Analytical Properties of Ultradiscrete Burgers Equation and Rule–184 Cellular Automaton

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Abstract

In this paper, we propose an ultradiscrete Burgers equation of which variables are all discrete. The equation is derived from discrete Burgers equation under ultradiscrete limit and reduces to an ultradiscrete diffusion equation through the Cole–Hopf transformation. Moreover, it becomes a cellular automaton (CA) under appropriate conditions and is identical to rule–184 CA in a specific case. We show shock wave solutions and asymptotic behaviors of the CA exactly via the diffusion equation. Finally, we propose a particle model expressed by the CA and discuss a mean flux of particles.

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I. INTRODUCTION

There are various discreteness of mathematical models to describe physical phenomena. For example, differential equation, difference equation, coupled map lattice and cellular automaton (CA) exist from fully continuous model to fully discrete one. Among them, CA is the most discrete model of which variables are all discrete [1]. Especially, its dependent variable takes on a finite set of discrete values. Many CA's have been proposed and used as a simulator of phenomenon and analyzed mathematically to grasp behavior of solutions. However, in the analysis, there often exist a difficulty peculiar to CA. For example, when we discuss linear stability or asymptotic behavior of difference equations, we often take continuous limit of the equations. In the case of CA, it is difficult to introduce such an approach due to the discreteness of dependent variable.

As an answer to the above problem, Tokihiro *et al* proposed a non-analytical limit named 'ultradiscrete limit';

$$\lim_{\varepsilon\to+0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \cdots), \tag{1}$$

where $\max(A, B, \cdots)$ returns the maximum element in $\{A, B, \cdots\}$ [2]. They showed the discrete Lotka–Volterra equation can reduce to the box and ball system under this limit. The former is a difference soliton equation with a continuous dependent variable [3]. The latter is a soliton CA defined by using boxes and balls [4]. Both have N–soliton solutions and those of box and ball system are obtained exactly from those of discrete Lotka–Volterra equation again by using the limit.

The ultradiscrete limit is not specific to soliton systems. Since Eq. (1) does not require integrability, it can be applied widely. Indeed, there exist examples where the ultradiscrete limit is applied to a chaotic equation and to an elliptic function [5,6].

In this paper, we apply the above ultradiscrete limit to a discrete analogue to the Burgers equation. Then, we obtain from the equation rule–184 CA of which rules are numbered following Wolfram [1]. The rule–184 CA (and its equivalence) is the only nonlinear CA preserving the number of 1's among CA's in the form

$$U_j^{t+1} = f(U_{j-1}^t, U_j^t, U_{j+1}^t), \tag{2}$$

where j is site number, t is time, U is 0 or 1 and f is a Boolean function. Moreover, solutions to rule–184 CA become steady at large enough t from any initial condition [7–9]. Due to these remarkable properties, the CA is often used as a base of traffic flow model [10,11].

Contents of this paper are as follows. In Sec. 2, using ultradiscrete limit, we show the discrete Burgers equation reduces to the ultradiscrete Burgers equation which can be CA under a specific condition. We call this CA Burgers CA. The discrete Burgers equation reduces to a linear discrete diffusion equation and Burgers CA also reduces to an ultradiscrete diffusion equation. In Sec. 3, we show Burgers CA with a specific parameter becomes rule–184 CA. Moreover, we derive shock wave solutions to Burgers CA obtained from ultradiscrete limit of discrete solutions. In Sec. 4, we show an asymptotic behavior of solutions to Burgers CA using ultradiscrete diffusion equation. Solution from any initial state becomes steady at large enough time. In Sec. 5, we propose a particle model expressed by Burgers CA. Mean flux of particles becomes constant at large enough time and the constant value depends only on density of particles. In Sec 6, we give concluding remarks and future problems. Throughout the results, we use properties of ultradiscrete diffusion equation, as we do for the continuous Burgers equation.

II. DISCRETE BURGERS EQUATION AND ITS ULTRADISCRETIZATION

First, we derive a discrete Burgers equation by using a discrete Cole–Hopf transformation. The continuous Burgers equation is

$$v_t = 2vv_x + v_{xx}. \tag{3}$$

It is well-known that this equation can be linearized through the Cole–Hopf transformation given by

$$v = \frac{f_x}{f}, \tag{4}$$

into the diffusion equation

$$f_t = f_{xx}. \tag{5}$$

To discretize Eq. (3), we utilize discrete analogues to Eqs. (4) and (5) [12]. Discretizing both time and space variables in Eq. (5), a discrete diffusion equation

$$\frac{f_j^{t+1} - f_j^t}{\Delta t} = \frac{f_{j+1}^t - 2f_j^t + f_{j-1}^t}{(\Delta x)^2}, \tag{6}$$

is obtained where Δt and Δx are lattice intervals in t and x respectively. Next we define a discrete analogue to the Cole–Hopf transformation

$$u_j^t \equiv c\frac{f_{j+1}^t}{f_j^t}, \tag{7}$$

where c is a constant. Rewriting Eq. (6) with u_j^t in place of f_j^t we obtain

$$u_j^{t+1} = u_{j-1}^t \frac{1 + \frac{1-2\delta}{c\delta}u_j^t + \frac{1}{c^2}u_j^t u_{j+1}^t}{1 + \frac{1-2\delta}{c\delta}u_{j-1}^t + \frac{1}{c^2}u_{j-1}^t u_j^t}, \tag{8}$$

where $\delta = \Delta t/(\Delta x)^2$. Assuming $v(j\Delta x, t\Delta t) = \dfrac{1}{\Delta x}\log\dfrac{u_j^t}{c}$ and taking limits $\Delta x \to 0$ and $\Delta t \to 0$, we obtain Eq. (3) from Eq. (8). Therefore, we can consider Eq. (8) is a discrete analogue to Burgers equation (3) and call Eq. (8) 'discrete Burgers equation'.

Next, we 'ultradiscretize' Eq. (8), that is, discretize a dependent variable u using Eq. (1). Let us introduce a transformation of variables and parameters as follows:

$$u_j^t = e^{U_j^t/\varepsilon}, \tag{9}$$

$$\frac{1-2\delta}{c\delta} = e^{-M/\varepsilon}, \tag{10}$$

$$c^2 = e^{L/\varepsilon}. \tag{11}$$

Then, Eq. (8) reduces to

$$\begin{aligned} U_j^{t+1} = U_{j-1}^t &+ \varepsilon\log(1 + \exp(\frac{U_j^t - M}{\varepsilon}) + \exp(\frac{U_j^t + U_{j+1}^t - L}{\varepsilon})) \\ &-\varepsilon\log(1 + \exp(\frac{U_{j-1}^t - M}{\varepsilon}) + \exp(\frac{U_{j-1}^t + U_j^t - L}{\varepsilon})). \end{aligned} \tag{12}$$

Taking a limit $\varepsilon \to +0$ and using the relation (1), we obtain

$$U_j^{t+1} = U_{j-1}^t + \max(0, U_j^t - M, U_j^t + U_{j+1}^t - L) \\ - \max(0, U_{j-1}^t - M, U_{j-1}^t + U_j^t - L). \tag{13}$$

Using identities

$$\max(A, B, \cdots) = -\min(-A, -B, \cdots), \tag{14}$$

$$\min(A, B, \cdots) + X = \min(A + X, B + X, \cdots), \tag{15}$$

the above equation becomes

$$U_j^{t+1} = U_j^t + \min(M, U_{j-1}^t, L - U_j^t) - \min(M, U_j^t, L - U_{j+1}^t). \tag{16}$$

If initial U and parameters M and L are all integer, then U for any t and j is always integer. Thus, we obtain an equation with all discrete variables by the ultradiscrete limit (1). We call Eq. (16) 'ultradiscrete Burgers equation'.

Under an appropriate condition, Eq. (16) becomes a CA. Assume that $M > 0$, $L > 0$ and $0 \le U_j^t \le L$ for any j at a certain t. Then, relations

$$\begin{aligned} &\min(M, U_{j-1}^t, L - U_j^t) \ge 0, \\ &\min(M, U_j^t, L - U_{j+1}^t) \ge 0, \\ &\min(M, U_{j-1}^t, L - U_j^t) + U_j^t = \min(M + U_j^t, U_{j-1}^t + U_j^t, L) \le L, \\ &\min(M, U_j^t, L - U_{j+1}^t) - U_j^t = \min(M - U_j^t, 0, L - U_{j+1}^t - U_j^t) \le 0, \end{aligned} \tag{17}$$

hold. Therefore, $0 \le U_j^{t+1} \le L$ holds for any j. This means Eq. (16) under the above condition is equivalent to a CA with a value set $\{0, 1, \cdots, L\}$. We call this CA 'Burgers CA', shortly BCA.

Moreover, introducing a transformation

$$f_j^t = \exp(F_j^t/\varepsilon), \tag{18}$$

an ultradiscrete Cole–Hopf transformation

$$U_j^t = F_{j+1}^t - F_j^t + \frac{L}{2}, \tag{19}$$

is obtained from Eq. (7) under the limit $\varepsilon \to +0$. Then, we obtain an ultradiscrete diffusion equation;

$$F_j^{t+1} = \max(F_{j-1}^t,\ F_j^t + \frac{L}{2} - M,\ F_{j+1}^t), \tag{20}$$

from Eq. (16). This equation can also be obtained from Eq. (6) with Eq. (18) under $\varepsilon \to +0$.

III. RELATION TO RULE–184 CA AND SHOCK WAVE SOLUTIONS OF BURGERS CA

In this section, we put a restriction, $L \le M$, on BCA for simplicity. Then, Eq. (16) reduces to

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t), \tag{21}$$

because any U_j^t satisfies $0 \le U_j^t \le L$ and is equal to or smaller than M.

Next, let us consider the case $L = 1$ for Eq. (21). The evolution rule for Eq. (21) is expressed symbolically by

$$\frac{U_{j-1}^t\, U_j^t\, U_{j+1}^t}{U_j^{t+1}} = \frac{000}{0}, \frac{001}{0}, \frac{010}{0}, \frac{011}{1}, \frac{100}{1}, \frac{101}{1}, \frac{110}{0}, \frac{111}{1}. \tag{22}$$

This rule is equivalent to that of rule–184 CA given by the following Boolean expression

$$U_j^{t+1} = (U_{j-1}^t \wedge \overline{U_j^t}) \vee (U_j^t \wedge U_{j+1}^t) \tag{23}$$

where $\wedge$, $\vee$ and $\overline{}$ denote AND, OR and NOT in Boolean operation respectively [1]. Therefore, we can conclude that BCA includes rule–184 CA as a special case. Note that various expressions using max and min functions can include the rule–184 CA. For example, replacing $x \wedge y$, $x \vee y$ and $\overline{x}$ with $\min(x, y)$, $\max(x, y)$ and $1 - x$ respectively in Eq. (23), we obtain

$$U_j^{t+1} = \max(\min(U_{j-1}^t, 1 - U_j^t), \min(U_j^t, U_{j+1}^t)), \tag{24}$$

which is equivalent to rule–184 CA if U is restricted to 0 or 1. However, Eqs. (21) and (24) are not equivalent if U can take an arbitrary integer value.

Then we derive solutions to Eq. (21) from shock wave solutions to discrete Burgers equation (8). Let us assume f_j^t has the following form;

$$f_j^t = 1 + \exp(kj + \omega t + \xi_0), \tag{25}$$

where k, ω and ξ_0 are constants. Substituting Eq. (25) into Eq. (6), we obtain a dispersion relation

$$\omega = \log(1 + \delta(e^k - 2 + e^{-k}). \tag{26}$$

Thus we obtain a solution

$$u_j^t = c\frac{f_{j+1}^t}{f_j^t} = c\frac{1 + \exp(k(j+1) + \omega t + \xi_0)}{1 + \exp(kj + \omega t + \xi_0)}. \tag{27}$$

This is a shock wave solution to discrete Burgers equation (8). From this solution, we obtain a shock wave solution to ultradiscrete Burgers equation (21) by the ultradiscrete limit. Assuming

$$k = \frac{K}{\varepsilon},\ \omega = \frac{\Omega}{\varepsilon},\ \xi_0 = \frac{\Xi_0}{\varepsilon}, \tag{28}$$

and noticing Eqs. (9) and (11), we obtain

$$U_j^t = \frac{L}{2} + \max(0, K(j+1) + \Omega t + \Xi_0) - \max(0, Kj + \Omega t + \Xi_0). \tag{29}$$

From Eq. (26) and the condition $L \leq M$, a dispersion relation

$$\Omega = |K| \tag{30}$$

is obtained. If $K > 0$, $\lim_{j\to-\infty} U_j^t = \frac{L}{2}$ and $\lim_{j\to+\infty} U_j^t = \frac{L}{2} + K$. If $K < 0$, $\lim_{j\to-\infty} U_j^t = \frac{L}{2} + K$ and $\lim_{j\to+\infty} U_j^t = \frac{L}{2}$. We can easily see that the above solution is a propagating wave with a speed -1 ($K > 0$) or $+1$ ($K < 0$) and its shape is like a step as shown in Fig. 1. Since any U_j^t must be an integer value from 0 to L, it is necessary for the above solution that L is an even positive integer, $|K| \leq L/2$ and Ξ_0 is an integer.

IV. ASYMPTOTIC BEHAVIOR OF BURGERS CA

About the rule–184 CA (22) with a periodic boundary condition, it is known that U_j^t at large enough t becomes a steady solution [7–9]. There are two types of such solutions, one is $U_j^{t+1} = U_{j-1}^t$ and the other is $U_j^{t+1} = U_{j+1}^t$. Which type is selected depends on the total number of 1's. So far such a behavior has been mainly derived by pattern analysis on 1–0 sequences. In this paper, since we obtain the relation between rule–184 CA and ultradiscrete Burgers equation reducible to ultradiscrete diffusion equation, we can derive the asymptotic behavior from analytic properties of the equations. Moreover, we can show Burgers CA as an extension of rule–184 CA has similar properties described above.

First let us assume the space site of Eq. (16) is periodic with period K, that is, $U_j^t = U_{j+K}^t$. Then, we can easily see that $\sum_{i=1}^{K} U_i^t$ is constant for t. Therefore, defining ρ by

$$\rho = \frac{1}{KL}\sum_{i=1}^{K} U_i^t, \tag{31}$$

ρ is constant. If we set initial value U_j^0 at an initial time $t = 0$, we can construct F_j^0 from an inverse relation of Eq. (19),

$$F_j^0 = \begin{cases} \sum_{i=0}^{j-1}(U_i^0 - \frac{L}{2}) & \text{if } j \geq 1 \\ U_0^0 - \frac{L}{2} - \sum_{i=0}^{j}(U_i^0 - \frac{L}{2}) & \text{otherwise} \end{cases}. \tag{32}$$

Note that F_j^0 has a freedom of constant and we set $F_0^0 = 0$ in the above equation. Moreover, F_j^0 is not periodic and

$$F_{j+K}^0 - F_j^0 = \sum_{i=j}^{j+K-1}(U_i^0 - \frac{L}{2}) = KL(\rho - \frac{1}{2}). \tag{33}$$

Then, we can calculate F_j^t for $t > 0$ using Eq. (20) and obtain U_j^t by Eq. (19). This U_j^t also satisfies Eq. (16) with the above initial value U_j^0. That is, we can grasp the dynamics of BCA by Eq. (20) in place of Eq. (16).

Next we show asymptotic behavior of U_j^t at large enough t. We can assume K is even without loss of generality because we can consider the period is $2K$ if K is odd.

Case 1 : $\dfrac{L}{2} \le M$

From Eq. (20), we obtain

$$F_j^t = \max(\max(F_{j-t}^0, F_{j-t+2}^0, \cdots, F_{j+t}^0), \max(F_{j-t+1}^0, F_{j-t+3}^0, \cdots, F_{j+t-1}^0) + \alpha), \tag{34}$$

where $\alpha = \dfrac{L}{2} - M$.

Case 1.1 : $\rho < \dfrac{1}{2}$

In this case, $F_{j+K}^0 < F_j^0$ holds from Eq. (33). Then

$$F_j^t = \max(\max(F_{j-t}^0, F_{j-t+2}^0, \cdots), \max(F_{j-t+1}^0, F_{j-t+3}^0, \cdots) + \alpha), \tag{35}$$

is derived for $t \ge \dfrac{K}{2}$. Therefore,

$$F_j^{t+1} = F_{j-1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j-1}^t, \tag{36}$$

are obtained. Figure 2(a) shows an example of evolution.

Substituting Eq. (36) into Eq. (20), we obtain

$$0 = \max(0, F_j^t - F_{j-1}^t + \alpha, F_{j+1}^t - F_{j-1}^t) = \max(0, U_{j-1}^t - M, U_{j-1}^t + U_j^t - L). \tag{37}$$

From this condition,

$$U_j^t \le M \qquad \text{and} \qquad U_j^t \le L - U_{j+1}^t, \tag{38}$$

hold for any j. In the case of rule–184 CA ($L = 1$, $M \ge 1$), the above condition means the sequence $U_1^t U_2^t \cdots U_K^t$ contains only 00, 01, 10 and not 11.

Case 1.2 : $\rho = \dfrac{1}{2}$

Since $F_{j+K}^0 = F_j^0$,

$$F_j^t = \begin{cases} \max(\max(F_2^0, F_4^0, \cdots, F_K^0), \max(F_1^0, F_3^0, \cdots, F_{K-1}^0) + \alpha) & \text{if } j-t \text{ is even} \\ \max(\max(F_1^0, F_3^0, \cdots, F_{K-1}^0), \max(F_2^0, F_4^0, \cdots, F_K^0) + \alpha) & \text{otherwise} \end{cases}, \tag{39}$$

is derived for $t \ge \dfrac{K}{2}$. Therefore, we obtain $F_j^{t+1} = F_{j\pm1}^t$ and $U_j^{t+1} = U_{j\pm1}^t$. Figure 2(b) shows an example of evolution. Substituting $F_j^{t+1} = F_{j\pm1}^t$ into Eq. (20), we get

$$U_j^t \le M \qquad L - U_{j+1}^t \le M \qquad \text{and} \qquad U_j^t = L - U_{j+1}^t, \tag{40}$$

for any j. In the case of rule–184 CA, the above condition means the sequence $U_1^t U_2^t \cdots U_K^t$ is $0101 \cdots 01$ or $1010 \cdots 10$.

Case 1.3 : $\rho > \dfrac{1}{2}$

By the similar discussion to Case 1.1, we obtain

$$F_j^{t+1} = F_{j+1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j+1}^t, \tag{41}$$

and

$$L - U_{j+1}^t \le U_j^t \qquad \text{and} \qquad L - U_j^t \le M, \tag{42}$$

for $t \ge \dfrac{K}{2}$. Figure 2(c) shows an example of evolution. In the case of rule–184 CA, the above condition means the sequence $U_1^t U_2^t \cdots U_K^t$ contains only 01, 10, 11 and not 00.

Case 2 : $\dfrac{L}{2} > M$

From Eq. (20), we obtain

$$F_j^t = \max(F_{j-t}^0, F_{j-t+1}^0 + \alpha, \cdots, F_j^0 + t\alpha, \cdots, F_{j+t-1}^0 + \alpha, F_{j+t}^0). \tag{43}$$

Case 2.1 : $\rho < \dfrac{M}{L}$

In this case, $F_{j+K}^t + K\alpha < F_j^t$ holds. Therefore,

$$F_j^t = \max(F_{j-t}^0, F_{j-t+1}^0 + \alpha, \cdots, F_{j-t+K-1}^0 + (K-1)\alpha), \tag{44}$$

is derived for $t \ge K$. Then, we obtain

$$F_j^{t+1} = F_{j-1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j-1}^t, \tag{45}$$

and

$$U_j^t \le M \qquad \text{and} \qquad U_j^t \le L - U_{j+1}^t. \tag{46}$$

Figure 3(a) shows an example of evolution.

Case 2.2 : $\dfrac{M}{L} \le \rho \le 1 - \dfrac{M}{L}$

In this case, since $|F^t_{j+K} - F^t_j| = KL|\rho - \frac{1}{2}| \le K\alpha$, we can derive

$$F^t_{j\pm K} \le F^t_j + K\alpha. \tag{47}$$

Therefore, we obtain

$$\begin{aligned} F^t_j = \max(&F^0_{j-K+1}, F^0_{j-K+2} + \alpha, \cdots, F^0_j + (K-1)\alpha, \\ &\cdots, F^0_{j+K-2} + \alpha, F^0_{j+K-1}) + (t - K + 1)\alpha, \end{aligned} \tag{48}$$

from Eq. (43) for $t \ge K$. Using this relation,

$$F^{t+1}_j = F^t_j + \alpha \qquad \text{and} \qquad U^{t+1}_j = U^t_j, \tag{49}$$

and

$$M \le U^t_j \le L - M, \tag{50}$$

hold for $t \ge K$. Figure 3(b) shows an example of evolution.

Case 2.3 : $\rho > 1 - \frac{M}{L}$

By the similar discussion to Case 2.1,

$$F^{t+1}_j = F^t_{j+1} \qquad \text{and} \qquad U^{t+1}_j = U^t_{j+1}, \tag{51}$$

and

$$L - U^t_{j+1} \le U^t_j \qquad \text{and} \qquad L - U^t_j \le M, \tag{52}$$

hold for $t \ge K$. Figure 3(c) shows an example of evolution.

V. PARTICLE MODEL EXPRESSED BY BURGERS CA

In rule–184 CA (22), the number of 1's is conserved for time and the evolution rule is interpreted as the following motion of particles [9]:

Each site can hold one particle at most. U_j^t denotes the number of particles at site j and time t. From t to $t+1$, particles move to their right site if the site is empty at t and do not move otherwise.

BCA (16) including rule–184 CA as a special case can express the following particle model:

Each site can hold L particles at most. U_j^t denotes the number of particles at site j and time t. From t to $t+1$, particles at site j can move to site $j+1$. The maximum number of movable particles is M. Under this restriction, they move to vacant space at site $j+1$ as many as possible.

According to the above rule, the number of movable particles at site j and time t is $\min(M, U_j^t, L - U_{j+1}^t)$. Therefore, U_j^{t+1} is calculated by Eq. (16). We can easily see from the above rule that the total number of particles is conserved.

Next let us consider a mean flux of particles [11]. If q^t denotes the mean flux, it is defined by

$$q^t = \frac{1}{KL}\sum_{i=1}^{K}\min(M, U_i^t, L - U_{i+1}^t). \tag{53}$$

From the results of the previous section, we can show q^t becomes constant at large enough t and the constant value depends only on the particle density ρ and not on the initial distribution of particles. For example, in Case 1.1 ($\frac{L}{2} \le M$, $\rho < \frac{1}{2}$), we get

$$q^t = \frac{1}{KL}\sum_{i=1}^{K} U_i^t = \rho \qquad (t \ge \frac{K}{2}), \tag{54}$$

since $\min(M, U_j^t, L - U_{j+1}^t) = U_j^t$ from Eq. (38). By similar discussions, in the case of $\frac{L}{2} \le M$,

$$q^t = \begin{cases} \rho & \text{if } \rho \le \frac{1}{2} \\ 1-\rho & \text{otherwise} \end{cases} \qquad (t \ge \frac{K}{2}), \tag{55}$$

and in the case of $\frac{L}{2} > M$,

$$q^t = \begin{cases} \rho & \text{if } \rho < \frac{M}{L} \\ \frac{M}{L} & \text{if } \frac{M}{L} \le \rho \le 1 - \frac{M}{L} \\ 1-\rho & \text{otherwise} \end{cases} \qquad (t \ge K). \tag{56}$$

Especially, we can show that q^t increases monotonically on t in the case of $\frac{L}{2} \leq M$. Using Eq. (16), $\sum_{i=1}^{K} U_i^t = \text{const.}$ and $\frac{L}{2} \leq M$, we obtain

$$\begin{aligned} q^{t+1} - q^t &= \frac{1}{KL} \sum_{i=1}^{K} \{\min(M, U_i^{t+1}, L - U_{i+1}^{t+1}) - \min(M, U_i^t, L - U_{i+1}^t)\} \\ &= \frac{1}{KL} \sum_{i=1}^{K} \max(0, g(U_i^t, U_{i+1}^t, U_{i+2}^t, U_{i+3}^t)) \geq 0, \end{aligned} \tag{57}$$

where

$$\begin{aligned} g(a_0, a_1, a_2, a_3) = &\min(2L, L + M + a_3, L + a_2 + a_3, M + a_1 + a_2 + a_3, a_0 + a_1 + a_2 + a_3) \\ &- \min(L + M + a_1, L + a_0 + a_1, 2M + a_1 + a_3, M + a_0 + a_1 + a_3). \end{aligned}$$

Figure 4 shows an evolution of q^t obtained from the same data as in Fig. 2(c). Since q^t is a finite value, q^t becomes constant at $t \gg 0$. Then $g(U_j^t, U_{j+1}^t, U_{j+2}^t, U_{j+3}^t) = 0$ is obtained for any j and we can derive the same results as in the previous section about the asymptotic behavior.

VI. CONCLUDING DISCUSSIONS

In this paper, the main results are as follows:

(i) The relation between Burgers equation and rule–184 CA is clarified via discrete and ultradiscrete Burgers equations. Under specific conditions, ultradiscrete Burgers equation can be Burgers CA including rule–184 CA.

(ii) Shock wave solutions exist in Burgers CA which is derived from discrete shock wave solutions under ultradiscrete limit.

(iii) Any solution to Burgers CA with periodic boundary condition becomes steady at large enough time. The sequence of U_j^t converges to a stable pattern shifting right or left, or to a static pattern. Only ρ decides which pattern is selected.

(iv) Burgers CA expresses an evolutional system of moving particles. The mean flux of particles becomes constant at large enough time. The constant value depends only on density of particles. In the specific case, the mean flux increases monotonically on time.

In the above results, linear diffusion equation obtained by Cole–Hopf transformation plays an important role. In the discrete Burgers equation, shock wave solutions and asymptotic behavior can be grasped through diffusion equation. In Burgers CA, corresponding results are obtained by parallel discussions. We can consider such a relation between discrete equation and CA can introduce a new viewpoint to discrete analysis.

On the other hand, Burgers CA and rule–184 CA are easy to analyze since they are related to discrete Burgers equation which can be analyzed exactly. There exist CA's in the form of Eq. (2) of which solutions show chaotic behavior. If we discuss such type of CA, it may be difficult to show what structure of CA is preserved in the corresponding discrete equation. This is left as a future problem.

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FIGURE CAPTIONS

Figure 1 : Shock wave solution. (a) $K > 0$, (b) $K < 0$.

Figure 2 : Time evolution from random initial state for $L = 3$, $M = 2$ and $K = 30$. Black, dark gray, light gray, white square denote a site with value 3, 2, 1, 0, respectively. (a) $\rho = 0.4$, (b) $\rho = 0.5$, (c) $\rho = 0.6$.

Figure 3 : Time evolution from random initial state for $L = 3$, $M = 1$ and $K = 30$. Black, dark gray, light gray, white square denote a site with value 3, 2, 1, 0, respectively. (a) $\rho = 0.3$, (b) $\rho = 0.4$, (c) $\rho = 0.7$.

Figure 4 : Monotonical increase of q^t. The same data as in Fig. 2(c) is used.

FIGURES

FIG. 1.

FIG. 2.

FIG. 3.

FIG. 4.

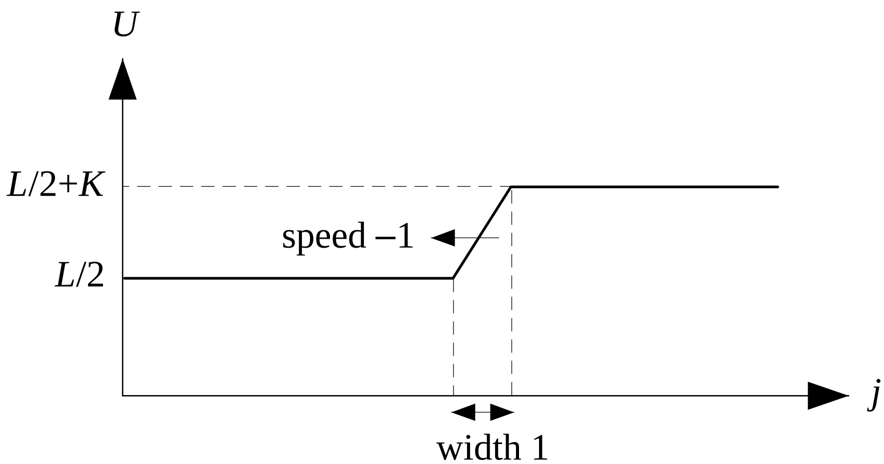

Figure 1 (a)

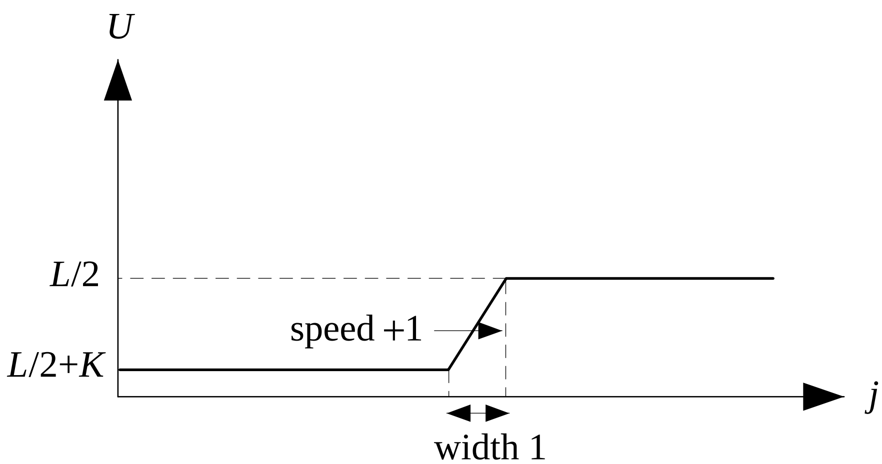

Figure 1 (b)

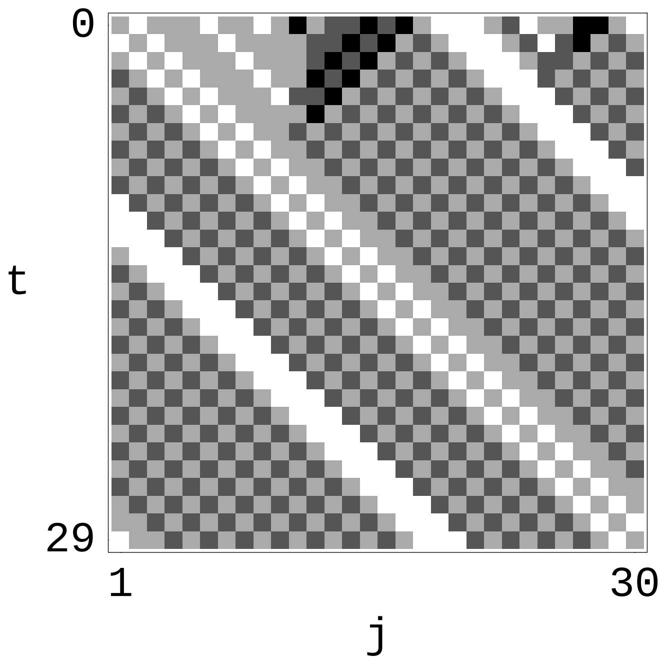

Figure 2 (a)

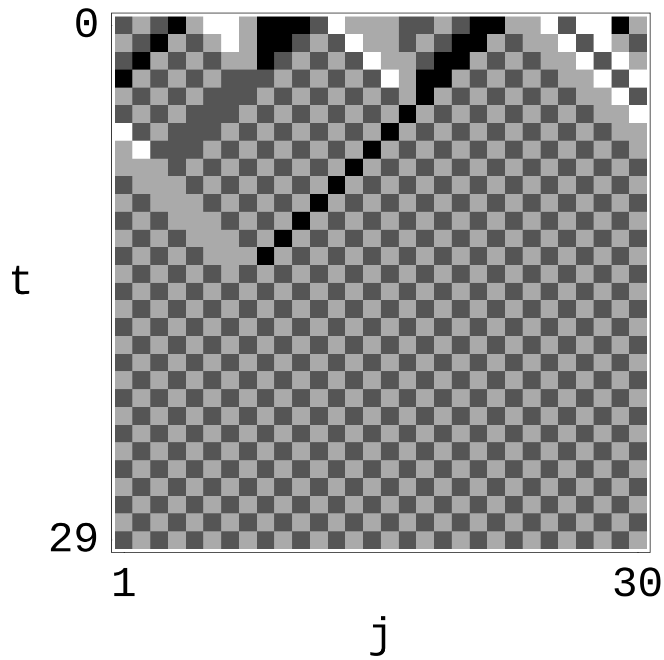

Figure 2 (b)

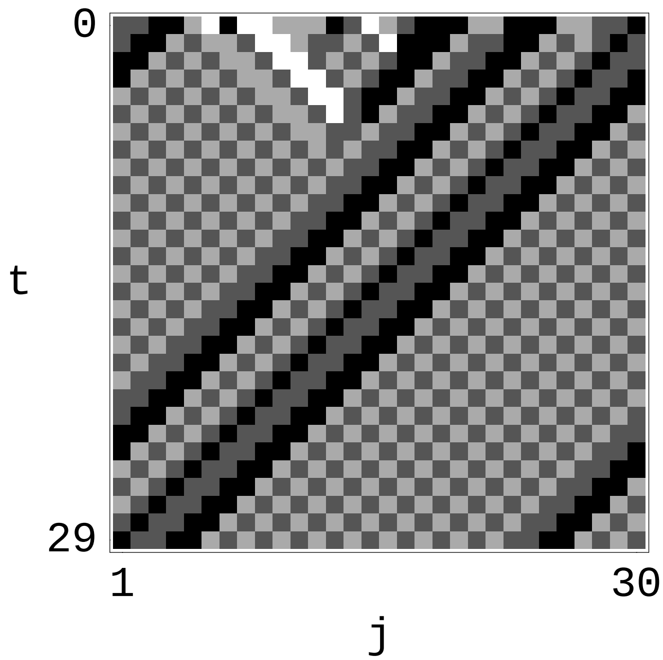

Figure 2 (c)

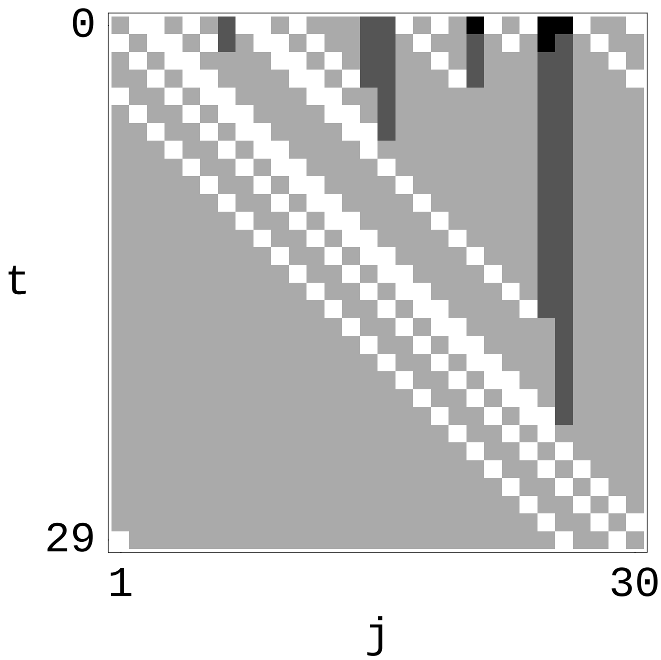

Figure 3 (a)

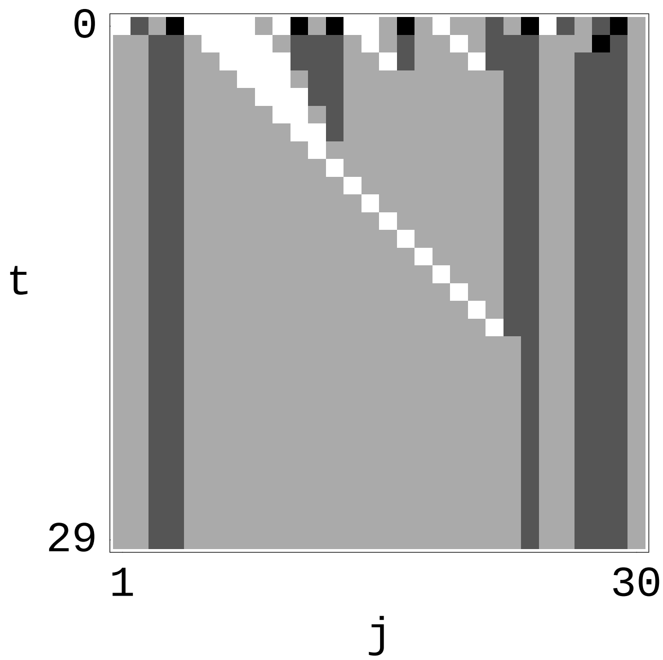

Figure 3 (b)

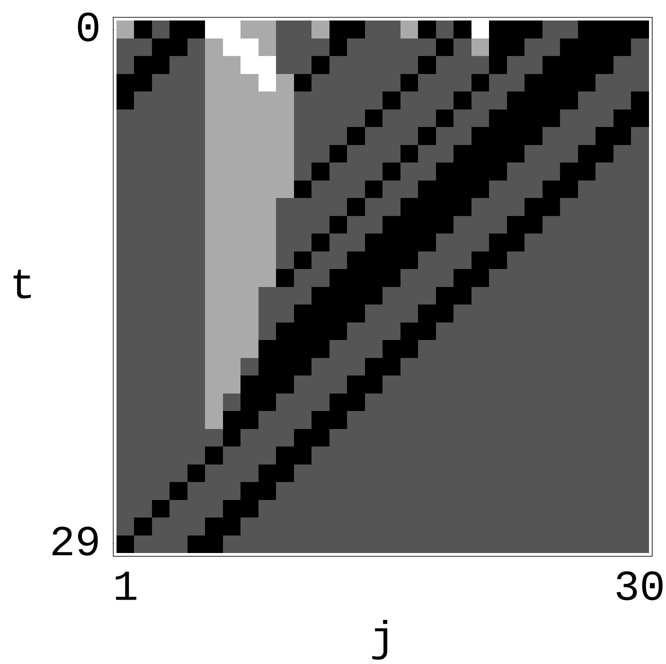

Figure 3 (c)

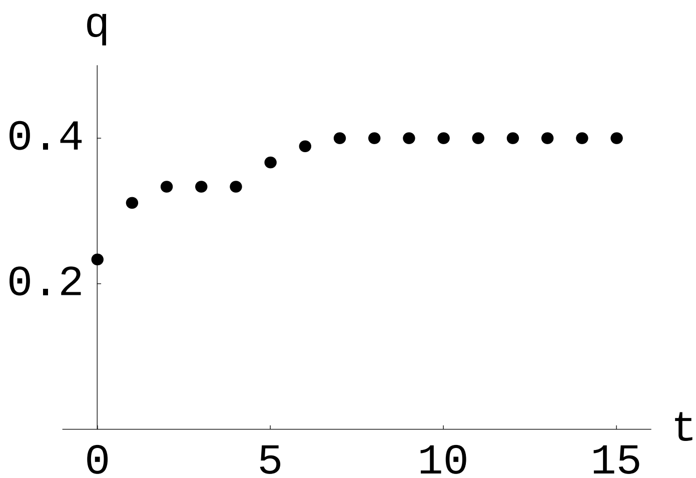

Figure 4