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TECHNICAL NOTE

Analytical solution of passive earth pressure

R. LANCELLOTTA*

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INTRODUCTION

It is well recognised that, when wall friction is present, a non-uniform stress field arises as well as a non-planar failure surface. This renders the problem of computing exact values of earth pressures non-trivial, and analytical solutions are not available in this case. In particular, when dealing with passive earth pressure, current practice relies on solutions provided by limit equilibrium methods with a curved (typically log-spiral) surface, but as these procedures are essentially of kinematical nature they are not conservative. In fact, should the assumed mechanism be admissible in kinematics terms, these solutions represent an upper bound of the exact solution. For this reason it is of interest to search for a statically admissible stress field, because this approach provides a conservative answer or the exact one (Calladine, 1985). In this respect, the numerical solution obtained by Sokolowski (1965), based on the method of characteristics, is actually of major interest, and the one most commonly used by designers is that of Caquot & Kerisel (1948) or Kerisel & Absi (1990).

This paper is intended to contribute to this problem by providing an analytical solution for earth pressure coefficients, based on the lower bound theorem of plasticity.

The expression obtained for the passive earth pressure coefficient is

$$K_p = \left[\frac{\cos\delta}{1-\sin\phi'}\left(\cos\delta + \sqrt{\sin^2\phi' - \sin^2\delta}\right)\right] e^{2\theta\tan\phi'} \quad (1)$$

where

$$2\theta = \sin^{-1}\left(\frac{\sin\delta}{\sin\phi'}\right) + \delta \quad (2)$$

This is in approximate agreement with Sokolowski's solution (see Fig. 1); it is of value in engineering practice, as it is a conservative estimate of the exact solution. It can further be inferred that, when the boundary condition of a smooth wall applies ($\delta = 0$), then equation (1) merges into Rankine's solution.

A further remark is concerned with the assumption of normality, when dealing with upper- and lower-bound theorems of plasticity. If a more realistic material behaviour requires a non-associated flow rule, it can be proved (see theorems of Radenkovic as quoted by Salençon, 1974) that the statically admissible solution, obtained for an associated flow rule, is still an approximate safe solution with respect to that obtained for a material obeying a non-associated flow rule.

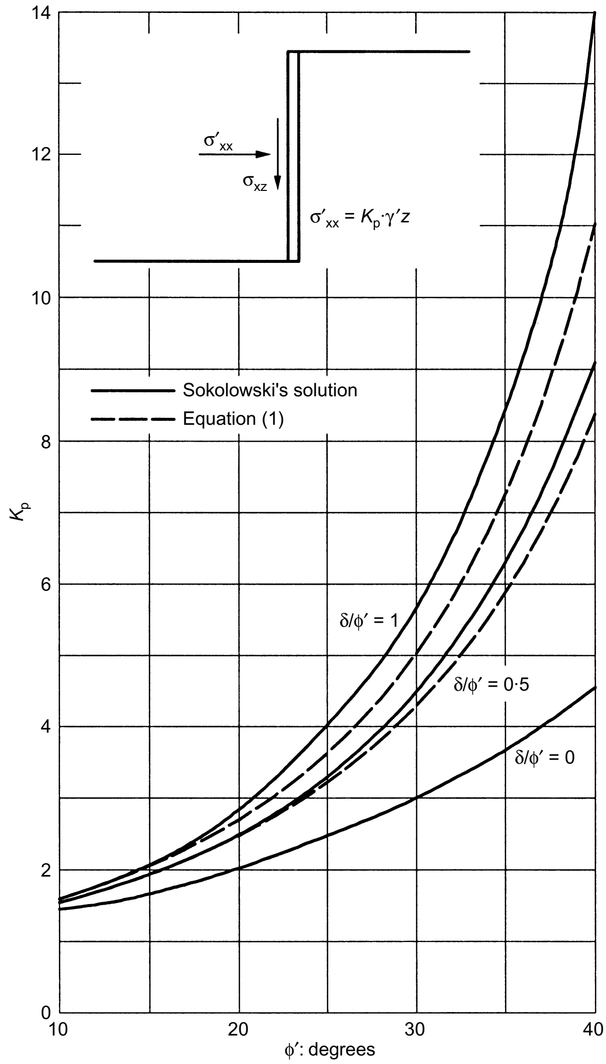

Fig. 1. Passive earth pressure coefficient

DISCONTINUITY CONDITIONS IN A LIMITING FIELD OF STRESS

Consider a plane of discontinuity between two states of stress relative to regions A and B in Fig. 2. In order to satisfy equilibrium, the Mohr circles of these regions must have a common point at X, and a relevant result that can be deduced from Fig. 2 is that there is a jump in the direction and magnitude of the major principal stress across the discontinuity.

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* Department of Structural and Geotechnical Engineering, Technical University of Turin, Italy.

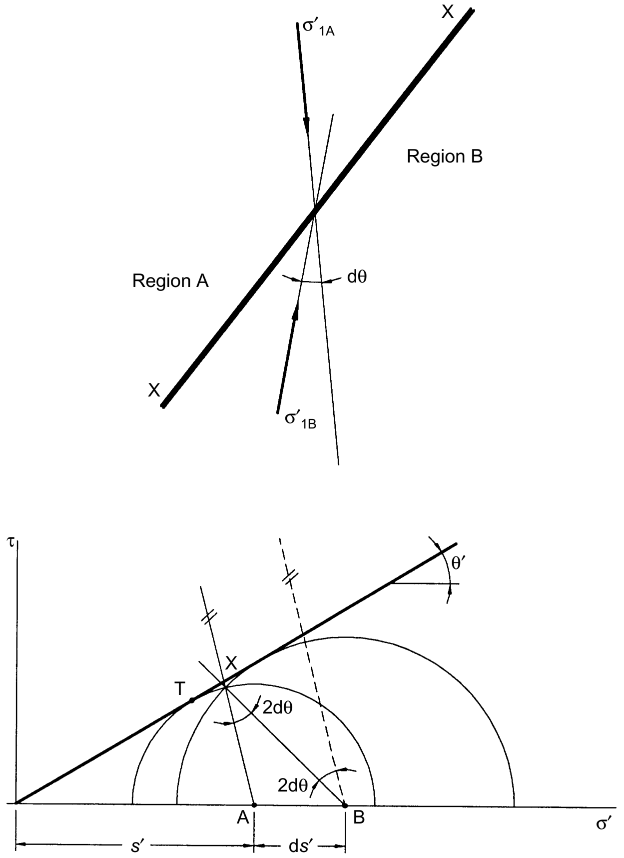

Fig. 2. Change of principal directions across a discontinuity, and shift of stress circle centres

By assuming small rotation, the shift in centre of the Mohr's circles is related to the rotation of principal direction through the equation

$$\frac{\mathrm{d}s'}{s'} = 2\mathrm{d}\theta \tan\phi' \tag{3}$$

where θ is the rotation of the principal directions, ϕ' is the angle of shear strength, and s' is the abscissa of the centre of the Mohr circle. Equation (3) can be proved by observing that, if $\mathrm{d}s' \to 0$, $\sin 2\theta \cong 2\theta$, the common point $\mathrm{X} \to \mathrm{T}$ and $\overline{\mathrm{BX}} \cong \overline{\mathrm{AX}} = s' \sin\phi'$. By applying the sine theorem to the triangle ABX, one gets $\overline{\mathrm{AX}}/\cos\phi' = \mathrm{d}s'/\sin 2\theta$, and equation (3) follows. If we consider a fan of stress discontinuities (Fig. 3), across which the rotation of principal direction assumes the finite value θ, then the shift between the two extreme Mohr circles is defined by

$$\frac{s_1'}{s_2'} = \mathrm{e}^{2\theta\tan\phi'} \tag{4}$$

In order to obtain the value of σ_{xx}' at the wall interface as a function of $\sigma_{zz}' = \gamma' z$, the overburden stress in region 2, referring to Fig. 3 the following relations apply:

$$\overline{\mathrm{OP}}_1 = \overline{\mathrm{OC}} + \overline{\mathrm{CP}}_1; \quad \overline{\mathrm{OP}}_1 = s_1'[\cos\delta + \sqrt{\sin^2\phi' - \sin^2\delta}] \tag{5}$$

where δ is the angle of wall friction and, since $\sigma_{xx}' = \overline{\mathrm{OP}}_1 \cos\delta$, it follows that

$$\sigma_{xx}' = \left[\frac{\cos\delta}{1-\sin\phi'}\frac{s_1'}{s_2'}(\cos\delta + \sqrt{\sin^2\phi' - \sin^2\delta})\right]\cdot\gamma' z \tag{6}$$

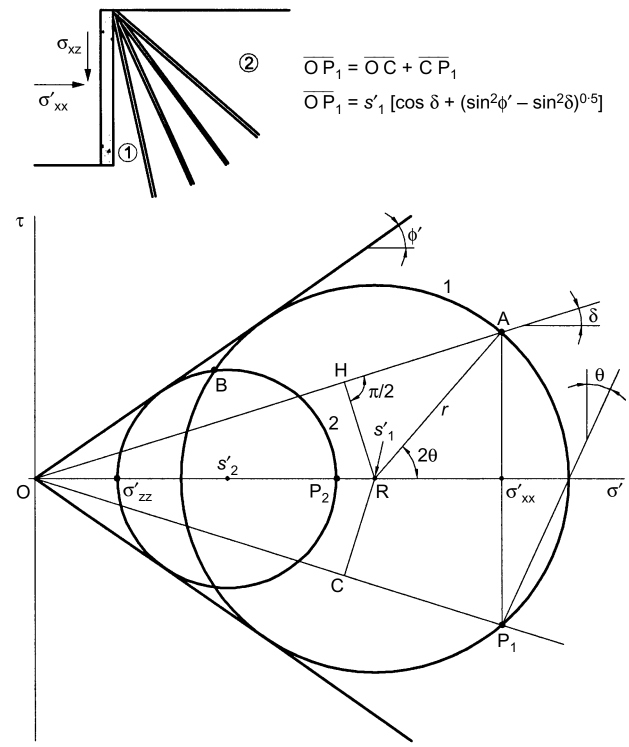

Fig. 3. Passive earth pressure

since

$$s_2' = \frac{\sigma_{zz}'}{1-\sin\phi'}$$

By inserting equation (4) into equation (6), equation (1) is obtained, with the additional condition

$$2\theta = \sin^{-1}\left(\frac{\sin\delta}{\sin\phi'}\right) + \delta$$

derived from circle 1 in Fig. 3, by observing that

$$\overline{\mathrm{HR}} = r\sin(2\theta - \delta) = \overline{\mathrm{OR}}\sin\delta = \frac{r}{\sin\phi'}\sin\delta \tag{7}$$

In order to evaluate equation (1), Table 1 provides a comparison with Sokolowski's solution, so that the conclusions anticipated in the introduction are justified.

For completeness we derive the corresponding expression for the active coefficient. Referring to Fig. 4, we obtain

$$s_1' = \frac{\sigma_{zz}'}{1+\sin\varphi'}; \quad \overline{\mathrm{OA}} = \overline{\mathrm{OH}} - \overline{\mathrm{AH}}$$

$$\overline{\mathrm{OH}} = s_2'\cos\delta; \quad \overline{\mathrm{AH}} = s_2'\sqrt{\sin^2\phi' - \sin^2\delta}$$

so that

$$\sigma_{xx}' = \left[\frac{\cos\delta}{1+\sin\phi'}\frac{s_2'}{s_1'}(\cos\delta - \sqrt{\sin^2\phi' - \sin^2\delta})\right]\gamma' z \tag{8}$$

Table 1. Values of passive earth pressure coefficient $K_p = \sigma_{xx}'/\sigma_{zz}'$, as given by equation (1) and compared with Sokolowski's (1965) solution

ϕ': degrees	20	20	30	30	40	40
δ	10	20	15	30	20	40
Sokolowski's solution	2·51	2·86	4·46	5·67	9·10	14·00
Equation (1)	2·48	2·74	4·29	5·03	8·37	11·03

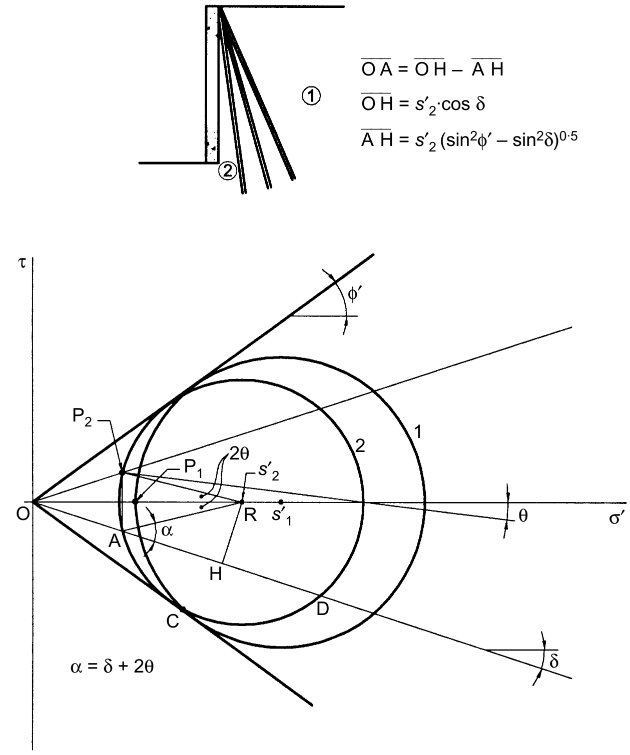

Fig. 4. Active earth pressure

By inserting equation (3) we obtain

$$K_a = \left[\frac{\cos\delta}{1+\sin\phi'}(\cos\delta - \sqrt{\sin^2\phi' - \sin^2\delta})\right]e^{-2\theta\tan\phi'} \tag{9}$$

where

$$2\theta = \sin^{-1}\left(\frac{\sin\delta}{\sin\phi'}\right) - \delta \tag{10}$$

Finally we add two remarks. First, equations (1) and (9) can be expressed as a single equation:

$$K_{p,a} = \left[\frac{\cos\delta}{1\mp\sin\phi'}(\cos\delta \pm \sqrt{\sin^2\phi' - \sin^2\delta})\right]e^{\pm 2\theta\tan\phi'} \tag{11}$$

with

$$2\theta = \sin^{-1}\left(\frac{\sin\delta}{\sin\phi'}\right) \pm \delta$$

where the upper sign applies for the passive coefficient and the lower sign for the active coefficient.

Second, the influence of a sloping backfill or sloping wall surface can be taken into account through an appropriate value of the rotation of principal directions: that is, it is reflected in the angle θ.

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