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Brief note

ANALYTICAL SOLUTION OF THE TIME FRACTIONAL FOKKER-PLANCK EQUATION

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A nonperturbative approximate analytic solution is derived for the time fractional Fokker-Planck (F-P) equation by using Adomian's Decomposition Method (ADM). The solution is expressed in terms of Mittag-Leffler function. The present method performs extremely well in terms of accuracy, efficiency and simplicity.

Key words: Fokker-Planck equation, Adomian Decomposition Method, fractional calculus, fractional differential equation.

1. Introduction

Recently, a great deal of interest has been focused on Adomian's Decomposition Method (ADM) and its applications to a wide class of physical problems containing fractional derivatives (Rida and Sherbiny, 2008; Saha Ray *et al*., 2008; Datta, 2007; Kaya, 2006; Jiang, 2005; Sutradhar, 2009). The decomposition method employed here is adequately discussed in the published literature (Wazwaz, 2002; Adomian, 1994; Ngarhasta *et al*. 2002), but it still deserves emphasis to point out the very significant advantages over other methods. The said method can also be an effective procedure for the solution of the time fractional Fokker-Planck equation by suitable choice of drift and fluctuation term.

The fractional differential equations have been used to model many physical and engineering processes such as frequency dependent damping behaviour of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation function for viscoelastic materials, etc. (Suarez and Shokooh, 1997; Glockle and Nonnenmacher, 1991). Moreover, phenomena in electromagnetics, acoustics, viscoplasticity, electrochemistry are also described by differential equations of fractional order (Podlubny, 1999; Shawagfeh, 2002).

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The Fokker-Planck (F-P) equation is one of the basic equations in the theory of stochastic processes (Markov process, for example) and has been the focus of many studies (Odibat and Momani, 2007). This parabolic differential equation describes the transition probability density function $f(x, t)$ and is given by Oksendal (2004).

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\{A(x)f\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\{B(x)f\} \tag{1.1}$$

where the coefficient A is called the drift term and the coefficient $B \geq 0$ is called the fluctuation term or the diffusion term.

In the present paper, we implemented the ADM to the time fractional F-P equation whose general form is given by

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\{B(x)f\} - \frac{\partial}{\partial x}\{A(x)f\}, \tag{1.2}$$

$$0 < a < 1, \qquad x \in R, \qquad t > 0$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional differential operator of order α (Podlubny, 1999; Oldham and Spanier, 1974). In these schemes, the solution is constructed in power series with easily computable components.

2. Mathematical aspects of fractional calculus

Many definitions of fractional calculus are used to solve the problems of fractional differential equations. The most frequently encountered definitions include the Riemann-Liouville, Caputo, Wely, Rize fractional operator. We introduced the following definitions (Podlubny, 1999; Oldham and Spanier, 1974).

Defn. 1: Let $\alpha \in R+$. The integral operator I^α defined on the usual Lebesgue space $L_1(a, b)$ by

$$I^\alpha f(x) = \frac{d^{-\alpha} f(x)}{dx^{-\alpha}} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \tag{2.1}$$

for $a \leq x \leq b$ is called the Riemann-Liouville fractional integral operator of order ($\alpha > 0$ and $\alpha \in R$).

Defn. 2: The Riemann-Liouville definition of fractional order derivative is

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-t)^{n-\alpha-1} f(t)dt \tag{2.2}$$

where n is an integer that satisfies $n-1 \leq \alpha < n$.

Defn. 3: A modified fractional differential operator D^α proposed by Caputo is given by

$$D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_o^x (x-t)^{n-\alpha-1} f^n(t)dt \tag{2.3}$$

where $\alpha(\alpha>0$ and $\alpha \in R)$ is the order of operation and n is an integer that satisfies $n-1 \leq \alpha < n$.

Defn. 4: A one-parameter function of the Mittag-Leffler type is defined by the series expansion (Podlubny, 1999; Oldham and Spanier, 1974; Miller and Ross, 1993).

$$E_B(x) = \sum_{r=0}^{\infty} \frac{x^r}{\Gamma(\beta r + 1)}, \qquad \beta > 0. \tag{2.4}$$

3. Analysis of the method

In this section, we consider Eq.(1.2) with the initial condition $f(x, 0) = \phi(x)$. The standard form of Eq.(1.2) in an operator form is

$$D_t^{\alpha}(f) = \frac{1}{2} L_{xx}\{B(x) f\} - L_x\{A(x) f\}, \tag{3.1}$$

$f(x,0) = \phi(x)$ where the operators are $L_{xx} = \frac{\partial^2}{\partial x^2}$, $L_x = \frac{\partial}{\partial x}$. Operating with I^{α} in both sides of Eq.(3.1) we get

$$f(x, t) = f(x, 0) + I^{\alpha}\left[\frac{1}{2} L_{xx}\{B(x) f\} - L_x\{A(x) f\}\right]. \tag{3.2}$$

The ADM assumes a series solution for $f(x, t)$ given by Wazwaz (2002), Adomian (1994), Datta (1993)

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t). \tag{3.3}$$

Substituting the decomposition series Eq.(3.3) into Eq.(3.2) yields

$$\sum_{n=0}^{\infty} f_n(x,t) = f(x,0) + I^{\alpha}\left[\frac{1}{2} L_{xx}\left\{B(x)\left(\sum_{n=0}^{\infty} f_n\right)\right\} - L_x\left\{A(x)\left(\sum_{n=0}^{\infty} f_n\right)\right\}\right]. \tag{3.4}$$

Identifying the zeros component $f_0(x, t)$ by $f(x, 0)$ the remaining components where $n \geq 0$ can be determined by using a recurrence relation (Wazwaz, 2002; Adomian, 1994; Adomian and Rach, 1993)

$$f_0(x,t) = f(x,0),$$

$$f_{n+1}(x,t) = I^{\alpha}\left[\frac{1}{2} L_{xx}\{B(x) f_n\} - L_x\{A(x) f_n\}\right], \qquad n \geq 0. \tag{3.5}$$

From this equation, the iterates are defined in the following recursive way (Wazwaz, 2002; Adomian, 1994; Datta 1993; Adomian and Rach, 1993)

$$f_0(x, t) = f(x, 0) = f_0(\text{say}),$$

$$f_1 = I^{\alpha}\left[\frac{1}{2}L_{xx}\left\{B(x)f_0\right\} - L_x\left\{A(x)f_0\right\}\right],$$

$$f_2 = I^{\alpha}\left[\frac{1}{2}L_{xx}\left\{B(x)f_1\right\} - L_x\left\{A(x)f_1\right\}\right], \tag{3.6}$$

$$f_3 = I^{\alpha}\left[\frac{1}{2}L_{xx}\left\{B(x)f_2\right\} - L_x\left\{A(x)f_2\right\}\right],$$

and so on.

Using the known f_0, all components $f_1, f_2, \ldots f_n$ etc. are determinable by using Eq.(3.6). Substituting these f_0, $f_1, f_2, \ldots$ etc. in Eq.(3.3) f is obtained.

Convergence of this method has been rigorously established by Cherruault (1989), Abbaoui and Cherruault (1994; 1995) and Himoun *et al*. (1999).

4. Application

Consider the time fractional F–P equation of order α with the initial condition $f(x, 0) = f_0 = x$. For simplicity, we take the drift term $A(x)$ equal to $(-x)$ and a constant fluctuation term $(2k)$. From the relations Eq.(3.6) we at once have

$$f_0 = f(x,0) = x,$$

$$f_1 = \frac{2xt^{\alpha}}{\Gamma(\alpha+1)},$$

$$f_2 = \frac{2^2 xt^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{3.7}$$

$$f_3 = \frac{2^3 xt^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$f_4 = \frac{2^4 xt^{4\alpha}}{\Gamma(4\alpha+1)},$$

and so on.

Substituting $f_0, f_1, f_2, f_3, \ldots$ into Eq.(3.3), we get the solution $f(x, t)$ in a series form by

$$f(x,\ t) = x\sum_{r=0}^{\infty}\frac{\left(2t^{\alpha}\right)^r}{\Gamma(r\alpha+1)} = xE_{\alpha}\left(2t^{\alpha}\right) \tag{3.8}$$

where $E_\alpha(x)$ is the Mittag-Leffler function defined by Eq.(2.4). Interestingly, as $\alpha \to 1$ we have from Eq.(3.8)

$$f(x,t) = xE_1(2t) = x\sum_{r=0}^{\infty}\frac{(2t)^r}{r!} = xe^{2t}. \tag{3.9}$$

Equation (3.9) is an exact solution to the standard form of Eq.(3.1) which can be easily verified through substitution. The decomposition series solutions generally converge very rapidly in real physical problems (Saha *et al*., 2008; Kaya, 2006; Wazwaz, 2002). It is also worth noting that a rapid stabilization to an acceptable accuracy is evident when numerical computation of the analytic approximation is carried out (Saha *et al*., 2008; Kaya, 2006; Ngarhasta *et al*., 2002; Shawagfeh, 2002).

5. Conclusions

The advantage of this global methodology lies in the fact that it not only leads to an analytical continuous approximation which is very rapidly convergent (Rida *et al*., 2008; Kaya, 2006; Wazwaz, 2002; Adomian and Rach, 1993), but also shows the dependence, giving insight into the character and behaviour of the solution just as in a closed form solution (Saha Ray *et al*., 2008; Kaya, 2006; Wazwaz, 2002; Suarez and Shokooh, 1997). The present analysis exhibits the applicability of the decomposition method to solve a fractional F-P equation. Furthermore, this method does not require any transformation techniques, linearization, discretization of the variables and it does not make closure approximation or smallness assumptions. Finally, we point out that if the conditions on one variable are better known than the others we consider the appropriate operator equation which can yield the solution without suffering traditional difficulty. This technique may be applied to the nonlinear partial differential equations such as the KdV equation, nonlinear Schrödinger equation, which will be considered in subsequent papers.

Nomenclature

A – drift term

B – fluctuation term

$\frac{d^{-q}}{dx^{-q}}$ – Riemann-Liouville integral operator of fractional order

$\frac{d^{q}}{dx^{q}}$ – Riemann-Liouville differential operator of fractional order

$E_B(x)$ – one parameter Mittag-Leffler function

$f(x,\ t)$ – transition probability density function

L_{xx} – second order derivative with respect to x

$\frac{\partial^\alpha}{\partial t^\alpha}$ – the fractional differential operator of order α

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