Hindawi
Journal of Function Spaces
Volume 2021, Article ID 6128586, 13 pages
https://doi.org/10.1155/2021/6128586

Research Article

Analytical Solution for Differential and Nonlinear Integral Equations via F_{ϖ_e}-Suzuki Contractions in Modified ϖ_e-Metric-Like Spaces

Hasanen A. Hammad,[1] **Manuel De la Sen**,[2] **and Hassen Aydi**[3,4,5]

[1]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
[2]Institute of Research and Development of Processes University of the Basque Country, 48940 Leioa, Bizkaia, Spain
[3]Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia
[4]China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[5]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa

Correspondence should be addressed to Hasanen A. Hammad; h.elmagd89@gmail.com
and Hassen Aydi; hassen.aydi@isima.rnu.tn

Received 7 July 2020; Accepted 1 February 2021; Published 15 February 2021

Academic Editor: Shanhe Wu

The aim of this manuscript is to present a new space, namely, a modified ϖ_e-metric-like space, and we establish some related fixed point results using extended F_{ϖ_e}-Suzuki and generalized F_{ϖ_e}-Suzuki contractions on the mentioned space. Here, we support our theoretical consequences in two ways: the first one consists of presenting illustrative examples and the second one consists of finding analytical solutions for some integral and differential equations in the context of the mentioned space.

1. Introduction and Elementary Discussions

Solving second-order differential equations after converting to integral equations using Green's function has become commonplace for many academic researchers because of its importance in theoretical and practical applications. One aspect of the solution of the differential and integral equations is the analytical solution, which in turn mainly supports the numerical solution used in solving dynamical systems.

There are many methods for obtaining analytical solutions, including the fixed point technique. For instance, many works have focused their attention on solving the Fredholm integral equation [1] analytically and numerically by this technique [2, 3].

Among the generalizations of the Banach principle [4], the notion of F-contractions was initiated by Wardkowski [5].

After two years, Piri and Kuman [6] made a slight change in the principle of Wardkowski and called it an F-Suzuki contraction. It has contributed significantly to upholding the reputation of the fixed point theory in many fields and has become a great weight in the functional analysis.

Definition 1. A mapping $A : \daleth \rightarrow \daleth$ defined on the metric space $(\daleth, d)$ is named as an

(i) F-contraction if there are $F \in \Pi$ and $\vartheta > 0$ so that

$$d(A\ell, A\hbar) > 0 \Rightarrow \vartheta + F(d(A\ell, A\hbar)) \leq F(d(\ell, \hbar)) \text{ for all } \ell, \hbar \in \daleth \tag{1}$$

(ii) F-Suzuki contraction if there are $F \in \Pi$ and $\vartheta > 0$ so that

$$\frac{1}{2}d(\ell, A\ell) < d(\ell, \hbar) \Rightarrow \vartheta + F(d(A\ell, A\hbar)) \leq F(d(\ell, \hbar)) \text{ for all } \ell, \hbar \in \daleth, \tag{2}$$

where Σ is the class of functions $F : (0,+\infty) \to \mathbb{R}$ so that

$(F_1)$for all $\mho, \Omega \in \mathbb{R}^+$ such that $\mho<\Omega, F(\mho) < F(\Omega)$

$(F_2)$for each positive real sequence $\{\mho_p\}$, $\lim_{p\to\infty}\mho_p = 0$ iff $\lim_{n\to\infty} F(\mho_p) = -\infty$

(F_3)there is $\omega \in (0, 1)$ so that $\lim_{\mho\to 0^+} \mho^{\omega} F(\mho) = 0$

There are developments to translate fixed point theorems into nonlinear integral equations and differential equations (for related works and developments, see [7–20]).

The nonlinear mapping in Banach contraction needs to be continuous. It is not applicable in the discontinuous case. In the past, Kannan [21]. was able to overcome this shortcoming by giving a fixed point result without the mapping being continuous. Variant works appeared to resolve this problem by adding conditions to the spaces (see [22–25]).

Among modern spaces, a b-metric-like space was introduced by Alghmandi et al. [26] as an extension of a b-metric, which was presented by Bakhtin [27], and a metric-like, which was presented by Amini-Harandi [28]. In [26], some fixed point theorems have been provided. In recent works, many contributions on fixed point results involving different contractive conditions are given (see [29–36]).

At the beginning of 2019, Parvaneh and Kadelburg [37] generalized the b-metric-like space by replacing the coefficient located in the third condition by a strictly increasing continuous function. They named it an extended b-metric-like space and studied on it some fixed point sequences for JSHR-contractive type mappings with some applications.

It is noted that in this section, we did not address definitions and mathematical theorems for two reasons: first, there are large basics related to the mentioned spaces, and second, access to the main results is direct.

According to the previous results, in this paper, we present fixed point consequences by using F_{ϖ_e}-Suzuki contractions in the class of modified ϖ_e-metric-like spaces. Under the framework of the mentioned space, we apply the theoretical results to find an analytical solution for nonlinear integral equations. On the other hand, some important examples to justify our theorems are discussed.

2. An Extended F_{ϖ_e}-Suzuki Contraction

We begin this section with definitions of metric-like and b-metric-like spaces.

Definition 2 (see [26]). Let $\daleth$ be a nonempty set. A function $\Delta : \daleth^2 \to \mathbb{R}^+$ is named as metric-like on $\daleth$, if for all $\mho_1, \mho_2, \mho_3 \in \daleth$:

$(\daleth_1)\Delta(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\daleth_2)\Delta(\mho_1, \mho_2) = \Delta(\mho_2, \mho_1)$

$(\daleth_3)\Delta(\mho_1, \mho_3) \leq \Delta(\mho_1, \mho_2) + \Delta(\mho_2, \mho_3)$

Definition 3 (see [26]). A b -metric-like on a nonempty set $\daleth$ is a function $\varpi : \daleth^2 \to \mathbb{R}^+$ so that for all $\mho_1, \mho_2, \mho_3 \in \daleth$ and a constant $s \geq 1$:

$(\varpi_1)\varpi(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\varpi_2)\varpi(\mho_1, \mho_2) = \varpi(\mho_2, \mho_1)$

$(\varpi_3)\varpi(\mho_1, \mho_3) \leq s[\varpi(\mho_1, \mho_2) + \varpi(\mho_2, \mho_3)]$

Here, $(\daleth, \varpi)$ is named as a b-metric-like space (with constant s).

For examples about metric-like and b-metric-like spaces, see [33–35].

Now, we will generalize Definition 3 as follows.

Definition 4. Let $\daleth$ be a nonempty set and $s : \daleth \times \daleth \to [1,\infty)$. A function $\varpi_e : \daleth^2 \to [0,\infty)$ is called a modified ϖ_e -metric-like if, for all $\mho_1, \mho_2, \mho_3 \in \daleth$:

$(\varpi_{e1})\varpi_e(\mho_1, \mho_2) = 0 \Rightarrow \mho_1 = \mho_2$

$(\varpi_{e2})\varpi_e(\mho_1, \mho_2) = \varpi_e(\mho_2, \mho_1)$

$(\varpi_{e3})\varpi_e(\mho_1, \mho_3) \leq s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]$

Here, $(\daleth, \varpi_e)$ is named as a modified extended b-metric-like space (simply, a modified ϖ_e-metric-like space).

Note that the class of modified ϖ_e-metric-like spaces is larger than the class of b-metric-like spaces by replacing the constant $s \geq 1$ of Definition 3 by a nonconstant function $s : \daleth \times \daleth \to [1,\infty)$ of Definition 4.

Example 5. Let $\daleth = [0,\infty)$. Define $\varpi_e : \daleth^2 \to [0,\infty)$ by

$$\varpi_e(\kappa, \mu) = \begin{cases} 0, & \text{if } \kappa = \mu = 0, \\ \dfrac{\mu}{1+\mu}, & \text{if } \kappa = 0,\ \mu \neq 0, \\ \dfrac{\kappa}{1+\kappa}, & \text{if } \mu = 0,\ \kappa \neq 0, \\ \kappa + \mu, & \text{if } \kappa \neq 0,\ \mu \neq 0. \end{cases} \tag{3}$$

Consider $s : \daleth^2 \to [1,\infty)$ as $s(\kappa, \mu) = 2 + 2\kappa + 2\mu$.

First, (ϖ_{e1}) and (ϖ_{e2}) are obvious. We need to prove (ϖ_{e3}). For this, let $\mho_1, \mho_2$, and $\mho_3$ in $\daleth$. We state the following cases.

Case 1. $\mho_1 = \mho_3 = 0$. Here, (ϖ_{e3}) holds.

Case 2. $\mho_1 = 0$ and $\mho_3 \neq 0$. Then,

$$\begin{aligned} \varpi_e(\mho_1, \mho_3) &= \frac{\mho_3}{1+\mho_3}, \\ s(\mho_1, \mho_3) &= 2 + 2\mho_3. \end{aligned} \tag{4}$$

Subcase 1. $\mho_2 = 0$. We have

$$\begin{aligned} \varpi_e(\mho_1, \mho_3) = \frac{\mho_3}{1+\mho_3} &\leq (2 + 2\mho_3)\left[0 + \frac{\mho_3}{1+\mho_3}\right] \\ &= s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \end{aligned} \tag{5}$$

Subcase 2. $\mho_2 \neq 0$. We have

$$\varpi_e(\mho_1, \mho_3) = \frac{\mho_3}{1+\mho_3} \leq (2+2\mho_3)\left[\frac{\mho_2}{1+\mho_2} + \mho_2 + \mho_3\right] = s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \tag{6}$$

Case 3. $\mho_3 = 0$ and $\mho_1 \neq 0$. Proceeding similarly as in Case 2, (ϖ_{e3}) holds.

Case 4. $\mho_1 \neq 0$ and $\mho_3 \neq 0$. Then,

$$\varpi_e(\mho_1, \mho_3) = \mho_1 + \mho_3, \quad s(\mho_1, \mho_3) = 2 + 2\mho_1 + 2\mho_3. \tag{7}$$

Subcase 1. $\mho_2 = 0$. We have

$$\varpi_e(\mho_1, \mho_3) = \mho_1 + \mho_3 \leq (2 + 2\mho_1 + 2\mho_3)\left[\frac{\mho_1}{1+\mho_1} + \frac{\mho_3}{1+\mho_3}\right] = s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \tag{8}$$

Subcase 2. $\mho_2 \neq 0$. We have

$$\varpi_e(\mho_1, \mho_3) = \mho_1 + \mho_3 \leq (2 + 2\mho_1 + 2\mho_3)[\mho_1 + \mho_2 + \mho_2 + \mho_3] = s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \tag{9}$$

On the other hand, $(\daleth, \varpi_e)$ is not a b-metric-like space. We argue by contradiction by assuming that $(\daleth, \varpi_e)$ is a b-metric-like space with a coefficient $s \geq 1$ (a constant). Then, for any real $\mu > 0$, we have

$$\varpi_e(\mu, \mu+1) \leq s[\varpi_e(\mu, 0) + \varpi_e(0, \mu+1)]. \tag{10}$$

That is,

$$2\mu + 1 \leq s\left[\frac{\mu}{1+\mu} + \frac{\mu+1}{2+\mu}\right]. \tag{11}$$

Letting $\mu \to \infty$, we get $+\infty \leq 2s$, which is a contradiction.

Example 6. Let $\daleth = \{0, 1, 2\}$. Define $\varpi_e : \daleth^2 \to [0,\infty)$ and $s : \daleth \times \daleth \to [1,\infty)$ as follows:

$$\begin{aligned} \varpi_e(0,0) &= \varpi_e(1,1) = \varpi_e(2,2) = 0, \\ \varpi_e(0,1) &= \varpi_e(1,0) = 12, \\ \varpi_e(0,2) &= \varpi_e(2,0) = 1, \\ \varpi_e(1,2) &= \varpi_e(2,1) = 3, \end{aligned} \tag{12}$$

and $s(\kappa, \mu) = 2 + \kappa + \mu$.

First, we show that ϖ_e is a modified ϖ_e-metric-like space. Trivially, the conditions (ϖ_{e1}) and (ϖ_{e2}) hold. For (ϖ_{e3}), we get

$$\varpi_e(0,1) = 12\,;\, s(0,1)[\varpi_e(0,2) + \varpi_e(2,1)] = 12. \tag{13}$$

Thus,

$$\varpi_e(0,1) \leq s(0,1)[\varpi_e(0,2) + \varpi_e(2,1)]. \tag{14}$$

Again,

$$\begin{aligned} \varpi_e(1,2) &= 3\,;\, s(1,2)[\varpi_e(1,0) + \varpi_e(0,2)] = 65, \\ \varpi_e(0,2) &= 1\,;\, s(0,2)[\varpi_e(0,1) + \varpi_e(1,2)] = 60. \end{aligned} \tag{15}$$

Hence, for all $\mho_1, \mho_2, \mho_3 \in \daleth$, $\varpi_e(\mho_1, \mho_3) \leq s(\mho_1, \mho_3)[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]$. Then, $(\daleth, \varpi_e)$ is a modified ϖ_e-metric-like space, but it is not a b-metric-like space because if we take $s = 2$ in the inequality (13), we get

$$\varpi_e(0,1) = 12\,;\, 2[\varpi_e(0,2) + \varpi_e(2,1)] = 8. \tag{16}$$

Thus,

$$\varpi_e(\mho_1, \mho_3) \nleq s[\varpi_e(\mho_1, \mho_2) + \varpi_e(\mho_2, \mho_3)]. \tag{17}$$

Definition 7. Let $\{\mho_\iota\}$ be a sequence in the modified ϖ_e-metric-like space $(\daleth, \varpi_e)$.

(a) If $\lim_{\iota\to\infty}\varpi_e(\mho_\iota, \mho) = \varpi_e(\mho, \mho)$, then $\{\mho_\iota\}$ is convergent to $\mho$

(b) $\{\mho_\iota\}$ is called Cauchy if $\lim_{\iota,j\to\infty}\varpi_e(\mho_\iota, \mho_j)$ exists and is finite

(c) If for each Cauchy sequence $\{\mho_\iota\}$, there is $\mho \in \daleth$, so that $\lim_{\iota,j\to\infty}\varpi_e(\mho_\iota, \mho_j) = \varpi_e(\mho, \mho) = \lim_{\iota\to\infty}\varpi_e(\mho_\iota, \mho)$; therefore, $(\daleth, \varpi_e)$ is said to be complete

Definition 8. A nonlinear self-mapping A on a modified ϖ_e-metric-like space $(\daleth, \varpi_e)$ is named as an extended F_{ϖ_e}-Suzuki contraction if there are $F_{\varpi_e} \in \Pi$ *and* $\vartheta > 0$ so that for $\mho, \mu \in \daleth$, the following condition holds:

$$\frac{1}{2}\varpi_e(\mho, A\mho) < \varpi_e(\mho, \mu) \Rightarrow \vartheta + F_{\varpi_e}(\varpi_e(A\mho, A\mu)) \leq F_{\varpi_e}(\varpi_e(\mho, \mu)), \tag{18}$$

such that $\lim_{n,m\to\infty}s(\mho_\iota, \mho_j) < 1/\eta$ for all $\mho_\circ \in \daleth$, where $0 < \eta < 1$. We consider here $\mho_\iota = A^\iota\mho_\circ$, $\iota = 1, 2, \cdots$, where Π is the set of continuous functions $F_{\varpi_e} : \mathbb{R}^+ \to \mathbb{R}$ so that

$(\mathfrak{J}_1)$For all $j, \ell \in \mathbb{R}^+$ with $j < \ell$, $F_{\varpi_e}(j) < F_{\varpi_e}(\ell)$

$(\mathfrak{J}_2)$For each positive real sequence $\{j_p\}$, $\lim_{p\to\infty}j_p = 0$ iff $\lim_{p\to\infty}F_{\varpi_e}(j_p) = -\infty$

$(\mathfrak{J}_3)$There is $\eta \in (0,1)$ so that $\lim_{j\to 0^+}j_\eta F_{\varpi_e}(j) = 0$

Now, we introduce our first theorem.

Theorem 9. *Let $(\daleth, \varpi_e)$ be a complete modified ϖ_e -metric-like space and A be an extended F_{ϖ_e} -Suzuki contraction mapping, then A admits a unique fixed point.*

Proof. Let $\mho_\circ \in \daleth$ and $\{\mho_\iota\}_{\iota=1}^{\infty}$ defined by $\mho_{\iota+1} = A\mho_\iota = A^{\iota+1}$ $\mho_\circ$. If there is $\iota \in \mathbb{N}$ so that $\varpi_e(\mho_\iota, A\mho_\iota) = 0$. It completes the proof. Otherwise, assume that $0 < \varpi_e(A_\iota, \Gamma\mho_\iota) = \varpi_e(\mho_\iota, \mho_{\iota+1}) = \varpi_e^\iota$; therefore, for all $\iota \in \mathbb{N}$,

$$\frac{1}{2}\varpi_e(\mho_\iota, A\mho_\iota) < \varpi_e(\mho_\iota, A\mho_\iota), \tag{19}$$

it yields or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho_\iota\right)\right) \le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)), \tag{20}$$

$$F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho_\iota\right)\right) \le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)) - \vartheta. \tag{21}$$

By the same method, one gets

$$\begin{aligned} F_{\varpi_e}(\varpi_e^\iota) &= F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho_\iota)) = F_{\varpi_e}\left(\varpi_e\left(A\mho_{\iota-1}, A^2\mho_{\iota-1}\right)\right) \\ &\le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, A\mho_{\iota-1})) - \vartheta \le F_{\varpi_e}(\varpi_e(\mho_{\iota-2}, A\mho_{\iota-2})) \\ &\quad - 2\vartheta \vdots \le F_{\varpi_e}(\varpi_e(\mho_\circ, A\mho_\circ)) - \iota\vartheta \text{ for all } \iota \ge 1. \end{aligned} \tag{22}$$

Taking $\iota \to \infty$ in (22), we have

$$\lim_{\iota\to\infty} F_{\varpi_e}(\varpi_e^\iota) = -\infty. \tag{23}$$

So, by $(\mathfrak{I}_2)$, we obtain

$$\lim_{\iota\to\infty} \varpi_e^\iota = 0. \tag{24}$$

Applying $(\mathfrak{I}_3)$, there is $\eta \in (0, 1)$ so that

$$\lim_{\iota\to\infty} (\varpi_e^\iota)^\eta F_{\varpi_e}(\varpi_e^\iota) = 0. \tag{25}$$

By (22), one writes for all $\iota \ge 1$,

$$(\varpi_e^\iota)^\eta\left(F_{\varpi_e}(\varpi_e^\iota) - F_{\varpi_e}(\varpi_e^\circ)\right) \le -\iota\vartheta(\varpi_e^\iota)^\eta \le 0. \tag{26}$$

Considering (24) and (25) and passing $\iota \to \infty$ in (26), one gets

$$\lim_{\iota\to\infty} \iota(\varpi_e^\iota)^\eta = 0. \tag{27}$$

By (27), there exists $\iota_1 \in \mathbb{N}$ so that $\iota(\varpi_e^\iota)^\eta \le 1$ for all $\iota \ge \iota_1$, or

$$\varpi_e^\iota \le \frac{1}{\iota^{1/\eta}} \text{ for all } \iota \ge \iota_1. \tag{28}$$

Consider the integers $m > \iota$. Applying (ϖ_3) and (28), one writes

$$\begin{aligned} \varpi_e(\mho_\iota, \mho_m) &\le s(\mho_\iota, \mho_m)[\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_m)] \\ &\le s(\mho_\iota, \mho_m)\varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot [\varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \varpi_e(\mho_{\iota+2}, \mho_m)] \\ &\le s(\mho_\iota, \mho_m)\varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \cdots + s(\mho_\iota, \mho_m)s(\mho_{\iota+1}, \mho_m)s \\ &\quad \cdot (\mho_{\iota+2}, \mho_m) \cdots s(\mho_{m-2}, \mho_m)s(\mho_{m-1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{m-1}, \mho_m) \le s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots s(\mho_\iota, \mho_m) \\ &\quad \cdot \varpi_e(\mho_\iota, \mho_{\iota+1}) + s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots s(\mho_{\iota+1}, \mho_m) \\ &\quad \cdot \varpi_e(\mho_{\iota+1}, \mho_{\iota+2}) + \cdots + s(\mho_1, \mho_m)s(\mho_2, \mho_m) \cdots \\ &\quad \cdot s(\mho_{m-1}, \mho_m)\varpi_e(\mho_{m-1}, \mho_m) = \sum_{\iota=1}^{\infty} \varpi_e(\mho_\iota, \mho_{\iota+1}) \\ &\quad \cdot \prod_{j=1}^{\iota} s(\mho_j, \mho_m) \le \sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}} \prod_{j=1}^{\iota} s(\mho_j, \mho_m) \\ &\le \sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}}\frac{1}{\eta} = \frac{1}{\eta}\sum_{\iota=1}^{\infty} \frac{1}{\iota^{1/\eta}}. \end{aligned} \tag{29}$$

Recall that $\sum_{\iota=1}^{\infty} 1/\iota^{1/\eta}$ converges, so $\varpi_e(\mho_\iota, \mho_m) \to 0$. Therefore, $\{\mho_\iota\}$ is a Cauchy sequence in the complete modified ϖ_e-metric-like space $(\daleth, \varpi_e)$; hence, there is $\mho^* \in \daleth$ such that $\mho_\iota \to \mho^*$, as $\iota \to \infty$. That is,

$$\lim_{\iota,m\to\infty} \varpi_e(\mho_\iota, \mho_m) = \lim_{\iota\to\infty} \varpi_e(\mho_\iota, \mho^*) = \varpi_e(\mho^*, \mho^*) = 0. \tag{30}$$

Next, if F_{ϖ_e} is continuous, then two short cases arise.

Case 1. For each $\iota \in \mathbb{N}$, there exists $j_\iota \in \mathbb{N}$ such that $\mho_{j_\iota} = A\mho^*$ and $j_\iota > j_{\iota-1}$, where $j_\circ = 0$. Therefore, one gets $\mho^* = \lim_{\iota\to\infty} \mho_{j_\iota} = \lim_{\iota\to\infty} A\mho^* = A\mho^*$.

Case 2. There is $\iota_\circ \in \mathbb{N}$ so that for all $\iota \ge \iota_\circ, \mho_\iota \ne A\mho^*$. It is clear that $1/2\varpi_e(\mho_\iota, A\mho^*) < \varpi_e(\mho_\iota, A\mho^*)$ for all $\iota \ge \iota_\circ$.

By (18), we have

$$\begin{aligned} \vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho_{\iota+1}, A^2\mho^*\right)\right) &= \vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho^*\right)\right) \\ &\le F_{\varpi_e}(\varpi_e(\mho_\iota, A\mho^*)). \end{aligned} \tag{31}$$

Since F_{ϖ_e} is continuous, we obtain at the limit $\iota \to \infty$,or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho^*, A^2\mho^*\right)\right) \le F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)), \tag{32}$$

$$F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)) \le F_{\varpi_e}(\varpi_e(\mho^*, \mho^*)) - \vartheta, \tag{33}$$

which is a contradiction due to $(\mathfrak{I}_1)$. Then, $\varpi_e(\mho^*, A\mho^*) = 0$, which means that $\mho^* = A\mho^*$.

The two cases above lead to the existence of a fixed point of A, i.e., $\mho^* = A\mho^*$.

Now, assume that $\mho_1^*$ and $\mho_2^*$ are so that $\mho_1^* = A\mho_1^* \ne \mho_2^* = A\mho_2^*$. We have $1/2\varpi_e(\mho_1^*, \mho_2^*) < \varpi_e(\mho_1^*, \mho_2^*)$, which implies by (18) that

$$\vartheta + F_{\varpi_e}(\varpi_e(\mho_1^*, \mho_2^*)) = \vartheta + F_{\varpi_e}(\varpi_e(A\mho_1^*, A\mho_2^*)) \leq F_{\varpi_e}(\varpi_e(\mho_1^*, \mho_2^*)). \tag{34}$$

It is again a contradiction.

The following examples verify all required hypotheses of Theorem 9.

Example 10. Let $\daleth = [0,\infty)$. Define $\varpi_e : \daleth^2 \to \mathbb{R}$ *by* $\varpi_e(\mho, \aleph) = (\mho + \aleph)^2$ and $s : \daleth \times \daleth \to [1,\infty)$ by $s(\mho, \aleph) = 1 + \mho + \aleph$, for all $\mho, \aleph \in \daleth$. Here, ϖ_e is a modified extended ϖ_e -metric-like space. Define $A : \daleth \to \daleth$ *as* $A\mho = (1/3)\mho$, for all $\mho \in \daleth$. It is clear that

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e\left(\mho, \frac{1}{3}\mho\right) = \frac{1}{2}\left(\mho + \frac{1}{3}\mho\right)^2 \\ &= \frac{16}{18}\mho^2 \leq \mho^2 \leq (\mho + \mu)^2 = \varpi_e(\mho, \mu).\end{aligned} \tag{35}$$

Consider, for all $\mho, \mu \in \daleth$,

$$\begin{aligned}F_{\varpi_e}(\varpi_e(A\mho, A\mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3}\mho, \frac{1}{3}\mu\right)\right) = F_{\varpi_e}\left(\left(\frac{1}{3}\mho + \frac{1}{3}\mu\right)^2\right) \\ &= F_{\varpi_e}\left(\frac{1}{9}(\mho + \mu)^2\right).\end{aligned} \tag{36}$$

Also,

$$F_{\varpi_e}(\varpi_e(\mho, \mu)) = F_{\varpi_e}\left((\mho + \mu)^2\right). \tag{37}$$

Let the function $F_{\varpi_e} \in \Pi$ be defined by $F_{\varpi_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$\begin{aligned}&F_{\varpi_e}(\varpi_e(A\mho, A\mu)) - F_{\varpi_e}(\varpi_e(\mho, \mu)) \\ &\quad = \ln\left(\frac{1}{9}(\mho + \mu)^2\right) - \ln\left((\mho + \mu)^2\right) = \ln\left(\frac{1/9(\mho + \mu)^2}{(\mho + \mu)^2}\right) \\ &\quad = \ln\left(\frac{1}{9}\right) = -2.197 \leq -2.\end{aligned} \tag{38}$$

Therefore, A is an extended F_{ϖ_e}-Suzuki contraction mapping with $\vartheta = 2$. Moreover, if $\mho_m = \{1/(m+1)\} \in \daleth$, we have

$$\lim_{i,m\to\infty} s(\mho_m, \mho_i) = \lim_{i,m\to\infty}\left(1 + \frac{1}{m+1} + \frac{1}{i+1}\right) = 1 < \frac{1}{\eta}, \tag{39}$$

for $\eta \in (0, 1)$. So, all hypotheses of Theorem 9 are satisfied, and A has 0 as a unique fixed point.

Example 11. Let $\daleth = \{1/3^{\iota-1} : \iota \in \mathbb{N}\} \cup \{0\}$. Suppose that $\varpi_e : \daleth \times \daleth \to 0,\infty)$ and $s : \daleth \times \daleth \to [1,\infty)$ are functions defined by $\varpi_e(\mho, \mu) = (\max\{\mho, \mu\})^2$ *and* $s(\mho, \mu) = 1 + \mu + \mho$, respectively, for all $\mu, \mho \in \daleth$. Then, the pair $(\daleth, \varpi_e)$ is a complete modified ϖ_e -metric-like space. Define a nonlinear mapping $A : \daleth \to \daleth$ *by*

$$A\mho = \begin{cases}\left\{\dfrac{1}{3^{2\iota}}\right\}, & \text{if} \mho \in \left\{\dfrac{1}{3^{2\iota-1}} ; \iota \in \mathbb{N}\right\}, \\ 0, & \text{if} \mho = 0.\end{cases} \tag{40}$$

We shall prove that a mapping A is an extended F_{ϖ_e} -Suzuki contraction with $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$, by showing the following cases.

Case 1. Let $\mho = 1/3^{2\iota-1}$ and $\mu = 1/3^{2m-1}$, for $m > \iota \geq 1$, one can write

$$\begin{aligned}\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e\left(\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right) = \frac{1}{2}\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right\}\right)^2 \\ &= \frac{1}{2}\left(\frac{1}{3^{2\iota-1}}\right)^2 < \left(\frac{1}{3^{2\iota-1}}\right)^2 \\ &= \left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2 = \varpi_e(\mho, \mu).\end{aligned} \tag{41}$$

Consider

$$\begin{aligned}F_{\varpi_e}(\varpi_e(A\mho, A\mu)) &= F_{\varpi_e}\left(\varpi_e\left(A\frac{1}{3^{2\iota-1}}, A\frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota}}, \frac{1}{3^{2m}}\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota}}, \frac{1}{3^{2m}}\right\}\right)^2\right) \\ &= F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota}}\right)^2\right) = \ln\left(\frac{1}{3^{2\iota}}\right)^2 = 2\ln\left(\frac{1}{3^{2\iota}}\right).\end{aligned} \tag{42}$$

Also,

$$\begin{aligned}F_{\varpi_e}(\varpi_e(\mho, \mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2m-1}}\right\}\right)^2\right) \\ &= F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota-1}}\right)^2\right) = 2\ln\left(\frac{1}{3^{2\iota-1}}\right).\end{aligned} \tag{43}$$

By subtracting (42) and (43), we find that

$$\begin{aligned}&F_{\varpi_e}(\varpi_e(A\mho, A\mu)) - F_{\varpi_e}(\varpi_e(\mho, \mu)) \\ &\quad = 2\left(\ln\left(\frac{1}{3^{2\iota}}\right) - \ln\left(\frac{1}{3^{2\iota-1}}\right)\right) = 2\left(\ln\left(\frac{1}{3^{2\iota}} \times 3^{2\iota} \cdot 3^{-1}\right)\right) \\ &\quad = -2\ln 3 < -2.\end{aligned} \tag{44}$$

Case 2. Let $\mho = 1/3^{2\iota-1}$ and $\mu = 0$. We have

$$\frac{1}{2}\varpi_e(\mho, A\mho) = \frac{1}{2}\left(\max\left\{\frac{1}{3^{2\iota-1}}, \frac{1}{3^{2\iota}}\right\}\right)^2 = \frac{1}{2}\left(\frac{1}{3^{2\iota-1}}\right)^2 < \left(\frac{1}{3^{2n-1}}\right)^2 = \left(\max\left\{\frac{1}{3^{2\iota-1}}, 0\right\}\right)^2 = \varpi_e(\mho, \mu). \tag{45}$$

Suppose that

$$\begin{aligned} F_{\varpi_e}(\varpi_e(A\mho, A\mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota}}, 0\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota}}, 0\right\}\right)^2\right) = F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota}}\right)^2\right) \\ &= \ln\left(\frac{1}{3^{2\iota}}\right)^2 = 2\ln\left(\frac{1}{3^{2\iota}}\right); \end{aligned} \tag{46}$$

also,

$$\begin{aligned} F_{\varpi_e}(\varpi_e(\mho, \mu)) &= F_{\varpi_e}\left(\varpi_e\left(\frac{1}{3^{2\iota-1}}, 0\right)\right) \\ &= F_{\varpi_e}\left(\left(\max\left\{\frac{1}{3^{2\iota-1}}, 0\right\}\right)^2\right) \\ &= F_{\varpi_e}\left(\left(\frac{1}{3^{2\iota-1}}\right)^2\right) = 2\ln\left(\frac{1}{3^{2\iota-1}}\right). \end{aligned} \tag{47}$$

By subtracting (46) and (47), we have the same inequality (44).

Case 3. Let $\mho = 0$ and $\mu = 1/3^{2m-1}$. The proof follows immediately as Case 2. Thus, A is an extended F_{ϖ_e}-Suzuki contraction mapping with $\vartheta = 2$. Here, 0 is the unique fixed point.

3. An Extended Generalized F_{ϖ_e}-Suzuki Contraction

Definition 12. A self-mapping A on a modified extended b-metric-like space $(\daleth, \varpi_e)$ is called an extended generalized F_{ϖ_e}-Suzuki contraction if there are $F_{\varpi_e} \in \Pi$ and $\vartheta > 0$ such that, if for all $\mho, \aleph \in \daleth$, the following hypothesis is satisfied

$$\begin{aligned} \frac{1}{2}\varpi_e(\mho, A\mho) < \varpi_e(\mho, \aleph) \Rightarrow \vartheta + F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) \\ \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \right.\right. \\ \left.\left. \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right). \end{aligned} \tag{48}$$

Remark 13.

(i) Every extended F_{ϖ_e}-Suzuki contraction is an extended generalized F_{ϖ_e}-Suzuki contraction

(ii) Suppose that A is an extended generalized F_{ϖ_e}-Suzuki contraction, by Definition 12, for all $\mho, \aleph \in \daleth$, we get $A\mho \neq A\aleph$ and $1/2\varpi_e(\mho, A\mho) < \varpi_e(\mho, \aleph)$. Thus,

$$\begin{aligned} F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) < \vartheta + F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) \\ \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \right.\right. \\ \left.\left. \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right) \end{aligned} \tag{49}$$

By condition $(\mathfrak{I}_1)$, for all $\mho, \aleph \in \daleth$ with $A\mho \neq A\aleph$, we have

$$\begin{aligned} \varpi_e(A\mho, A\aleph) \le \max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \right. \\ \left. \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}. \end{aligned} \tag{50}$$

Note that the inverse of the above remark is generally incorrect.

Example 14. Let $\daleth = [0, \infty)$. Define $\varpi_e : \daleth^2 \to \mathbb{R}$ *by* $\varpi_e(\mho, \aleph) = (\mho + \aleph)^2$ and $s : \daleth \times \daleth \to [1, \infty)$ *by* $s(\mho, \aleph) = 1 + \mho + \aleph$, for all $\mho, \aleph \in \daleth$. Here, ϖ_e is a modified extended ϖ_e-metric-like space. Define $A : \daleth \to \daleth$ as

$$A\mho = \begin{cases} 0, & \text{if } 0 \le \mho < 1, \\ \dfrac{1}{2}, & \text{if} \mho \ge 1. \end{cases} \tag{51}$$

Note that A is not an extended F_{ϖ_e}-Suzuki contraction. Indeed, for $0 \le \mho < 1$ and $\aleph = 1$, we can write $1/2\varpi_e(\mho, A\mho) = 1/2\varpi_e(\mho, 0) = 1/2\mho^2 < (\mho+1)^2 = \varpi_e(\mho, \aleph)$ and

$$\begin{aligned} &\max\left\{\varpi_e(\mho, 1), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(1, A1)}{1+\varpi_e(1, A1)}, \frac{\varpi_e(\mho, A1)+\varpi_e(1, A\mho)}{4(2+\mho)}\right\} \\ &= \max\left\{\varpi_e(\mho, 1), \frac{\varpi_e(\mho, 0)}{1+\varpi_e(\mho, 0)}, \frac{\varpi_e(1, A1)}{1+\varpi_e(1, A1)}, \frac{\varpi_e(\mho, A1)+\varpi_e(1, 0)}{4(2+\mho)}\right\} \ge \varpi_e(1, A1) = \varpi_e\left(1, \frac{1}{2}\right) \\ &= \frac{9}{4} > \frac{1}{4} = \varpi_e(A\mho, A\aleph). \end{aligned} \tag{52}$$

Let the function $F_{\varpi_e} \in \Pi$ be defined by $F_{\varpi_e}(\ell) = \ln(\ell)$, for $\ell > 0$. Then,

$$\begin{aligned}
&F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}(\varpi_e(\mho, \aleph)) \\
&\quad \le F_{\varpi_e}(\varpi_e(A\mho, A1)) - F_{\varpi_e}(\varpi_e(1, A1)) \\
&\quad = F_{\varpi_e}\left(\frac{1}{4}\right) - F_{\varpi_e}\left(\frac{9}{4}\right) = \ln\left(\frac{1}{4} \times \frac{4}{9}\right) = -\ln(9) < -2.
\end{aligned} \tag{53}$$

Therefore, A is an extended generalized F_{ϖ_e}-Suzuki contraction (for $\vartheta = 2$).

The following theorem is the main consequence of this part.

Theorem 15. *Let $(\daleth, \varpi_e)$ be an extended b -metric-like space and A be an extended generalized F_{ϖ_e} -Suzuki contraction self-mapping, then A has a unique fixed point, provided that $\lim_{i,m} s(\mho_i, \mho_m) \le 1/\eta$, for $0 < \eta < 1$.*

Proof. By the first lines of proof of Theorem 9, we build a sequence $\{\mho_\iota\}_{\iota=1}^{\infty}$ as $\mho_{\iota+1} = A\mho_\iota = A^{\iota+1}\mho_\circ$. Here, we consider $\iota \in \mathbb{N} \cup \{0\}$, $0 < \varpi_e(\mho_\iota, A\mho_\iota) = \varpi_e(\mho_\iota, \mho_{\iota+1})$, so

$$\frac{1}{2}\varpi_e(\mho_\iota, A\mho_\iota) < \varpi_e(\mho_\iota, A\mho_\iota). \tag{54}$$

Applying conditions (48) and (ϖ_{e3}), we get

$$\begin{aligned}
&\vartheta + F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) = \vartheta + F_{\varpi_e}(\varpi_e(A\mho_\iota, A(A\mho_\iota))) \\
&\quad \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho_\iota), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1 + \varpi_e(\mho_\iota, A\mho_\iota)},\right.\right. \\
&\quad \left.\left.\frac{\varpi_e(A\mho_\iota, A^2\mho_\iota)}{1 + \varpi_e(A\mho_\iota, A^2\mho_\iota)}, \frac{\varpi_e(\mho_\iota, A^2\mho_\iota) + \varpi_e(A\mho_\iota, A\mho_\iota)}{4s(\mho_\iota, A\mho_\iota)}\right\}\right) \\
&\quad = F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1 + \varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1 + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})},\right.\right. \\
&\quad \left.\left.\frac{\varpi_e(\mho_\iota, \mho_{\iota+2}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+1})}{4s(\mho_\iota, \mho_{\iota+1})}\right\}\right) \\
&\quad \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1 + \varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1 + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})},\right.\right. \\
&\quad \left.\left.\frac{s(\mho_\iota, \mho_{\iota+1})[\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})] + 2s(\mho_\iota, \mho_{\iota+2})\varpi_e(\mho_\iota, \mho_{\iota+1})}{4s(\mho_\iota, \mho_{\iota+2})}\right\}\right) \\
&\quad = F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \frac{\varpi_e(\mho_\iota, \mho_{\iota+1})}{1 + \varpi_e(\mho_\iota, \mho_{\iota+1})}, \frac{\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{1 + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})},\right.\right. \\
&\quad \left.\left.\frac{3\varpi_e(\mho_\iota, \mho_{\iota+1}) + \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})}{4}\right\}\right) \\
&\quad \le F_{\varpi_e}(\max\{\varpi_e(\mho_\iota, \mho_{\iota+1}), \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})\}).
\end{aligned} \tag{55}$$

Now, if $\varpi_e(\mho_\iota, \mho_{\iota+1}) < \varpi_e(\mho_{\iota+1}, \mho_{\iota+2})$, then

$$F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) - \vartheta, \tag{56}$$

which is a contradiction due to $(\mathfrak{I}_1)$, so we should write

$$F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) - \vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{57}$$

By the same manner,

$$F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, \mho_\iota)) - \vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{58}$$

From (57) and (58), one can write

$$F_{\varpi_e}(\varpi_e(\mho_{\iota+1}, \mho_{\iota+2})) \le F_{\varpi_e}(\varpi_e(\mho_{\iota-1}, \mho_\iota)) - 2\vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{59}$$

Repeating the same scenario, we have

$$F_{\varpi_e}(\varpi_e(\mho_\iota, \mho_{\iota+1})) \le F_{\varpi_e}(\varpi_e(\mho_\circ, \mho_1)) - \iota\vartheta, \quad \forall \iota \in \mathbb{N} \cup \{0\}. \tag{60}$$

The proof of Theorem 9, namely, relations (22)–(29), yields that $\{\mho_\iota\}$ is Cauchy sequence in $(\daleth, \varpi_e)$, which is complete; hence, there is $\mho^* \in \daleth$ so that $\mho_\iota \to \mho^*$ as $\iota \to \infty$. That is,

$$\lim_{\iota,m\to\infty} \varpi_e(\mho_\iota, \mho_m) = \lim_{\iota\to\infty} \varpi_e(\mho_\iota, \mho^*) = \varpi_e(\mho^*, \mho^*) = 0. \tag{61}$$

Now, if A is continuous, by (24) we get

$$\varpi_e(A\mho^*, \mho^*) = \lim_{\iota\to\infty} \varpi_e(A\mho_\iota, \mho_\iota) = \lim_{\iota\to\infty} \varpi_e(\mho_{\iota+1}, \mho_\iota) = 0. \tag{62}$$

Thus, $A\mho^* = \mho^*$; that is, $\mho^*$ is a fixed point of A.

Next, in the case that F_{ϖ_e} is continuous, we claim that

$$\varpi_e(\mho_m, \mho^*) \le \varpi_e(\mho^*, A\mho_m), \quad \forall m \in \mathbb{N} \cup \{0\}. \tag{63}$$

By the fact $1/2\varpi(\mho_m, A\mho_m) < \varpi(\mho_m, A\mho_m)$ and using (48), we obtain that

$$\begin{aligned}
&\vartheta + F_{\varpi_e}(\varpi(A\mho_m, A^2\mho_m)) \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1 + \varpi_e(\mho_m, A\mho_m)},\right.\right. \\
&\quad \left.\left.\frac{\varpi_e(A\mho_m, A^2\mho_m)}{1 + \varpi_e(A\mho_m, A^2\mho_m)}, \frac{\varpi_e(\mho_m, A^2\mho_m) + \varpi_e(A\mho_m, A\mho_m)}{4s(\mho_m, A\mho_m)}\right\}\right) \\
&\quad \le F\left(\max\left\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1 + \varpi_e(\mho_m, A\mho_m)}, \frac{\varpi_e(A\mho_m, A^2\mho_m)}{1 + \varpi_e(A\mho_m, A^2\mho_m)},\right.\right. \\
&\quad \left.\left.\frac{s(\mho_m, A\mho_m)[\varpi_e(\mho_m, A\mho_m) + \varpi_e(A\mho_m, A^2\mho_m)] + 2s(\mho_m, A\mho_m)\varpi_e(\mho_m, A\mho_m)}{4s(\mho_m, A\mho_m)}\right\}\right) \\
&\quad \le F\left(\max\left\{\varpi_e(\mho_m, A\mho_m), \frac{\varpi_e(\mho_m, A\mho_m)}{1 + \varpi_e(\mho_m, A\mho_m)}, \frac{\varpi_e(A\mho_m, A^2\mho_m)}{1 + \varpi_e(A\mho_m, A^2\mho_m)},\right.\right. \\
&\quad \left.\left.\frac{3\varpi_e(\mho_m, A\mho_m) + \varpi_e(A\mho_m, A^2\mho_m)}{4}\right\}\right) \\
&\quad \le F(\max\{\varpi_e(\mho_m, A\mho_m), \varpi_e(A\mho_m, A^2\mho_m)\}).
\end{aligned} \tag{64}$$

If $\varpi_e(\mho_m, A\mho_m) < \varpi_e(A\mho_m, A^2\mho_m)$, then we have

$$F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) \leq F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) - \vartheta, \quad (65)$$

a contradiction due to $(\mathfrak{I}_1)$. So, we should write

$$F_{\varpi_e}\left(\varpi\left(A\mho_m, A^2\mho_m\right)\right) \leq F_{\varpi_e}(\varpi_e(\mho_m, A\mho_m)) - \vartheta. \quad (66)$$

Since F_{ϖ_e} is continuous and strictly increasing, it follows that

$$\varpi_e\left(A\mho_m, A^2\mho_m\right) < \varpi_e(\mho_m, A\mho_m). \quad (67)$$

Now, to ensure the existence of a fixed point, two cases arise as follows.

Case 1. For each $\iota \in \mathbb{N}$, there is $j_\iota \in \mathbb{N}$ so that $\mho_{j_\iota} = A\mho^*$ and $j_\iota > j_{\iota-1}$, where $j_\circ = 1$. Then, we have $\mho^* = \lim_{\iota\to\infty}\mho_{j_\iota} = \lim_{\iota\to\infty}A\mho^* = A\mho^*$, i.e., $\mho^*$ is a fixed point of A.

Case 2. There is $\iota_\circ \in \mathbb{N}$ such that for all $\iota \geq \iota_\circ, \mho_{\iota+1} \neq A\mho^*$. It is clear that $1/2\varpi_e(\mho_\iota, A\mho^*) < \varpi_e(\mho_\iota, A\mho^*)$ for all $\iota \geq \iota_\circ$.

By (48) and (63), we get

$$\begin{aligned}
\vartheta + F_{\varpi_e}\left(\varpi_e\left(A\mho_\iota, A^2\mho^*\right)\right) &\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1+\varpi_e(\mho_\iota, A\mho_\iota)},\right.\right.\\
&\left.\left.\frac{\varpi_e\left(A\mho^*, A^2\mho^*\right)}{1+\varpi_e\left(A\mho^*, A^2\mho^*\right)}, \frac{\varpi_e\left(\mho_\iota, A^2\mho^*\right)+\varpi_e(A\mho^*, A\mho_\iota)}{4s(\mho_\iota, A\mho^*)}\right\}\right)\\
&\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \frac{\varpi_e(\mho_\iota, A\mho_\iota)}{1+\varpi_e(\mho_\iota, A\mho_\iota)}, \frac{\varpi_e\left(A\mho^*, A^2\mho^*\right)}{1+\varpi_e\left(A\mho^*, A^2\mho^*\right)},\right.\right.\\
&\left.\left.\frac{2\varpi_e(\mho_\iota, A\mho^*)+\varpi_e\left(A\mho^*, A^2\mho^*\right)+\varpi_e(\mho_\iota, A\mho_\iota)}{4}\right\}\right)\\
&\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mho_\iota, A\mho^*), \varpi_e(\mho_\iota, A\mho_\iota), \varpi_e\left(A\mho^*, A^2\mho^*\right)\right\}\right)\\
&< F_{\varpi_e}(\max\{\varpi_e(\mho_\iota, A\mho^*), \varpi_e(\mho_\iota, A\mho_\iota), \varpi_e(\mho^*, A\mho^*)\}).
\end{aligned} \quad (68)$$

Since F_{ϖ_e} is continuous, we find at the limit $\iota \to \infty$, or

$$\vartheta + F_{\varpi_e}\left(\varpi_e\left(\mho^*, A^2\mho^*\right)\right) \leq F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)), \quad (69)$$

$$F_{\varpi_e}(\varpi_e(\mho^*, A\mho^*)) \leq F_{\varpi_e}(\varpi_e(\mho^*, \mho^*)) - \vartheta, \quad (70)$$

which is a contradiction. So, $\varpi_e(\mho^*, A\mho^*) = 0$, which leads to $\mho^* = A\mho^*$.

The two cases above ensure the existence of a fixed point of A.

To ensure the uniqueness, suppose that $\mho^*, \upsilon^*$ are distinct fixed points of A. Hence, $1/2\varpi_e(\mho^*, \upsilon^*) < \varpi_e(\mho^*, \upsilon^*)$, which implies that

$$\begin{aligned}
\vartheta + F_{\varpi_e}(\varpi_e(\mho^*, \upsilon^*)) &= \vartheta + F_{\varpi_e}(\varpi_e(A\mho^*, A\upsilon^*))\\
&\leq F_{\varpi_e}\left(\max\left\{\varpi_e(\mho^*, \upsilon^*), \frac{\varpi_e(\mho^*, A\mho^*)}{1+\varpi_e(\mho^*, A\mho^*)},\right.\right.\\
&\left.\left.\frac{\varpi_e(\upsilon^*, A\upsilon^*)}{1+\varpi_e(\upsilon^*, A\upsilon^*)}, \frac{\varpi_e(\mho^*, A\upsilon^*)+\varpi_e(\upsilon^*, A\mho^*)}{4s(\mho^*, \upsilon^*)}\right\}\right)\\
&\leq F_{\varpi_e}(\varpi_e(\mho^*, \upsilon^*)),
\end{aligned} \quad (71)$$

which is a contradiction again. Hence, the fixed point is unique.

In the following, we justify all required hypotheses of Theorem 15.

Example 16. Suppose that $\daleth = [0,\infty)$. Define functions $\varpi_e : \daleth^2 \to \mathbb{R}$ and $s : \daleth \times \daleth \to [1,\infty)$ *by* $\varpi_e(\mho, \aleph) = (\mho+\aleph)^2$ and $s(\mho, \aleph) = 1 + \mho + \aleph$, respectively. Then, $(\daleth, \varpi_e)$ is a complete modified ϖ_e -metric-like space. Define $A : \daleth \to \daleth$ by

$$A\mho = \begin{cases} 0, & \text{if } \mho\in\left[0, \dfrac{1}{4}\right),\\ \left\{\dfrac{1}{4^\iota}\right\}, & \text{if } \mho\in\left[\dfrac{1}{4}, \infty\right), \iota \in \daleth. \end{cases} \quad (72)$$

Define the function $F_{\varpi_e} \in \Pi$ by $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$ and $\vartheta > 0$. We state the following.

Case 1. Let $\mho = 1/4^{\iota-1}$ and $\aleph = 1/4^{m-1}$, for $m > \iota \geq 2$. Now, for $\iota = 1$ and $m = 2$, we have $\mho = 1$ and $\aleph = 1/4$. Therefore,

$$\begin{aligned}
\frac{1}{2}\varpi_e(\mho, A\mho) &= \frac{1}{2}\varpi_e(1, A1) = \frac{1}{2}\left(1+\frac{1}{4}\right)^2 = \frac{25}{32} < \frac{25}{16}\\
&= \left(1+\frac{1}{4}\right)^2 = \varpi_e(\mho, \aleph).
\end{aligned} \quad (73)$$

Let

$$\begin{aligned}
F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) &= F_{\varpi_e}\left(\varpi_e\left(A1, A\frac{1}{4}\right)\right) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{4}, \frac{1}{16}\right)\right)\\
&= F_{\varpi_e}\left(\frac{5}{16}\right)^2 = 2\ln\left(\frac{5}{16}\right) = -2.326,
\end{aligned} \quad (74)$$

as well as,

$$\begin{aligned}
&F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right)\\
&= F_{\varpi_e}\left(\max\left\{\varpi_e\left(1, \frac{1}{4}\right), \frac{\varpi_e(1, A1)}{1+\varpi_e(1, A1)}, \frac{\varpi_e(1/4, A(1/4))}{1+\varpi_e(1/4, A(1/4))},\right.\right.\\
&\left.\left.\frac{\varpi_e(1, A(1/4))+\varpi_e(1/4, A1)}{4s(1, 1/4)}\right\}\right) = F_{\varpi_e}\left(\max\left\{\frac{25}{16}, \frac{25}{41}, \frac{25}{281}, \frac{353}{2304}\right\}\right)\\
&= F_{\varpi_e}\left(\frac{25}{16}\right) = \ln\left(\frac{25}{16}\right) = 0.0446.
\end{aligned} \quad (75)$$

So, we get

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right) = -2.326 - 0.044 = -2.37 < -2. \quad (76)$$

Case 2. Let $\mho = 1/4$ and $\aleph = 0$. So, we get

$$\frac{1}{2}\varpi_e(\mho, A\mho) = \frac{1}{2}\varpi_e\left(\frac{1}{4}, \frac{1}{16}\right) = \frac{25}{512} < \frac{1}{16} = \left(\frac{1}{4}+0\right)^2 = \varpi_e(\mho, \aleph). \quad (77)$$

Consider

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) = F_{\varpi_e}\left(\varpi_e\left(A\frac{1}{4}, A0\right)\right) = F_{\varpi_e}\left(\varpi_e\left(\frac{1}{16}, 0\right)\right) = F_{\varpi_e}\left(\frac{1}{16}\right)^2 = 2\ln\left(\frac{1}{16}\right) = -5.545. \quad (78)$$

Additionally,

$$F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right)$$
$$= F_{\varpi_e}\left(\max\left\{\varpi_e\left(\frac{1}{4}, 0\right), \frac{\varpi_e(1/4, A(1/4))}{1+\varpi_e(1/4, A(1/4))}, \frac{\varpi_e(0, A0)}{1+\varpi_e(0, A0)}, \frac{\varpi_e(1/4, A0)+\varpi_e(0, A(1/4))}{4s(1/4, 0)}\right\}\right)$$
$$= F_{\varpi_e}\left(\max\left\{\frac{1}{16}, \frac{25}{281}, 0, \frac{17}{1280}\right\}\right) = F_{\varpi_e}\left(\frac{25}{281}\right) = \ln\left(\frac{25}{281}\right) = -2.419. \quad (79)$$

Subtracting the two relations, we have

$$F_{\varpi_e}(\varpi_e(A\mho, A\aleph)) - F_{\varpi_e}\left(\max\left\{\varpi_e(\mho, \aleph), \frac{\varpi_e(\mho, A\mho)}{1+\varpi_e(\mho, A\mho)}, \frac{\varpi_e(\aleph, A\aleph)}{1+\varpi_e(\aleph, A\aleph)}, \frac{\varpi_e(\mho, A\aleph)+\varpi_e(\aleph, A\mho)}{4s(\mho, \aleph)}\right\}\right) = -5.545 + 2.419 = -3.126 < -2. \quad (80)$$

From the above, we deduce that the mapping A is an extended generalized F_{ϖ_e}-Suzuki contraction with $\vartheta = 2$. Moreover, if $\mho_m = \{1/4^m\} \in \daleth$, we have

$$\lim_{i,m\to\infty} s(\mho_m, \mho_i) = \lim_{i,m\to\infty}\left(1 + \frac{1}{4^m} + \frac{1}{4^i}\right) = 1 < \frac{1}{\eta}, \quad (81)$$

for $\eta \in (0, 1)$. Hence, the requirements of Theorem 15 hold; therefore, A has a unique fixed point. Here, it is 0.

4. Supportive Applications

This part is considered as the strength of the paper, where we use the results presented in Theorems 9 and 15 to get the analytical solutions both of the Fredholm integral equation and the second-order differential equation, respectively. For this purpose, we will divide this section into two parts as follows.

4.1. Analytical Solution of Fredholm Integral Equation. Let the Fredholm integral equation given by

$$\mho(\eta) = \int_u^v \Phi(\eta, \zeta, \mho(\zeta))d\zeta, \quad (82)$$

for all $\eta, \zeta \in [u, v]$, where $F_{\varpi_e} : [u, v] \to \mathbb{R}$ and $\Phi : [u, v] \times u, v] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\daleth = C([u, v], \mathbb{R})$ be the set of all real continuous functions defined on $[u, v]$, endowed with

$$\varpi_e(\mho, \aleph) = \left(\|\mho + \aleph\|_\infty\right)^2 \text{ for all} \mho, \aleph \in \daleth, \quad (83)$$

where $\|\mho\|_\infty = \sup_{\eta\in[u,v]}\{|\mho(\eta)|e^{-\eta\vartheta}\}$ with $s(\mho, \aleph) = 1 + |\mho| + |\aleph|$, where $s : \daleth \times \daleth \to [1,\infty)$. Note that $(\daleth, \varpi_e)$ is a complete modified ϖ_e-metric-like space.

Now, the following is the main result of this part.

Theorem 17. *Let A be self-mapping on the complete modified ϖ_e -metric-like space $(\daleth, \varpi_e)$. Assume that*

(i) for each $\eta, \zeta \in [u, v]$ and $\mho, \aleph \in \daleth$,

$$\frac{1}{2}\left(\left\|\mho(\eta) + \int_u^v \Phi(\eta, \zeta, \mho(\zeta))d\zeta\right\|_\infty\right)^2 \le \left(\|\mho(\eta)\|_\infty + \|\aleph(\eta)\|_\infty\right)^2 \quad (84)$$

(ii) for all $\eta, \zeta \in [u, v]$, there is a constant $\vartheta \in \mathbb{R}^+$ such that

$$|\Phi(\eta, \zeta, \mho(\zeta)) + \Phi(\eta, \zeta, \aleph(\zeta))| \le \frac{e^{-\vartheta/2}}{(v-u)}(|\mho(\zeta) + \aleph(\zeta)|) \quad (85)$$

Then, there exists a solution of the problem (82).

Proof. Consider the nonlinear self-mapping $A : \daleth \rightarrow \daleth$ given as

$$A\mho(\eta) = \int_u^v \Phi(\eta, \zeta, \mho(\zeta)) d\zeta. \tag{86}$$

Clearly, if $\mho^* = A\mho^*$, then it is a solution of the problem (82).

Let $\mho, \aleph \in \daleth$, so, by condition (i), we deduce that $1/2\varpi_e(\mho(\eta), A\mho(\eta)) < \varpi_e(\mho(\eta), \aleph(\eta))$. After applying the condition (ii), for any $\mho(\eta), \aleph(\eta) \in \daleth$, we can write

$$\begin{aligned}
|A\mho(\eta) + A\aleph(\eta)|^2 &\leq \left(\int_u^v |\Phi(\eta, \zeta, \mho(\zeta)) + \Phi(\eta, \zeta, \aleph(\zeta))| d\zeta\right)^2 \\
&\cdot \left(\int_u^v \frac{e^{-\vartheta/2}}{(v-u)} (|\mho(\zeta) + \aleph(\zeta)|) d\zeta\right)^2 \\
&\leq \left(\int_u^v \frac{e^{-\vartheta/2}}{v-u} \times \sqrt{e^{-2\vartheta\eta} \times e^{2\vartheta\eta}} \times (|\mho(\zeta) + \aleph(\zeta)|) d\zeta\right)^2 \\
&\leq \frac{e^{-\vartheta}}{(v-u)^2} \times \varpi_e(\mho, \aleph) \times e^{2\vartheta\eta} \left(\int_u^v d\zeta\right)^2 \leq e^{-\vartheta} \varpi_e(\mho, \aleph) \times e^{2\vartheta\eta},
\end{aligned} \tag{87}$$

so we have

$$\left(|A\mho(\eta) + A\aleph(\eta)| \times e^{-\vartheta\eta}\right)^2 \leq e^{-\vartheta} \varpi_e(\mho, \aleph), \tag{88}$$

which leads to

$$\left(\|A\mho(\eta) + A\aleph(\eta)\|_\infty\right)^2 \leq e^{-\vartheta} \varpi_e(\mho, \aleph). \tag{89}$$

It yields that

$$\varpi_e(A\mho(\eta), A\aleph(\eta)) \leq e^{-\vartheta} \varpi_e(\mho, \aleph). \tag{90}$$

Taking $F_{\varpi_e}(\ell) = \ln(\ell)$ for $\ell > 0$, one gets or

$$\ln(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq \ln\left(e^{-\vartheta} \varpi_e(\mho, \aleph)\right), \tag{91}$$

$$\vartheta + \ln(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq \ln(\varpi_e(\mho, \aleph)). \tag{92}$$

Equivalently,

$$\vartheta + F_{\varpi_e}(\varpi_e(A\mho(\eta), A\aleph(\eta))) \leq F_{\varpi_e}(\varpi_e(\mho, \aleph)). \tag{93}$$

By Theorem 9, A admits a fixed point, which is a solution of the problem (82).

4.2. Analytical Solution of Second-Order Differential Equation. Consider the second-order differential equation given as follows:

$$\begin{cases} \mho''(\eta) = -\Phi(\eta, \mho(\eta)), & \eta \in [0, \gamma], \\ \mho(0) = \mho(\eta) = 0, \end{cases} \tag{94}$$

where $\Phi : [0, \gamma] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

Resolving the problem (94) is equivalent to resolving the following integral equation:

$$\mho(\eta) = \int_0^\eta \phi(\eta, \zeta) \Phi(\zeta, \mho(\zeta)) d\zeta, \quad \forall \eta \in [0, \gamma], \tag{95}$$

where ϕ is Green's function defined by

$$\phi(\eta, \zeta) = \begin{cases} \eta(1-\zeta), & \text{if } 0 \leq \eta \leq \zeta \leq \gamma, \\ \zeta(1-\eta), & \text{if } 0 \leq \zeta \leq \eta \leq \gamma, \end{cases} \tag{96}$$

and Φ is a function as in Theorem 17. Hence, if $\mho \in C([0, \gamma])$, then $\mho$ is a solution of the problem (94) if and only if $\mho$ is a solution of the problem (95).

Let $\daleth = C([0, \gamma], \mathbb{R})$ be the set of all continuous functions defined on $[0, \gamma]$, and define a norm $\|\mho\|_\vartheta = \max_{\eta \in 0, \gamma]}\{|\mho(\eta)| e^{-1/2\eta\vartheta}\}$, for arbitrary $\eta \geq 1$. Obviously, $\|\cdot\|_\vartheta$ is equivalent to the maximum norm $\|\cdot\|$ on $\daleth$, and $\daleth$ is endowed with the extended generalized ϖ_{e_ϑ}-metric-like as

$$\varpi_{e_\vartheta}(\mho, \aleph) = \left(\|\mho + \aleph\|_\vartheta\right)^2 = \max_{\eta \in 0, \gamma]} \left\{|\mho(\eta) + \aleph(\eta)|^2 e^{-\eta\vartheta}\right\} \text{ for all } \mho, \aleph \in \daleth \text{ and } e^{\eta\vartheta} \geq 1. \tag{97}$$

Then, $(\daleth, \varpi_e)$ is a complete modified ϖ_e-metric-like space with $s(\mho, \aleph) = 1 + |\mho| + |\aleph|$. Our main theorem is as follows.

Theorem 18. *Suppose that $(\daleth, \varpi_e)$ is a complete modified b -metric-like space and A is a nonlinear self-mapping on $\daleth$, then (95) possesses a unique solution $\mho \in C([0, \gamma], \mathbb{R})$, if*

(a1) for each $\eta, \zeta \in [0, \gamma]$ and $\mho, \aleph \in \daleth$,

$$\frac{1}{2}\left\|\mho(\eta) + \int_0^\eta \phi(\eta, \zeta) \Phi(\zeta, \mho(\zeta)) d\zeta\right\|_\vartheta \leq \|\mho + \aleph\|_\vartheta \tag{98}$$

(a2) $\Phi \in C([0, \gamma] \times \mathbb{R})$ and $\phi \in C([0, \gamma] \times [0, \gamma])$

(a3) Φ satisfies

$$|\Phi(\zeta, \mho(\zeta)) + \Phi(\zeta, \aleph(\zeta))|^2 \leq \vartheta e^{-\vartheta} \psi(\mho, \aleph), \tag{99}$$

for all $\zeta\in[0,\gamma]$ *and* $\mho, \aleph \in \mathbb{R}$, *where*

$$\psi(\mho,\aleph)=\max\left\{|\mho+\aleph|^2, \frac{|\mho+A\mho|^2}{1+|\mho+A\mho|^2}, \frac{|\aleph+A\aleph|^2}{1+|\aleph+A\aleph|^2}, \frac{|\mho+A\aleph|^2+|\aleph+A\mho|^2}{4(1+|\mho|+|\aleph|)}\right\} \quad (100)$$

(a4) $\max \int_0^\eta \phi(\eta,\zeta)d\zeta \le 1$, *for all* $\eta\in[0,\gamma]$

Proof. Consider on the set $\daleth$, the mapping A as

$$A\mho(\eta)=\int_0^\eta \phi(\eta,\zeta)\Phi(\zeta,\mho(\zeta))d\zeta, \quad (101)$$

for all $\eta\in[0,\gamma]$ and $\mho\in\daleth$. The solution of (95) is also a fixed point of A on $\daleth$. By condition (a1) and the definition of A, we can write $1/2\varpi_{e_\vartheta}(\mho(\eta),A\mho(\eta)) < \varpi_{e_\vartheta}(\mho(\eta),\aleph(\eta))$.

Let $\mho, \aleph \in \daleth$. By the hypotheses (a2)-(a4), we have

$$\begin{aligned}(|A\mho(\eta)+A\aleph(\eta)|)^2 &= \left|\int_0^\eta \phi(\eta,\zeta)[\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))]d\zeta\right|^2\\ &\le \int_0^\eta |\phi(\eta,\zeta)|^2|\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))|^2 d\zeta\\ &\le \int_0^\eta |\Phi(\zeta,\mho(\zeta))+\Phi(\zeta,\aleph(\zeta))|^2 d\zeta\\ &\le \int_0^\eta \vartheta e^{-\vartheta}\psi(\mho(\zeta),\aleph(\zeta))d\zeta\\ &\le \vartheta e^{-\vartheta}\int_0^\eta e^{\zeta\vartheta}\max\left\{|\mho+\aleph|^2e^{-\zeta\vartheta}, \frac{|\mho+A\mho|^2e^{-2\zeta\vartheta}}{1+|\mho+A\mho|^2e^{-\zeta\vartheta}}, \frac{|\aleph+A\aleph|^2e^{-2\zeta\vartheta}}{1+|\aleph+A\aleph|^2e^{-\zeta\vartheta}}, \frac{|\mho+A\aleph|^2+|\aleph+A\mho|^2}{4(1+|\mho|+|\aleph|)}e^{-\zeta\vartheta}\right\}d\zeta\\ &\le \vartheta e^{-\vartheta}\int_0^\eta e^{\zeta\vartheta}\max\left\{\varpi_{e_\vartheta}(\mho,\aleph), \frac{\varpi_{e_\vartheta}(\mho,A\mho)}{1+\varpi_{e_\vartheta}(\mho,A\mho)}, \frac{\varpi_{e_\vartheta}(\aleph,A\aleph)}{1+\varpi_{e_\vartheta}(\aleph,A\aleph)}, \frac{\varpi_{e_\vartheta}(\mho,A\aleph)+\varpi_{e_\vartheta}(\aleph,A\mho)}{4s(\mho,\aleph)}\right\}d\zeta\\ &= \vartheta e^{-\vartheta}\psi(\mho,\aleph)\int_0^\eta e^{\zeta\vartheta}d\zeta = \vartheta e^{-\vartheta}\psi(\mho,\aleph)\times\left(\frac{e^{\eta\vartheta}}{\vartheta}-1\right)\\ &\le \vartheta e^{-\vartheta}\psi(\mho,\aleph)\times\frac{e^{\eta\vartheta}}{\vartheta} \le e^{-\vartheta}\psi(\mho,\aleph)e^{\eta\vartheta}.\end{aligned} \quad (102)$$

Hence, for all $\mho, \aleph \in \daleth$,

$$(|A\mho(\eta)+A\aleph(\eta)|)^2\times e^{-\eta\vartheta} \le e^{-\vartheta}\psi(\mho,\aleph), \quad (103)$$

which yields

$$\varpi_{e_\vartheta}(A\mho,A\aleph)=\max_{\eta\in 0,\gamma]}\left\{(|A\mho(\eta)+A\aleph(\eta)|)^2\times e^{-\eta\vartheta}\right\} \le e^{-\vartheta}\psi(\mho,\aleph). \quad (104)$$

That is,

$$\vartheta + \ln \varpi_{e_\vartheta}(A\mho,A\aleph) \le \ln \psi(\mho,\aleph). \quad (105)$$

Defining the function $F_{\varpi_e}(\alpha)=\ln(\alpha),\alpha>0$ in (105), such that $F_{\varpi_e}\in\Pi$, we have

$$\vartheta + F_{\varpi_e}(\varpi_e(A\mho,A\aleph)) \le F_{\varpi_e}\left(\max\left\{\varpi_e(\mho,\aleph), \frac{\varpi_e(\mho,A\mho)}{1+\varpi_e(\mho,A\mho)}, \frac{\varpi_e(\aleph,A\aleph)}{1+\varpi_e(\aleph,A\aleph)}, \frac{\varpi_e(\mho,A\aleph)+\varpi_e(\aleph,A\mho)}{4s(\mho,\aleph)}\right\}\right). \quad (106)$$

Hence, all requirements of Theorem 15 hold and A is an extended generalized F-Suzuki contraction; hence, Γ possesses a fixed point $\mho\in\daleth$, which is a solution of the problem (95).

5. Conclusion

A modified ϖ_e-metric-like space is presented, and related fixed point results via it are discussed. Nontrivial examples are conducted for supporting the mentioned space and theorems. Thereafter, by using a fixed point technique, a simple and efficient solution for the integral and differential equations is found in the setting of a modified ϖ_e-metric-like space. A lot of authors connected fixed point techniques and classical integral equations in various abstract spaces such as metric spaces, b-metric spaces, and partial metric spaces. We also follow the same method in the new space. In the literature, our obtained applications are an extension and/or a generalization of many existing classical integral and differential equations. The observed results of this paper open new framework research avenues for

(i) fixed point techniques for solving Volterra-Fredholm integral equation in a modified ϖ_e-metric-like space

(ii) collocation-type methods for Volterra-Hammerstein integral equations in modified ϖ_e-metric-like spaces

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

Acknowledgments

The authors thank the Spanish Government and the European Fund of Regional Development FEDER for Grant RTI2018-094336-B-I00 (MCIU/AEI/FEDER, UE) and the Basque Government for Grant IT1207-19.

References

[1] E. I. Fredholm, "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, vol. 27, pp. 365–390, 1903.

[2] M. D. Rus, "A note on the existence of positive solution of Fredholm integral equations," *Fixed Point Theory*, vol. 5, pp. 369–377, 2004.

[3] M. I. Berenguer, M. V. F. Munoz, A. I. G. Guillem, and M. R. Galan, "Numerical treatment of fixed point applied to the nonlinear Fredholm integral equation," *Fixed Point Theory and Applications*, vol. 2009, no. 1, Article ID 735638, 2009.

[4] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[5] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, 2012.

[6] H. Piri and P. Kuman, "Some fixed point theorems concerning F-contraction in complete metric spaces," *Fixed Point Theory and Applications*, vol. 2014, no. 1, 11 pages, 2014.

[7] D. Singh, V. Chauhan, P. Kumam, and V. Joshi, "Some applications of fixed point results for generalized two classes of Boyd–Wong's F-contraction in partial b-metric spaces," *The Mathematical Scientist*, vol. 12, no. 2, pp. 111–127, 2018.

[8] P. S. Kumari, K. Zoto, and D. Panthi, "d-Neighborhood system and generalized F-contraction in dislocated metric space," *Springer Plus*, vol. 4, no. 1, p. 368, 2015.

[9] R. Batra and S. Vashistha, "Fixed points of an F-contraction on metric spaces with a graph," *International Journal of Computer Mathematics*, vol. 91, pp. 1–8, 2014.

[10] R. Batra, S. Vashistha, and R. Kumar, "A coincidence point theorem for F-contractions on metric spaces equipped with an altered distance," *The Journal of Mathematics and Computer Science*, vol. 4, pp. 826–833, 2014.

[11] M. Cosentino and P. Vetro, "Fixed point results for F-contractive mappings of Hardy-Rogerstype," *Univerzitet u Nišu*, vol. 28, pp. 715–722, 2014.

[12] H. A. Hammad and M. De la Sen, "A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations," *Mathematics*, vol. 7, no. 7, p. 634, 2019.

[13] M. Sarwar, M. B. Zada, and S. Radenovic, "Rational type inequality with applications to Voltera–Hammerstein nonlinear integral equations," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 21, no. 5, pp. 465–473, 2018.

[14] M. B. Zada, M. Sarwar, and H. K. Nashine, "Solution of infinite system of ordinary differential equations and fractional hybrid differential equations via measure of noncompactness," *Journal of Taibah University for Science*, vol. 3, no. 1, pp. 1119–1127, 2019.

[15] M. B. Zada and M. Sarwar, "Common fixed point theorems for rational F R $F_{\mathcal{R}}$ -contractive pairs of mappings with applications," *Journal of Inequalities and Applications*, vol. 2019, no. 1, 11 pages, 2019.

[16] M. Shoaib, M. Sarwar, K. Shah, and P. Kumam, "Fixed point results and its applications to the systems of non-linear integral and differential equations of arbitrary order," *Journal of Nonlinear Science and. Applications*, vol. 9, no. 6, pp. 4949–4962, 2016.

[17] H. A. Hammad, D. M. Albaqeri, and R. A. Rashwan, "Coupled coincidence point technique and its application for solving nonlinear integral equations in RPOCbML spaces," *Egyptian Mathematical Society*, vol. 28, no. 1, p. 8, 2020.

[18] H. A. Hammad, H. Aydi, and M. De la Sen, "Generalized dynamic process for an extended multi-valued F-contraction in metric-like spaces with applications," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3817–3825, 2020.

[19] V. Gupta, W. Shatanawi, and N. Mani, "Fixed point theorems for $$(\psi,\beta) $$(ψ, β)-Geraghty contraction type maps in ordered metric spaces and some applications to integral and ordinary differential equations," *Journal of Fixed Point Theory and Applications*, vol. 19, no. 2, pp. 1251–1267, 2017.

[20] P. Borisut, P. Kumam, V. Gupta, and N. Mani, "Generalized (ψ,α,β)-weak contractions for initial value problems," *Mathematics*, vol. 7, no. 3, p. 266, 2019.

[21] R. Kannan, "Some results on fixed points," *Bulletin of the Calcutta Mathematical Society*, vol. 60, pp. 71–76, 1968.

[22] A. C. M. Ran and M. C. B. Reuring, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of American Mathematical Society*, vol. 132, pp. 1435–1443, 2004.

[23] W. A. Kirk, P. S. Srinavasan, and P. Veeramani, "Fixed points for mapping satisfying cyclical contractive conditions," *Fixed Point Theory*, vol. 4, pp. 79–89, 2003.

[24] W. Shatanawi and M. Postolache, "Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[25] P. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis*, vol. 75, no. 4, pp. 2154–2165, 2012.

[26] M. A. Alghmandi, N. Hussain, and P. Salimi, "Fixed point and coupled fixed point theorems on b-metric-like spaces," *Journal of Inequalities and Applications*, vol. 402, 25 pages, 2013.

[27] I. A. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Funct. Anal. Ulianowsk Gos. Ped. Inst.*, vol. 30, pp. 26–37, 1989.

[28] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications.*, vol. 2012, no. 1, p. 204, 2012.

[29] H. Aydi, A. Felhi, and S. Sahmim, "Common fixed points via implicit contractions on b-metric-like spaces," *Journal of Nonlinear Sciences & Applications (JNSA)*, vol. 10, no. 4, pp. 1524–1537, 2017.

[30] H. K. Nashine and Z. Kadelburg, "Existence of solutions of cantilever beam problem via α-β-FG-contractions in b-metric-like spaces," *Univerzitet u Nišu*, vol. 31, pp. 3057–3074, 2017.

[31] H. Aydi and E. Karapinar, "Fixed point results for generalized α-ψ-contractions in metric-like spaces and applications," *Electronic Journal of Differential Equations*, vol. 133, pp. 1–15, 2015.

[32] H. Alsulami, S. Gülyaz, E. Karapinar, and I. M. Erhan, "An Ulam stability result on quasi-b-metric-like spaces," *Open Mathematics*, vol. 14, no. 1, pp. 1087–1103, 2016.

[33] H. A. Hammad and M. De la Sen, "Solution of nonlinear integral equation via fixed point of cyclic α_L^{ψ}-rational contraction mappings in metric-like spaces," *Bulletin of the Brazilian Mathematical Society, New Series*, vol. 51, pp. 81–105, 2020.

[34] H. A. Hammad and M. De la Sen, "Generalized contractive mappings and related results in b-metric like spaces with an application," *Symmetry*, vol. 11, no. 5, p. 667, 2019.

[35] H. A. Hammad and M. De la Sen, "A solution of Fredholm integral equation by using the cyclic η s q -rational contractive mappings technique in b-metric-like spaces," *Symmetry*, vol. 11, no. 9, article 1184, 2019.

[36] V. Gupta, N. Mani, and N. Sharma, "Fixed point theorems for (ψ,β)-mappings satisfying generalized C- condition and its application to boundary value problem," *Computational and Mathematical Methods*, vol. 1, no. 4, pp. 1–12, 2019.

[37] V. Parvaneh and Z. Kadelburg, "Fixed points of JSHR-contractive type mappings in extended b-metric-like spaces," *Vietnam Journal of Mathematics*, vol. 47, pp. 387–401, 2019.