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Analytical Solution of the Frenet-Serret Systems of Circular Motion Bodies

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Abstract

In this paper, the general Frenet-Serret system of circular motion body with constant velocity is analytically solved in three dimensional space. The tangent, normal, and binormal vectors are found by reducing the system into a high order ordinary differential equation. Solving this equation gives a closed form of those vectors. A special case of four dimensional Frenet-Serret system is also solved in this work.

Keywords: Frenet-Serret, high order ODE, Tangent, Normal, Binormal, Curvature, Torsion, circular orbits

1. Introduction

The Frenet-Serret frame is one of the most important tools that analyze and describe the properties of a particle along differentiable curves in Eucledian space [1,10].

Frenet and Serret [2] adapted the frame to curves by directly expressing the changes in derivatives of the tangent, normal and binormal vectors in terms of the frame. A few decades later, after the result of Frenet and Serret, their theory was extended to surfaces [3], also an n-dimensional vector calculus formulations of the system is developed [4]. Moreover, extensions to the frame have been proposed using quaternion-formulations [1]. In applications, studying the Frenet-Serret systems is of great importance in applied mathematics, physics, engineering and many fields of science [5-19].

One of the most important applications of the Frenet-Serret frames is understanding the kinematic properties of circular bodies, like the circular orbits in black hole space [10,11]. In this case, understanding the frame is useful in studying the properties of these orbits and provides interpretation of their geometry.

The general three dimensional Frenet-Serret system to be discussed in this paper is defined by:

$$\begin{bmatrix} T'(t) \\ U'(t) \\ V'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ U(t) \\ V(t) \end{bmatrix} \tag{1}$$

Where T, U, V are the tangent, normal and binormal vector fields respectively, t is the time, $\kappa_1 = T' \bullet U$, $\kappa_2 = T' \bullet V$, and τ is the torsion.

Studying of systems like (1) has been carried out in both analytical and numerical approaches as in [20-25]. This System will be analytically solved in this paper for bodies of circular motion with constant velocities.

2. Analysis and results

2.1 The Three Dimensional System

Considering a circular motion body with constant velocity leads to a constant curvature and torsion, hence, differentiating the third equation of (1) twice and differentiating the first and the second equations once with respect to time give

$$V''' = -\kappa_2 T'' - \tau U'' \tag{2}$$

$$T'' = \kappa_1 U' + \kappa_2 V' \tag{3}$$

and
$$U'' = -\kappa_1 T' + \tau V'. \tag{4}$$

Substituting (3) and (4) into (2) gives:

$$V''' = -\kappa_2(\kappa_1 U' + \kappa_2 V') - \tau(-\kappa_1 T' + \tau V') \tag{5}$$

which is written as:

$$V''' = \kappa_1 \tau T' - \kappa_1 \kappa_2 U' - (\kappa_2^2 + \tau^2) V' \tag{6}$$

But

$$T' = \kappa_1 U + \kappa_2 V \tag{7}$$

and

$$U' = -\kappa_1 T + \tau V. \tag{8}$$

Therefore, substituting (7) and (8) in (6) yields to

$$V''' = \kappa_1^2 \kappa_2 T + \kappa_1^2 \tau U + (-\kappa_2^2 - \tau^2) V' \tag{9}$$

hence,

$$V''' = \kappa_1^2(\kappa_2 T + \tau U) - (\kappa_2^2 + \tau^2) V' \tag{10}$$

but from (1)**,**

$$\kappa_2 T + \tau U = -V'. \tag{11}$$

Substituting (11) in (10) gives

$$V''' = -\kappa_1^2 V' - (\kappa_2^2 + \tau^2) V' \tag{12}$$

which is

$$V''' + (\kappa_1^2 + \kappa_2^2 + \tau^2) V' = 0 \tag{13}$$

The characteristic equation of the homogeneous ordinary differential equation (13) is

$$r^3 + (\kappa_1^2 + \kappa_2^2 + \tau^2) r = 0. \tag{14}$$

In addition to the trivial solution, The solution of (14) is $r = \pm i\sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2}$, where $i = \sqrt{-1}$, hence ,the solution of V is

$$V(t) = C_1 + C_2 \cos(\sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2})t + C_3 \sin(\sqrt{\kappa_1^2 + \kappa_2^2 + \tau^2})t \tag{15}$$

Now, as V is known , the following system has to be solved for T and U ,

$$\begin{aligned} T' &= \kappa_1 U + \kappa_2 V \\ U' &= -\kappa_1 T + \tau V \end{aligned} \tag{16}$$

From (16)

$$U'' = -\kappa_1 T' + \tau V' \tag{17}$$

Substituting the first equation of (16) into (17) gives

$$U'' = -\kappa_1^{\,2} U - \kappa_1\kappa_2 V + \tau V' \tag{18}$$

which is

$$U'' + \kappa_1^{\,2} U = F(t) \tag{19}$$

where

$$\begin{aligned} F(t) &= -\kappa_1\kappa_2[C_1 + C_2\cos(\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2})t + C_3\sin(\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2})t] \\ &+ \tau\,[-C_2\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2}\sin(\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2})t + C_3\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2}\cos(\sqrt{\kappa_1^{\,2}+\kappa_2^{\,2}+\tau^2})t] \end{aligned} \tag{20}$$

If $\alpha = \sqrt{\kappa_1^{\,2} + \kappa_1^{\,2} + \tau^2}$ **,** then

$$F(t) = -\kappa_1\kappa_2 C_1 + [\tau C_3\alpha - \kappa_1\kappa_2 C_2]\cos(\alpha t) - [\tau C_2\alpha + \kappa_1\kappa_2 C_3]\sin(\alpha t) \tag{21}$$

Using the variation of parameters method, the solution of (19) is

$$U = \frac{-\kappa_2 C_1}{\kappa_1} + A_2\cos\kappa_1 t + A_3\sin\kappa_1 t + \frac{\tau C_2\alpha + \kappa_1\kappa_2 C_3}{\kappa_2^{\,2}+\tau^2}\sin(\alpha t) + \frac{\kappa_1\kappa_2 C_2 - \tau C_3\alpha}{\kappa_2^{\,2}+\tau^2}\cos(\alpha t) \tag{22}$$

Now, as U and V are known, finding T is obvious by solving the first equation of (16).

2.2 The four Dimensional System

Consider the following well-known four dimensional Frenet-Serret system [4] :

$$\begin{bmatrix} T' \\ U' \\ V' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \\ W \end{bmatrix}. \tag{23}$$

It is clear that

$$U'' = -\kappa_1 T' + \kappa_2 V' \tag{24}$$

and

$$U^{(4)} = -\kappa_1 T''' + \kappa_2 V''' \tag{25}$$

So

$$U^{(4)} = -\kappa_1(\kappa_1 U'') + \kappa_2(-\kappa_2 U'' + \kappa_3 W'') \tag{26}$$

And hence,

$$U^{(4)} = -\kappa_1^{\,2} U'' - \kappa_2^{\,2} U'' + \kappa_2\kappa_3 W'' \tag{27}$$

But $W'' = -\kappa_3 V'$. Hence, from from (24),

$$V' = \frac{1}{\kappa_2}[U'' + \kappa_1 T'] \tag{28}$$

So

$$W'' = -\kappa_3[\frac{1}{\kappa_2}(U'' + \kappa_1 T')] \tag{29}$$

But $T' = \kappa_1 U$, so

$$W'' = -\frac{\kappa_3}{\kappa_2}[U'' + \kappa_1\kappa_1 U] \tag{30}$$

So

$$U^{(4)} = -\kappa_1{}^2 U'' - \kappa_2{}^2 U'' + \kappa_2\kappa_3(\frac{-\kappa_3}{\kappa_2}[U'' + \kappa_1{}^2 U]) \tag{31}$$

therefore

$$U^{(4)} + (\kappa_1{}^2 + \kappa_2{}^2 + \kappa_3{}^2)U'' + \kappa_1{}^2\kappa_3{}^2 U = 0 \tag{32}$$

Equation (32) is a homogeneous fourth order ordinary differential equations. Solving it for U makes the solution of system (23) obvious.

Conclusions and Future Perspectives

In this paper, the Frenet-Serret system (1) is efficiently reduced to a homogeneous third order ordinary differential equation which is solved for the binormal vector field. The normal vector field is obtained by solving a linear system of first order ordinary differential equations, while the tangent vector field can be found by solving a simple linear ordinary differential equation. A special case of four dimensional Frenet-Serret system when the torsion is *zero* has been analytically solved. As a next step, a circular motion bodies with non-constant velocities will be under consideration.

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