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Analytical solutions for fully developed laminar flow of some viscoelastic liquids with a Newtonian solvent contribution

D.O.A. Cruz [a], F.T. Pinho [b], P.J. Oliveira [c,*]

[a] *Depart. Eng. Mecânica, Universidade Federal do Pará-UFPa, Campus Universitário do Guamá, 66075-900 Pará, Brasil*
[b] *Centro de Estudos de Fenómenos de Transporte, DEM, Universidade do Minho, Campus de Azurém, 4800-058 Guimarães, Portugal*
[c] *Depart. Eng. Electromecânica, Unidade Materiais Têxteis e Papeleiros, Universidade da Beira Interior, 6201-001 Covilhã, Portugal*

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Abstract

We present analytical solutions for fully developed pipe and channel flows of two viscoelastic fluids possessing a Newtonian solvent, where the polymer contribution is either described by the Phan-Thien–Tanner (PTT) or FENE-P models. We derive in detail the pipe flow solution for the PTT fluid, and present the final solutions for the remaining three cases. This constitutes an important addition to existing results where the presence of a solvent with Newtonian characteristics has been consistently overlooked, as it posed considerable difficulties to the task of obtaining a closed form solution. In addition, interesting aspects of the solutions are discussed.

Keywords: Analytical solution; Pipe flow; Channel flow; Fully developed; PTT; Solvent viscosity; FENE-P; Viscoelastic

1. Introduction

The quest for analytical solutions of the most frequently used viscoelastic rheological models in relatively simple flows is, in our view, a matter of great importance but one which has been largely overlooked. As a justification for this statement we would like to highlight out the major relevance of the Poiseuille flow solution in the fluid mechanics of Newtonian fluids: it has a pedagogical motivation, appearing in all books of fluid mechanics as the typical example of a closed-form analytical solution to the Navier-Stokes equations; it is employed in practical experimental apparatus as a means to obtaining the viscosity of Newtonian liquids, for example in capillary-tube viscometers; and finally, it is often used to check numerical solutions, as a simple limiting test case, or to impose boundary conditions in the inlet or outlet of pipes and channels in complex geometries. A final point is that analytical solutions (when available) provide the simplest and most efficient way to perform parametric investigations of the effects of independent variables on output variables.

Similar motivations are valid when the fluid, instead of following the simple linear stress/strain relationship of the Newtonian model, follows more complex differential constitutive models typical of non-Newtonian media possessing viscoelasticity. In the early days of Rheology, when some of the constitutive equations still in use today were devised, like the type A and B fluids developed by Oldroyd [1], such a need was realised, but a relatively simple graphical procedure was proposed by Oldroyd [2] and later improved upon by Walters [3,4] to solve fully developed flows in pipes or channels for any fluid model, provided the steady viscosity function $\eta(\dot{\gamma})$ was known in terms of the shear rate $\dot{\gamma}$.

The indirect procedure devised by Walters [3] consisted in eliminating the shear stress from the constitutive and motion equations in order to obtain an explicit equation giving the radial position r as a function of the shear rate $\dot{\gamma}$. In this way the main independent variable is not r but the shear rate itself. Then by integrating the definition $\dot{\gamma} = \mathrm{d}u/\mathrm{d}r$ he was able to obtain the desired velocity profile $u(\boldsymbol{r})$ as $u = \int_{\dot{\gamma}_w}^{\dot{\gamma}} \dot{\gamma}(\mathrm{d}r/\mathrm{d}\dot{\gamma})\,\mathrm{d}\dot{\gamma}$, where the wall shear rate $\dot{\gamma}_w$ was taken as the main independent parameter. This approach can obviously be applied to both Generalised-Newtonian Fluids (GNF) and also to viscoelastic fluid models. In the latter case the viscosity function $\eta(\dot{\gamma})$ is not known a priori, but can be derived out for many models, leading to complex viscosity variations. No attempt was made to introduce such

* Corresponding author. Fax: +351 275329952.
E-mail addresses: doac@ufpa.br (D.O.A. Cruz), fpinho@dem.uminho.pt (F.T. Pinho), pjpo@ubi.pt (P.J. Oliveira).

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viscosity variations, arising from relevant differential viscoelastic models, into the above procedure and solve it analytically for the axial velocity component. An example of application of Walters' procedure is provided by the work of Van Schaftingen and Crochet [5] who derived an "implicit" solution for the Johnson/Segalman model having a solvent viscosity.

For many generalized Newtonian fluid models (GNF), analytical solutions for fully developed flows were derived during the 1950 and 1960s and are compiled in the book of Bird et al. [6]; these include the power-law model, the viscoplastic Bingham model, Casson's model, and other models for yield stress and non-yield stress fluids.

For differential viscoelastic fluid models, some work to obtain analytical solutions has been conducted in the past, for example the paper already referred to above by Van Schaftingen and Crochet [5] for the Johnson-Segalman fluid [7] and Yoo and Choi [8] and Schleiniger and Weinacht [9] for the Giesekus model [10]. However, it should be pointed out that exact, explicit analytical solutions were only possible when the Newtonian solvent contribution was neglected, while a constructive procedure, along the lines set forth by Walters, was deemed necessary for the non-zero solvent viscosity cases.

More recently, and as direct motivation to the present contribution, a number of analytical solutions [11,12] were derived for the affine version ($\xi = 0$) of the model developed by Phan-Thien and Tanner [13] whose constitutive equation is:

$$f(tr(\boldsymbol{\tau}))\cdot\boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} = \eta_p(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$$
$$(\text{with } f(\boldsymbol{\tau}) = 1 + \varepsilon\frac{\lambda}{\eta_p}tr(\tau)) \tag{1}$$

to be solved for the extra "polymeric" stress tensor $\boldsymbol{\tau}$. In this equation, $\overset{\nabla}{\boldsymbol{\tau}}$ denotes Oldroyd's upper convected derivative, $f(\boldsymbol{\tau})$ is a function of stress invariants which follows here the linear form proposed in the original paper, λ is the constant relaxation time, η_p is a constant parameter equal to the polymer contribution to the zero-shear rate viscosity, and ε is another constant parameter of the model related to its extensional properties. Generally, the lower the value of ε, the higher is the uniaxial extensional viscosity predicted by this PTT model; when ε tends to zero, the above equation reduces to the well-known Oldroyd-B model [1,6]. However, contrary to this model the PTT predicts a shear-thinning viscosity $\eta(\dot{\gamma})$ whose limit for vanishing shear rate is $\eta_0 = \eta_p + \eta_s$. The amount of Newtonian solvent contribution is therefore controlled by the constant solvent viscosity in the momentum equation:

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nabla\cdot\boldsymbol{\tau}_s + \nabla\cdot\boldsymbol{\tau} = -\nabla p + \eta_s\nabla\cdot\nabla\boldsymbol{u} + \nabla\cdot\boldsymbol{\tau} \tag{2}$$

measured by a non-dimensional solvent viscosity parameter $\beta = \eta_s/\eta_o$. In the present problem the acceleration $D\boldsymbol{u}/Dt$ is zero and the continuity equation,

$$\nabla\cdot\boldsymbol{u} = \boldsymbol{0} \tag{3}$$

will be satisfied identically by the assumed axial velocity distribution $u(\boldsymbol{r})$.

In conclusion, most exact closed-form analytical solutions of practical differential viscoelastic models in pipe flow, like the solution of Oliveira and Pinho [11], assume a vanishing solvent viscosity contribution. The objective of the present work is to derive an exact solution for the same problem, involving the affine-PTT and FENE-P models, but where the solvent viscosity is finite. It is also important to emphasize the point that the present analytical solutions can be viewed as pertaining to a viscoelastic liquid that is not necessarily a polymer solution, but one with a retardation time that would effectively take, in the current analysis, the role of the solvent viscosity, η_s.

2. Analytical derivation for the PTT model

We consider a fully developed flow in a circular-cross section pipe of radius R, in which the only non-zero velocity component is the axial component $u(\boldsymbol{r})$ depending solely on the radial coordinate $\boldsymbol{r}$. A generalisation for the planar case of the channel flow is given without detailed derivation at the end of this section. The axial momentum equation (from Eq. (2)) in cylindrical coordinates is the only relevant momentum equation,

$$0 = -p_z + \eta_s\frac{\partial}{r\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial}{r\partial r}(r\tau_{rz}) \tag{4}$$

and since the pressure gradient $p_z \equiv dp/dz$ is a constant, Eq. (4) can be integrated once to give:

$$0 = -\frac{p_z r}{2} + \eta_s\frac{du}{dr} + \tau_{rz} \tag{5}$$

The constitutive equations for the polymeric stress components are obtained from Eq. (1) and follow straight from our previous work [11]:

$$f(\tau_{zz})\cdot\tau_{rr} = 0 \Rightarrow \tau_{rr} = 0 \tag{6}$$

$$f(\tau_{zz})\cdot\tau_{zz} = 2\lambda\tau_{rz}\frac{du}{dr} \tag{7}$$

$$f(\tau_{zz})\cdot\tau_{rz} = \eta_p\frac{du}{dr} \tag{8}$$

with the trace of the stress tensor reduced only to $\tau_{kk} = \tau_{zz}$, so that:

$$f(\boldsymbol{\tau}) = 1 + \left(\frac{\varepsilon\lambda}{\eta_p}\right)\tau_{zz} \tag{9}$$

Still following our previous work, a direct relationship between the normal and the shear stress can be obtained by dividing Eq. (7) by Eq. (8), therefore eliminating the stress function $f(\tau)$:

$$\tau_{zz} = \frac{2\lambda}{\eta_p}\tau_{rz}^2 \tag{10}$$

The key point at this stage, and where the present derivation departs from that of the previous work, is to recognise that no further substitution of τ_{rz} in terms of $\boldsymbol{r}$ is required by invoking the momentum equation, but rather that Eq. (8) ought to be solved

as a cubic equation for the shear stress, after using (9) and (10):

$$\left(1+\left(\frac{\varepsilon\lambda}{\eta_p}\right)\frac{2\lambda}{\eta_p}\tau_{rz}^2\right)\cdot\tau_{rz}=\eta_p\frac{du}{dr}=\frac{\eta_p}{\eta_s}\left(\frac{p_z r}{2}-\tau_{rz}\right) \quad (11)$$

For the last equality on the right-hand side, the momentum Eq. (5) was expressed explicitly in terms of the local shear rate du/dr. After rearranging the various terms in (11) we arrive at the standard form for a cubic equation:

$$\tau_{rz}^3+a_1\tau_{rz}^2+a_2\tau_{rz}+a_3=0 \Leftrightarrow \tau_{rz}^3+3A\tau_{rz}-2B=0 \quad (12)$$

with ($a_1=0$, $a_2=3A$, $a_3=-2B$):

$$A=\frac{\eta_p^2}{6\varepsilon\lambda^2}\left(1+\frac{\eta_p}{\eta_s}\right) \quad \text{and} \quad B=\left(\frac{\eta_p^2}{4\varepsilon\lambda^2}\frac{\eta_p}{\eta_s}\frac{p_z}{2}\right)r\equiv Cr \quad (13)$$

The real solution to this cubic equation is:

$$\tau_{rz}=\sqrt[3]{B+\sqrt{A^3+B^2}}+\sqrt[3]{B-\sqrt{A^3+B^2}} \quad (14)$$

where it is noted that B is a function of the independent coordinate variable r, a feature made clearer by introducing the definition for C in Eq. (13). Since B and C are negative (because $p_z<0$) care should be exercised while evaluating the cubic root in the second term on the right-hand side of (14) in order to avoid the two conjugate imaginary solutions. The result of the first cubic root on the r.h.s. of (14) is always real.

Having obtained an explicit solution for the polymeric shear stress component, we may go back to the momentum Eq. (5) and re-write it as an equation for the shear rate:

$$\frac{du(r)}{dr}=\frac{p_z r}{2\eta_s}-\frac{1}{\eta_s}\tau_{rz}=\frac{p_z r}{2\eta_s}-\frac{1}{\eta_s}\left(\sqrt[3]{Cr+\sqrt{A^3+(Cr)^2}}+\sqrt[3]{Cr-\sqrt{A^3+(Cr)^2}}\right) \quad (15)$$

where A and C are constant parameters defined by Eqs. (13) which depend only on the basic material constants of the PTT model. Upon integration and imposition of the no-slip boundary condition (that is, $u(\boldsymbol{r})=0$ for $r=R$) we arrive, after some rather cumbersome manipulations whose details are omitted, to the following expression for the velocity profile in fully developed pipe flow:

$$u(r)=\left(-\frac{p_z R^2}{4\eta_s}\right)\left(1-\left(\frac{r}{R}\right)^2\right)+\frac{3}{8C\eta_s}\left\{-\left(CR+\sqrt{A^3+(CR)^2}\right)^{1/3}\left(-3CR+\sqrt{A^3+(CR)^2}\right)\right\}$$
$$+\frac{3}{8C\eta_s}\left\{\left(Cr+\sqrt{A^3+(Cr)^2}\right)^{1/3}\left(-3Cr+\sqrt{A^3+(Cr)^2}\right)\right\}$$
$$+\frac{3}{8C\eta_s}\left\{\left(CR-\sqrt{A^3+(CR)^2}\right)^{1/3}\left(+3CR+\sqrt{A^3+(CR)^2}\right)\right\}$$
$$-\frac{3}{8C\eta_s}\left\{\left(Cr-\sqrt{A^3+(Cr)^2}\right)^{1/3}\left(+3Cr+\sqrt{A^3+(Cr)^2}\right)\right\} \quad (16)$$

This long expression can be written under a compact form with help of some function definitions (rather like statement functions). If we define:

$$F^{\pm}(X)=\left(CX\pm\sqrt{A^3+(CX)^2}\right)^{1/3} \quad \text{and}$$
$$G^{\pm}(X)=3CX\pm\sqrt{A^3+(CX)^2} \quad (17)$$

then we may write (16) as:

$$u(r)=(2U_N/\beta)\left(1-\left(\frac{r}{R}\right)^2\right)+\frac{3}{8C\eta_s}\{F^+(R)G^-(R)-F^+(r)G^-(r)+F^-(R)G^+(R)-F^-(r)G^+(r)\} \quad (18)$$

where we chose to define a characteristic velocity as:

$$U_N=\frac{-p_z R^2}{8\eta_0} \quad (19)$$

The first term in either Eq. (16) or Eq. (18) is clearly the parabolic velocity variation due to the Newtonian solvent contribution and therefore U_N has the meaning of the average velocity for the pipe flow of a Newtonian fluid subjected to the same pressure gradient $-p_z$.

At this point we are in possession of an exact explicit solution for the problem of PTT pipe flow, including a Newtonian solvent contribution, when the pressure gradient is given:

- the velocity profile $u(\boldsymbol{r})$ is given by Eq. (16);
- the shear stress due to the polymer $\tau_{rz}(\boldsymbol{r})$ is given by Eq. (14);
- and the axial normal stress $\tau_{zz}(\boldsymbol{r})$ is determined from Eq. (10).

If required, the total shear stress can be easily obtained by summing the solvent contribution, $\eta_s du/dr$ with du/dr from Eq. (15), to the polymer stress τ_{rz}. In addition, the local shear viscosity can be calculated directly from its definition:

$$\eta=\eta_s+\frac{\tau_{rz}}{(du/dr)} \quad (20)$$

Another quantity of interest is the average or bulk velocity which can be obtained by integrating the velocity distribution over the pipe cross-section:

$$U \equiv \frac{1}{\pi R^2}\int_0^R (2\pi r)u(r)\,\mathrm{d}r = U_1 + U_2 + U_3 + U_4 \tag{21}$$

With $u(\boldsymbol{r})$ given by Eq. (18), we obtain the following four contributions to the average velocity:

$$U_1 = \frac{-p_z R^2}{8\eta_s},$$

$$U_2 = \frac{3}{8\eta_s C}\{F^+(R)G^-(R) + F^-(R)G^+(R)\},$$

$$U_3 = \frac{9}{320R^2\eta_s C^3}\left\{F^+(R)\left[3\sqrt{A^3+B_R^2}(A^3+4B_R^2) - B_R(A^3+28B_R^2)\right] - 3A^5\right\},$$

$$U_4 = \frac{-9}{320R^2\eta_s C^3}\left\{F^-(R)\left[3\sqrt{A^3+B_R^2}(A^3+4B_R^2) + B_R(A^3+28B_R^2)\right] + 3A^5\right\} \tag{22}$$

with $F^{\pm}$ being the function defined in Eq. (17) and here $B_R = CR$ according to Eq. (13).

2.1. Solution for channel flow

The solution for the planar (x, y) geometry of fully developed channel flow aligned with x can be derived following the same procedure and the final result for the velocity profile is:

$$u(y) = \left(-\frac{p_x H^2}{2\eta_s}\right)\left(1 - \left(\frac{y}{H}\right)^2\right) + \frac{3}{8\eta_s C}\{\cdots\} \tag{23}$$

with the term inside the curled brackets exactly like the corresponding one in Eq. (16), except that the lateral coordinate is now y, instead of r, and the half-width of the channel is H, instead of R. The polymeric shear stress τ_{xy} is given by an expression like (14) with the same definition for A as in (13), while B and C take a slightly different definition:

$$B \equiv Cy \quad \text{and} \quad C \equiv \frac{\eta_p^2}{4\varepsilon\lambda^2}\frac{\eta_p}{\eta_s}p_x \tag{24}$$

The solution for the axial normal stress τ_{xx} follows exactly from the equivalent to Eq. (10) and the shear rate is given by Eq. (15), with proper changes, except that there is no 1/2 in the first term containing the pressure gradient. Finally, the average velocity follows from the equivalent to Eq. (21):

$$U \equiv \frac{1}{H}\int_0^H u(y)\,\mathrm{d}y = U_1 + U_2 + U_3 + U_4 \tag{25}$$

while U_1, U_2, U_3 and U_4 follow expressions similar to (22) with the necessary modifications:

$$U_1 = \frac{-p_x H^2}{3\eta_s},$$

$$U_2 = \frac{3}{8\eta_s C}\{F^+(H)G^-(H) + F^-(H)G^+(H)\},$$

$$U_3 = \frac{9}{280H^2\eta_s C^3}\left\{F^+(H)\left[8A^3 + B_H(-19B_H + 9\sqrt{A^3+B_H^2})\right] - 8A^{7/2}\right\},$$

$$U_4 = \frac{9}{280H^2\eta_s C^3}F^-(H)\left[8A^3 + B_H(19B_H - 9\sqrt{A^3+B_H^2})\right] + 8A^{7/2} \tag{26}$$

where $B_H = CH$ with C from Eq. (24)

3. Solution for the FENE-P model

In this section we deal with the analytical solution for the FENE-P model [14], another very popular differential constitutive equation often employed in computational rheology. While the PTT model was derived from network theory and is therefore more appropriate for melts, the FENE-P model was rooted in kinetic theory and was initially developed to represent the behaviour of dilute polymer solutions. In the FENE-P model a molecule is represented by a single dumbbell, whose connector follows a non-linear spring law possessing limited extension, without consideration for excluded volume effects and hydrodynamic interaction, and the resulting constitutive equation for the polymer stress can be written as [14,15]:

$$Z(tr(\boldsymbol{\tau}))\cdot\boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} - \lambda\left(\boldsymbol{\tau} - \frac{b}{b+2}nkT\boldsymbol{I}\right)\frac{\mathrm{D}\ln Z}{\mathrm{D}t} = \frac{b}{b+2}nkT\lambda(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}}) \tag{27}$$

where Z is a function of the trace of the stress tensor,

$$Z = 1 + \frac{3}{b}\left(\frac{b}{b+2} + \frac{tr(\boldsymbol{\tau})}{3nkT}\right) \tag{28}$$

b and nkT are parameters of the model, and we note that the improvements introduced by Bird et al. ([15], pp. 88–91) are already accounted for. A Newtonian solvent contribution must be added to τ, exactly as in Eq. (2) above, in order to obtain the total extra stress acting on a fluid element. In Eqs. (27) and (28) the parameter b measures the extensibility of the dumbbell and the zero-shear rate polymer viscosity is given by $\eta_p = nkT\lambda b/(b+5)$. In spite of their quite different origins, the final equations for the two models, Eq. (1) for the PTT and (27) for the FENE-P, do show already some similarities (the recent paper of Tanner and Nasseri [16], where similarities among various constitutive equations are discussed, is highly recommended in this respect). In the case of the fully developed rectilinear flows considered here, those similarities are even more striking leading to an exact

equivalence in the sense of a parameter to parameter match, as found by Oliveira [17] who gives the analytical solution for slit and pipe flow of the FENE-P model *without* a solvent viscosity. Fully developed conditions imply that DZ/Dt = 0 and in this case the constitutive Eq. (27) for each stress component becomes:

$$Z(\tau_{zz}) \cdot \tau_{zz} = 2\lambda\tau_{rz}\frac{\mathrm{d}u}{\mathrm{d}r} \tag{29}$$

$$Z(\tau_{zz}) \cdot \tau_{rz} = \left(\frac{b+5}{b+2}\right)\eta_\mathrm{p}\frac{\mathrm{d}u}{\mathrm{d}r} \tag{30}$$

together with $\tau_{rr} = 0$, giving:

$$Z(\tau_{zz}) = 1 + \frac{3}{b+2}\left(1 + \frac{(b+2)\lambda\tau_{zz}}{3(b+5)\eta_\mathrm{p}}\right) = \left(\frac{b+5}{b+2}\right)\left(1 + \frac{(b+2)\lambda\tau_{zz}}{3(b+5)^2\eta_\mathrm{p}}\right) \tag{31}$$

Those equations are to be compared with Eqs. (7), (8), and (9), respectively, for the affine-PTT fluid. Since the momentum equation remains unchanged (Eq. (4)) it is straightforward to show, by equating the last term of (31) to Eq. (9), that the results of Section 2 remain valid provided the following substitutions are made:

$$f \to \left(\frac{b+2}{b+5}\right)Z; \quad \lambda \to \left(\frac{b+2}{b+5}\right)\lambda;$$

$$\varepsilon \to \frac{1}{b+5}; \quad \eta_\mathrm{p} \to \eta_\mathrm{p} \tag{32}$$

These relations thus define an equivalence between FENE-P and PTT results, allowing for the same expressions to be employed with both models.

4. Results

We shall concentrate here on results for the PTT model. Oliveira and Pinho [11] have already studied with some detail the influence of elasticity, as measured by either a Deborah number De or the group $\sqrt{\varepsilon}De$, on the main characteristics of the PTT pipe flow solution. Therefore, the interest here is to seek the influence of the solvent viscosity ratio β, which may vary between $\beta = 0$ (PTT without solvent viscosity; solution given by Oliveira and Pinho [11]) and $\beta = 1$ (Newtonian fluid), on the velocity and stress profiles. As noted above two Deborah numbers may be formed, depending on the value chosen for the velocity scale: $De_\mathrm{N} = \lambda U_\mathrm{N}/R$ with $U_\mathrm{N} = -p_z R^2/8\eta_0$ based on the primary variables; and $De = \lambda U/R$ with U being the average velocity which can only be determined after knowing the basic solution.

Fig. 1 shows profiles of the velocity distribution, scaled with U_N, for increasing values of β at constant $\varepsilon = 0.25$ and $De_\mathrm{N} = 1$. The interest is to verify that the proper limits are reproduced by the present solution. For $\beta = 1$ the Newtonian solution shown by the round symbols in Fig. 1 is recovered; in fact, the solid line corresponds to Eq. (16) with $\beta = 0.9999$. For β tending to zero the solution of Oliveira and Pinho [11] is recovered, shown by the square symbols marked with $\beta = 0$. Eq. (16) with $\beta = 0.001$ gives a profile in close agreement with the result of Oliveira and Pinho but it is remarkable that a small discrepancy for such a low solvent viscosity contribution is still visible. For a smaller β, Eq. (16) agrees very well with Oliveira and Pinho, thus indirectly confirming the correctness of the present solution. Otherwise, Fig. 1 shows that the flow rate in the pipe rises considerably when the concentration of the viscoelastic polymer increases, on account of strong shear-thinning (an effect similar to drag reduction, but here due to shear-thinning in laminar flow). With polymer solutions, a concentration parameter is usually defined as $c = \eta_\mathrm{P}/\eta_\mathrm{S}$ (e.g. [18]) giving $c = (1/\beta) - 1$.

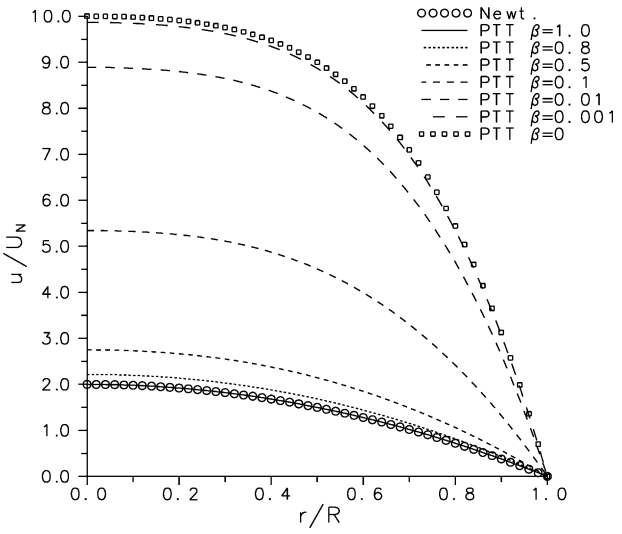

Fig. 1. Solution in terms of velocity profiles and influence of the viscosity ratio $\beta = \eta_\mathrm{s}/\eta_0$ for $\varepsilon = 0.25$ and $De_\mathrm{N} = 1$. Note: $U_\mathrm{N} = (-\mathrm{d}p/\mathrm{d}z)R^2/8\eta_0$.

Corresponding profiles for the polymeric stress contribution are shown in Fig. 2(a), for the axial normal component τ_{zz}, and Fig. 2(b), for the tangential component τ_{zr}. Stresses are here normalised with the stress scale $\eta_\mathrm{p}U_\mathrm{N}/R$, appropriate for the polymeric stresses, and as expected those stress components increase when β decreases, meaning higher polymer concentration. The solution of Oliveira and Pinho [11], shown by the symbols with $\beta = 0$, is approached for very low values of β; it is worth noting that in that work the stress scale was taken as $4\eta_\mathrm{p}U/R$ giving a unit shear stress value at the wall, for $r = R$. With the stress scale chosen in terms of the primary velocity scale based directly on the imposed pressure gradient, both stress components increase, in absolute value, when β decreases; with $\beta = 1$ the polymeric stresses vanish.

When the velocity is normalised with the average velocity in the pipe for the conditions studied here, then the level of discrepancy between results at low β (say $\beta = 0.01$) and the solution of Oliveira and Pinho seems less accentuated, as shown by the velocity profiles in Fig. 3 for $De_\mathrm{N} = 1$ ($De = 6.3$) and $\varepsilon = 0.25$. U is given by Eqs. (21) and (22) and could also be calculated by numerical integration of the velocity profile with Simpson's rule, following the definition (21); it was found that values of U obtained with these two methods agreed very closely (errors less than 0.04–0.4%), but for very small β (say $\beta < 0.0001$) Eqs. (21) and (22) give erroneous answers due to approximation errors generated by subtractive cancellation (even when double precision arithmetic is used to evaluate those equations).

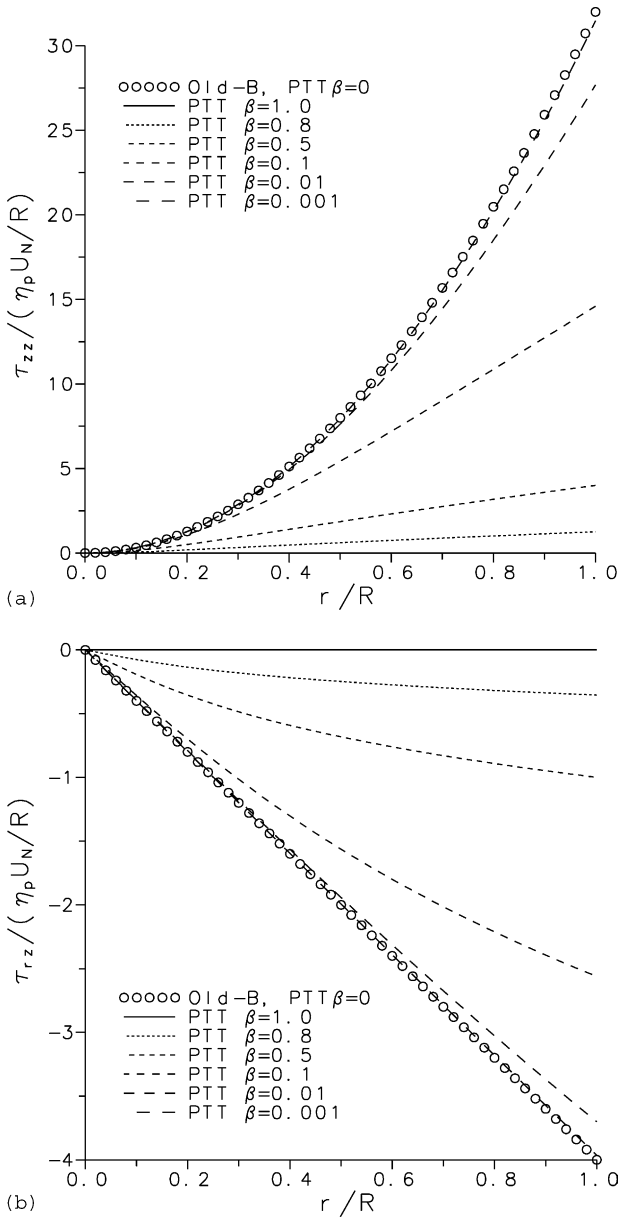

Fig. 2. Normal (a) and tangential (b) polymer stress components, for $\varepsilon = 0.25$ and $De_N = 1$.

The ratio U_N/U of the Newtonian average velocity for the given pressure gradient and the actual average velocity is a useful quantity to evaluate, not only because it gives a direct relation between the two Deborah numbers at play De_N/De, but also because it represents a non-dimensional pressure drop being directly proportional to the Fanning friction coefficient, defined in the usual way:

$$fRe = 16\frac{U_N}{U} \tag{33}$$

(see [11]). Fig. 4 shows the variation of U_N/U with the group $\sqrt{\varepsilon}De$, having β as a parameter. Eqs. (13)–(16) all show that the solution depends on $\sqrt{\varepsilon}De$ (or, equivalently, on $\sqrt{\varepsilon}De_N$) not on ε and De separately. Results of Oliveira and Pinho valid for $\beta = 0$ are shown by circular symbols in Fig. 4, offering a useful check

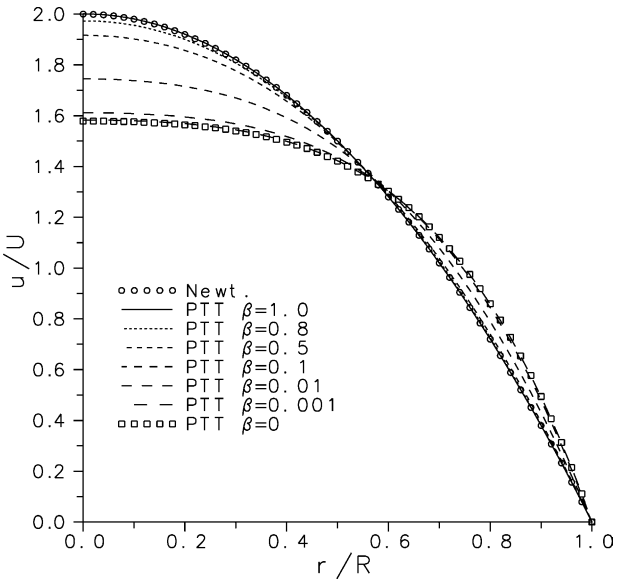

Fig. 3. Analytical solution in terms of velocity profiles scaled with the average velocity.

for the lower limiting conditions $\beta \to 0$. There is a very quick initial decrease of U_N/U due to shear thinning, followed by a gradual levelling out at higher εDe when "elastic" effects have saturated.

The presence of a non-zero solvent viscosity affects primarily and directly the shear viscosity function of the PTT model and so it seems relevant to look at the radial variation of shear viscosity and shear rate across the pipe section. Profiles of the shear viscosity are shown in Fig. 5, with η from Eq. (20), while the absolute shear rate $\dot{\gamma} \equiv |\mathrm{d}u(r)/\mathrm{d}r|$ can be obtained from Eq. (15). When β is nonzero, η becomes limited from below by η_s,

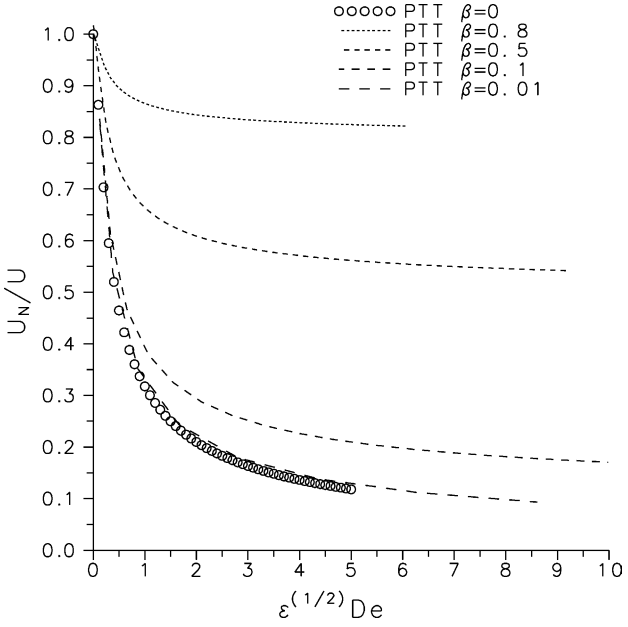

Fig. 4. Variation of the ratio between the two velocity scales with elasticity, having $\beta = \eta_s/\eta_0$ as a parameter.

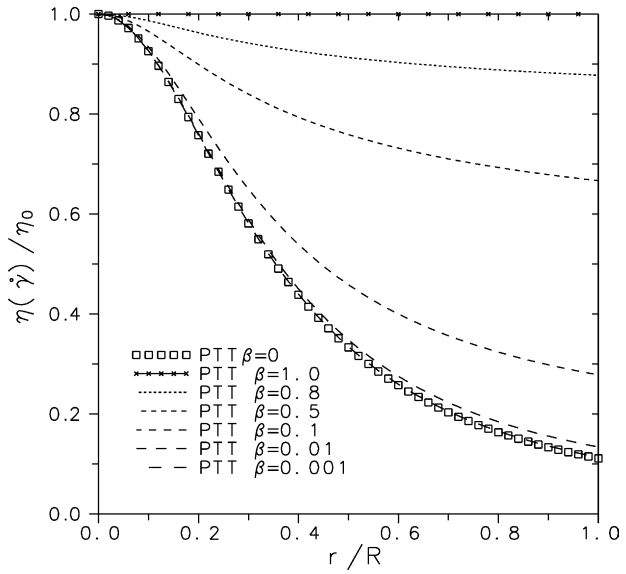

Fig. 5. Radial profiles of the shear viscosity.

a limitation occurring in regions of high shear rate. Hence the profiles of η should be accordingly modified, with a tendency towards uniformity across the pipe as β is raised; such behaviour is well illustrated by Fig. 5. The absolute values of the shear rate (whose variation is not shown) increase monotonically from zero at the centre, to a maximum at the wall, and are greatly reduced when β is increased (values of $\dot{\gamma}$ at the wall increasing from 4 for $\beta = 1$, to 35.2 for $\beta = 0.001$).

A final check of the present analytical solution is provided by plotting the data from Fig. 5 as η versus $\dot{\gamma}$, where $\dot{\gamma}$ is the shear rate at position r (Eq. (15)), and comparing the resulting variation with the known shear viscosity function of the PTT model. Such comparison is valid because steady pipe flow is

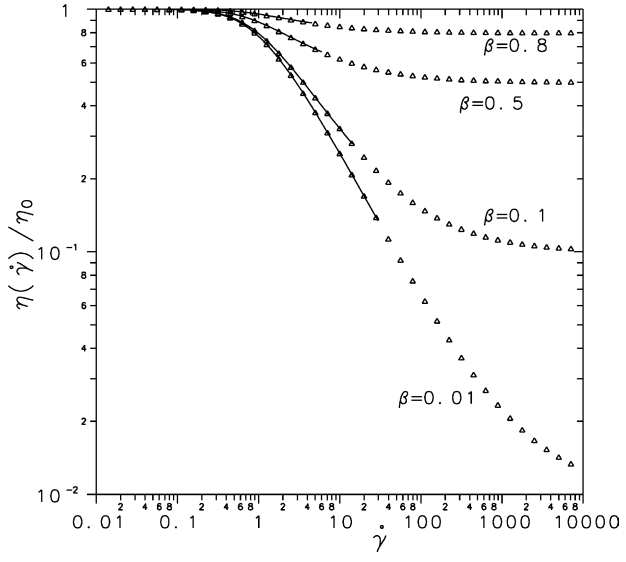

Fig. 6. Shear viscosity at a given radial position plotted as a function of the local shear rate (lines) and comparison with the viscometric viscosity function for the PTT (symbols).

essentially a shear flow with a varying local rate of shear $\dot{\gamma}(r) = |\mathrm{d}u(r)/\mathrm{d}r|$. The shear viscosity function is given by many authors (for example Oliveira and Pinho):

$$\eta(\dot{\gamma}) = \eta_s + \frac{\eta_p}{1 + (\alpha - 1)^2/3\alpha} \quad (\alpha \equiv (\theta + \sqrt{\theta^2 - 1})^{1/3};$$

$$\theta \equiv 1 + 27\varepsilon(\lambda\dot{\gamma})^2) \tag{34}$$

and Fig. 6 shows the resulting plot. In this figure, the symbols pertain to the material function above (Eq. (34)) and the lines to the present analytical solution. A perfect agreement is observed with the plot in log-log scale showing more clearly the usual viscosity decay, limited by the solvent viscosity, of a standard material function representation of $\log(\eta)$ versus $\log(\dot{\gamma})$.

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