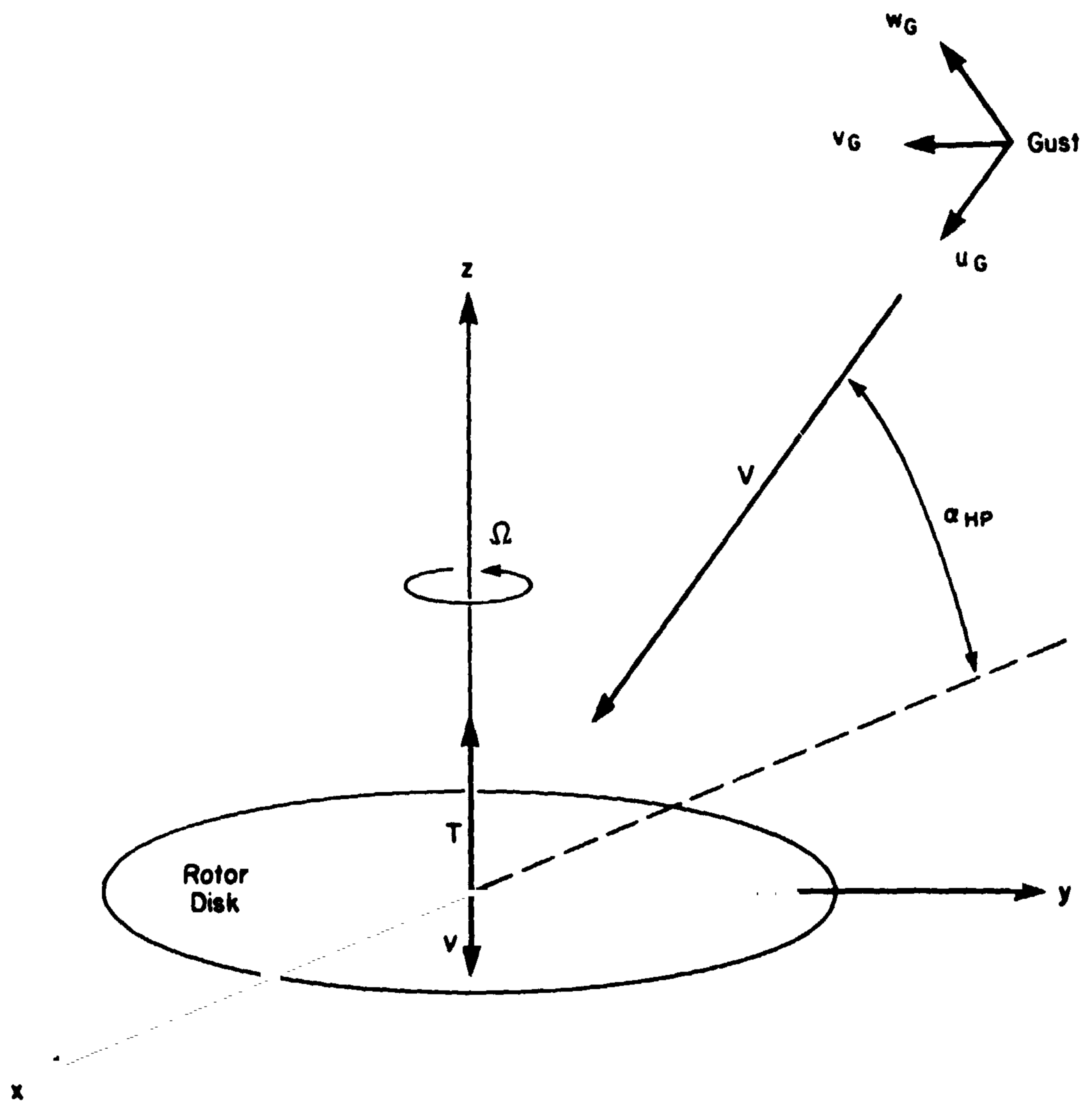


Figure 11. Notation and sign conventions for rotor velocity and orientation (V and α_{HP}), induced velocity (v), and aerodynamic gust velocity components (u_G, v_G, w_G).

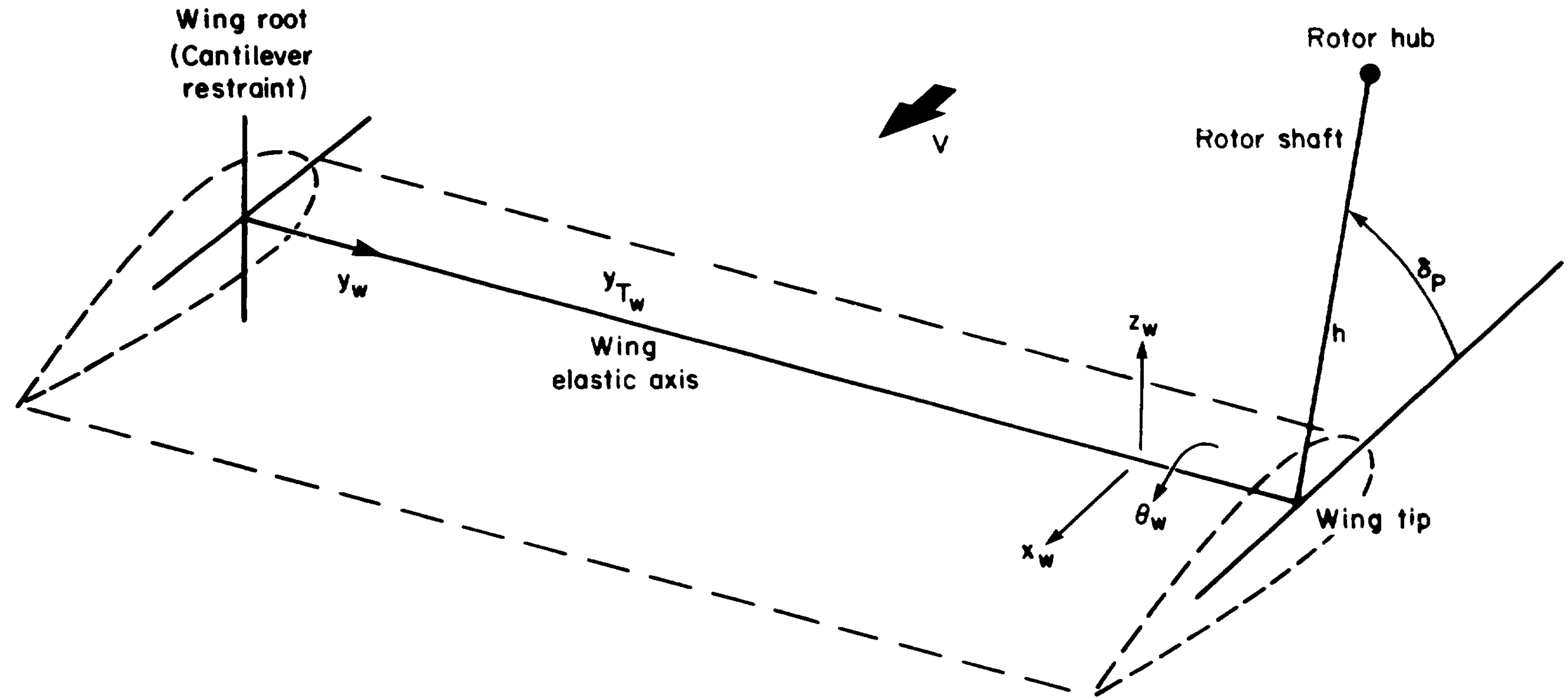


Figure 12. Geometry of cantilever wing and rotor shaft orientation.

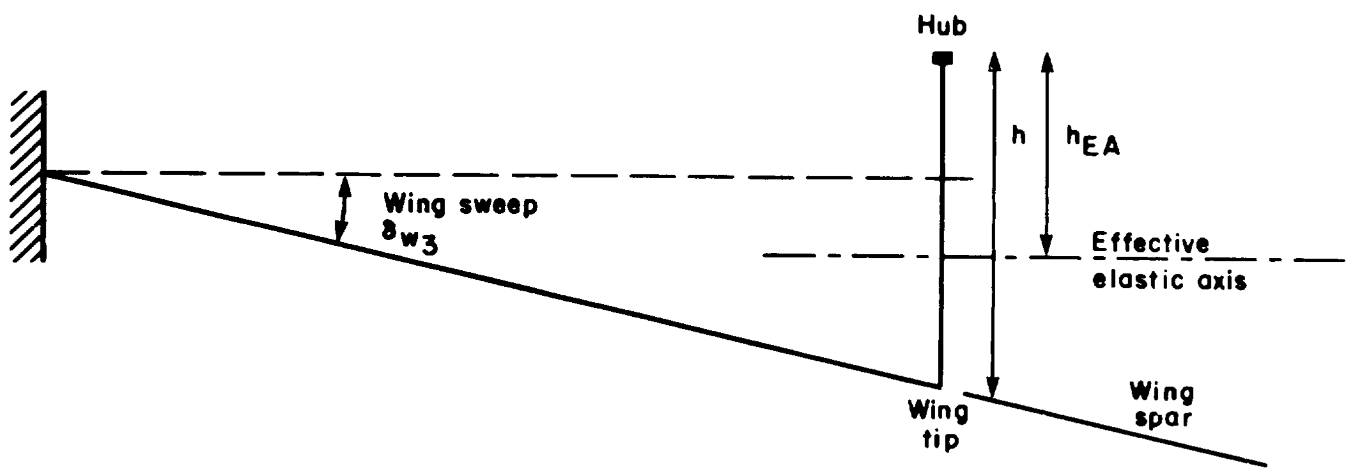


Figure 12. Geometry of wing effective elastic axis.

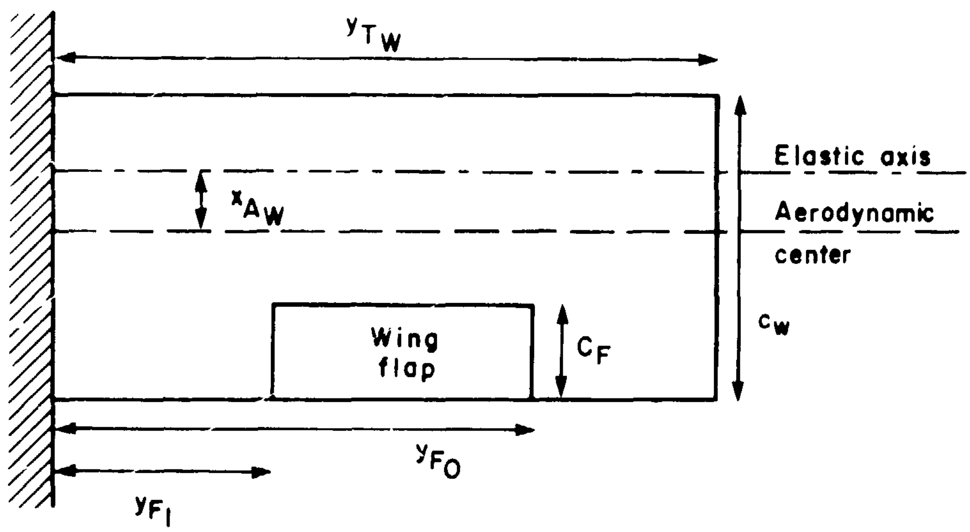


Figure 14. Geometry of wing and wing flap aerodynamics.