Research Article

Analytical Solution of System of Volterra Integral Equations Using OHAM

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In this work, a reliable technique is used for the solution of a system of Volterra integral equations (VIEs), called optimal homotopy asymptotic method (OHAM). The proposed technique is successfully applied for the solution of different problems, and comparison is made with the relaxed Monto Carlo method (RMCM) and hat basis function method (HBFM). The comparisons show that the present technique is more suitable and reliable for the solution of a system of VIEs. The presented technique uses auxiliary function containing auxiliary constants, which control the convergence. Moreover, OHAM does not require discretization like other numerical methods and is also free from small or large parameter.

1. Introduction

Differential and integral equations have their own importance and a lot of applications. These equations have been used in many fields, such as control, economics, electrical engineering, medicine, and so on, and also the system of these equations arises in modeling of different phenomena in different fields of science and technology. The exact solutions of most of the nonlinear systems are difficult to obtain; therefore, the researchers used an alternative approach to find their approximate solutions, such as the residual power series method [1], homotopy analysis method [2], wavelet-Galerkin method [3], Adomian decomposition method [4], modified reproducing kernel method [5], new homotopy perturbation method [6], Chebyshev wavelet method [7], homotopy perturbation method [8], fixed-point method [9], hat basis functions [10], modified variational iterative method [11], relaxed Monte

Carlo method [12], variational iterative method [13], collocation method [14], modified tanh-coth method [15], and generalized hypergeometric solutions [16].

In this work, OHAM is used [17, 18] for the solution of a system of VIEs. It is motivated by the aspiration to acquire an exciting solution of a system of VIEs by using the proposed method. The novelty of the presented technique is its flexible convergence and it provides us with a convenient way to control and adjust the approximation series. Many researchers used the proposed technique for different types of problems in literature. Iqbal et al. implemented it for Klein–Gordon equations [19] and singular Lane–Emden type equation [20], Sheikholeslami et al. used OHAM for the investigation of the laminar viscous flow [21], Hashmi et al. obtained solution of Fredholm integral equations [22], and Nawaz et al. obtained optimum solutions of fractional-order Zakharov–Kuznetsov Equations [23], three-dimensional Volterra integral equations [24], and fractional integro-

differential equations [25]. We apply the method to obtain the approximate solution of a system of VIEs and test some numerical examples to show the effectiveness and accuracy of the method.

This paper is organized as follows. Section 2 gives the basic idea of OHAM for a system of VIEs. The application of OHAM to the system of VIEs is given in Section 3, and conclusions of the paper are given in Section 4.

2. Basic Idea of OHAM

In this section, we discuss the formulation of OHAM for solving the system of Volterra integral equations of the second kind following the procedure outlined in [17, 18]. Let us consider the system of VIEs of the form

$$W(x) = G(x) + \int_0^x K(x,s)w(s)\mathrm{d}s, \quad 0 \le x \le 1, \tag{1}$$

where

$$
\begin{aligned}
W(x) &= [w_1(x), w_2(x), \dots, w_n(x)]^\mathrm{T}, \\
G(x) &= [g_1(x), g_2(x), \dots, g_n(x)]^\mathrm{T}, \\
K(x,s) &= [\kappa_{ij}(x,s)], \quad i,j = 1,2,\dots,n.
\end{aligned}
\tag{2}
$$

In equation (1), $K(x,s)$ is the kernel, $G(x)$ are given functions, and $W(x)$ is the unknown solution of the system. Consider the ith equation of (1):

$$w_i(x) = g_i(x) + \int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_j(s)\mathrm{d}s, \quad i = 1,2,\dots,n. \tag{3}$$

First, we construct homotopy: $\psi_i(w_i(x,q);q)$: $\mathfrak{R} \times [0,1] \longrightarrow \mathfrak{R}$, for equation (3), such that

$$\psi_i(w_i(x,q);q) = (1-q)\{w_i(x) - g_i(x)\} = H(q)\left(w_i(x) - g_i(x) - \int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_j(s)\mathrm{d}s\right), \quad i = 1,2,\dots,n, \tag{4}$$

where $H(q) = \sum_{p=1}^m c_p q^p$ is an auxiliary function, c_p are auxiliary constants, $0 \le q \le 1$ is an embedding parameter, and $H(0) = 0$.

For $q = 0$, equation (4) becomes

$$\psi_i(w_i;0) = w_i - g_i = 0, \tag{5}$$

and for $q = 1$,

$$\psi_i(w_i;1) = H(1)\left(w_i(x) - g_i(x) - \int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_j(s)\mathrm{d}s\right), \quad i = 1,2,\dots,n. \tag{6}$$

When q approaches from zero to 1, then $w_i(x,0)$ is continuously deformed to $w_i(x,1)$.

For approximate solution of equation (3), using Taylor's series expansion about q, one can get

$$w_i^\infty(x,q,c_p) = w_{i0}(x) + \sum_{\kappa=1}^\infty w_{i\kappa}(x,c_p)q^\kappa, \quad i,p = 1,2,\dots,n. \tag{7}$$

Substituting equation (7) into equation (4) and comparing the by comparing the coefficient of the like powers of q, one can get a series of problems.

$$q^0: w_{i0}(x) = g_i(x), \quad i = 1,2,\dots,n,$$

$$q^1: w_{i1}(x) = -c_1\left(\int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_{j0}(s)\mathrm{d}s\right), \quad i = 1,2,\dots,n. \tag{8}$$

For k^{th} order problem for $w_{ik}(x)$, $i = 1,2,\dots,n$, it becomes

$$
\begin{aligned}
q^k: w_{ik}(x) = {}& w_{ik-1}(x) - c_k\left(\int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_{j0}(s)\mathrm{d}s\right) \\
& + \sum_{l=1}^{k-1} c_l\left[w_{ik-l}(x) - \int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_{jk-1}(s)\mathrm{d}s\right], \\
& k = 2,3,\dots.
\end{aligned}
\tag{9}
$$

At $q = 1$, equation (7) converges to the series solution:

$$w_i^\infty(x,c_p) = w_{i0}(x) + \sum_{\kappa=1}^\infty w_{i\kappa}(x,c_p), \quad i = 1,2,\dots,n. \tag{10}$$

m^{th} order approximation is

$$
\begin{aligned}
w_i^m(x,c_p) = {}& w_{i0}(x) + \sum_{\kappa=1}^m w_{i\kappa}(x,c_p), \\
& i = 1,2,\dots,n, \ p = 1,2,3,\dots,m.
\end{aligned}
\tag{11}
$$

We define the residual on substituting equation (11) into equation (3).

$$R_i(x,c_p) = w_i^m(x,c_p) - g_i(x) - \int_0^x \sum_{j=1}^n \kappa_{ij}(x,s)w_j^m(s,c_p)\mathrm{d}s. \tag{12}$$

To find c_p, $p = 1,2,3,\dots$, we use the least squares method as follows:

$$\varphi_i(c_p) = \int_a^b R_i^2(x, c_p)\,dx, \qquad (13)$$

where limit of integration is the domain of the problem. Differentiating equation (13) with respect to the constant involved, we get

$$\frac{\partial \varphi_i}{\partial c_p} = 0, \quad i, p = 1, 2, \ldots. \qquad (14)$$

One can easily obtain the values of c_p from equation (14) for which the mth order approximation given in equation (11) is well determined.

3. System of VIEs

In this section, the consistency and reliability of OHAM are verified by some numerical problems, and the obtained results are compared with other methods in the form of tables; these tables clearly show the dominance of the proposed technique over these methods.

3.1. Problem 1. Consider the following system of VIEs [12]:

$$w_1(x) = -\frac{1}{3}(x^4 + x^3) + x^2 + 1 + \int_0^x (x-s)^3 w_1(s)\,ds + \int_0^x (x-s)^2 w_2(s)\,ds,$$

$$w_2(x) = -\frac{1}{4}\left(\frac{x^7}{105} + x^5 + x^4\right) - x^3 + x + 1 + \int_0^x (x-s)^4 w_1(s)\,ds + \int_0^x (x-s)^3 w_2(s)\,ds, \quad x \in [0,1]. \qquad (15)$$

Equation (15) has exact solutions $w_1(x) = x^2 + 1$ and $w_2(x) = -x^2 + x + 1$.

Applying the proposed algorithm discussed in Section 2, one can get different order solutions, as follows.
Zeroth-order solution:

$$w_{10}(x) = 1 - \frac{1}{3}x^2(-3 + x + x^2), \qquad (16)$$

$$w_{20}(x) = 1 + x - x^3 - \frac{x^4}{4} - \frac{x^5}{4} - \frac{x^7}{420}, \qquad (17)$$

$$w_{11}(x) = \frac{c_1 x^3}{151200}\left(-50400 - 50400x + 720x^4 + 405x^5 + x^7\right), \qquad (18)$$

$$w_{21}(x) = \frac{c_1 x^4}{1663200}\left(-415800 + x\left(-415800 - 3960x^2 + x^3\left(3465 + 1705x + 3x^3\right)\right)\right), \qquad (19)$$

$$w_{12}(x) = \frac{1}{908107200}x^3\left(-302702400\left(c_1 + c_1^2 + c_2\right) - 302702400\left(c_1 + c_1^2 + c_2\right)x + 4324320\left(c_1 + 2c_1^2 + c_2\right)x^4 \right.$$
$$\left. + 2432430\left(c_1 + 2c_1^2 + c_2\right)x^5 + 6006\left(c_1 + 2c_1^2 + c_2\right)x^7 - 7098c_1^2 x^8 - 2639c_1^2 x^9 - 3c_1^2 x^{11}\right), \qquad (20)$$

$$w_{22}(x) = \frac{1}{1297296000}x^4\left(-324324000\left(c_1 + c_1^2 + c_2\right) - 324324000\left(c_1 + c_1^2 + c_2\right)x - 3088800\left(c_1 + c_1^2 + c_2\right)x^3 \right.$$
$$\left. + 2702700\left(c_1 + 2c_1^2 + c_2\right)x^4 + 1329900\left(c_1 + 2c_1^2 + c_2\right)x^5 + 2340\left(c_1 + 2c_1^2 + c_2\right)x^7 - 2925c_1^2 x^8 - 1005c_1^2 x^9 - c_1^2 x^{11}\right), \qquad (21)$$

$$w_{13}(x) = \frac{1}{22230464256000}x^3\left(-7410154752000\left(c_2 + c_1(1 + c_1)^2 + 2c_1c_2 + c_3\right)\right.$$
$$- 7410154752000\left(c_2 + c_1(1 + c_1)^2 + 2c_1c_2 + c_3\right)x + 105859353600\left(c_2 + c_1 + c_1^2(4 + 3c_1) + 4c_1c_2 + c_3\right)x^4$$
$$+ 59545886400\left(c_2 + c_1 + c_1^2(4 + 3c_1) + 4c_2\right)x^5 + 147026880\left(c_2 + c_1 + c_1^2(4 + 3c_1) + 4c_1c_2 + c_3\right)x^7$$
$$- 173759040c_1\left(c_1(2 + 3c_1) + 2c_2\right)x^8 - 64602720c_1\left(c_1(2 + 3c_1) + 2c_2\right)x^9 - 73440c_1\left(c_1(2 + 3c_1) + 2c_2\right)x^{11}$$
$$\left. + 68544c_1^3 x^{12} + 19125c_1^3 x^{13} + 13c_1^3 x^{15}\right), \qquad (22)$$

$$w_{23}(x) = \frac{1}{46930980096000}x^4\left(-11732745024000\left(c_2 + c_1\left(1 + c_1\right)^2 + 2c_1c_2 + c_3\right)\right.$$
$$- 11732745024000\left(c_2 + c_1\left(1 + c_1\right)^2 + 2c_1c_2 + c_3\right)x - 111740428800\left(c_2 + c_1\left(1 + c_1\right)^2 + 2c_1c_2 + c_3\right)x^3$$
$$+ 97772875200\left(c_2 + c_1 + c_1^2\left(4 + 3c_1\right) + 4c_1c_2 + c_3\right)x^4 + 48110462400\left(c_2 + c_1 + c_1^2\left(4 + 3c_1\right) + 4c_1c_2 + c_3\right)x^5 \tag{23}$$
$$+ 84651840\left(c_2 + c_1 + c_1^2\left(4 + 3c_1\right) + 4c_1c_2 + c_3\right)x^7 - 105814800c_1\left(c_1\left(2 + 3c_1\right) + 2c_2\right)x^8$$
$$\left. - 36356880c_1\left(c_1\left(2 + 3c_1\right) + 2c_2\right)x^9 - 36176c_1\left(c_1\left(2 + 3c_1\right) + 2c_2\right)x^{11} + 31331c_1^3x^{12} + 8227c_1^3x^{13} + 5c_1^3x^{15}\right).$$

Substituting equations (16)–(23) into $w_i = \sum_{k=0}^{3} w_{ik}$, for $i = 1, 2$, one can get the third-order OHAM solution. The approximate solution contains auxiliary constants; using the technique discussed in Section 2, one can get the following values of the auxiliary constants.

	c_1	c_2	c_3
For $w_1(x)$	−1.0002105730135733	$1.62089016814963810 \times 10^{-7}$	0
For $w_2(x)$	−1.0001674339294635	$9.10706259849821210 \times 10^{-8}$	0

(24)

Using these constants, the third-order OHAM solution becomes

$$w_1(x) \approx 1 + x^2\left(1 + x\left(2.586682803444161 \times 10^{-11} + 2.586682803444161 \times 10^{-11}x - 2.177931206041872 \times 10^{-9}x^4\right.\right.$$
$$- 1.225086304103284 \times 10^{-9}x^5 - 3.024904453268861 \times 10^{-12}x^7 + 4.942292985147856 \times 10^{-9}x^8$$
$$+ 1.837519186786403 \times 10^{-9}x^9 + 2.08888122787343 \times 10^{-12}x^{11} - 3.085284629176499 \times 10^{-9}x^{12} \tag{25}$$
$$\left.\left. - 8.608495059086216 \times 10^{-10}x^{13} - 5.851526053235074 \times 10^{-13}x^{15}\right)\right),$$

$$w_2(x) \approx 1 + x - x^3 + 8.797558100309516 \times 10^{-12}x^4 + 8.797558100309516 \times 10^{-12}x^5$$
$$+ 8.378634873750055 \times 10^{-14}x^7 - 5.548306011675386 \times 10^{-10}x^8$$
$$- 2.730118834996797 \times 10^{-10}x^9 - 4.803728167284256 \times 10^{-13}x^{11}$$
$$+ 1.133324718039621 \times 10^{-9}x^{12} + 3.893987492756097 \times 10^{-10}x^{13} \tag{26}$$
$$+ 3.874614420651065 \times 10^{-13}x^{15} - 6.67932785302346 \times 10^{-10}x^{16}$$
$$- 1.753880509617439 \times 10^{-10}x^{17} - 1.06592956704597 \times 10^{-13}x^{19}.$$

3.2. Problem 2. Consider the following system of VIEs [10]:

$$w_1(x) = -3\cos(x) + 3 + \int_0^x \sin(x - s)w_1(s)ds - \int_0^x \cos(x - s)w_2(s)ds,$$
$$w_2(x) = e^x + \frac{1}{3}x^x - 1 - \int_0^x w_1(s)ds - \int_0^x e^{x-s}w_2(s)ds. \tag{27}$$

The system in equation (27) has exact solutions $w_1(x) = x^2$ and $w_2(x) = x$. Using the proposed algorithm, one can get the following third-order approximate solution and the auxiliary constants:

	c_1	c_2	c_3
For $w_1(x)$	-0.7995466949308285	0.002890722834173066	-0.001253980860449977
For $w_2(x)$	-0.8102088316012146	0.010500689249646119	0.0015000804520277254

(28)

$$w_1(x) = 27.09222491836092 + (-4.848664727375189 - 4.818444983672184x)x$$
$$+ e^x(1.068431841695209 + (0.3176884240001817 - 0.12778253683480223x)x)$$
$$+ (-28.160656760056128 + x(-1.1491588724698 + 1.10079844163505x))\cos(x)$$
$$+ (4.611703334149606 + x(-10.082823921491624 + (-0.03194563420870056 + 0.03194563420870056x)x))\sin(x),$$

(29)

$$w_2(x) = 6.376428434455357 + x(-4.892276735153841 + x(5.424128884377468 + 1.9640922450512805)x)$$
$$+ e^x - (11.610657907670205 + x(2.2942522881388965 + (0.1783715512035034 - 0.0886420248706046x)x)$$
$$+ (5.234229473214848 - 4.324561216828857x)\cos(x) + (19.533243571514003 + (0.5318521492236276$$
$$- 0.19944455595886038x)x)\sin(x).$$

(30)

3.3. Problem 3. Consider the following system of VIEs:

$$w_1(x) = x - \frac{x^5}{3} - \frac{x^4}{4} + \frac{x^3}{3} + \int_0^x (x^2 - s)w_1(s)\mathrm{d}s + \int_0^x (x^2 - s)w_2(s)\mathrm{d}s,$$

(31)

$$w_2(x) = x^2 - \frac{x^3}{2} - \frac{x^4}{3} + \int_0^x xw_1(s)\mathrm{d}s + \int_0^x xw_2(s)\mathrm{d}s, \quad x \in [0, 1].$$

The system in equation (31) has exact solutions $w_1(x) = x$ and $w_2(x) = x^2$. Using the proposed algorithm, one can get the following auxiliary constants and third-order approximate solution.

	c_1	c_2	c_3
For $w_1(x)$	-1.0483193039800733	0.000792340895704867	-0.0000437460995528
For $w_2(x)$	-1.0678469758951452	0.002263150079957279	-0.0002815239537518

(32)

$$w_1(x) = x\big(1. + x^2 - 0.00007771018797754492 + x(0.00005828264098329574 + x(0.0003818527204535183$$
$$+ x(0.0005069042207927407 + x(-0.0008216358845389711 + x(-0.001512942573949648$$
$$+ x(-0.0007978851119508088 + x(0.0018700147338272261 + x(0.0017717351539895954$$
$$+ x(0.00034944073528392453 + (-0.001148881532032241 - 0.0005926311682586243x)x))))))))),$$

(33)

$$w_2(x) = x^2\big(1. + x(0.0004504668772854076 + x(0.0003003112515236363 + x(-0.0008403580896928913$$
$$+ x(-0.0023530026511400376 + x(-0.0007914072605337691 + x(0.0020684412093087273$$
$$+ x(0.003660087185341051 + x(0.0011269789420038446 + x(-0.0012737677715320215$$
$$+ (-0.001792475116204039 - 0.0006263697932730246x)x)))))))))$$

(34)

TABLE 1: The exact solution and third-order OHAM solution for problem 1.

x	Exact $w_1(x)$	Exact $w_2(x)$	OHAM $w_1(x)$	OHAM $w_2(x)$
0	1	1	1	1
0.1	1.01	1.099	1.01	1.099
0.2	1.04	1.192	1.04	1.192
0.3	1.09	1.273	1.09	1.273
0.4	1.16	1.336	1.16	1.336
0.5	1.25	1.375	1.25	1.375
0.6	1.36	1.384	1.36	1.384
0.7	1.49	1.357	1.49	1.357
0.8	1.64	1.288	1.64	1.288
0.9	1.81	1.171	1.81	1.171
1	2	1	2	1

TABLE 2: The exact solution and third-order OHAM solution for problem 2.

x	Exact $w_1(x)$	Exact $w_2(x)$	OHAM $w_1(x)$	OHAM $w_2(x)$
0	0	0	-8.88×10^{-16}	0
0.1	0.01	0.1	0.010025	0.100038
0.2	0.04	0.2	0.040058	0.200077
0.3	0.09	0.3	0.090071	0.300055
0.4	0.16	0.4	0.160071	0.399968
0.5	0.25	0.5	0.250088	0.499851
0.6	0.36	0.6	0.360159	0.599761
0.7	0.49	0.7	0.490290	0.699745
0.8	0.64	0.8	0.640481	0.799798
0.9	0.81	0.9	0.810602	0.899817
1	1	1	1.000450	0.999532

TABLE 3: Comparison of absolute errors of relaxed Monte Carlo method (RMCM) [12] with $k = 8, h = 0.2$, and $N = 100$ and third-order OHAM.

x	RMCM [12] $w_1(x)$	RMCM [12] $w_2(x)$	OHAM $w_1(x)$	OHAM
0	0	0	0	0
0.1	3.52×10^{-10}	4.87×10^{-11}	2.79776×10^{-14}	8.88178×10^{-16}
0.2	1.10×10^{-8}	7.55×10^{-10}	2.17382×10^{-13}	1.53211×10^{-14}
0.3	1.47×10^{-9}	3.75×10^{-9}	3.61045×10^{-13}	5.15143×10^{-14}
0.4	1.86×10^{-7}	4.01×10^{-8}	1.81966×10^{-12}	9.83658×10^{-14}
0.5	6.55×10^{-7}	1.77×10^{-7}	1.41989×10^{-11}	1.56253×10^{-12}
0.6	4.45×10^{-7}	4.33×10^{-8}	5.23852×10^{-11}	7.48757×10^{-12}
0.7	1.38×10^{-7}	8.08×10^{-8}	1.29325×10^{-10}	2.25802×10^{-11}
0.8	1.53×10^{-5}	6.18×10^{-6}	2.20657×10^{-10}	4.67144×10^{-11}
0.9	6.38×10^{-5}	2.37×10^{-5}	2.58731×10^{-10}	6.76399×10^{-11}
1	2.09×10^{-5}	2.88×10^{-6}	5.19127×10^{-10}	1.3096×10^{-10}

TABLE 4: Comparison of absolute errors of hat basis functions (HBF) [10] with $n = 64$ and third-order OHAM.

x	HBF [10] $w_1(x)$	HBF [10] $w_2(x)$	OHAM $w_1(x)$	OHAM $w_2(x)$
0	0	0	8.8817×10^{-16}	0
0.1	7.700×10^{-4}	7.460×10^{-4}	2.51322×10^{-5}	3.75983×10^{-5}
0.2	1.434×10^{-3}	1.533×10^{-3}	5.85541×10^{-5}	7.68171×10^{-5}
0.3	2.054×10^{-3}	2.313×10^{-3}	7.12685×10^{-5}	5.54488×10^{-5}
0.4	2.641×10^{-3}	3.085×10^{-3}	7.10555×10^{-5}	3.19823×10^{-5}
0.5	3.103×10^{-3}	3.844×10^{-3}	8.84445×10^{-5}	1.48950×10^{-4}
0.6	3.647×10^{-3}	4.583×10^{-3}	1.58546×10^{-4}	2.38864×10^{-4}
0.7	4.089×10^{-3}	5.296×10^{-3}	2.98618×10^{-4}	2.55278×10^{-4}
0.8	4.535×10^{-3}	5.970×10^{-3}	4.81177×10^{-4}	2.01705×10^{-4}
0.9	4.998×10^{-3}	6.599×10^{-3}	6.02408×10^{-4}	1.82698×10^{-4}
1	5.390×10^{-3}	7.170×10^{-3}	4.45527×10^{-4}	4.68227×10^{-4}

TABLE 5: The exact solution and third-order OHAM solution for problem 3.

| x	Exact $w_1(x)$	OHAM $w_1(x)$	$w_1(x)$ Absolute errors	Exact $w_2(x)$	OHAM $w_2(x)$	$w_2(x)$ Absolute errors
0.	0	0	0.	0	0	0
0.1	0.1	0.0999999	6.76544×10^{-8}	0.01	0.010000	4.69687×10^{-7}
0.2	0.2	0.200000	3.88365×10^{-7}	0.04	0.040003	3.66185×10^{-6}
0.3	0.3	0.299999	6.09156×10^{-7}	0.09	0.090010	1.08757×10^{-5}
0.4	0.4	0.400000	2.25143×10^{-7}	0.16	0.160019	1.93236×10^{-5}
0.5	0.5	0.500002	2.49502×10^{-6}	0.25	0.250021	2.10611×10^{-5}
0.6	0.6	0.600004	4.18880×10^{-6}	0.36	0.360008	8.03933×10^{-6}
0.7	0.7	0.700002	1.61514×10^{-6}	0.49	0.489986	1.39203×10^{-5}
0.8	0.8	0.799997	3.36620×10^{-6}	0.64	0.639988	1.22626×10^{-5}
0.9	0.9	0.900002	2.19350×10^{-6}	0.81	0.810036	3.63494×10^{-5}
1	1	0.999987	1.34560×10^{-5}	1	0.999929	7.10952×10^{-5}
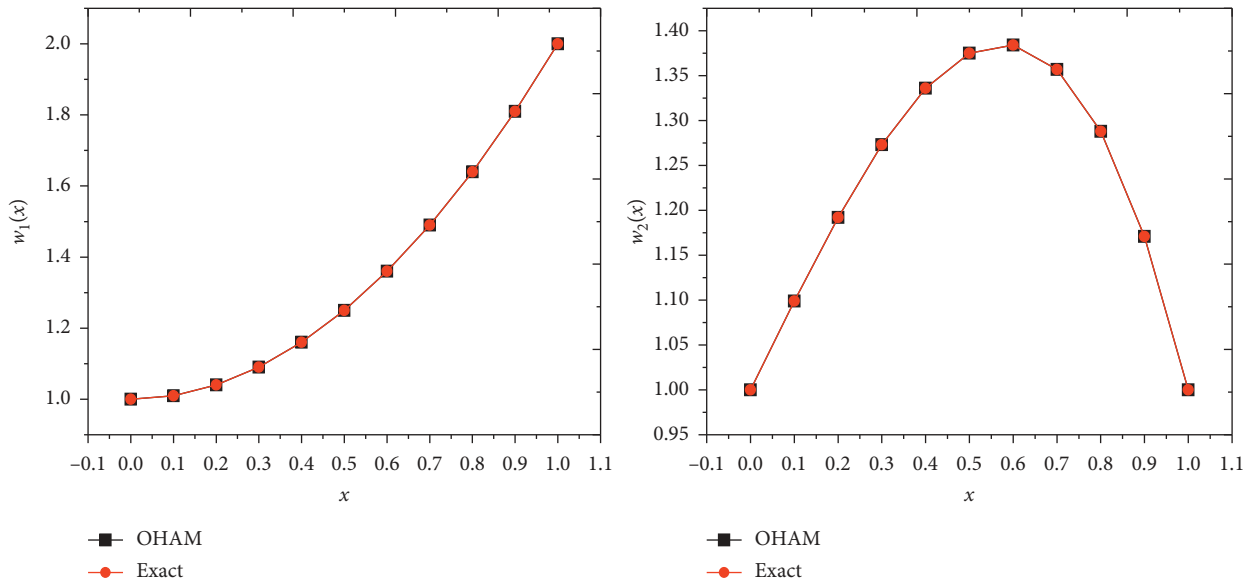
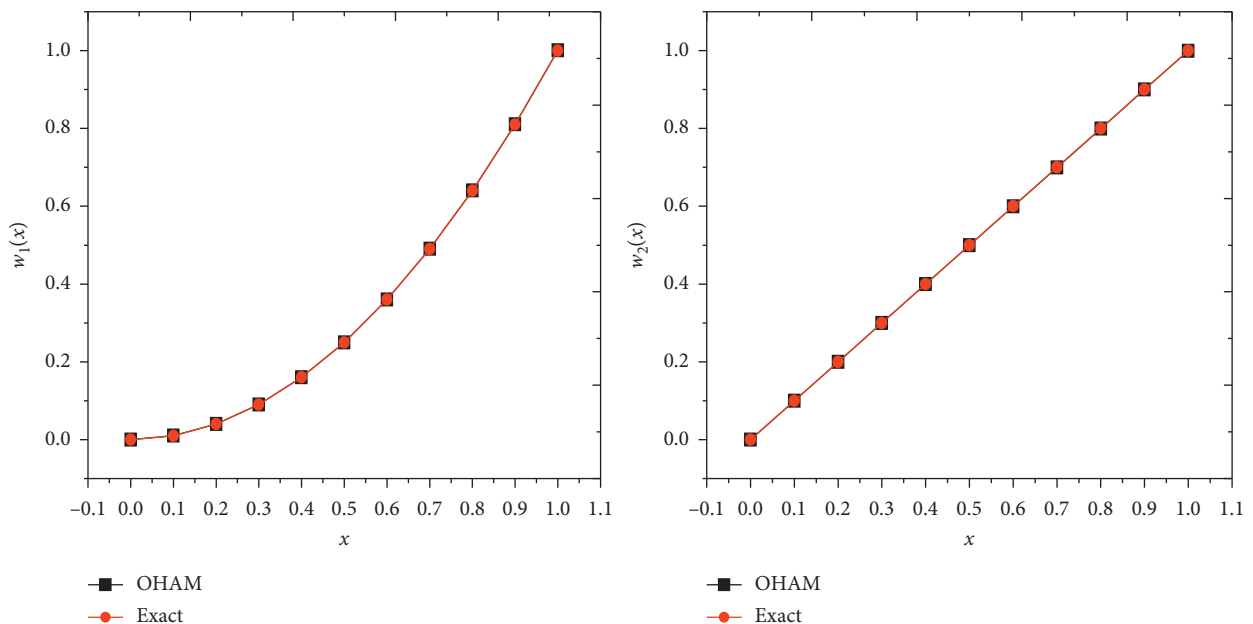FIGURE 1: Plots of exact and OHAM solutions for problem 1.

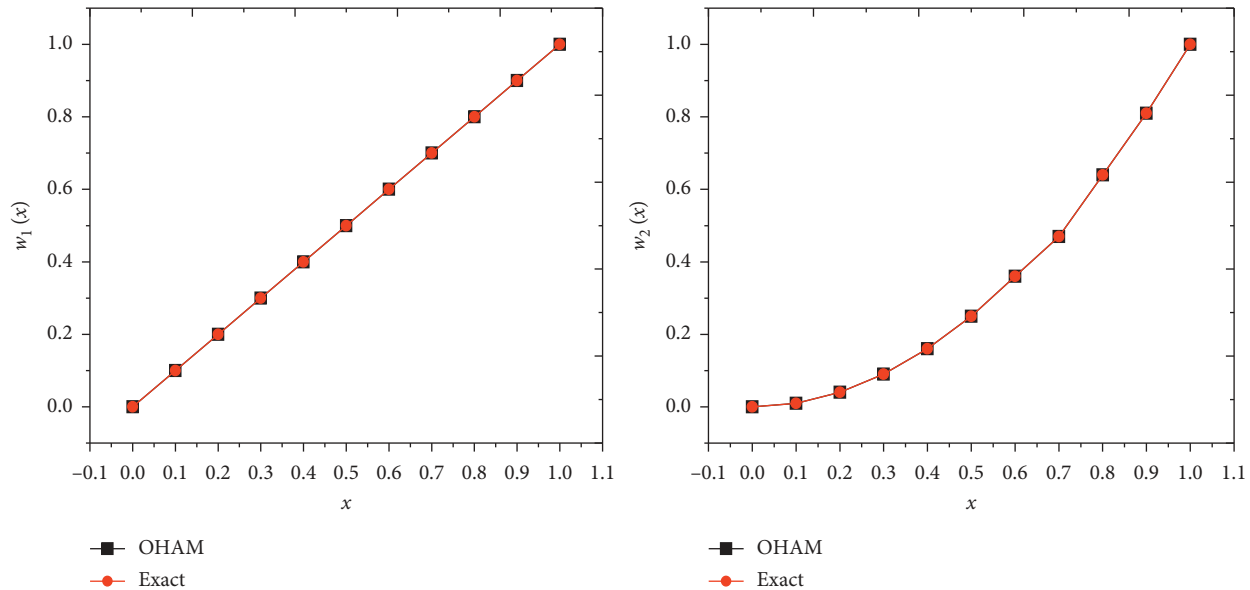FIGURE 2: Plots of exact and OHAM solutions for problem 2.

FIGURE 3: Plots of exact and OHAM solutions for problem 3.

4. Conclusions

Tables 1 and 2 show the exact and third-order OHAM solutions for problems 1 and 2, respectively. Third-order OHAM solutions for problem 1 are compared with the relaxed Monte Carlo method (RMCM) with $k = 8$, $h = 0.2$, and $N = 100$ in Table 3, while comparison of third-order OHAM solutions is made with hat basis functions (HBF) [10] with $n = 64$ in Table 4. The OHAM solution, exact solution, and absolute errors of OHAM for problem 3 are shown in Table 5. Figures 1–3 show the plot of OHAM and exact solutions for problems 1–3, respectively. The graphical representation and tables clearly show the reliability and consistency of the proposed method for the system of VIEs.

In this paper, the simple and easy algorithm of OHAM is successfully implemented to the system of VIEs. The obtained results confirmed the reliability of the proposed method. The accuracy of the result is a point of interest; moreover, if we increase the order of approximation, the results get closer to the exact solutions. The fast convergence and accuracy of the proposed technique are valid reasons for researcher to use the OHAM for different problems arising in various fields of science and technology.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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